COL106
Data Structures and Algorithms

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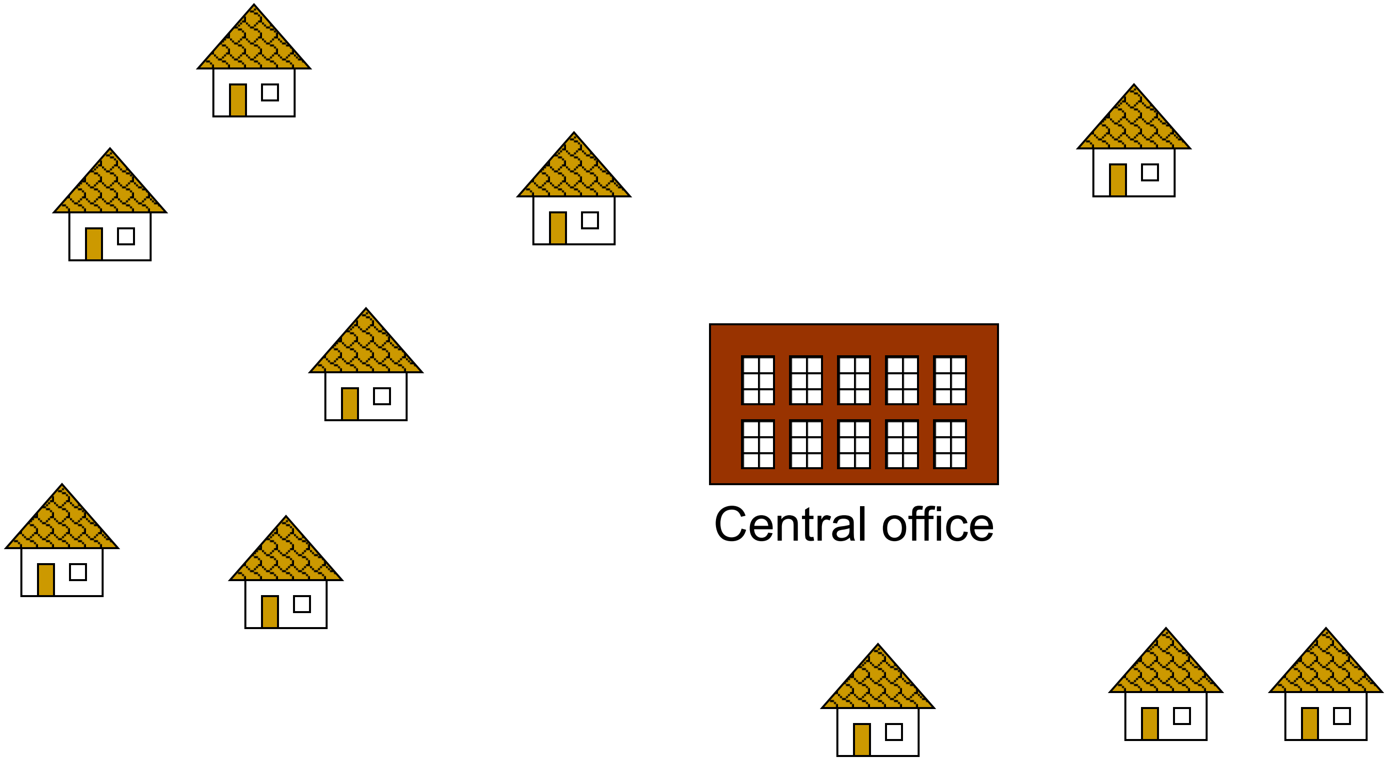
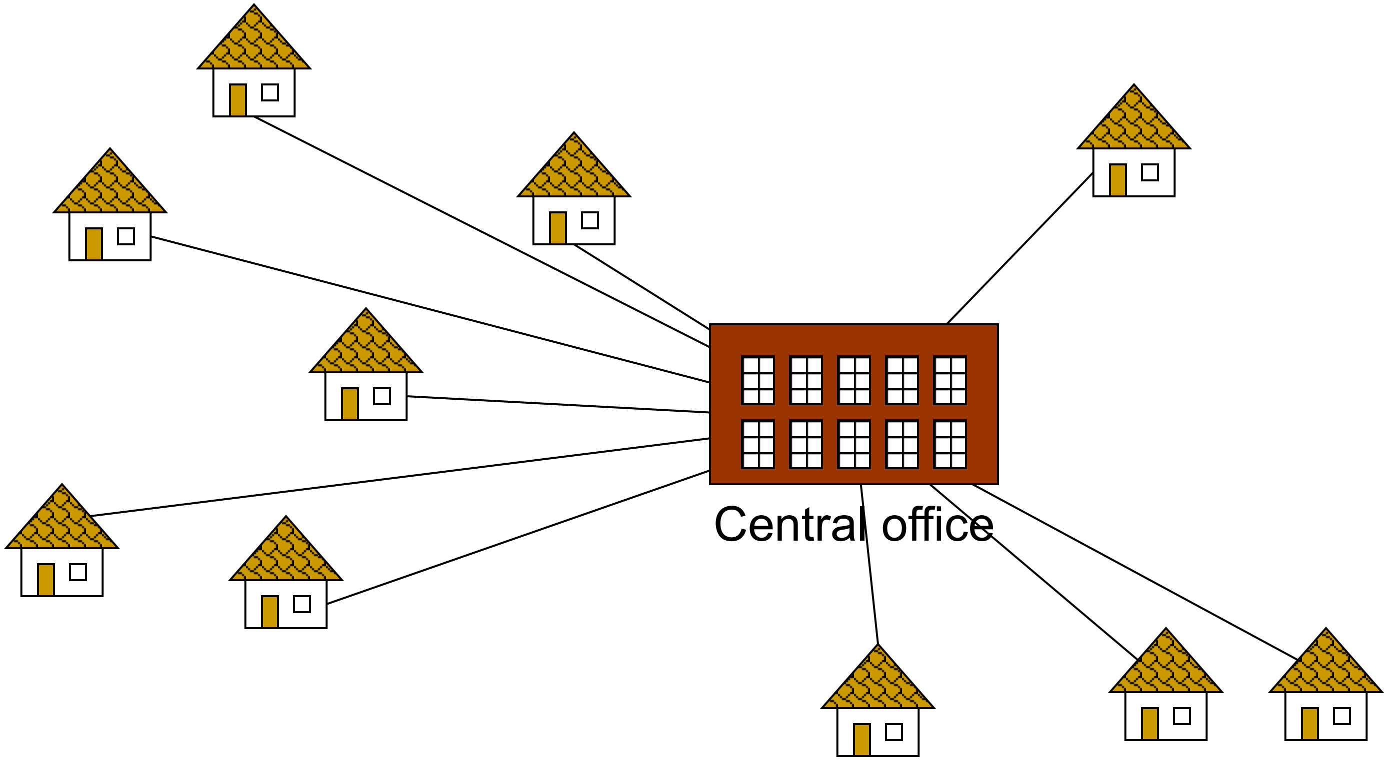
Minimum Spanning Trees

Based on slides by: Guy Kortsarz, Rutgers University, Rose Hoberman, CMU, Longin Jan Latecki, Temple University

Problem: Laying Telephone Wire

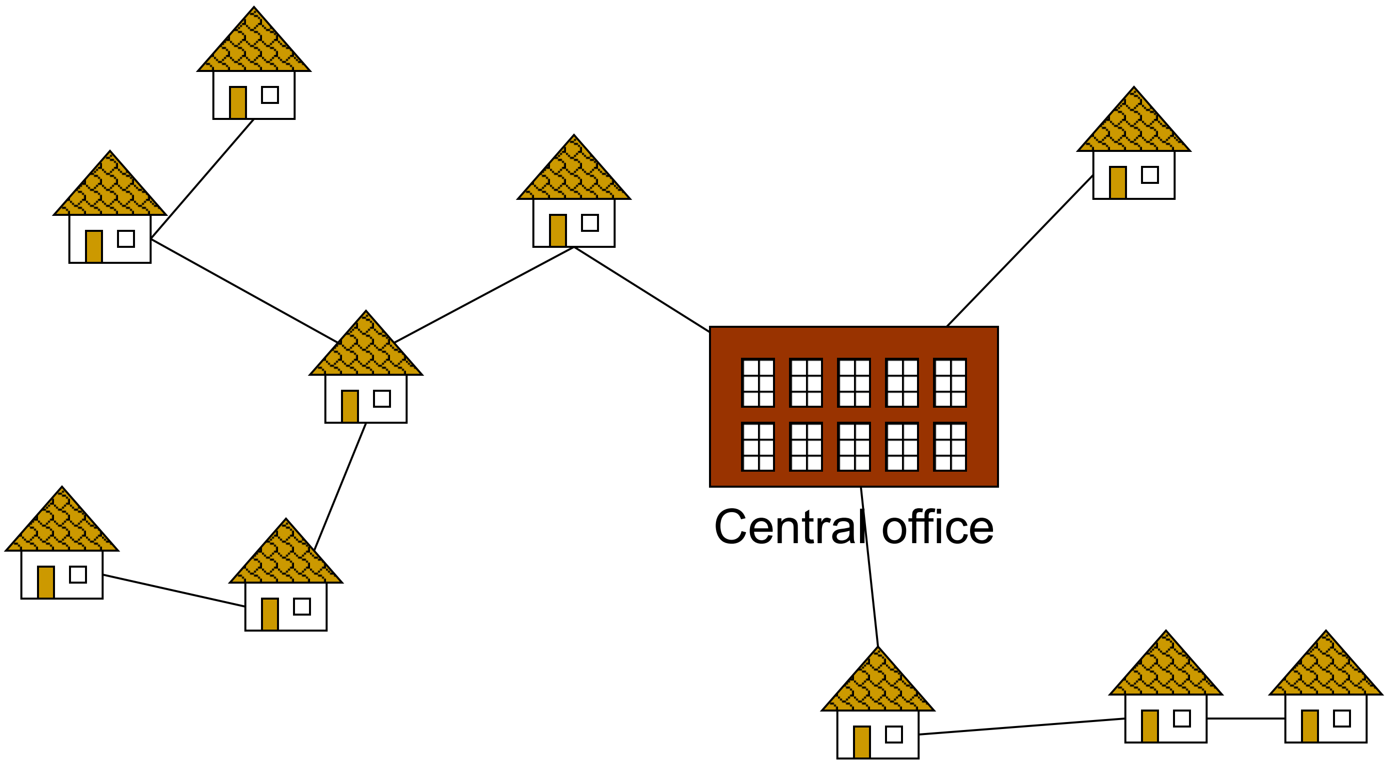


Wiring: Naive Approach



Expensive!

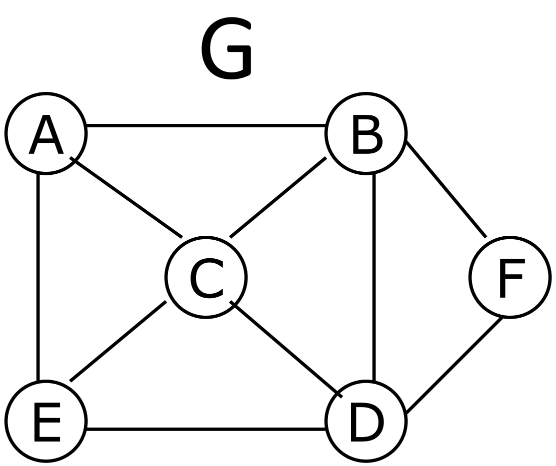
Wiring: Better Approach

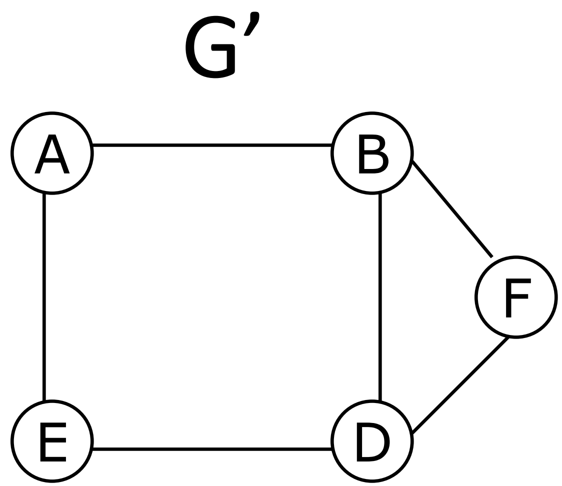


Minimize the total length of wire connecting the customers

Definitions

- Given a graph G = (V, E)
- A subgraph of G is a graph G’ with a subset of vertices and a subset of edges
- G’ = (V’, E’) where $V' \subseteq V, E' \subseteq E$
- Examples:





Spanning Tree of a Graph

- T is a spanning tree of a graph G = (V, E) if
  - It is a subgraph of G
  - It has all the vertices V of G
  - All the vertices are connected
  - It has no cycles

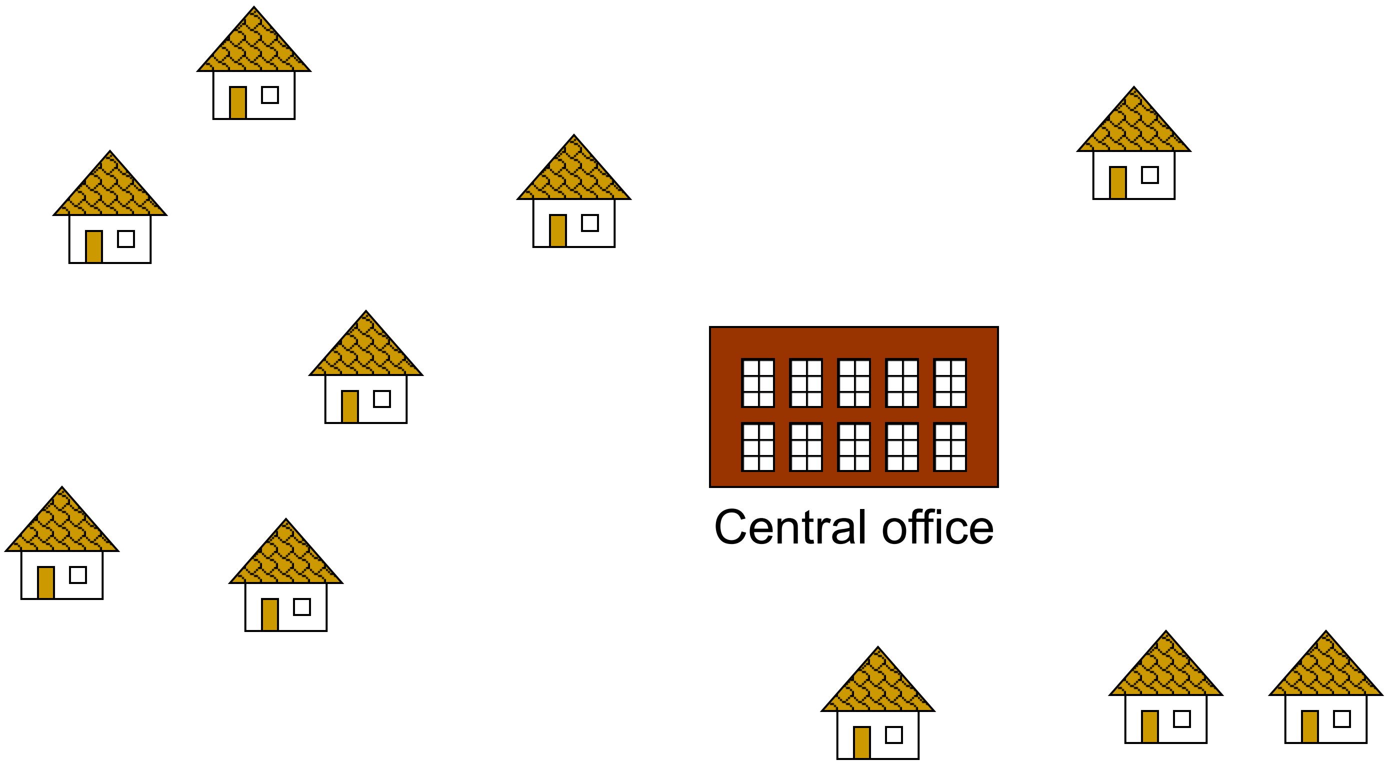
Claim: If T is a spanning tree of G, then it has exactly |V| - 1 edges

Proof: A tree with n vertices has n-1 edges

Minimum Spanning Tree

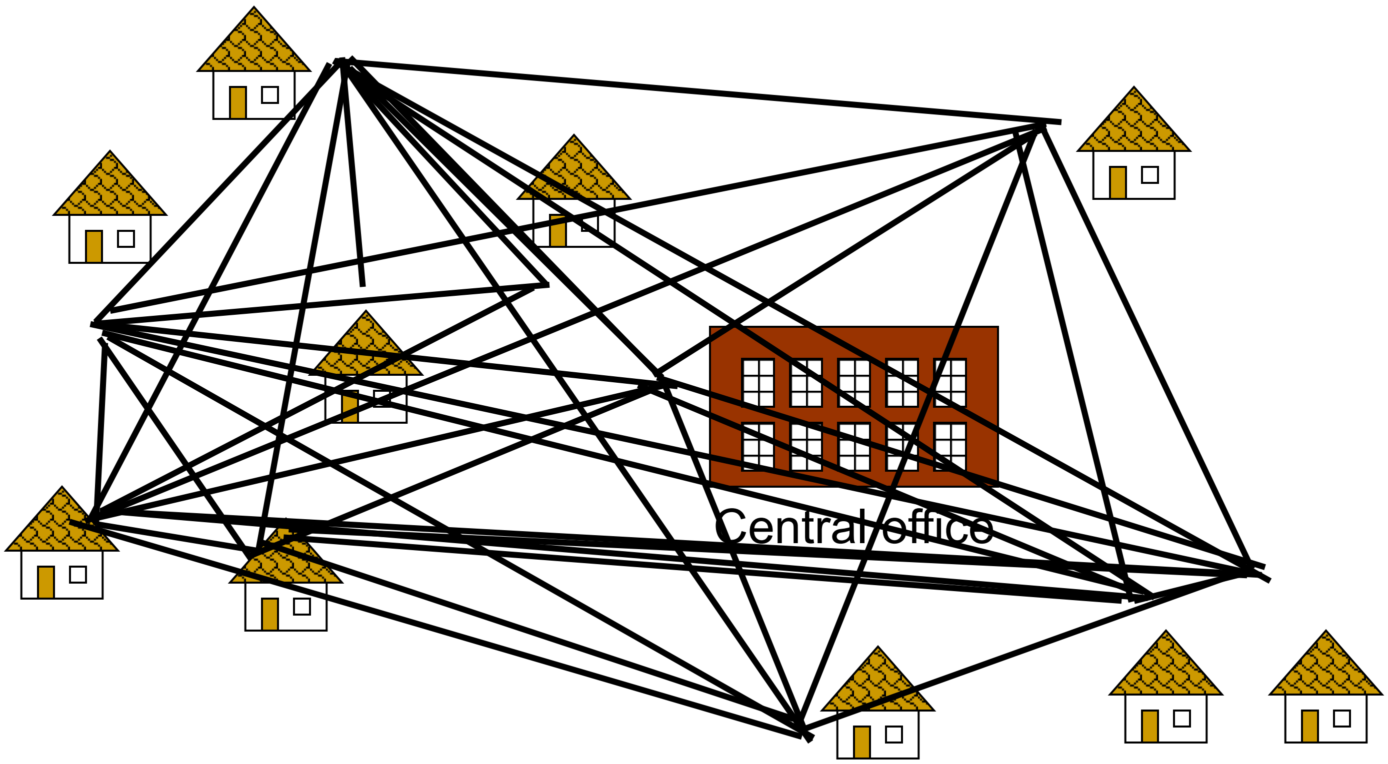
- Given a undirected weighted graph G = (V, E, W) where $W: E \rightarrow R^+$
- Cost of a spanning tree T = (V, E') is given by $W(T) = \sum_{e \in E'} W(e)$
- T is the minimum cost spanning tree if and only iff T is a spanning tree of G and has the minimum cost

Problem: Laying Telephone Wire



Can it be formulated as a MST problem?
What will be the corresponding graph?

Problem: Laying Telephone Wire



Can it be formulated as a MST problem?

What will be the corresponding graph?

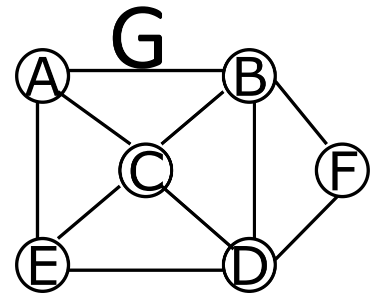
How to Find a MST?

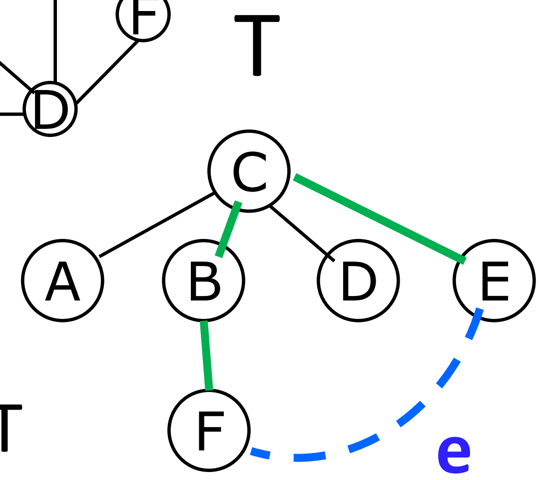
- Prims Algorithm for MST: $O(|E| + |V| \log |V|)$

- Today, we will cover Kruskal's algorithm for finding the MST

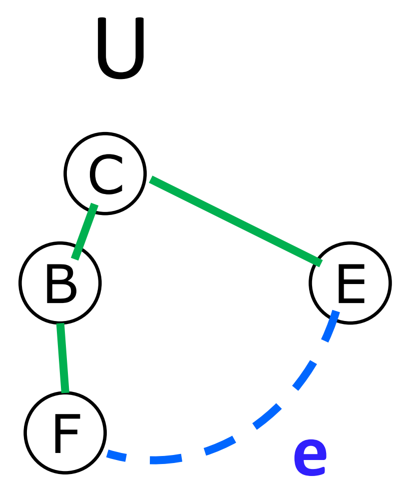
- Let us revisit their properties

Cycle Property



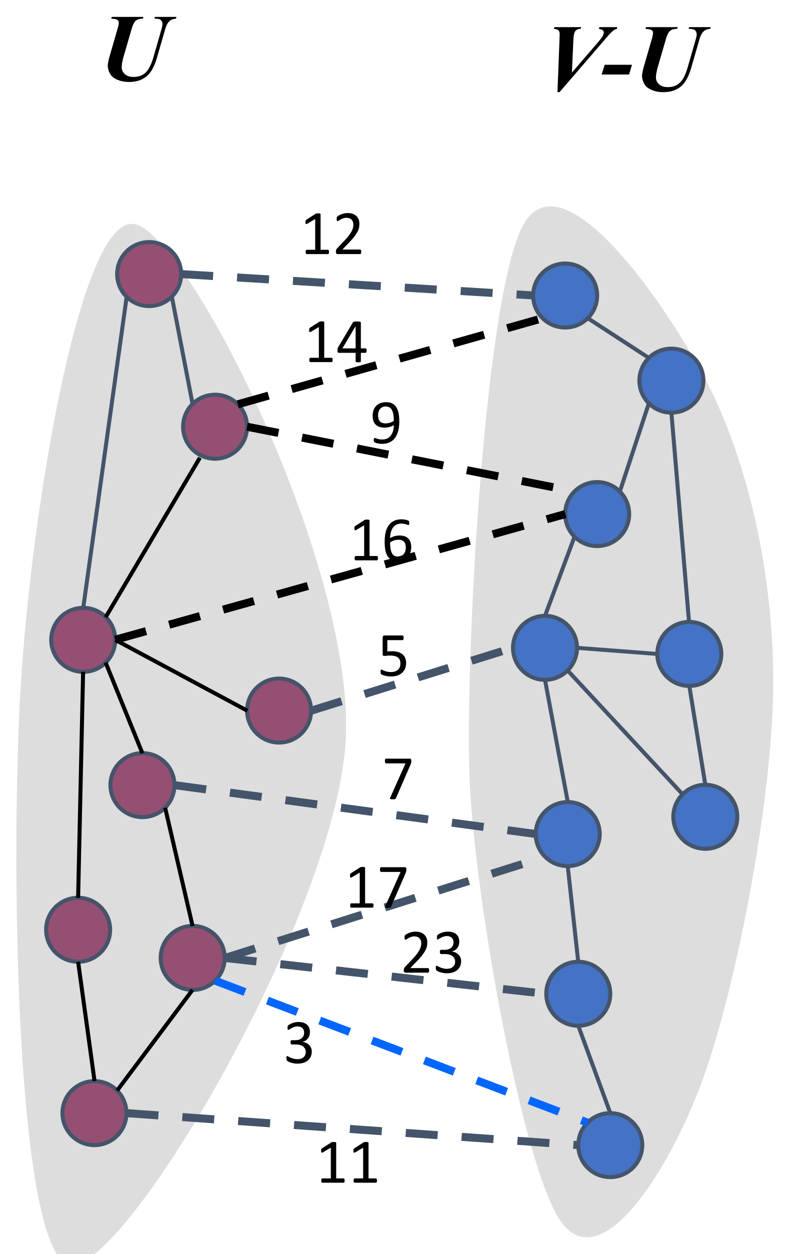


- Let **T** be a MST of a weighted undirected weighted graph G
- Let **e** be an edge of G that is not in T
- Let **U** be the cycle formed by adding e in T
- For every edge **f** of **U**, $W(f) \leq W(e)$
- Why?
- Removing any single edge in U will make it a spanning tree again



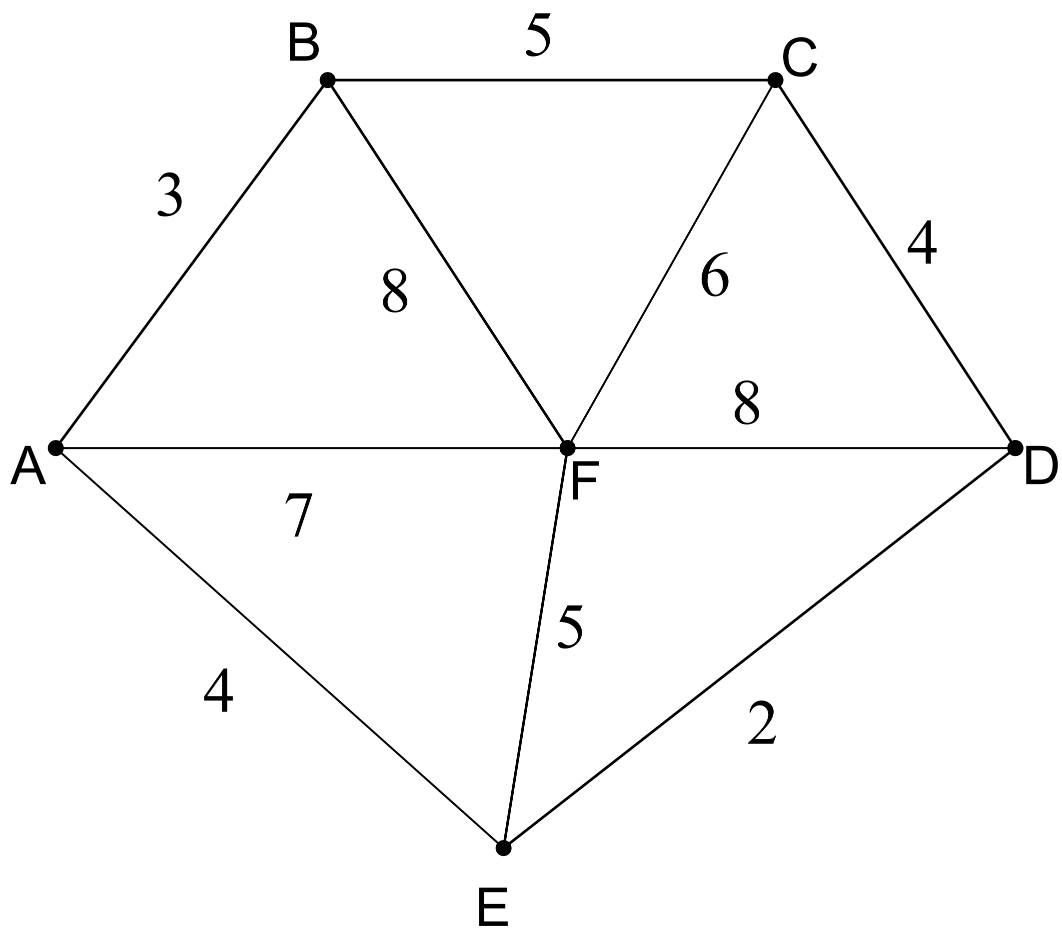
Cut Property

- Consider a partition of the vertices of G into subsets U and V-U
- Let e be an edge of minimum weight across (U, V-U)
- There is a minimum spanning tree of G containing edge e



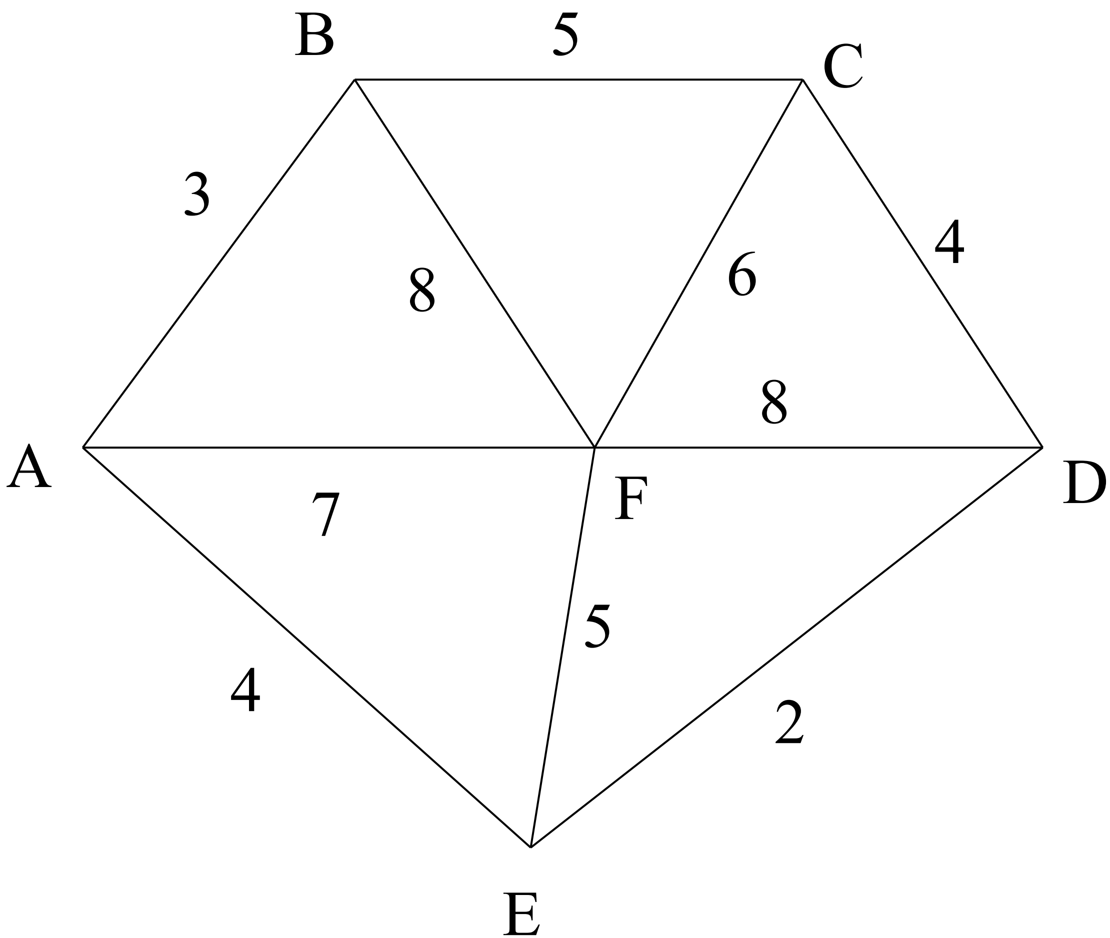
Outline of Kruskal’s Algorithm for Finding MST

- Sort the edges in ascending order of their costs
- Keep on adding the edge to the forest in this order
- As long as they don’t make a cycle
- After adding $|V|-1$ edges we have the MST
- Why?



B
5
C
3
8
6
4
8
A
F
D
7
5
4
2
E

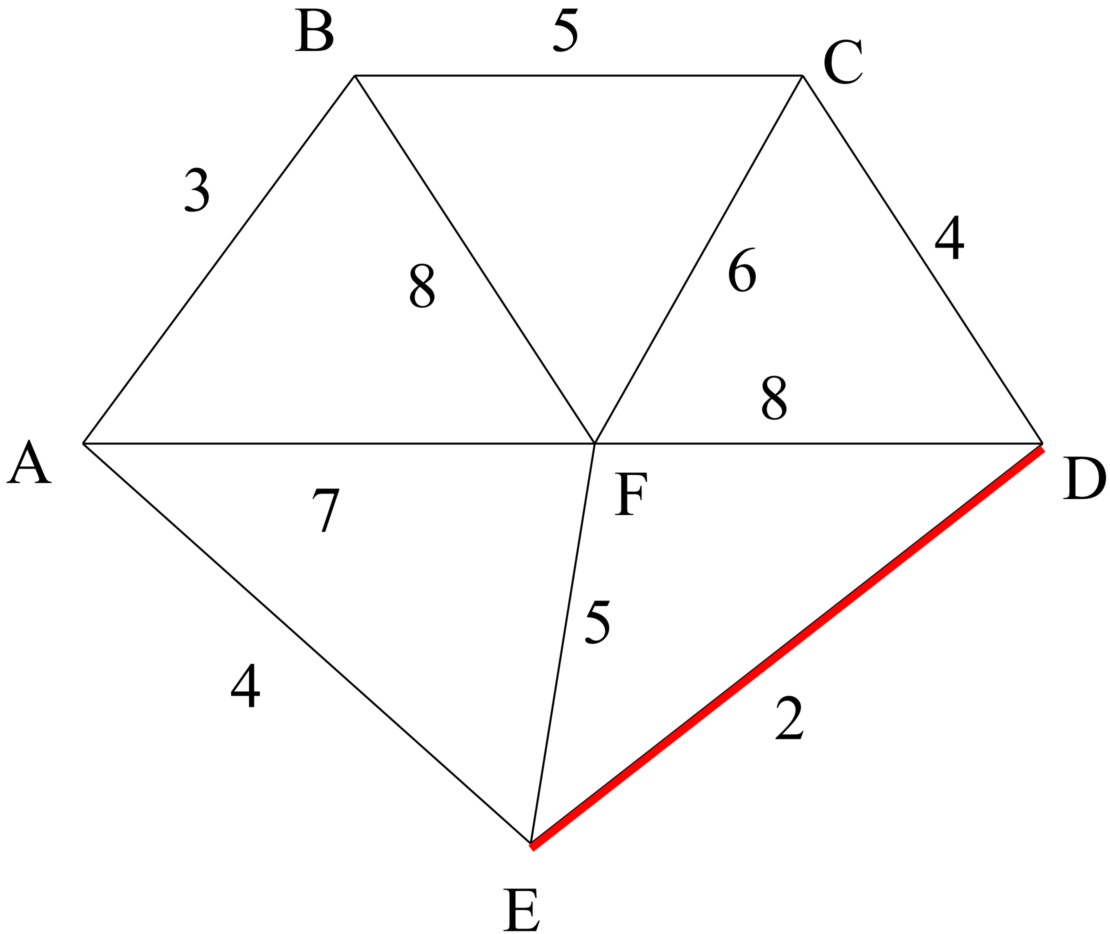
Kruskal's Algorithm



List the edges in increasing costs:

ED 2
AB 3
AE 4
CD 4
BC 5
EF 5
CF 6
AF 7
BF 8
CF 8

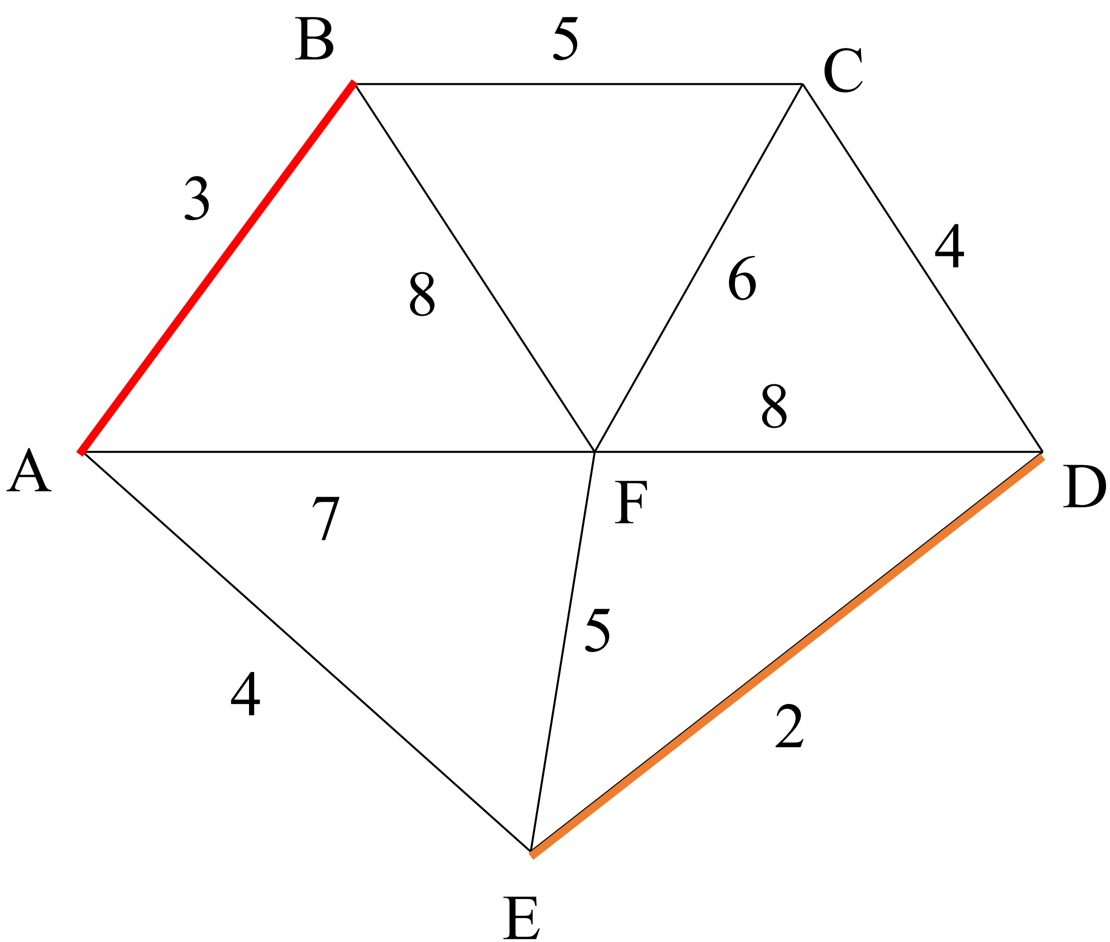
Kruskal's Algorithm



Select the shortest edge in the network

ED 2

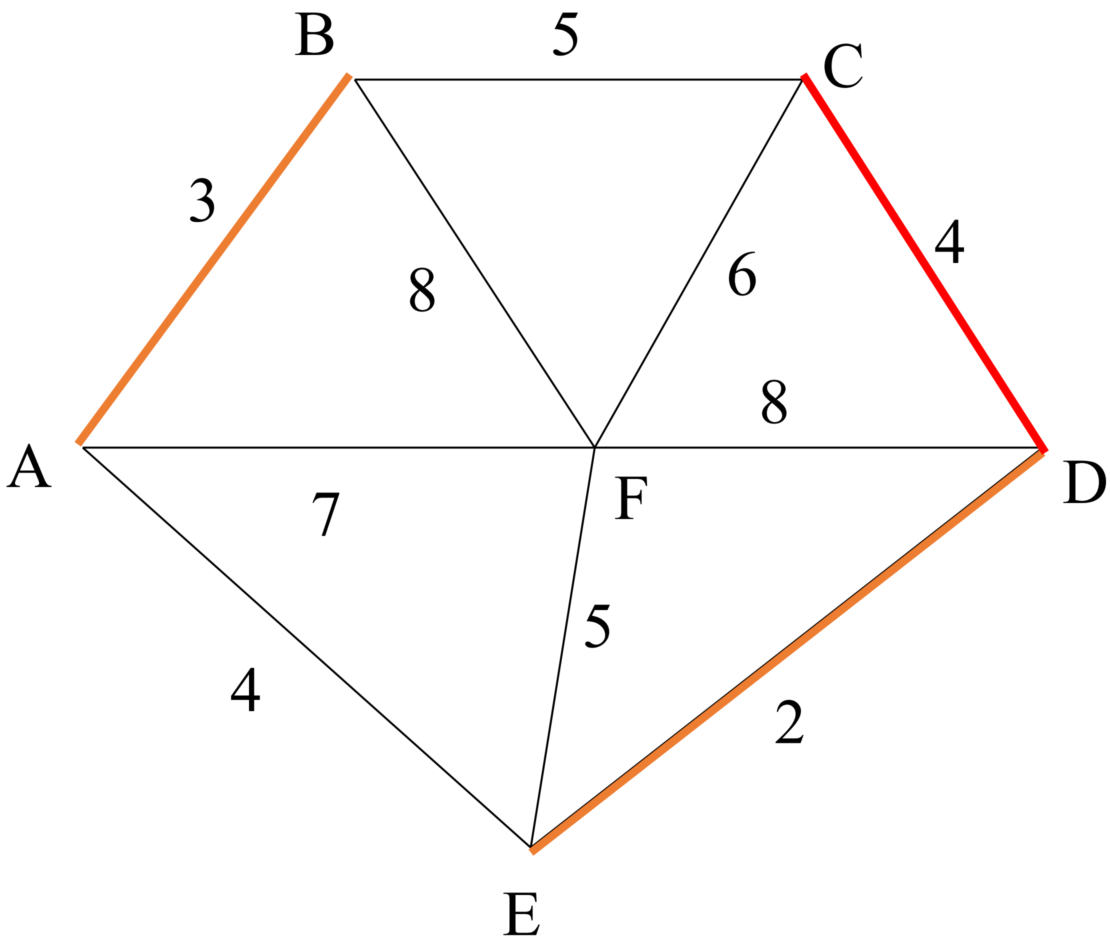
Kruskal’s Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3

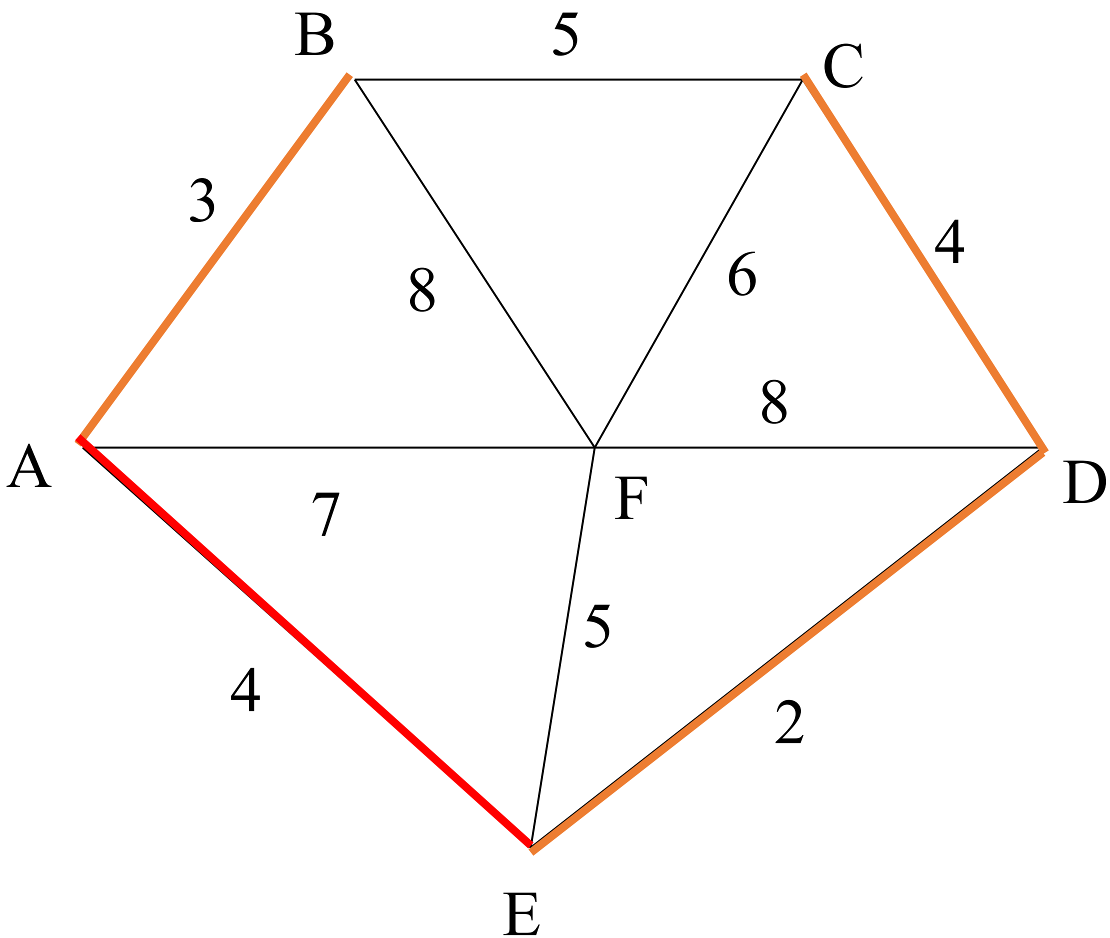
Kruskal's Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3
CD 4 (or AE 4)

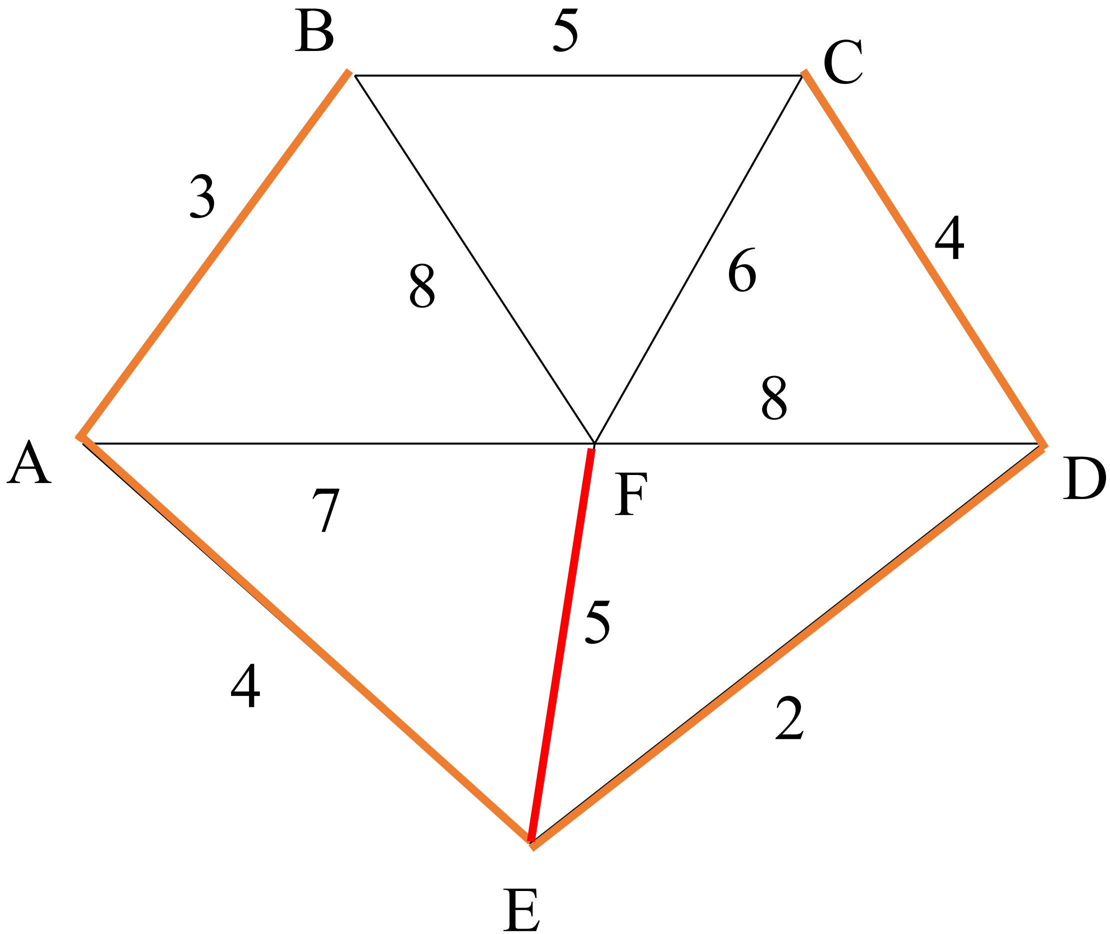
Kruskal's Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3
CD 4
AE 4

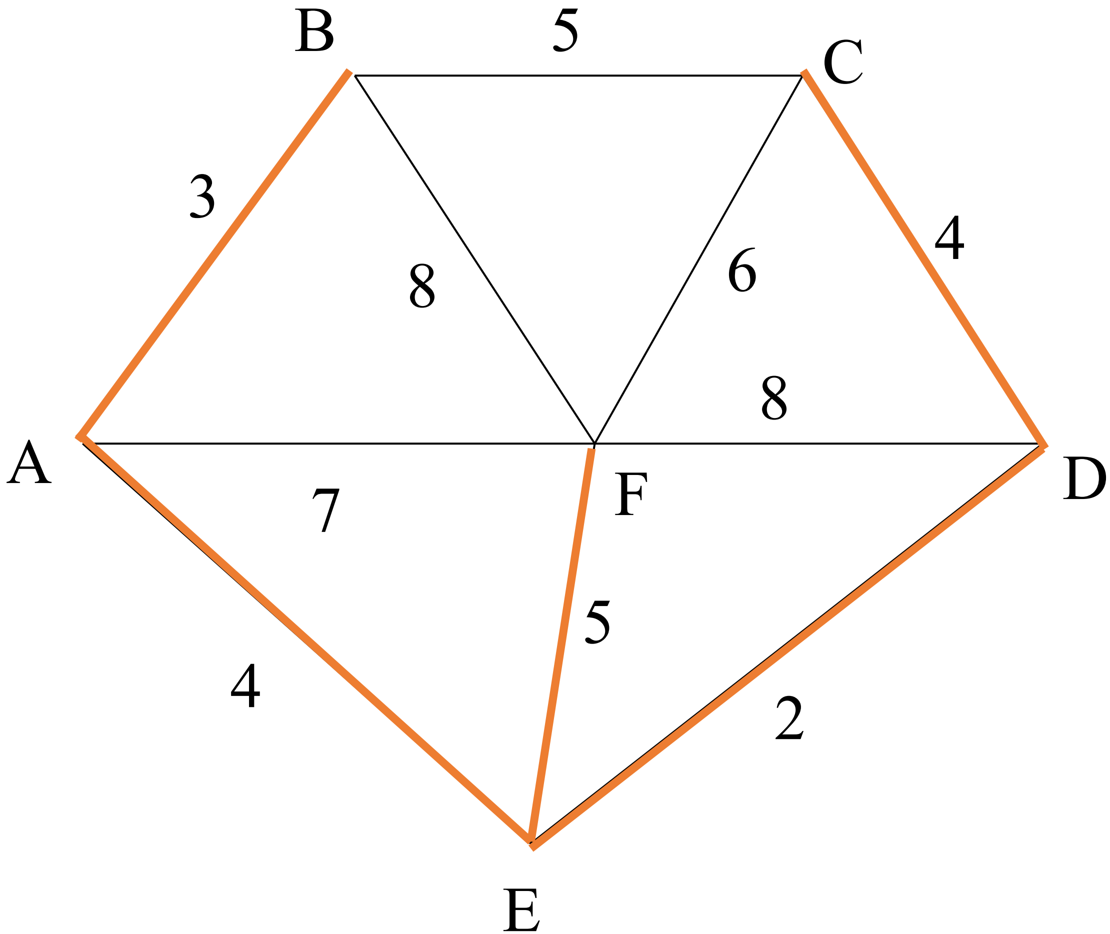
Kruskal's Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3
CD 4
AE 4
BC 5 – forms a cycle
EF 5

Kruskal's Algorithm



All vertices have been connected.

The solution is

ED 2
AB 3
CD 4
AE 4
EF 5

Total weight of tree: 18

Kruskal's MST Algorithm

Input: Weighted undirected graph G = (V, E, W)

Output: A MST T of the graph

1. $T \leftarrow \varnothing$
2. Sort E into non-decreasing order by w — **O(|E|log|E|)**
3. **for** each (u, v) taken from the sorted E **do**
4.   **if** (T U {(u, v)} does not form a cycle) **?**
5. $T \leftarrow T \cup \{(u, v)\}$ — **O(1) * (|V|-1)**
6. **end for**
7. **return** T

How to Check for Cycle in T

Input: Weighted undirected graph G = (V, E, W)

Output: A MST T of the graph

$T \leftarrow \varnothing$

Sort E into non-decreasing order by w   **O(|E|log|E|)**

for each (u, v) taken from the sorted E **do**

  if (T U {(u, v)} does not form a cycle) **|E|times**

    $T \leftarrow T \cup \{(u, v)\}$ **O(1) * (|V|-1)**

end for

return T

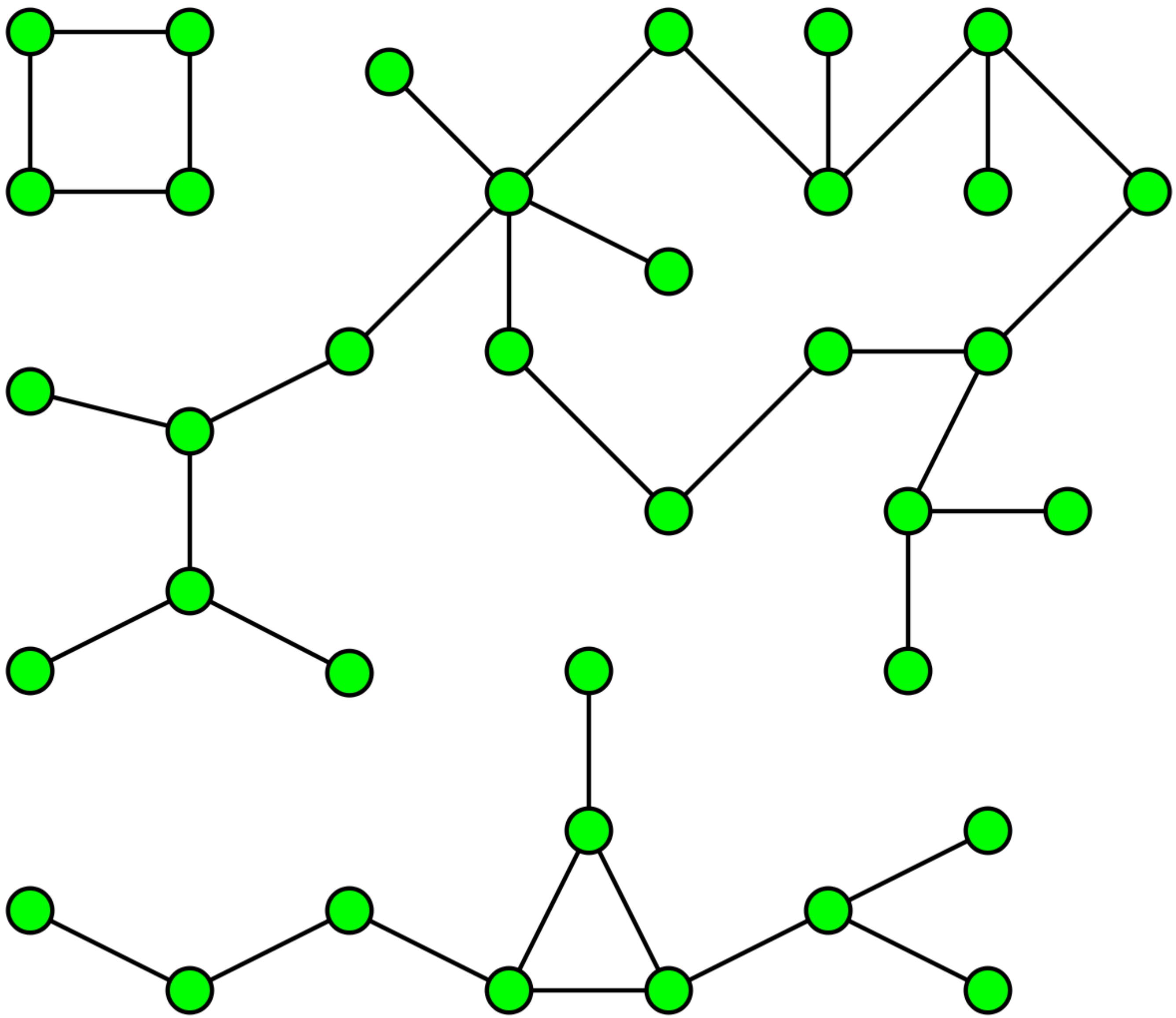
How to Efficiently Check for Cycle in a Graph?

- Need not start from scratch each time
- We need an incremental algorithm
- Can maintain some data structures corresponding to T
- At every step, T changes by union of 1 edge
- Update the data structures efficiently

Connected Components in an Undirected Graph

- Connectivity induces an equivalence relationship on the graph nodes
- Reflexivity: v is connected to v for all v
- Symmetry: if u is connected to v then v is connected to u
- Transitivity: if u is connected to v and v is connected to w, then u is also connected to w
- The set of vertices are partitioned into connected components

How Many Connected Components?



Can Represent Using Sets

- Operation needed
  - S = Initialize(u): Initializes the set with singleton vertex u
  - S = Find(u): Returns the set id of the vertex u
  - S = Union(S1, S2): Takes the union of the two sets

Implementing Union Find using Heaps

- S = Find(u): Returns the set id of the vertex u
- S = Union(S1, S2): Takes the union of the two sets
- S = Initialize(u): Initializes the set with vertex u
- Implement each set as a separate heap
- Smallest vertex id at the root = identity of the set
- find(u): traverse to the root and return the id of root
  - O(log(|V|)
- union(S1, S2): May use binomial heaps
  - Or make smaller height tree the child of the larger height tree
  - O(1)

Kruskal's MST Algorithm

1. T $\leftarrow \varnothing$
2. **for** each vertex v $\in$ V
3.   **do** initialize(v)

   O(V)

4. Sort E into non-decreasing order by *w*
5. **for** each (u, v) taken from the sorted list

   O(|E|log|E|)

6.   **do if** find(u) $\neq$ find(v)

   O(|E|)

7.   **then** T $\leftarrow$ T $\cup$ {(u, v)}
8.   union(find(u), find(v))
9. **return** T

   O(log|V|)

Total runtime: O(|E|log|E| + |E|log|V|) = O(|E|log|E|)

Kruskal's Algorithm Invariant

- After each iteration, every tree in the forest is a MST of the vertices it connects

- Algorithm terminates when all vertices are connected into one tree

Correctness of Kruskal’s

Lemma: If the input graph was connected, then Kruskal’s algorithm ends up finding a spanning tree

Proof: By contradiction. Assume Kruskal’s algo doesn’t return a spanning tree.

1. The algorithm starts with singleton vertices (no edges) and never adds an edge that results into a cycle. So in the worst case, the algorithm might have returned a forest.
2. Let a component U1 be disconnected from a component U2
3. Since, the original graph was connected, there must be an edge (u, v) between U1 and U2.
4. This edge must have been considered by Kruskal’s algorithm. In the final result, u and v were disconnected, they must be disconnected even at the sage when this edge was considered
5. Adding this edge could not have created a cycle. Contradiction!!

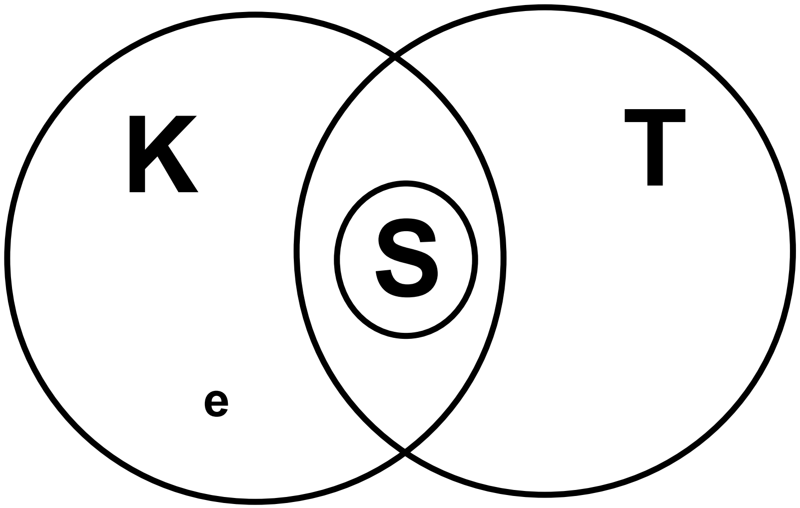
Correctness of Kruskal’s

But is this a *minimum* spanning tree?

Suppose it wasn't.

- There must be point at which it fails, and in particular there must a single edge whose insertion first prevented the spanning tree from being a minimum spanning tree.

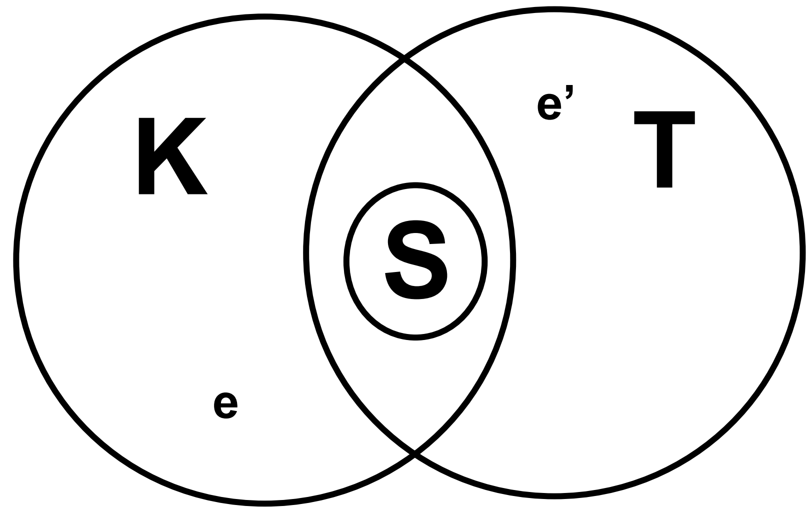
Correctness of Kruskal's



- Let **e** be this first errorful edge.
- Let **K** be the Kruskal spanning tree
- Let **S** be the set of edges chosen by Kruskal's algorithm *before* choosing **e**
- Let **T** be a MST containing all edges in **S**, but not **e**.

Correctness of Kruskal’s

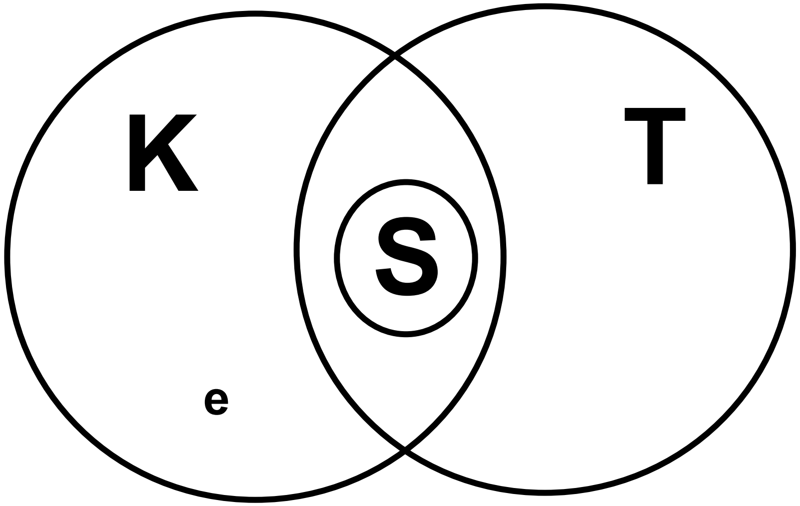
Lemma: w(**e’**) ≥ w(**e**) for all edges **e’** in **T - S**



Proof (*by contradiction*):

- Assume there exists some edge **e’** in **T - S**, w(**e’**) < w(**e**)
- Kruskal’s must have considered **e’** before **e**
- **e’** cannot be in K else it would have been in S
- However, since **e’** is not in **K**, it must have been discarded because it caused a cycle with some of the other edges only in **S**.
- But **e’ + S** is a subgraph of **T**, which means it cannot form a cycle **...Contradiction**

Correctness of Kruskal’s



- Let **e** be this first errorful edge.
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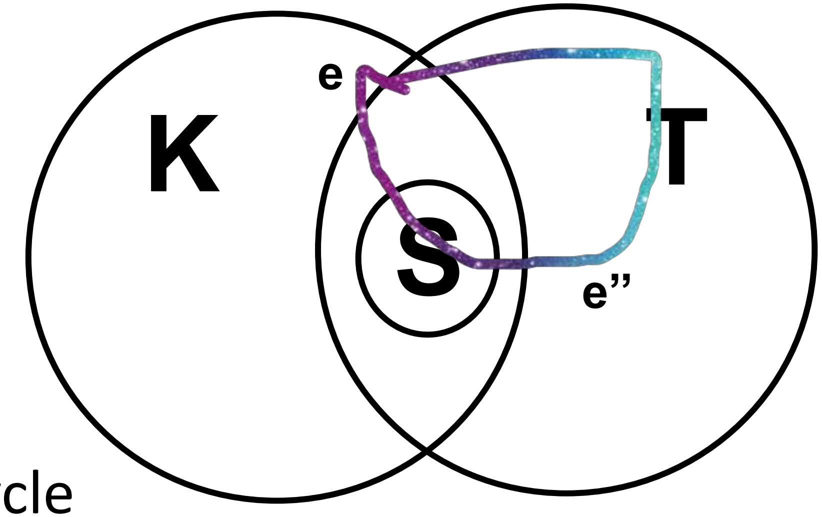
Correctness of Kruskal's



- Inserting edge **e** into **T** will create a cycle
- There must be an edge on this cycle which is not in **K** (*why??*). Call this edge **e''**
- **e''** must be in **T - S,** so (by our lemma) w(**e''**) >= w(**e**)
- We could form a new spanning tree **T'** by swapping **e** for **e''** in **T** (*prove this is a spanning tree*).
- w(**T'**) is clearly no greater than w(**T)**
- But that means **T'** is a MST
- And yet it contains all the edges in **S**, and also **e**

...Contradiction

Thank You