The exam is open book and open notes. The weightage is 40 marks and the duration is 2 hours. Good luck!

1. Following an announcement from the Director, the students of IITD are super excited to come to the campus for the Holi-2022 semester. The office of the DoS sets out to assign n hostel rooms $r_1, r_2, ...r_n$ to n students $s_1, s2, ...s_n$ in such a way that every room is assigned to exactly one student. Each student s_i submits a subset $A_i \in \{r_1, r_2, ...r_n\}$ of hostel rooms which he/she is willing to take. However, some students are picky and submit "special requests" to be assigned specific rooms if possible. Each special request is a pair (s_i, r_j) where $r_j \in A_i$. (Note that a student may submit 0, 1 or more special requests).

   A special request is said to be *infeasible* if it is impossible to allocate s_i to r_j and still allocate a room to each student. Now,

   (a) **(5 points)** Suppose that $A_i = \{r_1, r_2, ...r_i\}$ for all i. Is it possible to assign a room to every student? Also identify all possible infeasible special requests.

   (b) **(5 points)** Suppose that each A_i is such that it is possible to assign a room to every student s_i. Find the maximum possible no. of infeasible student requests as a function of n and prove your answer.

2. **(8 points)** Recall that $\mathcal{S}_{\mathbb{N}}$ denotes the set of bijective functions from $\mathbb{N}$, the set of natural numbers, to itself, and that $2^{\mathbb{N}}$ denotes the powerset of $\mathbb{N}$. Do $\mathcal{S}_{\mathbb{N}}$ and $2^{\mathbb{N}}$ have the same cardinality? Prove your answer.

3. **(10 points)** Consider the problem of finding the maximum of an array $a_1, \ldots, a_n$ of n distinct real numbers. All of us are smart enough to write a program to do this. However, there is a caveat. You get the numbers one after the other in one of the $n!n!$ possible orders, and on receiving every number a_i you must either say "maximum", or ignore the number. Note that this decision must depend only on the sequence of previously seen numbers (including the current number). You succeed if and only if you call "maximum" exactly once, and the number that you call maximum is indeed the maximum number in the array. Here is an algorithm that attempts to achieve this.

   Ignore the first m numbers. Let T be the maximum of the first m numbers. Thereafter, ignore numbers that are less than T. As soon as you get a number greater than T, say "maximum". Then ignore the remaining numbers. Note that the above algorithm involves the parameter m . For simplicity, assume that the array of numbers is $1, 2, \ldots, n$.Call a permutation $b_1, \ldots, b_n$ of these numbers *favorable* if the algorithm, on input correctly identifies n as the maximum element. Derive an expression for $F(n, m)$, the number of favorable permutations, as a function of n, m. (Hint: First find the number of favorable permutations in which $b_k = n$). Your expression can involve the application of the Harmonic function H given by $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

4. **(12 points)** State whether each of the following statements about groups is true or false, and provide a short justification.

   (a) Let $G = \{a + \sqrt{2}b | a, b \in \mathbb{Q}, a, b \neq 0\}$. Then $(G, \times)$ is a group.

   (b) For every $n \in \mathbb{N} \cup \{0\}$. There exists a group $(G, *)$ such that $|G| = 2^n$ and $g^-1 = g$ for all $g \in G$.

   (c) For every $n, k \in \mathbb{N} \cup \{0\}$ such that $k \leq n$, there exists a group of size $n!$ that has a subgroup of size $k! \cdot (n - k)$.

   (d) For every prime number p, the group $(\mathbb{Z}_p, +_p)$ where $\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$ denotes addition mod p has exactly two subgroups.

   (e) Let f be a homomorphism from a group $(G, *)$ to a group $(H, \#)$. If $(G, *)$ is a commutative group, then $(\mathrm{Im}(f), \#)$ is necessarily a commutative group.

   (f) Let f be a homomorphism from a group $(G, *)$ to a group $(H, \#)$. If $(\mathrm{Im}(f), \#)$ is a commutative group, then $(G, *)$ is necessarily a commutative group.

2) We know that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

so we prove that $\mathbb{N} \nsim 2^{\mathbb{N}}$

If possible let $\mathbb{N} \sim 2^{\mathbb{N}}$

$\rightarrow \exists\ g : \mathbb{N} \rightarrow 2^{\mathbb{N}}$ surjective function

let A_g be a set of natural numbers such that

$$A_g = \{a : a \notin g(a)\}$$

Since $A_g \subseteq 2^{\mathbb{N}}$, and g is surjective

$\rightarrow \exists\ u \in \mathbb{N}$, such that $g(u) = A_g$

If $\quad u \in A_g \rightarrow u \notin g(u) \rightarrow u \notin A_g$

$\rightarrow u \in A_g \rightarrow u \notin A_g$ contradiction

---

Soln) If x is the maximum of first m integers. Then all integers from 1 to m positions are less than x. $\rightarrow$ No. less than x left are $x-m$. We want them to come before n in the next part of the sequence.

Total ways:- $(x-m)! \left(\sum_{i=0}^{x-m} \frac{1}{i!} \right)$

Probability x is smallest of first $m \rightarrow \frac{m(x-1)!}{(x-m-1)!} \frac{1}{n!}$

Probability of success:- $(x-m)! \left(\sum \frac{1}{i!} \right) \frac{m(x-1)!}{n!(x-m-1)!} \frac{1}{(n-m)!}$