Q1

4 Points

Soon after the launch of Facebook, on one fine day, a data scientist shared the following observations with Mark Zuckerberg.

1. It is possible for every user to reach the profile of every other user through the list of friends. That is, for every pair (S, T) of distinct users, there exists a sequence of users $S = P_0, P_1, \ldots, P_{m-1}, P_m = T$ for some $m \in \mathbb{N}$ such that for all i, P_{i-1} and P_i are friends.
2. There does not exist a sequence of distinct users $P_1, \ldots, P_m$ such that $m \geq 3$, P_{i-1} and P_i are friends for all i, and P_1, P_m are friends too.
3. For every $k \in \{2, \ldots, 2021\}$, there exists exactly one user who has exactly k friends, and no user has more than 2021 friends.

"Interesting," said Mark, "but how many users do we have at the moment?" The data scientist did not have this answer, but after thinking for a minute, she managed to find it from the above observations. What is the answer, and how did she find it?

Let us construct a graph $G = (V, E)$ where each vertex v represents a person P_i for some i. If an edge exists between persons, then they are "friends".

According to observation 1, we know that the graph G is connected, because there exists a walk between every pair of vertices in G.

According to observation 2, we know that the graph G is acyclic, because if there existed a cycle then there must exist a sequence of users $P1, ...P_m$ with $m \geq 3$ such that P_{i-1} and P_i are friends, i.e. exists an edge between them and also exists an edge between P_1 and P_m, which would form a cycle by definition.

We know that if a graph G is connected and acyclic, then it must be a tree and hence must have exactly $n-1$ edges, where n is the size of the vertex set, as we had proved in class.

We recall that every tree has exactly 2 leaf nodes.

Besides this, note that here, the degree of a vertex is analogous to the number of friends a user has. We know two vertices have degree 1 (since they are leaves). No vertex has degree 0 since G is connected. Also, we are given exactly one vertex has exactly degree k for $k \in \{2, 3, ...2021\}$, and no vertex has degree > 2021. We know that by the handshake lemma,

$$\sum_{n} d_v = 2|E|$$

where d_v is the degree of the vertices and $|E|$ is the size of the edge set. We have, $|E| = n - 1$ and the LHS is simply, 1 + $\sum_{i=1}^{2021} i = 1 + \frac{(2021)(2022)}{2} = 2043232$
So we get,

$$2(n-1) = 2043232$$

Finally we get,

$$\boxed{n = 1021617}$$

This is probably the process the data scientist used to get the number of users.

Q2

6 Points

Let $k, n \in \mathbb{N}$. Consider a matrix A with k rows and n columns in which each of the numbers $\{0, 1, \ldots, nk-1\}$ appears exactly once. We call a set $S \subseteq \{0, 1, \ldots, nk-1\}$ $nice$ if it satisfies the following two properties.

1. No two distinct elements in S are in the same column of A.
2. The difference of no two distinct elements in S is divisible by n (in other words, no two distinct elements of S are congruent mod n).

Prove that given any such matrix A, there always exists a nice set S with $|S| = n$.

We will prove the above claim by construction.

Let $B = \{0, 1, 2, ...nk - 1\}$. Let $B_i = \{b \in B | b \mod n = i\}$. We observe that $|B_i| = k \ \forall\ i$, since there exists exactly one element

1.

We can model the network of friends as an undirected graph, with a vertex corresponding to each user, and an edge (P1, P2) iff P1 and P2 are friends.

Since there is a path from every user to every other user, the graph is connected.

Since there are no cyclical paths, the graph is acyclic.

Suppose there are N users. The graph shall be a connected acyclic graph with N-1 edges. By handshake lemma, the sum of degrees of all vertices shall be 2(N-1)

Since the graph is connected, every vertex must have degree at least 1. There are no vertices of degree more than 2021, and one vertex for every degree from 2 to 2021. Let the number of nodes of degree 1 be k. The sum of degrees of all vertices = k + (2 + 3 + … + 2021). The number of vertices N shall be k + 2020.

Sum of degrees for vertices = 2(k+2020 - 1) = k+(2 +3 + … + 2021)

-> k = (2 + 3 + … + 2021) - 4038

-> N = (2 + 3 + … + 2021) - 2018

-> N = 2041212

2.

Let G be a bipartite graph with bipartition (L, R) defined as follows.

L={a_1,…,a_n} is a set of n vertices (where a_i represents the i'th column of A).

R={b_0,…,b_{n-1}} is a set of n vertices (where b_j represents the remainder j mod n).

Edge set E is defined as follows: For each i,j, {a_i,b_j} \in E if the i'th column of A contains a number that is j mod n.

We first claim that the above bipartite graph has a perfect matching using Hall’s theorem.

Consider any subset T of L. We claim that |T| <= |N(T)|, where N(T) is the set of neighbours of vertices in T.

The columns indexed by elements of T contain k*|T| (distinct) numbers. All these numbers, mod n, have one of |N(T)| remainders. But for every j in {0,..,n-1}, exactly k elements of A are j mod n. Therefore, k*|T| <= k*|N(T)|. Thus |T| <= |N(T)|.

Thus, by Halls theorem, the graph contains a perfect matching, say M.

Construct a set S as follows. For each a_i, look at its partner under M, say b_j. Since {a_I,b_j} \in E, the i'th column of M is guaranteed to contain a number which is j mod n. Include one such number in S. Thus, we include one number from each column of A in S. Moreover, since each b_j is matched to exactly one a_i, we include in S exactly one number that is j mod n.

Thus, S is a nice set.