COL202: Discrete Mathematical Sturctures. I semester, 2017-18.
Quiz 1
27 July 2017
Maximum Marks: 4

Name	**Ent. No.**

Important: Keep your answer within the box. Anything written outside the box will be treated as rough work. Do your rough work on the flip side of this sheet.

Q. A roller coaster has n carriages each with 2 seats. n married couples, let each couple comprise one man and one woman to make things simple, want to ride the roller coaster. Each carriage must have one man and one woman but no couple must be seated with each other in the same carriage. In how many ways can they be seated?

$n! \cdot (D_n)$

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Let's say the couples are $\{(x_i, y_i) : 1 \leq i \leq n\}$. To pair them up let us put the x_is in the positions given by their indexes, i.e., $x_1, x_2, \ldots, x_n$ and permute the y_is. Let us count the number of bad cases, i.e., cases where all couples are *not* separated.

Now let us S_i be the set of permutations in which one couple i paired up, i.e., y_i is in position i. Clearly the set of all bad permutations is $\cup_{i=1}^{n} S_i$. And the size of S_i is $(n-1)!$ since we have fixed the pair i and left the rest free. So is $\sum_{i=1}^{n} |S_i| = n \cdot (n-1)!$ the answer? No it isn't because that would mean the number of permutations where no man and wife are together is 0, which is not true since we could simply pair x_i with $y_{i+1 \bmod n+1}$ to satisfy the requirement.

Note that $|\cup_{i=1}^{n} S_i|$ is less than $\sum_{i=1}^{n} |S_i|$ since for any i j such that $i \neq j$, S_i and S_j are not disjoint. So the sum principle cannot be used. However we can count the number of permutations in which both couples i and j are paired up, let us call this set S_{ij}. The size of this set is $(n-2)!$ and there are $\binom{n}{2}$ such sets. So let us subtract $\sum_{i,j:i\neq j} |S_{ij}|$ from $\sum_{i=1}^{n} |S_i|$. But now we have caused another kind of problem. Consider a permutation in which three couples i, j and k were paired. This permutation was added three times, via S_i, S_j and S_k and removed three times via S_{ij}, S_{jk} and S_{ik}, so it is gone. We need to bring it back, so we consider all the sets S_{ijk} where 3 couples are paired and add it back. This causes a problem for permutations with four couples paired and we need to subtract those sets and so forth. This is an instance of the general principle known as the Inclusion Exclusion principle which says that

Given a collection of sets $A_1, A_2, \ldots, A_n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i_i \neq i_2} |A_{i_1} \cap A_{i_2}| \cdots + (-1)^{k-1} \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} |A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}| \cdots + (-1)^{n-1} |A_1 \cap A_2 \cdots \cap A_n|.$$

How do we prove this? Either directly by arguing as we argued above or by induction. Let us omit the proof here, please look it up or try to work it out yourself. Note also that this principle reduces to sum principle if the sets are non-intersecting.

Coming back to our permutations, let us find the general term, i.e., the size of set $S_{i_1 i_2 \cdots i_k}$ where $i_1, i_2, \cdots, i_k$ are distinct. The size of this set is $(n-k)!$ because the remaining couples are left free to be seated in any way and number of such sets of k couples are $\binom{n}{k}$ which is the number of ways of choosing k couples from n. Putting this into the inclusion exclusion principle we get that if S' is the set of bad permutations then

$$|S'| = \sum_{k=1}^{n} (-1)^{k-1} (n-k)! \binom{n}{k} = \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}.$$

Therefore the size of the set of good permutations, S, is given by

$$|S| = n! \left(1 - \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!}\right).$$

Noting that $\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!} \approx \frac{1}{e}$ as n tends to infinity, we see that

$$|S| \approx n!\left(1 - \frac{1}{e}\right).$$

Funny how e shows up in the most unexpected of places. To finish the solution to this problem we further note that the x_is could be seated in the roller coaster in $n!$ ways and since there are two ways of seating two people in a car the total number of seatings is $n! \cdot 2^n \cdot |S|$.