# Data Structures and Algorithms

## Week 10 - More on SCCs, Topological Sort

**Subodh Sharma and Rahul Garg**

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# Strongly Connected Components

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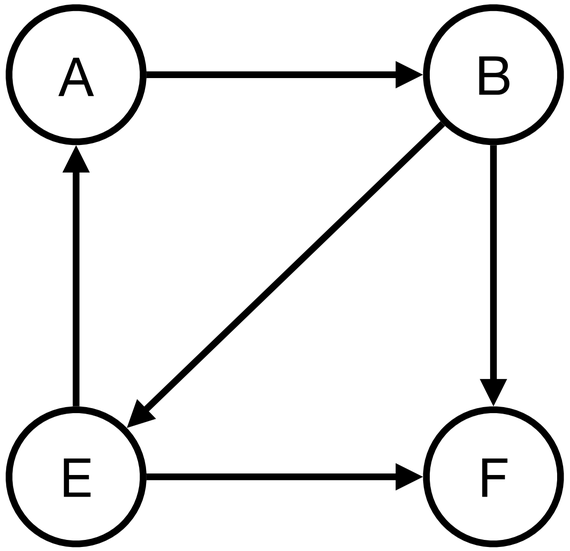
- A graph is said to be strongly connected - If every vertex is reachable from every other vertex
- The binary relation of being strongly connected is an **equivalence relation**
  - That is it is reflexive, symmetric and transitive
- Strongly connected component of a directed graph G is also **maximal**
- **Used in Abstractions!** SCCs in a graph can be **condensed** into single vertices leading to the formation of a **DAG**

# Strongly Connected Components



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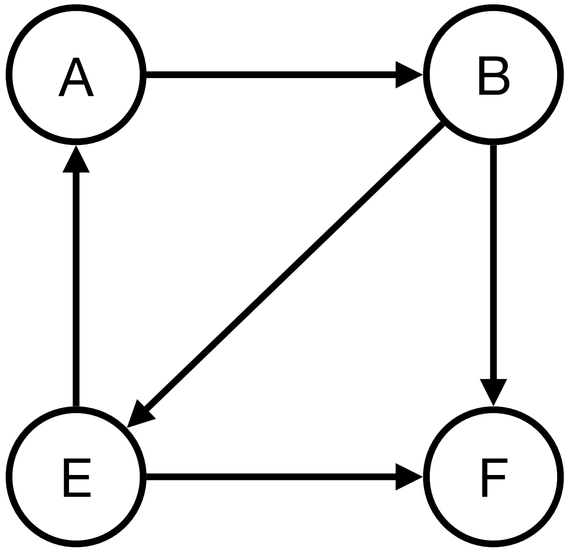


- SCC:  ({A,B,E}. {() .. })
- Use of DFS to find SCCs — Robert Tarjan 1972 (also discovered Splay and Fibonacci Heaps)

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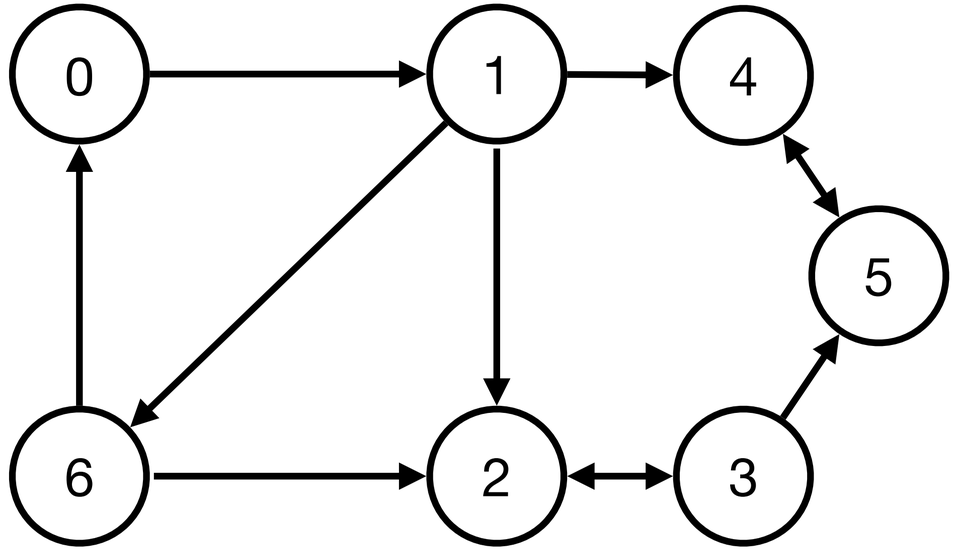
## Tarjan's Algorithm



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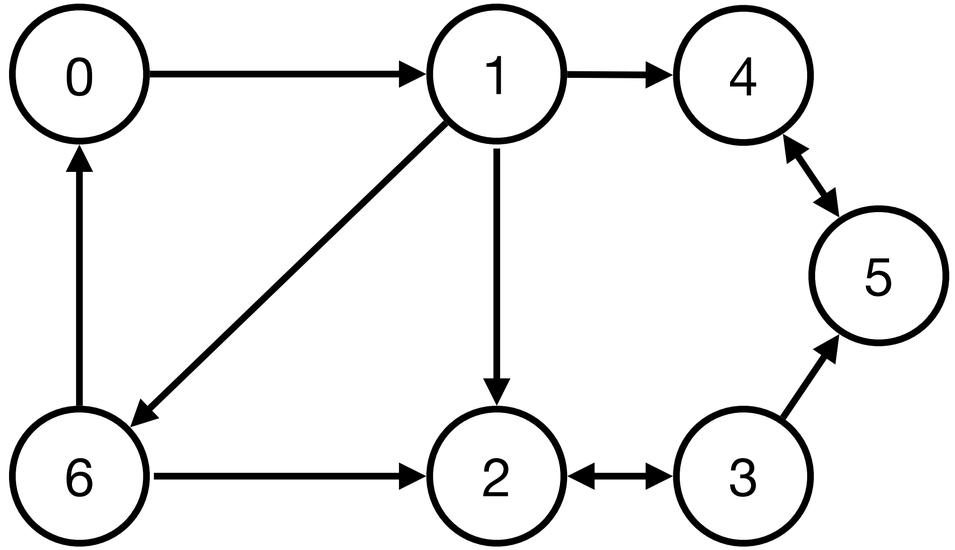

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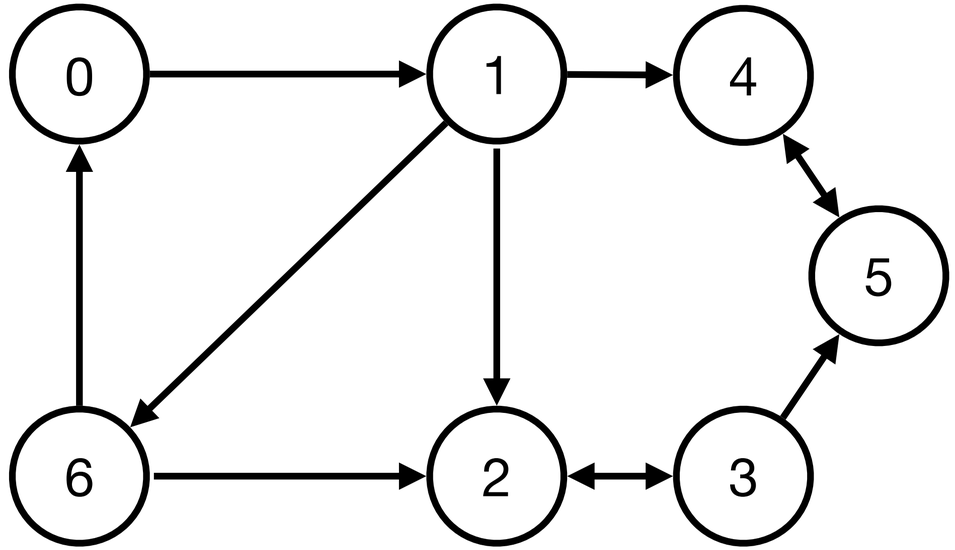

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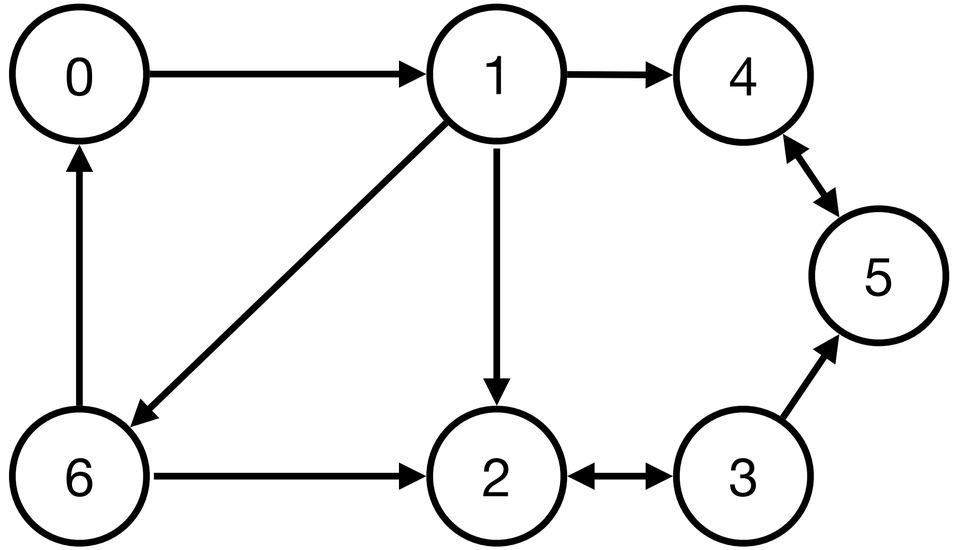
  - **Output:** subgraph with vertices of SCC



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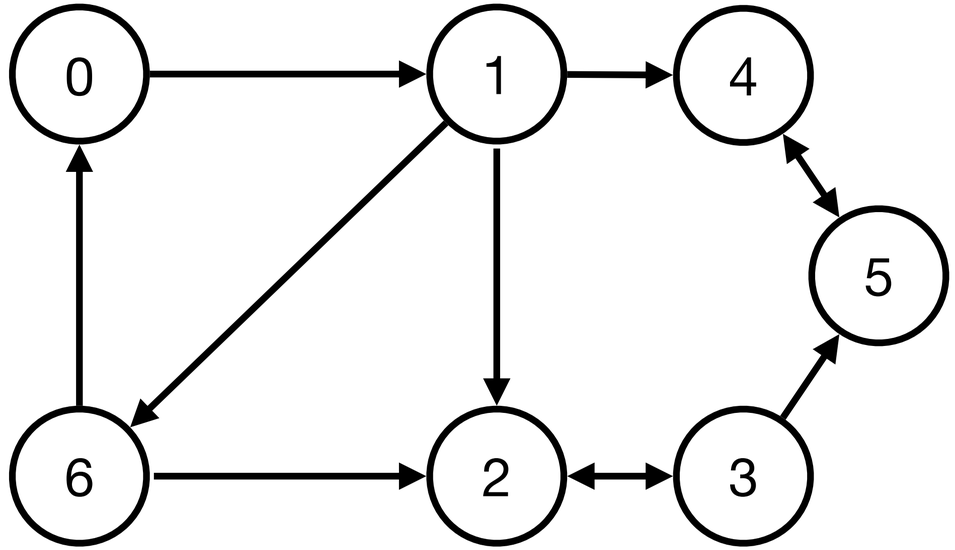
  - Each vertex appears in exactly one SCC of the graph



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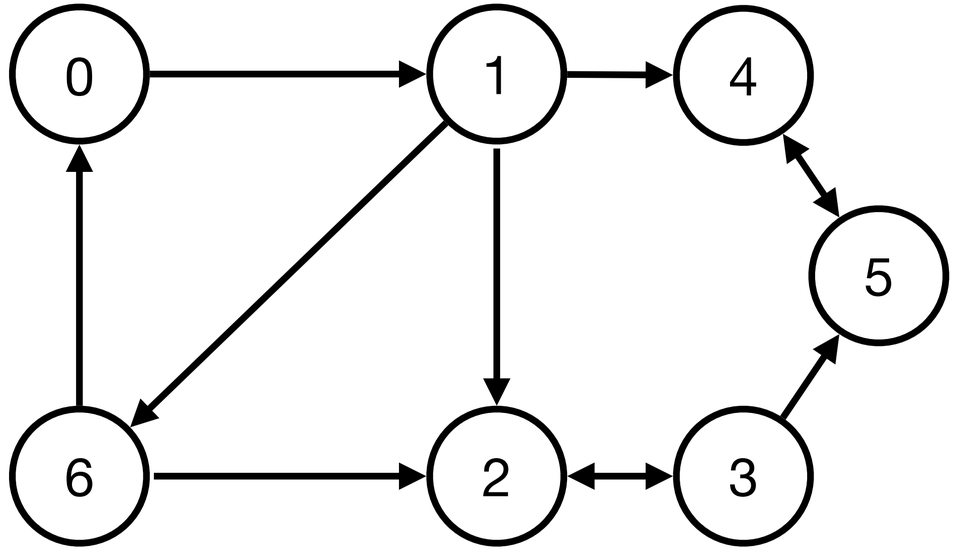


- **Key Observations:**
  - **Input:** Directed Graph G
  - **Output:** subgraph with vertices of SCC
  - Each vertex appears in exactly one SCC of the graph
  - Use of DFS + idea of **low-link values**

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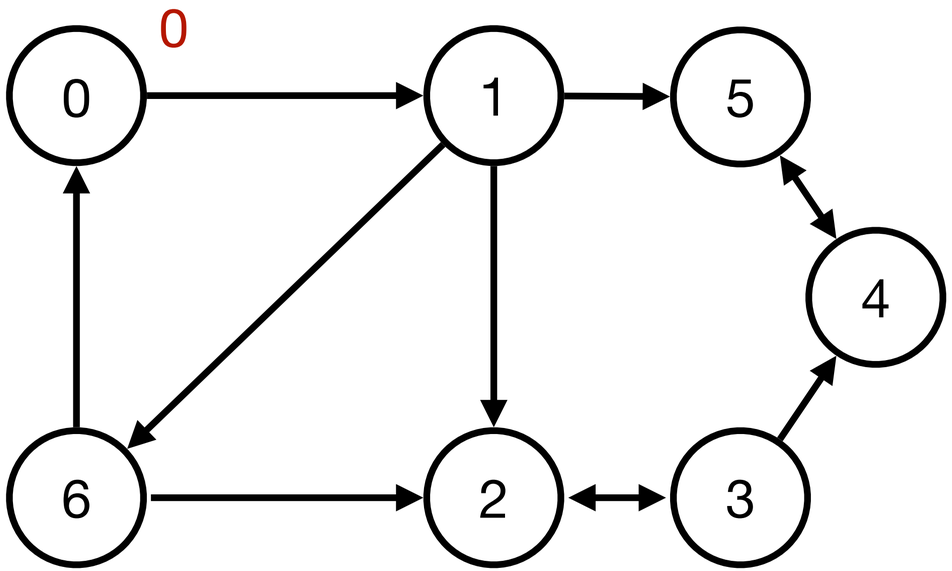

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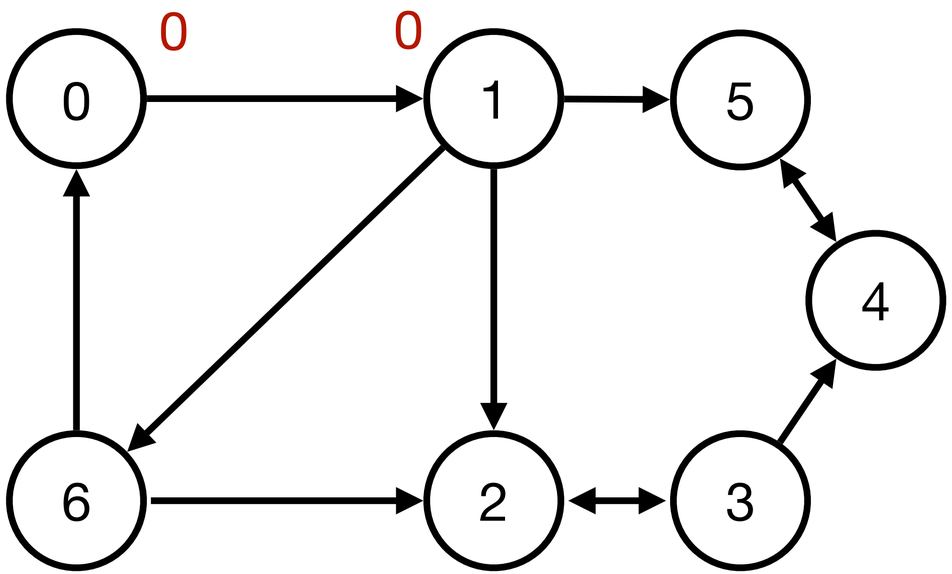

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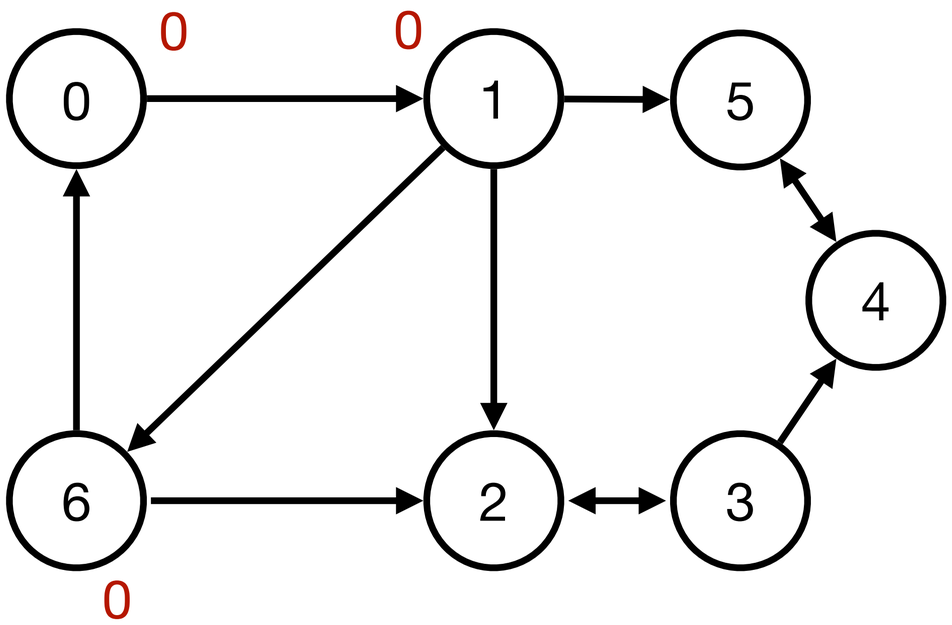

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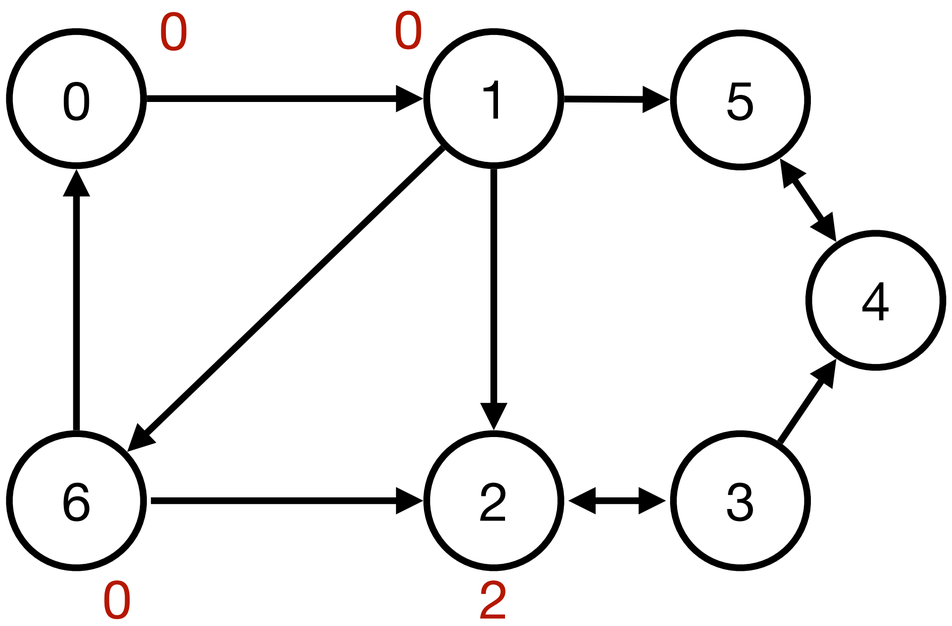

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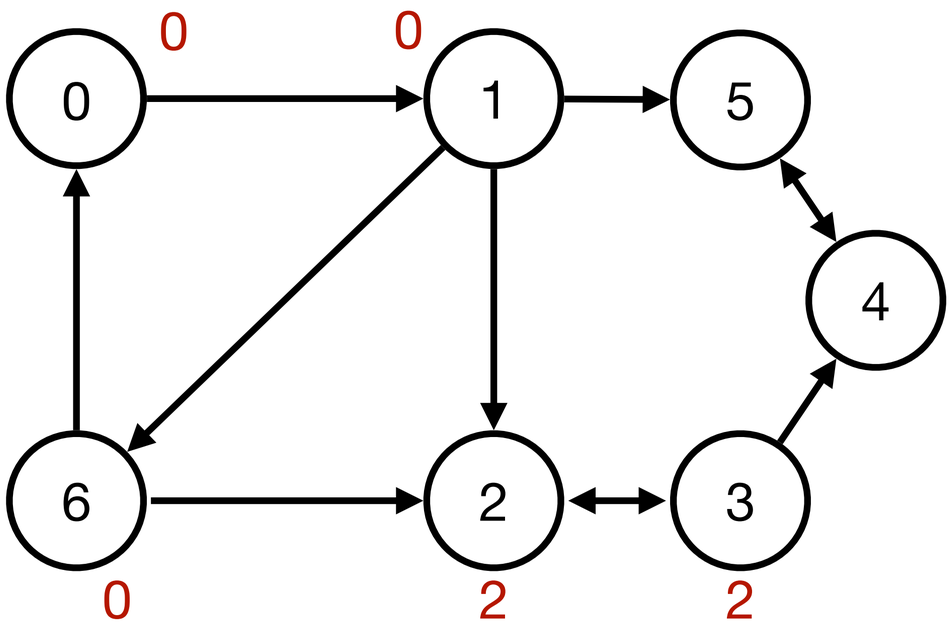

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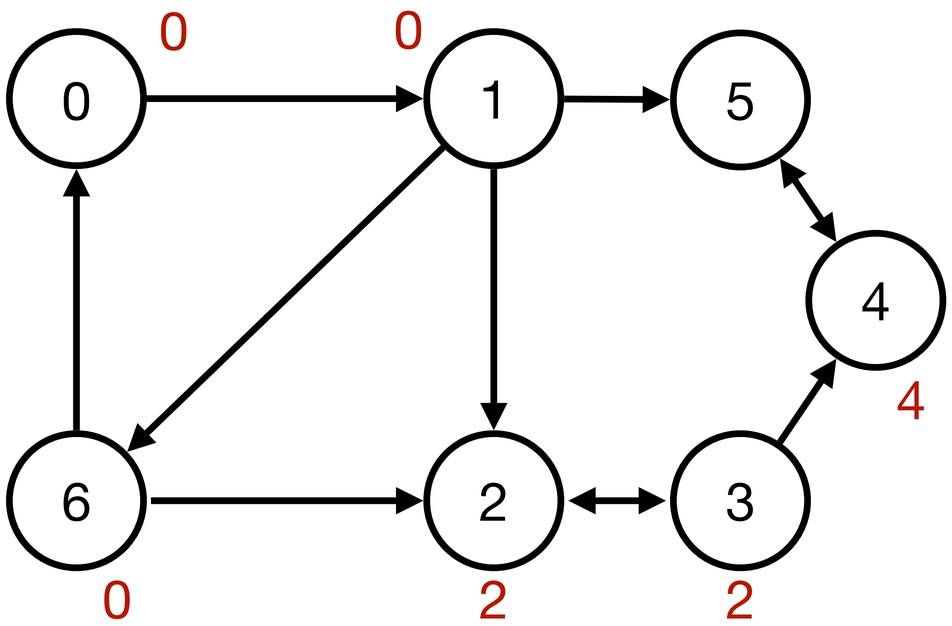

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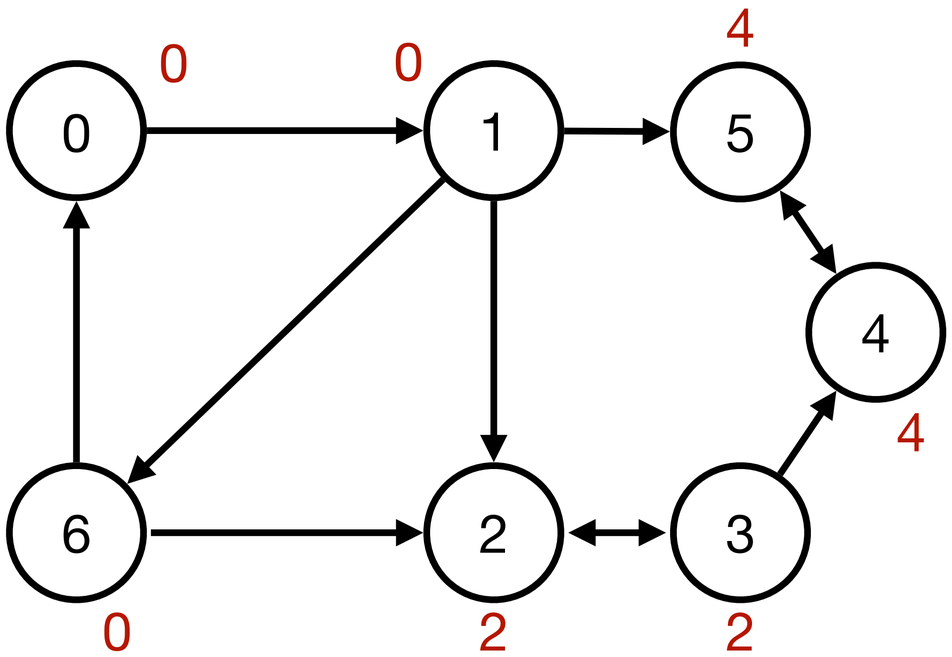

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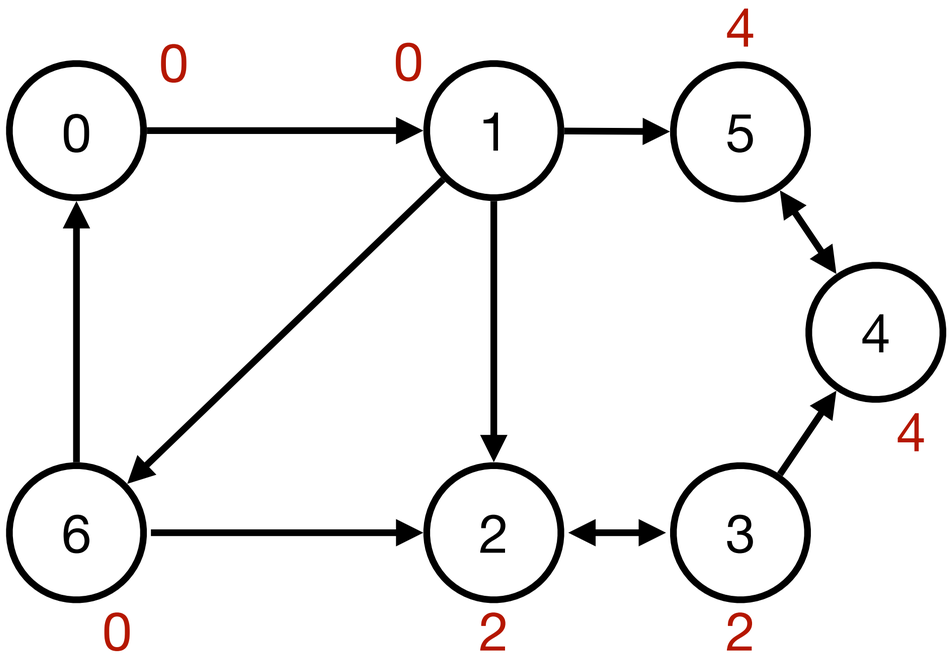

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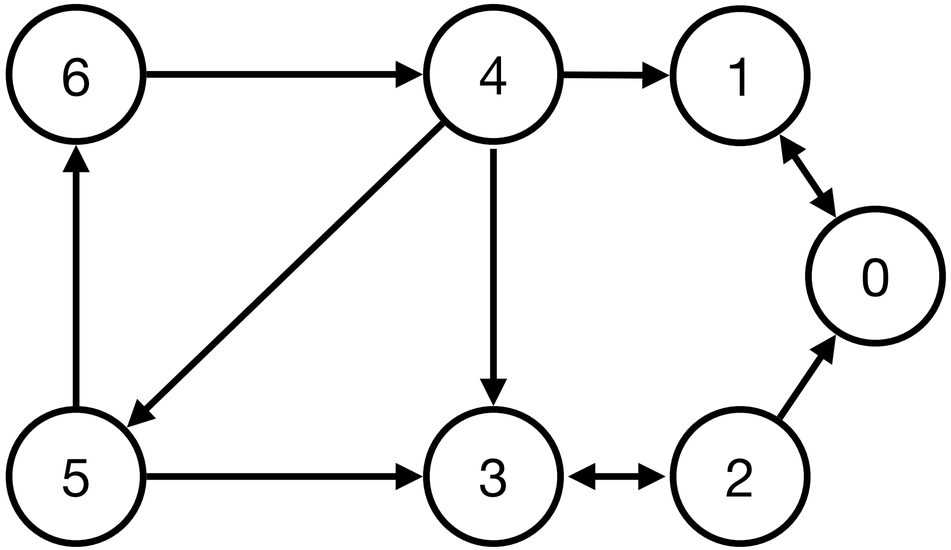

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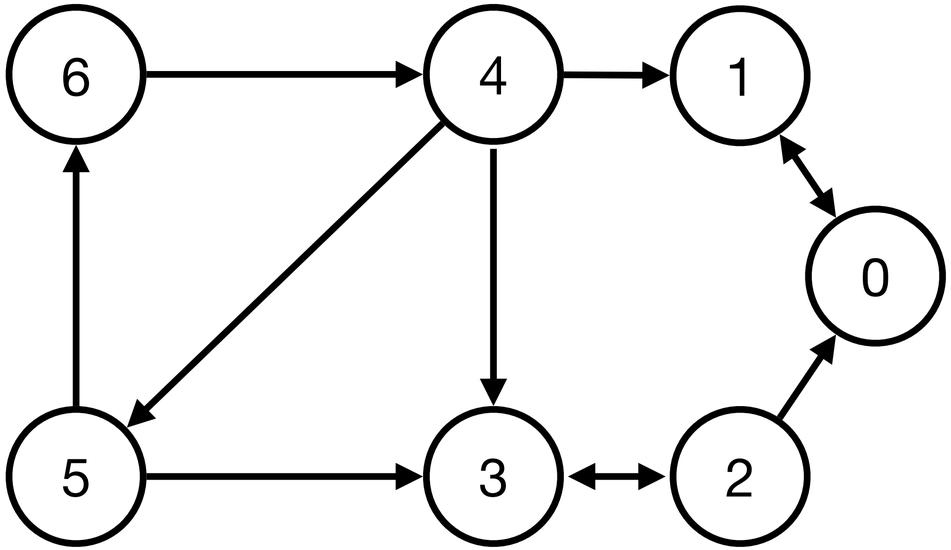

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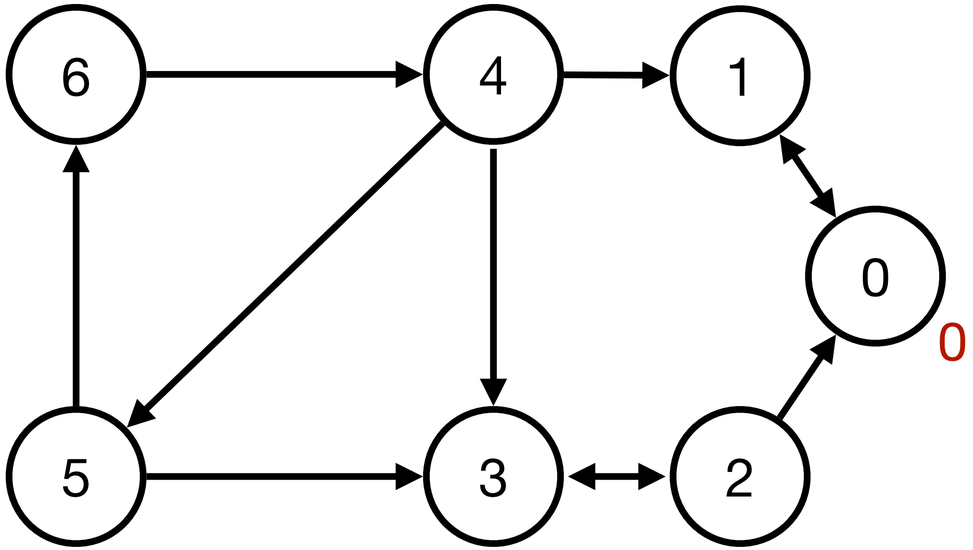

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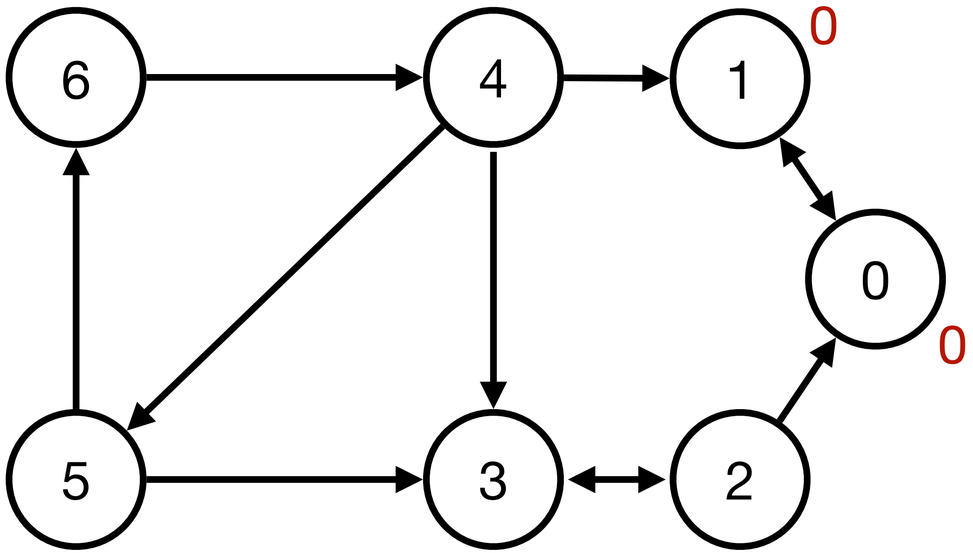

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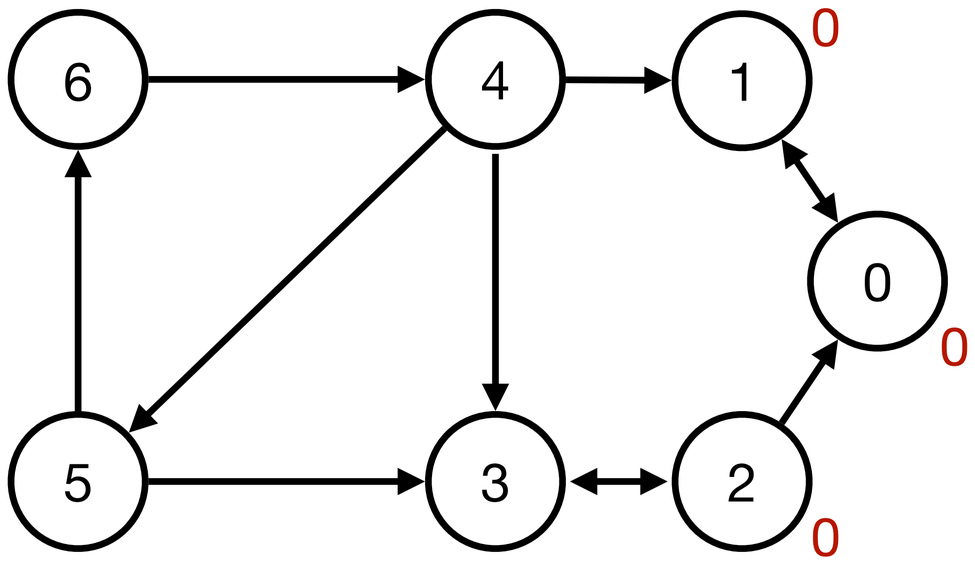

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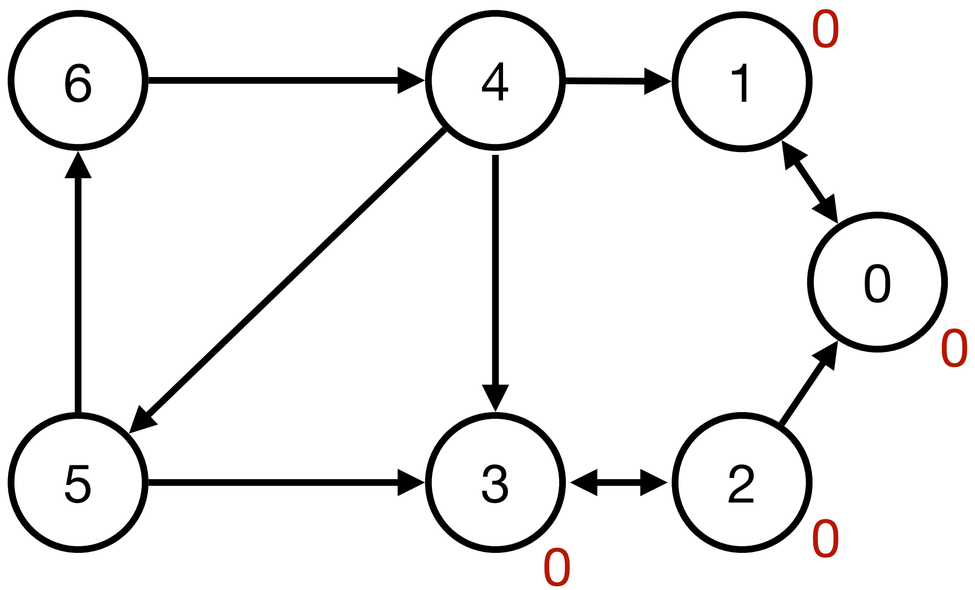

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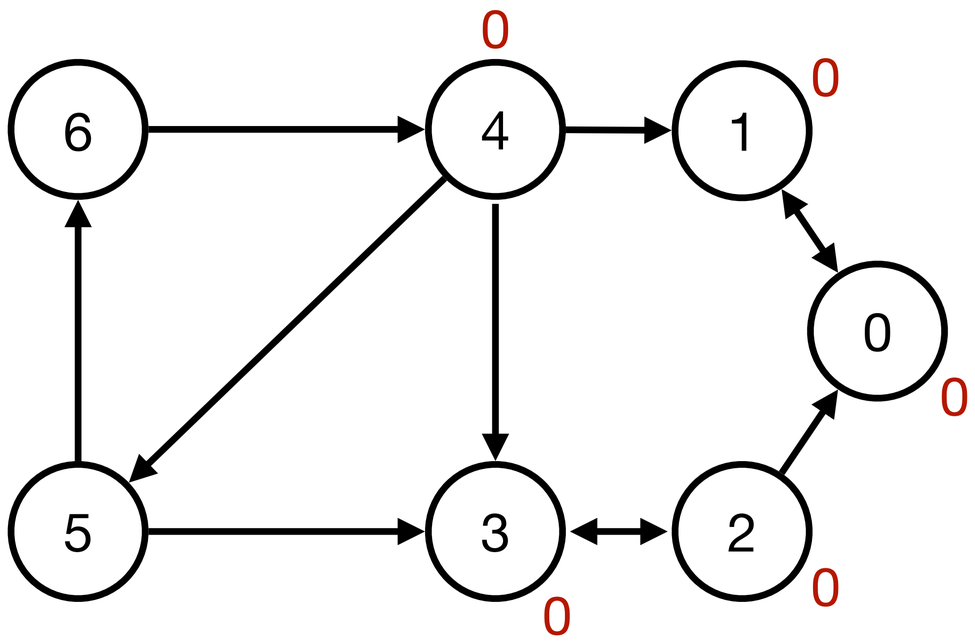

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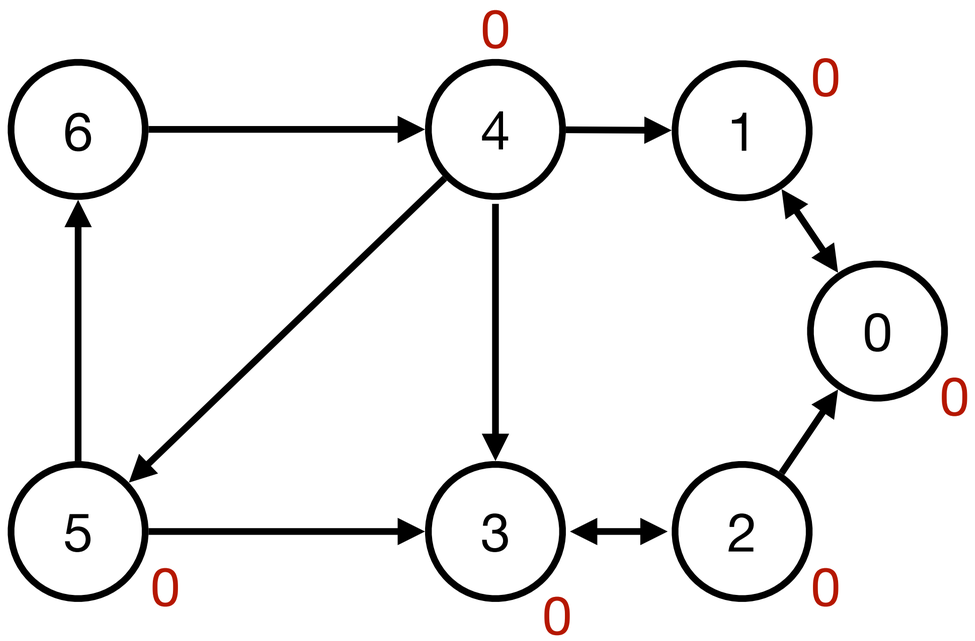

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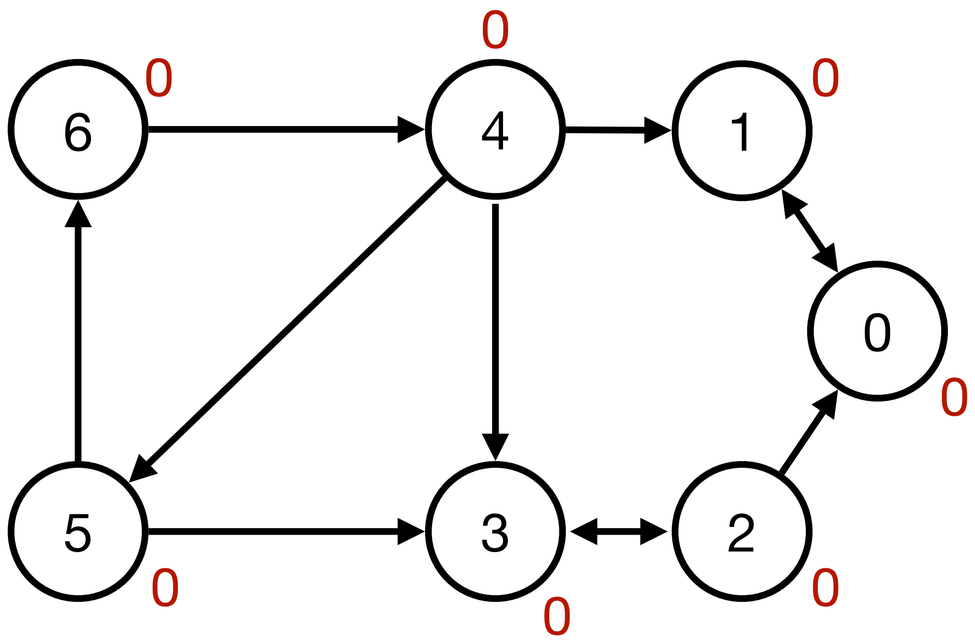

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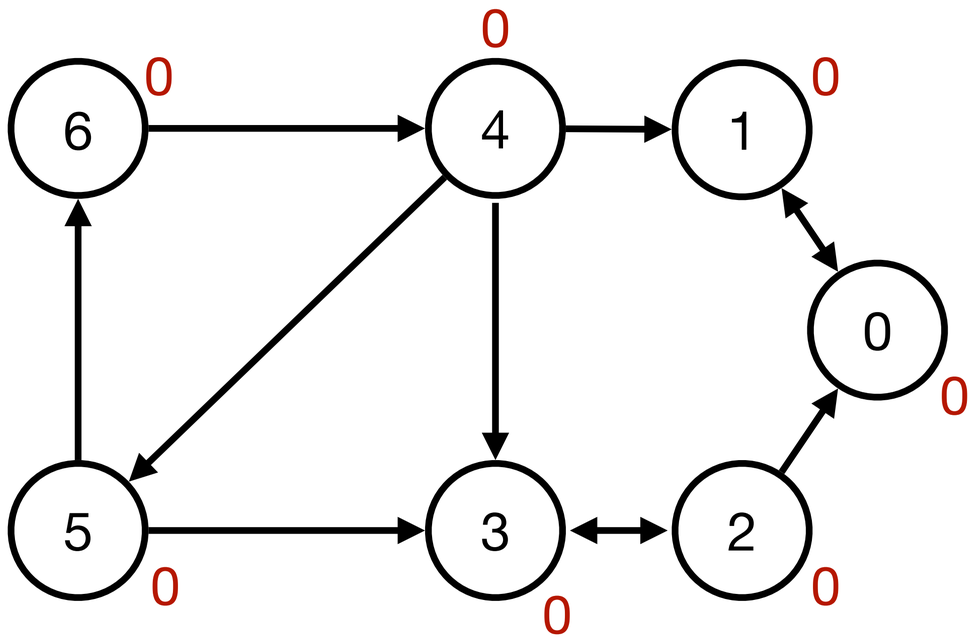

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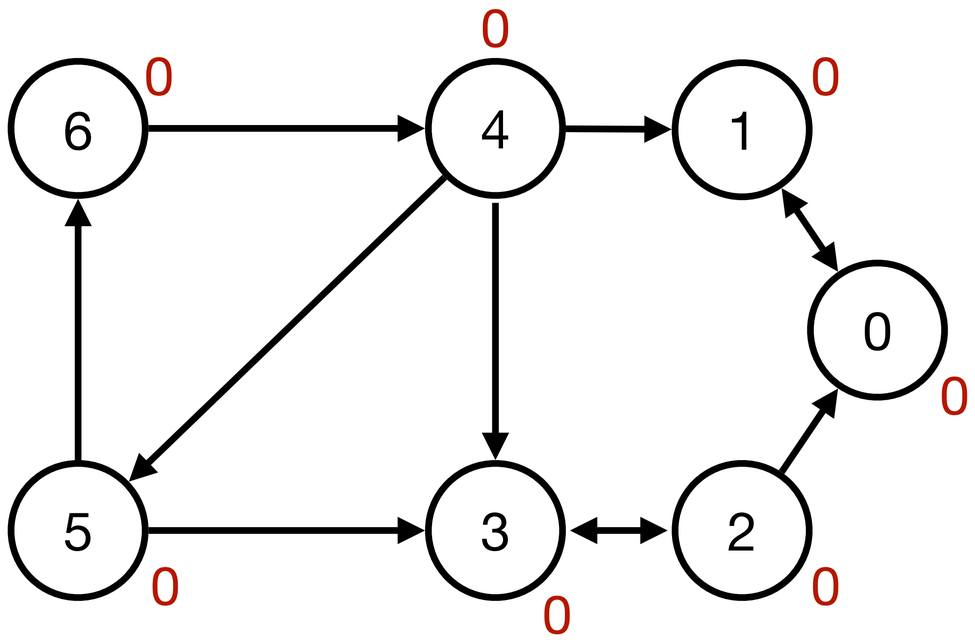


- **Incorrect SCC was computed**

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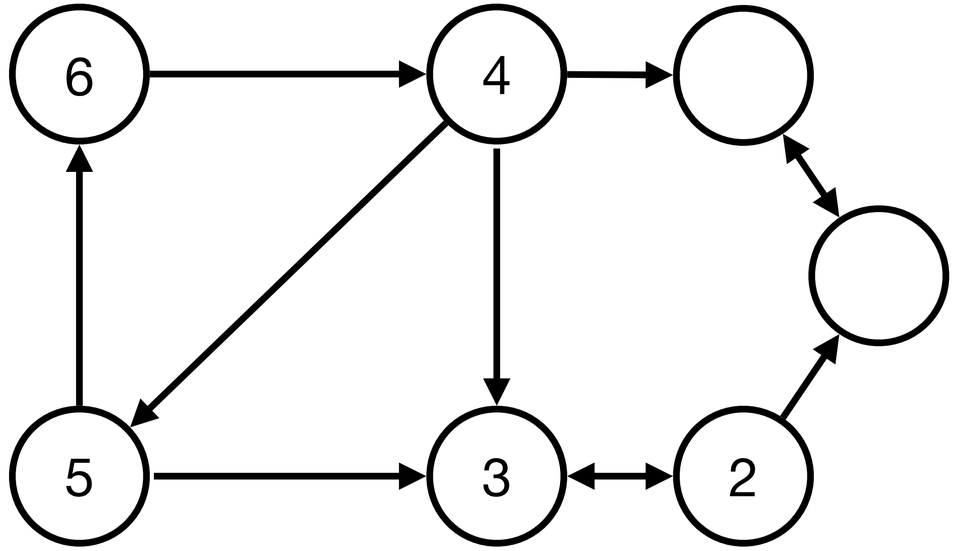


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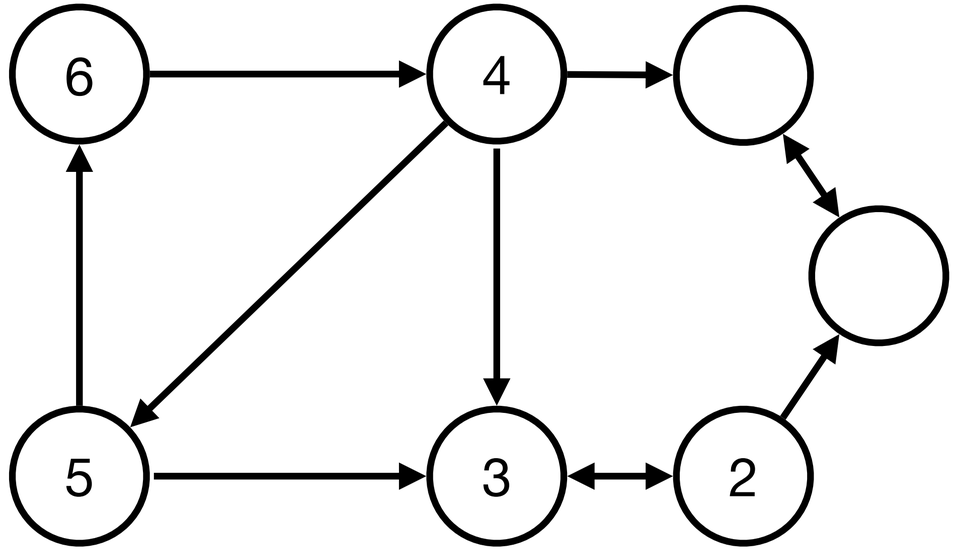


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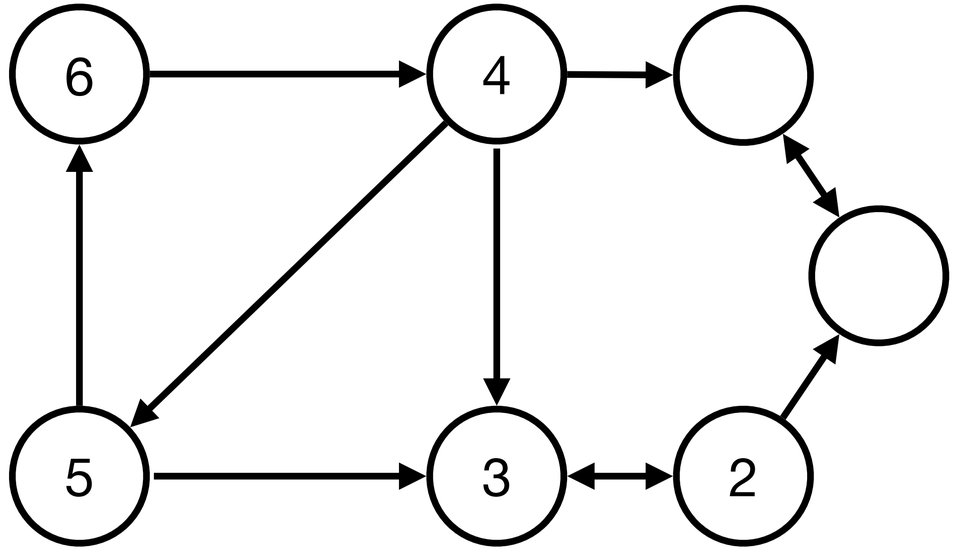


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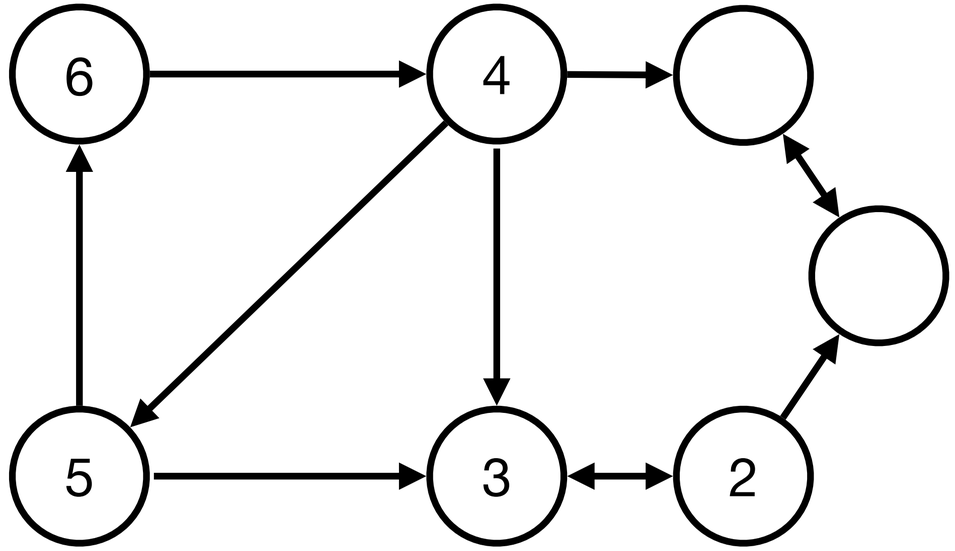


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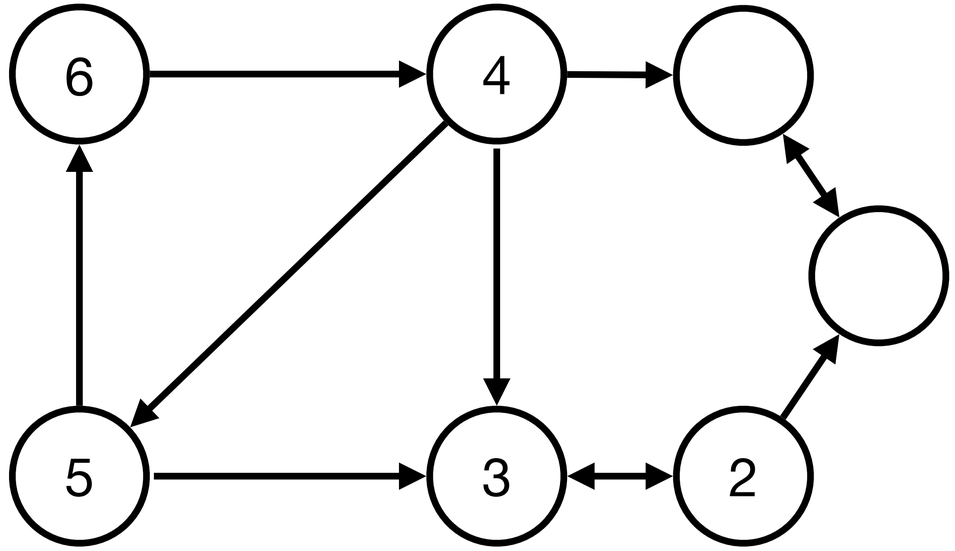


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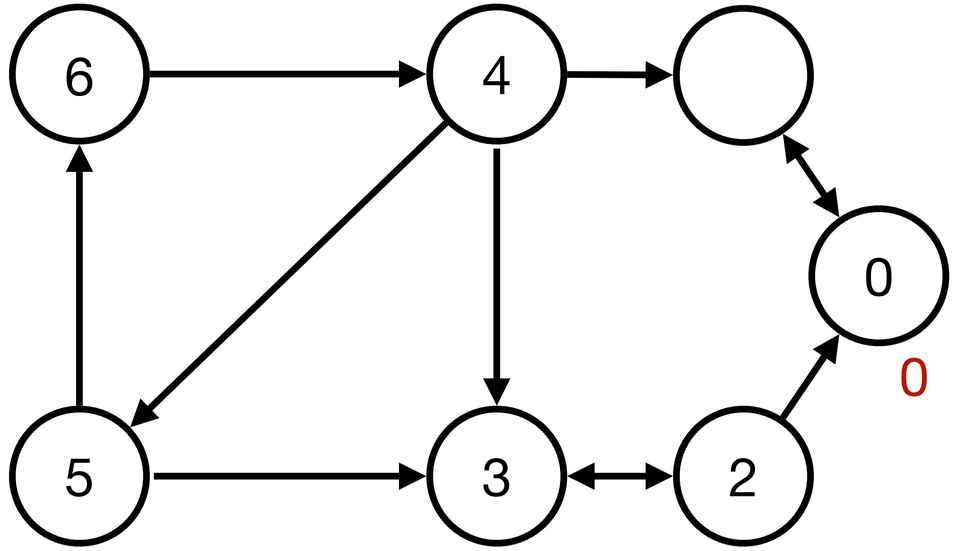


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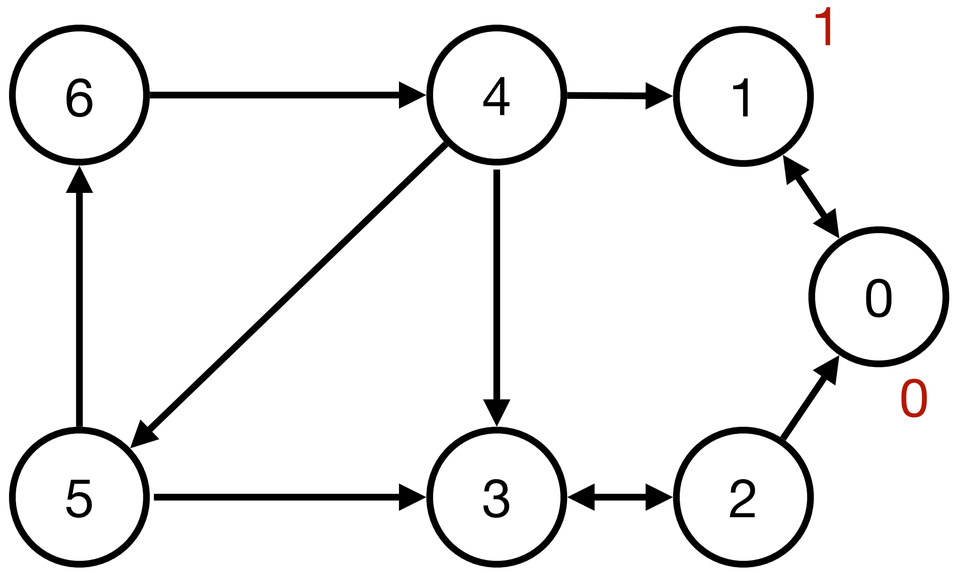


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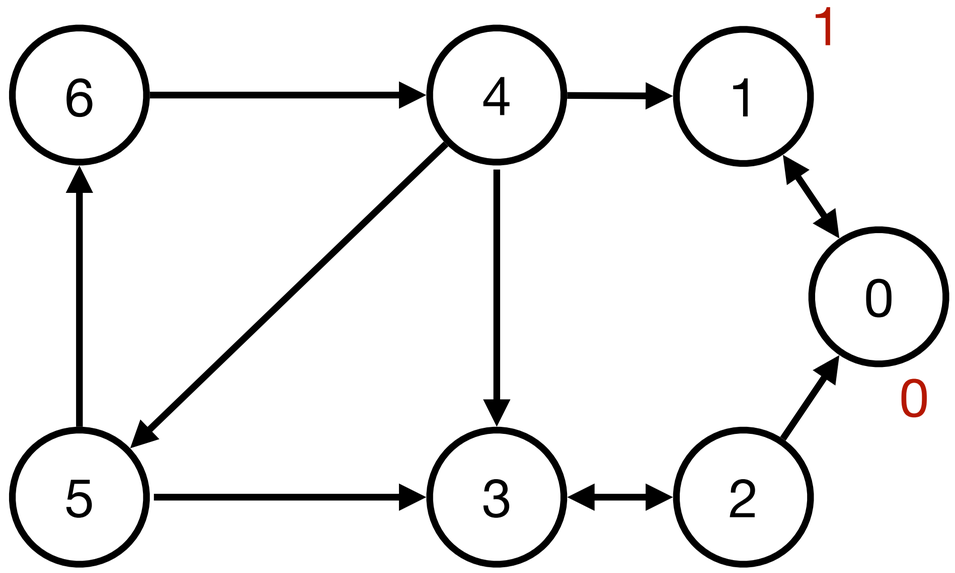


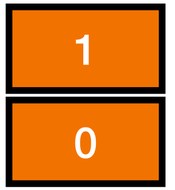


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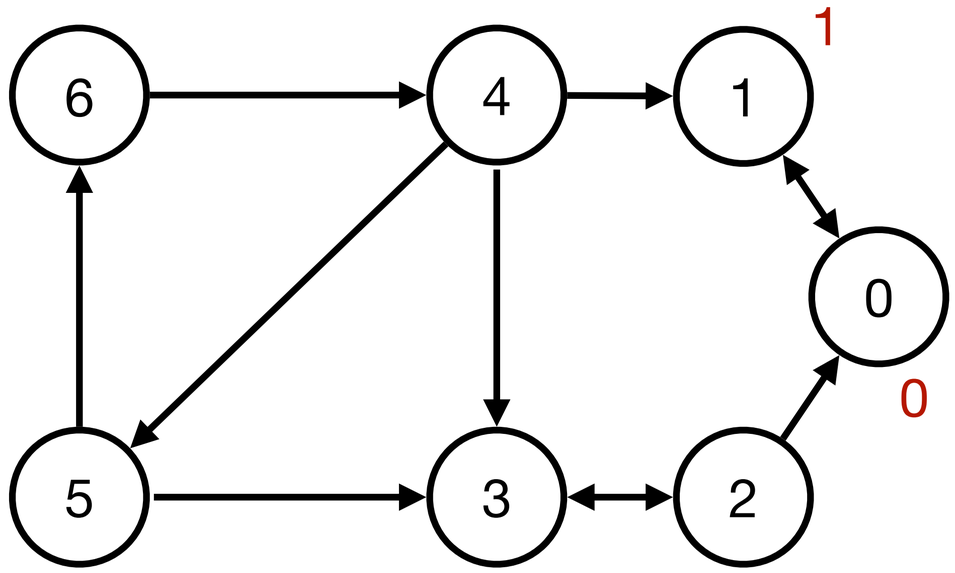


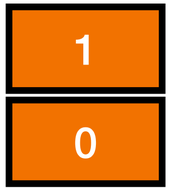


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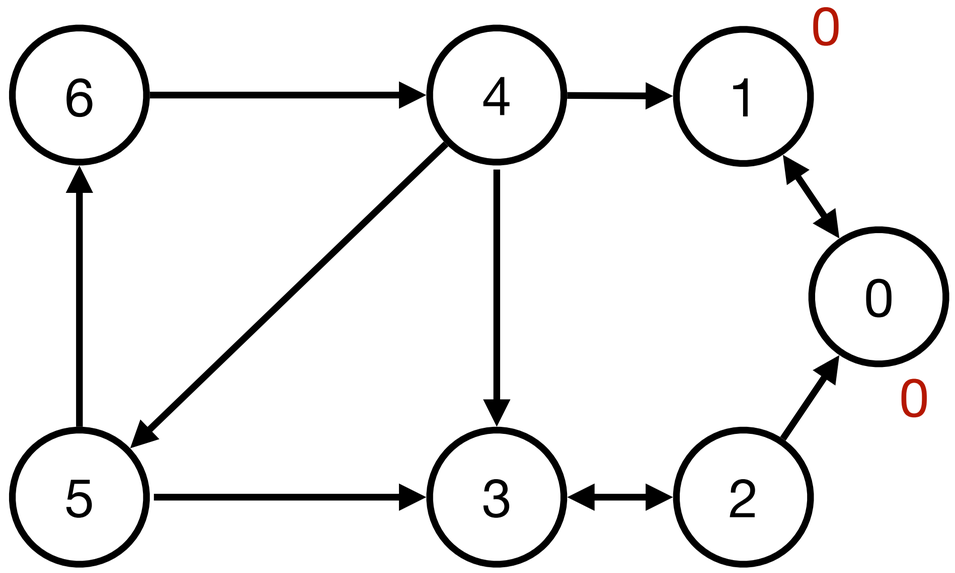


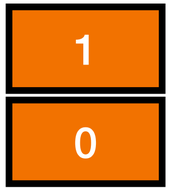


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# Strongly Connected Components

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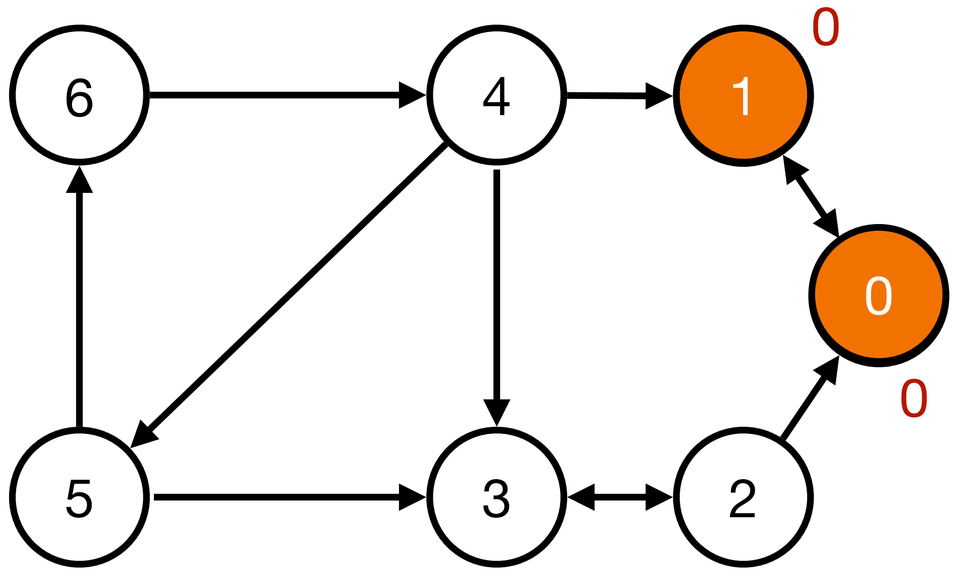


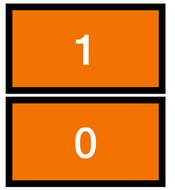


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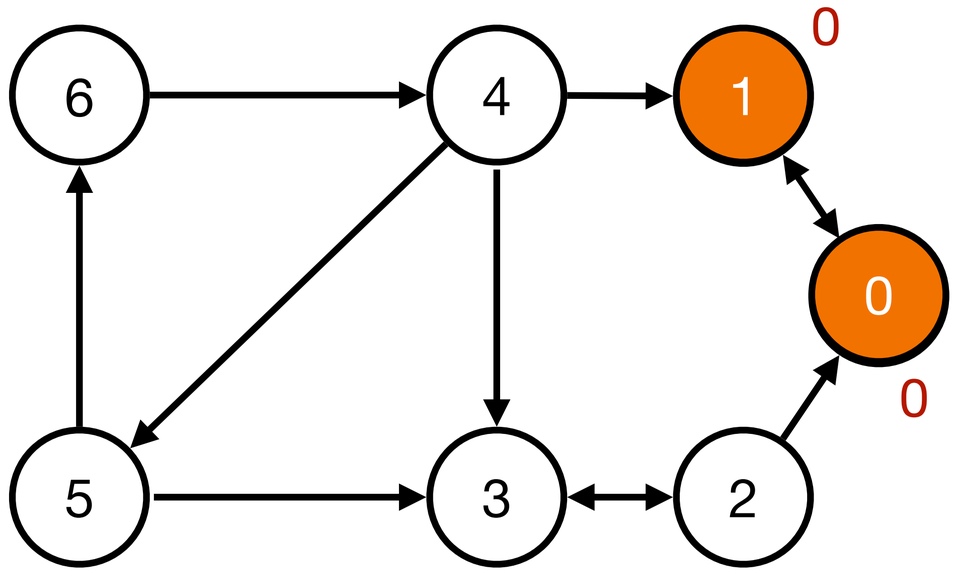


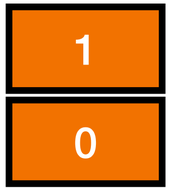


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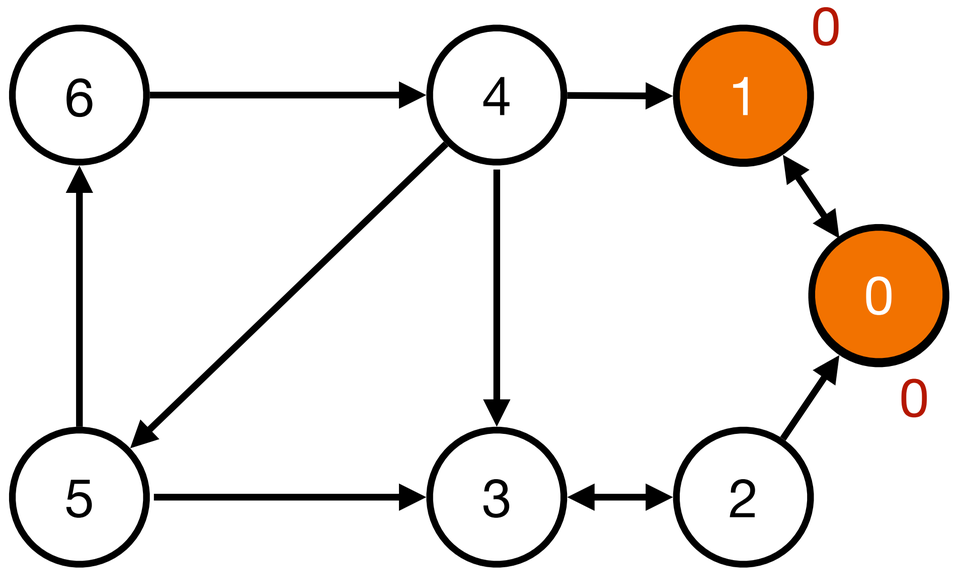


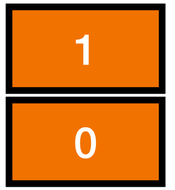


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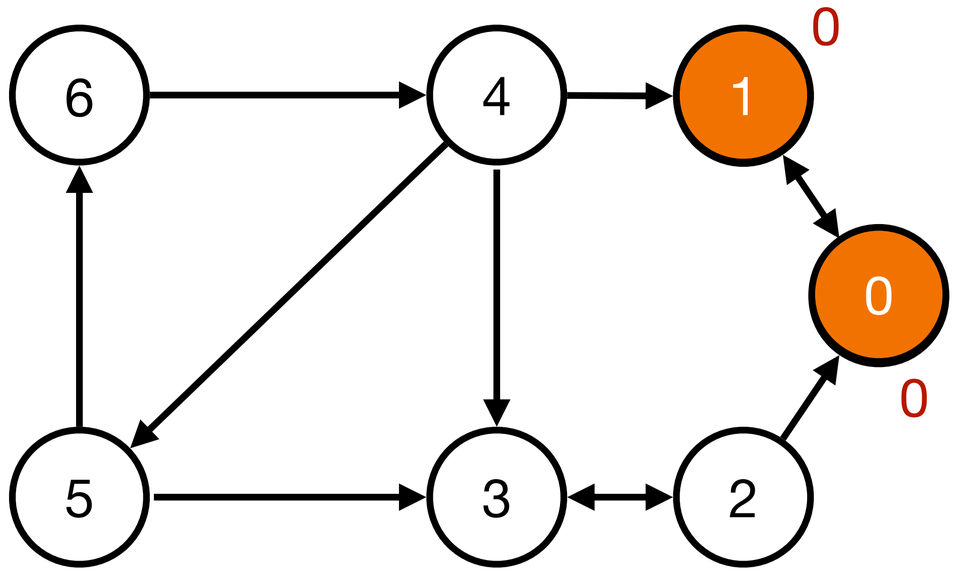


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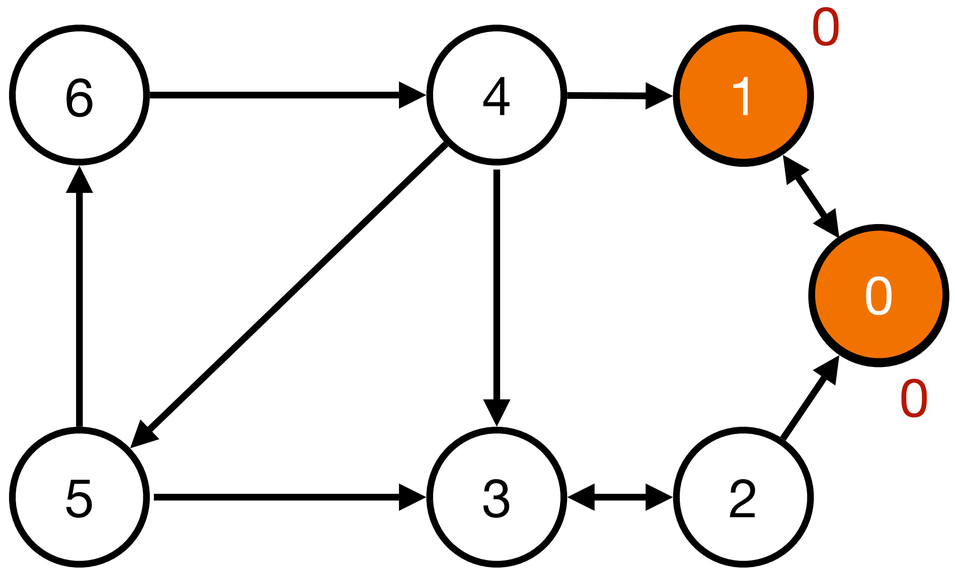


low[1] = min(low[1], ids[0])

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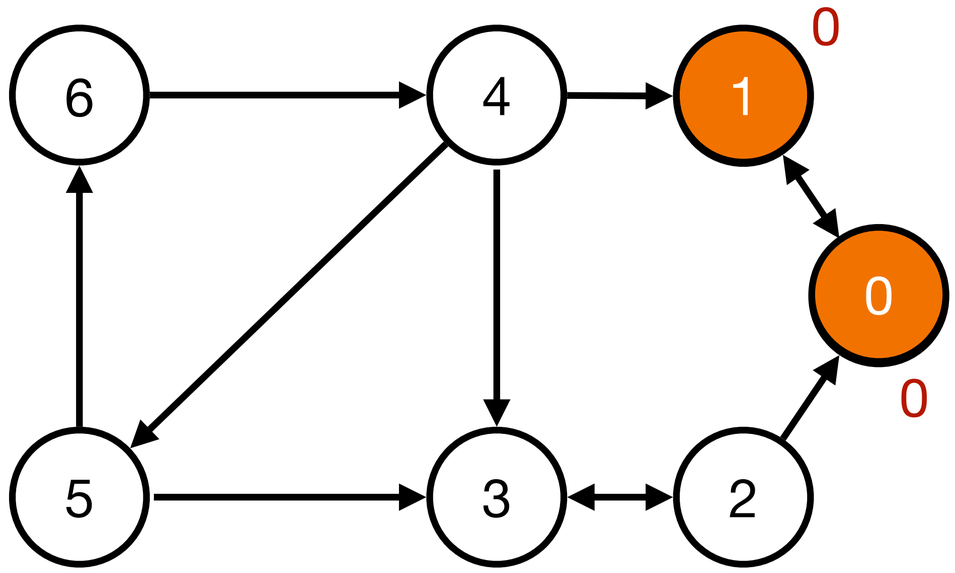


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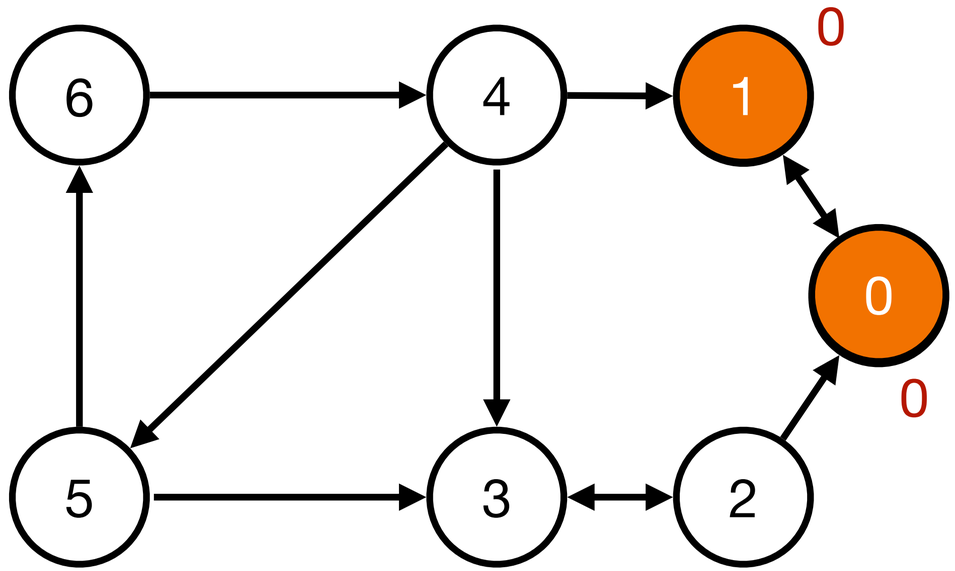

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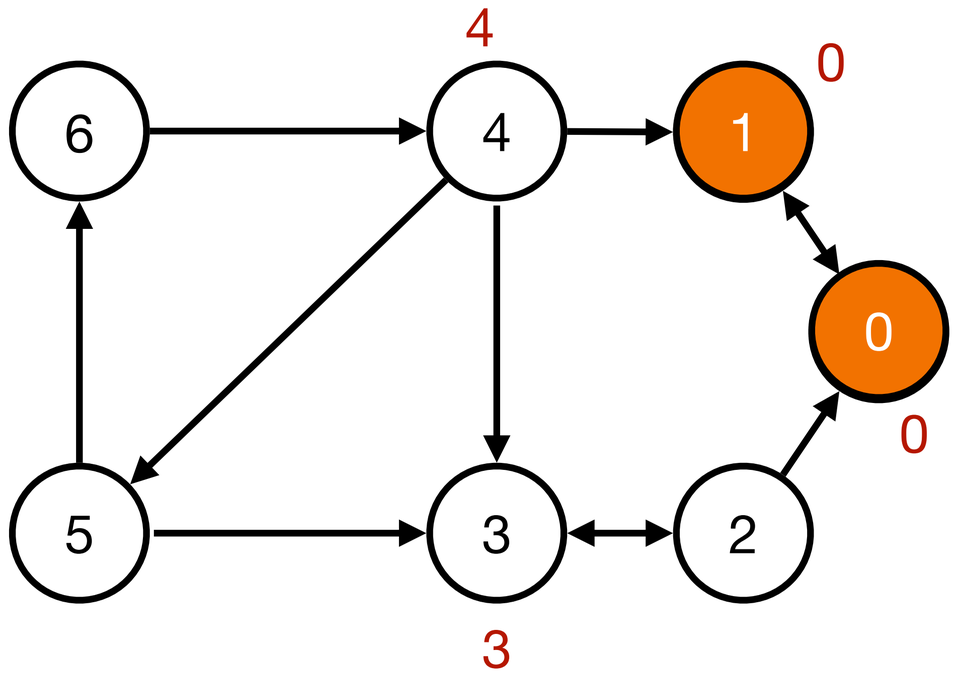

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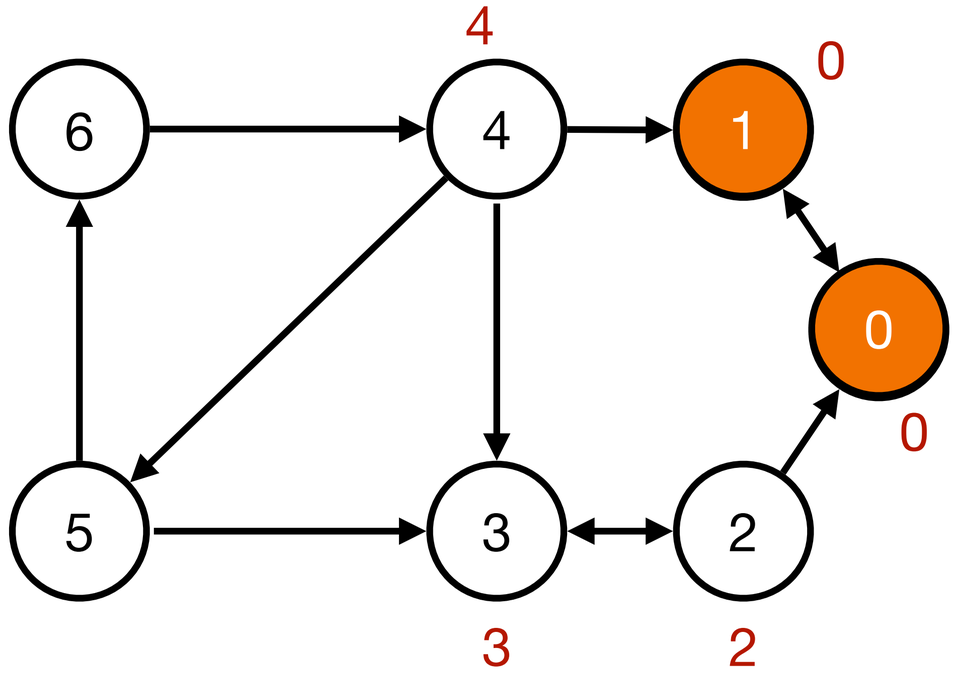

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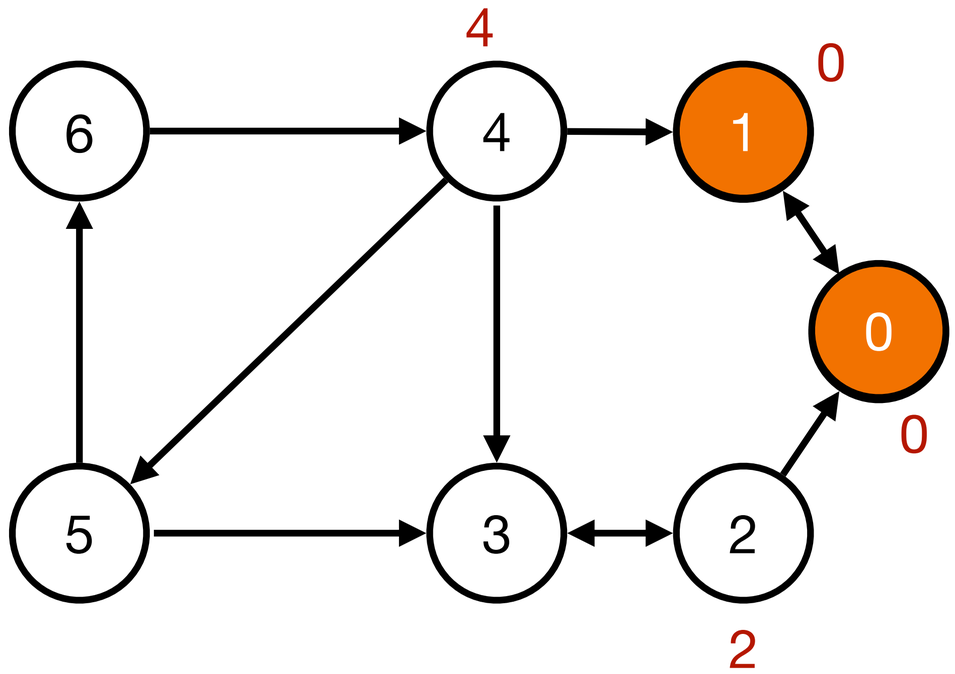
## Tarjan's Algorithm



# Strongly Connected Components

## Tarjan's Algorithm

- **Invariant of Tarjan's Alg:** A node remains on the stack **iff** there exists a path from it to a node on the stack
  - Prevents the LL values of different SCCs from interfering with each other