

Name: \_\_\_\_\_

Entry number: \_\_\_\_\_

There are 2 questions for a total of 10 points.

1. Answer the following questions.

- (a) (1 point) State true or false (no reasons required): The probability of having an empty bin when throwing  $k$  distinguishable balls randomly into  $n$  distinguishable bins is the same as the probability of having an empty bin when throwing  $k$  indistinguishable balls randomly into  $n$  distinguishable bins.

(a) True**This is for explanation. You were not expected to write this.**

Note that the randomness is over the random choices made when throwing the balls. Suppose there is an  $n$ -sided dice that is rolled every time the ball is supposed to be thrown randomly. The sample space is  $\{1, \dots, n\} \times \{1, \dots, n\} \times \dots \times \{1, \dots, n\}$  (all possible dice rolls) and the number of favourable cases is the same in both the scenarios.

- (b) (4 points) Suppose you flip a biased coin that turns heads with probability  $p$ . What is the probability that you get even number of heads in  $n$  coin tosses. You have to give a concise expression.

(b)  $\frac{1}{2} + \frac{1}{2} \cdot (1 - 2p)^n$ **Solution:** Let  $Q$  denote this probability. Then we have

$$Q = \binom{n}{0} \cdot p^0 \cdot (1-p)^n + \binom{n}{2} \cdot p^2 \cdot (1-p)^{n-2} + \binom{n}{4} \cdot p^4 \cdot (1-p)^{n-4} + \dots$$

We know that

$$((1-p) + p)^n = \binom{n}{0} \cdot p^0 \cdot (1-p)^n + \binom{n}{1} \cdot p^1 \cdot (1-p)^{n-1} + \binom{n}{2} \cdot p^2 \cdot (1-p)^{n-2} + \binom{n}{3} \cdot p^3 \cdot (1-p)^{n-3} + \dots$$

and

$$((1-p) - p)^n = \binom{n}{0} \cdot p^0 \cdot (1-p)^n - \binom{n}{1} \cdot p^1 \cdot (1-p)^{n-1} + \binom{n}{2} \cdot p^2 \cdot (1-p)^{n-2} - \binom{n}{3} \cdot p^3 \cdot (1-p)^{n-3} + \dots$$

Adding the last two equations, we get  $1 + (1 - 2p)^n = 2 \cdot Q$ . This gives  $Q = \frac{1}{2} + \frac{1}{2} \cdot (1 - 2p)^n$ .

2. (5 points) A fair coin is tossed repeatedly until two consecutive heads are tossed. Find the probability that the coin was tossed 11 times. Show calculations in the space below.

2.  $\frac{55}{2048}$

**Solution:** The total number of outcomes of  $n$  fair coin tosses are  $2^n$ . We will count the total number of favourable cases. If  $n \geq 3$ , then any favourable case, can be denoted by a string in  $\{H, T\}^n$  that ends with  $THH$  and does not have consecutive  $H$ 's at any previous position. Let  $a_m$  denote the number of  $H/T$  strings of length  $m$  that does not have consecutive  $H$ 's. Then we have the following recurrence relation  $\forall m \geq 2, a_m = a_{m-1} + a_{m-2}$ . Also,  $a_1 = 2$  and  $a_2 = 3$ . The recurrence relation is true since strings of length  $m$  not having consecutive  $H$ 's is made of strings of length  $m-1$  that do not have consecutive  $H$ 's followed by a  $T$  and strings of length  $m-2$  that do not have consecutive  $H$ 's followed by  $TH$ . Note that the solution of the recurrence relation is  $\forall n \geq 1, a_n = F_{n+2}$ , where  $F_k$  denotes the  $k^{th}$  Fibonacci number ( $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55, F_{11} = 89, F_{12} = 144, \dots$ ).

Note that the number of favourable coin tosses  $\forall n \geq 4$  is given by  $a_{n-3} = F_{n-1}$ . Let  $Q_n$  denote the probability the coin was tossed  $n$  times. Then we have  $Q_1 = 0, Q_2 = 1/4, Q_3 = 1/8$ , and  $\forall n \geq 4, Q_n = \frac{F_{n-1}}{2^n}$ . In general, we can write  $\forall n \geq 1, Q_n = \frac{F_{n-1}}{2^n}$ . So,  $Q_{11} = \frac{F_{10}}{2^{11}} = \frac{55}{2048}$ .