Q1

6 Points

A *permutation* of a set S is a bijection from S to S. Given a function $f : S \longrightarrow S$, the associated relation R_f on S is $R_f = \{(x, y) \in S \times S \mid f(x) = y\}$. Note that the transitive closure R_f^+ of the relation R_f is the following relation.

$$R_f^+ = \{(x, y) \in S \times S \mid \exists m \in \mathbb{N}\, \exists z_0, \ldots, z_m \in S \text{ such that}$$

$$x = z_0,\, z_m = y, \text{ and } \forall i\; z_i = f(z_{i-1})\},$$

or equivalently,

$$R_f^+ = \{(x, y) \in S \times S \mid \exists m \in \mathbb{N}\; y = f^m(x)\},$$

where f^m denotes the function f composed m times. Henceforward, we assume that f is a permutation of a finite set S with $|S| = n$. We wish to prove that R_f^+ is an equivalence relation. It is transitive, by definition, already.

Q1.1

3 Points

Prove that R_f^+ is reflexive.

Let us consider any element $x \in S$. We claim that $\exists m \leq n$, where $n = |S|$ such that $f^m(x) = x$. Suppose such an m does not exist, i.e. $f^m(x) \neq x\; \forall m \leq n$.

Note that since f is a bijection, each element of S is mapped to by exactly one element of S under f. Under the above contrapositve assumption we make the following claim: $f^k(x) \neq f^m(x)\; \forall k < m \leq n$. The base case $m = 2$ is trivial to check. Suppose it is true for some m. Then for $m + 1$, suppose $f(f^m(x)) = y$. If $y = f^k(x)$ for some $k < m$, then by bijectivity of f, $f^m(x) = f^{k-1}(x)$ which violates the inductive hypothesis. Hence the claim is true.

Now since $f^k(x) \neq f^m(x)\; \forall k < m \leq n$, after n applications of f on x we must get an element distinct from all the previous elements.

But this means $|S| = n + 1$, which is not possible. Hence our assumption is false and so $\exists m \leq n$, where $n = |S|$ such that $f^m(x) = x$. Thus, R_f^+ is reflexive.

Q1.2

3 Points

Prove that R_f^+ is symmetric.

Suppose $(x, y) \in R_f^+$, i.e. exists $m_1 \in \mathbb{N}$ such that $f^{m_1}(x) = y$. There may exist more than one m_1 but by WOP we choose the smallest such m_1. Now, since f is a bijection, f^{-1} exists and so $(f^{-1})^{m_1}(y) = x$.

By the previous claim, we know that there exists $m_2 \in \mathbb{N}$ such that $f^{m_2}(y) = y$. Apply f^{-1} on both sides of the equation m_1 times to get, $f^{m_2 - m_1}(y) = (f^{-1})^{m_1}(y) = x$, while noting that $m_2 > m_1$ because otherwise the minimality of m_1 is contradicted. So, $\exists m = m_2 - m_1$ such that $f^m(y) = x$, and so R_f^+ is symmetric.

Q2

4 Points

Next, we want to count the number of permutations f of a set of a given size such that R_f^+ has a given number of equivalence classes. Let Q_k^n denote the set of permutations f of the set $\{1, \ldots, n\}$ such that R_f^+ has exactly k equivalence classes. Let $b(n, k) = |Q_k^n|$. We will now derive the following recurrence for the numbers $b(n, k)$.

$$b(n, k) = b(n-1, k-1) + (n-1) \cdot b(n-1, k).$$

Q2.1

1 Point

Prove that $b(n-1, k-1)$, the first term on the right-hand-side, is exactly the number of permutations $f \in Q_k^n$, such that $f(n) = n$.

Suppose we have $f(n) = n$, then the equivalence class of $[n]_R = \{n\}$, since $f^m(n) = n \; \forall \; n \in \mathbb{N}$. We need to create $k-1$ equivalence classes of the remaining $n-1$ elements of the set and so $b(n-1, k-1)$ is exactly the number of permutations $f \in Q_k^n$ with $f(n) = n$.

Q2.2

3 Points

Prove that $(n-1) \cdot b(n-1, k)$, the second term on the right-hand-side, is exactly the number of permutations $f \in Q_k^n$, such that $f(n) \neq n$.

Suppose we have $f(n) \neq n$. Then we should create k equivalence classes from the remaining $n-1$ elements

Quiz 3

GRADED

STUDENT

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TOTAL POINTS

8 / 10 pts

QUESTION 1

	(no title)	**6** / 6 pts
1.1	(no title)	**3** / 3 pts
1.2	(no title)	**3** / 3 pts

QUESTION 2

	(no title)	**2** / 4 pts
2.1	(no title)	**1** / 1 pt
2.2	(no title)	**1** / 3 pts

1.1.

Let x be an arbitrary element of S. We need to prove $(x,x) \in R_f^+$.

Let $x_0 = x$ and $x_I = f^i(x)$ for each $I \in N$. Since $|S|=n$, some two of $x_0, x_1, \ldots, x_n$ must be equal. Say $x_i = x_j$, where $i<j$. Applying f^{-1} $j-i$ times, we get $x = x_0 = x_{j-i} = f^{j-i}(x)$, where $j-I>0$.

Thus, $(x,x) \in R_f^+$.

1.2

Suppose $(x,y) \in R_f^+$. We need to prove $(y,x) \in R_f^+$.

From part 1, for every x in S there exists a k in N such that $f^k(x) = x$. Also, since $(x,y) \in R_f^+$, there exists some l in N such that $y = f^l(x)$. Let m be a multiple of k greater than l. Then $f^{m-l}(y) = f^m(x) = x$. Thus, $(y,x) \in R_f^+$.

2.1.

$P^* = Q^n_k$ intersection $\{f \mid f$ is a bijection from S to S and $f(n) = n\}$ is in bijective correspondence with Q_{n-1}^{k-1} as follows. f in P^* is mapped g in Q_{n-1}^{k-1}, where $g(x) = f(x)$ for all x in $\{1,\ldots,n-1\}$. R_g^+ has $k-1$ equivalence classes, namely, the equivalence classes of R_f^+ other than $\{n\}$ (nothing else is in the equivalence class of n). Note that every g in Q_{n-1}^{k-1} has a unique pre-image f in P^* under this mapping, where $f(x) = g(x)$ for all x in $\{1,\ldots,n-1\}$, and $f(n) = n$.

2.2

For a fixed z in $\{1,\ldots,n-1\}$ let us count the size of the set $P_z = Q^n_k$ intersection $\{f \mid f$ is a bijection from S to S and $f(n) = z\}$. P_z is in bijective correspondence with Q^{n-1}_k as follows. f in P_z is mapped to g in Q_{n-1}_k, where $g(f^{-1}(n)) = z$, and for all other x in $\{1,\ldots,n-1\}$, $g(x) = f(x)$. Note that the equivalence classes of R_g^+ are the same as those of R_f^+, except that n disappears from its equivalence class, which contains more elements (eg. z). Given a g in Q^{n-1}_k, the unique f in P_z which maps to g is given by $f(g^{-1}(z)) = n$, $f(n) = z$, and for all other x in $\{1,\ldots,n\}$, $f(x) = g(x)$. Putting together the facts $|P_z| = |Q^{n-1}_k|$, P_z's are disjoint, and their union is Q^n_k, the claim stands proved.