Name: ______________________________

Entry number: ________________________

There are 2 questions for a total of 10 points.

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1. (6 points) Prove or disprove: Given any 17 natural numbers, it is possible to choose 5 whose sum is divisible by 5.

Solution: We will prove the above statement. For $i \in \{0, 1, 2, 3, 4\}$ let S_i denote the sub-collection of the numbers that leave a remainder i when divided by 5. We break the proof into the following two cases:

Case 1 : At least one of S_0, S_1, S_2, S_3, S_4 is empty.
In this case, there is one sub-collection that has at least 5 numbers (otherwise the sum of numbers in all the sub-collections cannot exceed 16). WLOG, let this sub-collection be S_i. Consider any five numbers from S_i. These numbers can be written as: $5k_1 + i, 5k_2 + i, 5k_3 + i, 5k_4 + i, 4k_5 + i$ for some natural numbers $k_1, ..., k_5$ since S_i contains numbers that leave remainder i when divided by i. The sum of these numbers is $5(k_1 + k_2 + k_3 + k_4 + k_5) + 5i$ which is divisible by 5.

Case 2 None of the sets are empty.
In this case, consider one number from each of the sub-collections S_0, S_1, S_2, S_3, S_4. These numbers can be written as $5k_0, 5k_1+1, 5k_2+2, 5k_3+3, 5k_4+4$ for some natural numbers $k_0, ..., k_4$ from the definition of $S_0, ..., S_4$. The sum of these numbers is $5(k_0 + k_1 + k_2 + k_3 + k_4) + 10$ which is divisible by 5.

2. (4 points) Prove or disprove: For any positive integer n, $n^5 - 5n^3 + 4n$ is always divisible by 5.

Solution: We will prove the above statement. Note that $n^5 - 5n^3 + 4n$ can be factored as follows:

$$n^5 - 5n^3 + 4n = n(n^2 - 1)(n^2 - 4) = (n - 2)(n - 1)n(n + 1)(n + 2).$$

We can now consider the following cases:

1. n leaves remainder 0 when divided by 5: In this case, the number is divisible by 5.
2. n leaves remainder 1 when divided by 5: In this case, $(n - 1)$ is divisible by 5 and hence the number is divisible by 5.
3. n leaves remainder 2 when divided by 5: In this case, $(n - 2)$ is divisible by 5 and hence the number is divisible by 5.
4. n leaves remainder 3 when divided by 5: In this case, $(n + 2)$ is divisible by 5 and hence the number is divisible by 5.
5. n leaves remainder 4 when divided by 5: In this case, $(n + 1)$ is divisible by 5 and hence the number is divisible by 5.

Since $n^5 - 5n^3 + 4n$ is divisible by 5 in all the above cases (representing all possibilities for number n), we conclude that $n^5 - 5n^3 + 4n$ is divisible by 5 for all positive integers n.