Name: ____________________

Entry number: ____________________

There are 2 questions for a total of 10 points.

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1. (5 points) Prove or disprove: $[-1, 1]$ has the same cardinality as $(1, 3) \cup (4, 6)$.

Solution: We give injective mappings from $[-1, 1]$ to $(1, 3) \cup (4, 6)$ and from $(1, 3) \cup (4, 6)$ to $[-1, 1]$ to show that the cardinality of $[-1, 1]$ is the same as the cardinality of $(1, 3) \cup (4, 6)$.

Claim 1: There is an injective mapping from $[-1, 1]$ to $(1, 3) \cup (4, 6)$.

Proof. Consider the function $f : [-1, 1] \rightarrow (1, 3) \cup (4, 6)$ defined as:

$$f(x) = 0.5 \cdot x + 2$$

For any inputs $a, b \in [-1, 1]$ $f(a) = f(b)$ implies that $0.5 \cdot a + 2 = 0.5 \cdot b + 2$ which implies that $a = b$. This shows that f is injective. $\square$

This shows that $|[-1, 1]| \leq |(1, 3) \cup (4, 6)|$.

Claim 2: There is an injective mapping from $(1, 3) \cup (4, 6)$ to $[-1, 1]$.

Proof. Consider the function $g : (1, 3) \cup (4, 6) \rightarrow [-1, 1]$ defined as:

$$g(x) = 0.1 \cdot x$$

This function is injective since for any $a, b \in (1, 3) \cup (4, 6)$, $f(a) = f(b)$ implies that $0.1 \cdot a = 0.1 \cdot b$ which implies that $a = b$. This shows that g is injective. $\square$

The above claim shows that $|(1, 3) \cup (4, 6)| \leq |[-1, 1]|$.

Using Schröeder-Bernstein Theorem, we conclude that $|[-1, 1]| = |(1, 3) \cup (4, 6)|$.

2. Let A, B, C be non-empty sets, and let $g : A \rightarrow B$ and $h : A \rightarrow C$ and let $f : A \rightarrow B \times C$ defined as:

$$f(x) = (g(x), h(x)).$$

Answer the following:

(a) ($1/2$ point) State true or false: If f is onto, then both g and h are onto.

(a) **True**

(b) ($1/2$ point) State true or false: If g and h are onto, then f is onto.

(b) **False**

(c) ($1/2$ point) State true or false: If at least one of g, h are one-to-one, then f is one-to-one.

(c) **True**

(d) ($1/2$ point) State true or false: If g and h are not one-to-one, then f is not one-to-one.

(d) **False**

(e) (3 points) Give reasons for your answer to part (b).

Solution: We give a counter example. Let $A = B = \{1, 2\}$ and let $g(x) = x$ and $h(x) = x$. Note that both g and h here are onto functions but f is not onto since $(1, 2)$ does not have a pre-image.

Reason for part (d) (You were not required to give this):
We give a counterexample. Consider $A = \{0, 1, 2, 3\}$, $B = \{0, 1\} = C$. Functions g and h are defined as follows: $g(0) = g(1) = 0$ and $g(2). = g(3) = 1$ and $h(0) = h(2) = 0$ and $h(1) = h(3) = 1$. Hence g and h are not one-to-one. However, note that in this case $f(0) = (0, 0), f(1) = (0, 1), f(2) = (1, 0)$, and $f(3) = (1, 1)$ which is a one-to-one function.