Name: ______________________

Entry number: ______________________

There are 4 questions for a total of 15 points.

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1. Answer the following questions on Propositional Logic.

(a) (1/2 point) Fill the truth-table below:

| $P$ | $Q$ | $R$ | $P \leftrightarrow Q$ | $Q \vee \neg R$ | $(P \leftrightarrow Q) \rightarrow (Q \vee \neg R)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | T |
| T | F | T | F | F | T |
| F | T | T | F | T | T |
| T | F | F | F | T | T |
| F | T | F | F | T | T |
| F | F | T | T | F | F |
| F | F | F | T | T | T |

(b) (1 point) Consider the following two compound proposition $P, Q$:

$$P : (A \vee B) \rightarrow C \quad \text{and} \quad Q : (\neg C \rightarrow \neg A) \vee (\neg C \rightarrow \neg B)$$

Which of the following describe the relationship between $P$ and $Q$? Circle all the correct choices and show your reasoning in the space below.

(a) $P$ and $Q$ are equivalent

(b) $P \rightarrow Q$

(c) $Q \rightarrow P$

**Solution:** We solve this using a truth table.

| **A** | **B** | **C** | **P** | **Q** |
|---|---|---|---|---|
| F | F | F | T | T |
| F | F | T | T | T |
| F | T | F | F | T |
| T | F | F | F | T |
| T | T | F | F | F |
| T | F | T | T | T |
| F | T | T | T | T |
| T | T | T | T | T |

- $P$ and are not equivalent since the columns for $P$ and $Q$ do not match.
- $Q \rightarrow P$ does not hold since in the third row, $Q$ evaluates to $T$ but $P$ evaluates to $F$.
- $P \rightarrow Q$ holds since there is no row in which $P$ is $T$ but $Q$ is $F$.

So, the correct answer is option (b).

2. Answer the following questions on Predicate Logic.

   (a) (4 ½ points) Consider the following predicates:

   1. $B(x)$: $x$ is brilliant.
   2. $S(x)$: $x$ studies hard.
   3. $L(x)$: $x$ is lucky.
   4. $C(x, y)$: $x$ clears the final exam of course $y$.
   5. $G(x, y)$: $x$ gets an A grade in course $y$.
   6. $J(x)$: $x$ sleeps too much.

   Express each of the statements using quantifiers and the predicates given above. The domain of variable $x$ in the above predicates is the set of all students of COL202 and domain of variable $y$ is the set of all courses being taught at IIT Delhi during Semester-I-2018-19.

| | **Statement** | **Quantified expression** |
|---|---|---|
| $S_1$ | Everyone who clears any final exam studies hard or is brilliant or is lucky. | $\forall x\ [\exists y\ C(x, y) \rightarrow (S(x) \vee B(x) \vee L(x))]$ |
| $S_2$ | Everyone who gets an A in some course has cleared the final exam of some course. | $\forall x\ [\exists y\ G(x, y) \rightarrow \exists z\ C(x, z)]$ |
| $S_3$ | No one is lucky. | $\forall x\ [\neg L(x)]$ |
| $S_4$ | Anyone who sleeps too much does not study hard. | $\forall x\ [J(x) \rightarrow \neg S(x)]$ |
| $S_5$ | If everyone gets an A in some course, then everyone who sleeps too much is brilliant. | $(\forall x\ \exists y\ G(x, y)) \rightarrow (\forall p\ (J(p) \rightarrow B(p)))$ |

   (b) (1 point) Consider the quantified expressions $S_1, ..., S_5$ obtained in the previous part. Use the expressions obtained in the previous part to replace $S_1, ..., S_5$ below and then determine whether it makes a valid argument form. Answer "yes" or "no". You do not need to give explanation for this problem.

$S_1$
$S_2$
$S_3$
$S_4$
$\therefore S_5$

(b) **True**

**Reason (You were not supposed to give this)**: We obtain the following argument form by replacing the $S_1, ..., S_5$ above.

$\forall x\ [\exists y\ C(x,y) \rightarrow (S(x) \vee B(x) \vee L(x))]$
$\forall x\ [\exists y\ G(x,y) \rightarrow \exists z\ C(x,z)]$
$\forall x\ [\neg L(x)]$
$\forall x\ [J(x) \rightarrow \neg S(x)]$

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$\therefore (\forall x\ \exists y\ G(x,y)) \rightarrow (\forall w\ (J(w) \rightarrow B(w)))$

We will show that the above argument form is valid using rules of inference:

1. $\forall x\ [\exists y\ C(x,y) \rightarrow (S(x) \vee B(x) \vee L(x))]$ (Premise)
2. $\forall x\ [\exists y\ G(x,y) \rightarrow \exists z\ C(x,z)]$ (Premise)
3. $\forall x\ [\neg L(x)]$ (Premise)
4. $\forall x\ [J(x) \rightarrow \neg S(x)]$ (Premise)
5. $\exists y\ C(s,y) \rightarrow (S(s) \vee B(s) \vee L(s))$ for an arbitrary student $s$
   (From (1) using Universal instantiation)
6. $\exists y\ G(s,y) \rightarrow \exists z\ C(s,z)$
   (From (2) using Universal instantiation)
7. $\exists y\ G(s,y) \rightarrow (S(s) \vee B(s) \vee L(s))$
   (From (5) and (6) using modus ponens)
8. $\forall x\ [\exists y\ G(x,y) \rightarrow (S(x) \vee B(x) \vee L(x))]$
   (From (7) using Universal generalization)
9. $\forall x\ [(\forall y\ \neg G(x,y)) \vee S(x) \vee B(x) \vee L(x)]$
   (From (8) using De Morgan's law for quantifiers and $p \rightarrow q \equiv p \vee q$)
10. $(\forall y\ \neg G(s,y)) \vee S(s) \vee B(s) \vee L(s)$ for an arbitrary student $s$
    (From (9) using Universal generalization)
11. $\neg L(s)$
    (From (3) using Universal generalization)
12. $(\forall y\ \neg G(s,y)) \vee S(s) \vee B(s)$
    (Resolvent of (10) and (11))
13. $\forall x\ [\neg J(x) \vee \neg S(x)]$
    (From (4) using $p \rightarrow q \equiv p \vee q$)
14. $\neg J(s) \vee \neg S(s)$
    (From (13) using Universal generalization)
15. $(\forall y\ \neg G(s,y)) \vee \neg J(s) \vee B(s)$
    (Resolvent of (12) and (14))
16. $\exists x\ [(\forall y\ \neg G(x,y)) \vee \neg J(s) \vee B(s)]$
    (From (15) using existential generalization)
17. $\forall w\ \exists x\ [(\forall y\ \neg G(x,y)) \vee \neg J(w) \vee B(w)]$
    (From (16) using universal generalization)
18. $(\exists x\ \forall y \neg G(x,y)) \vee (\forall w\ (J(w) \rightarrow B(w)))$
    (From (18) using $p \rightarrow q \equiv p \vee q$)
19. $(\forall x\ \exists y\ G(x,y)) \rightarrow \forall w\ (J(w) \rightarrow B(w))$
    (From (19) using $p \rightarrow q \equiv p \vee q$ and De Morgan's law for quantifiers)

Note that step (16) is a correct but a bit unconventional application of existential generalization. In general, if we have a statement $P(s) \vee Q(s)$ that holds for arbitrary $s$ in the domain, then $P(s) \vee (\exists x\ Q(x)) \equiv \exists x\ [P(s) \vee Q(x)]$ also holds for an arbitrary element $s$ of the domain. This is the fact that we have used here.

(c) (2 ½ points) Consider the quantified expressions $S_1, ..., S_4$ obtained in part (a). Use the expressions obtained in part (a) to replace $S_1, ..., S_4$ below and then determine whether it makes a valid argument form. Explain your answer. (*If your answer is "yes", then you need to show all steps while using rules of inference*)

$$\begin{array}{l} S_1 \\ S_2 \\ S_3 \\ S_4 \\ \hline \therefore \forall x\ [(\exists y\ G(x,y)) \rightarrow (J(x) \rightarrow B(x))] \end{array}$$

**Solution:** We obtain the following argument form by replacing the $S_1, ..., S_4$ above.

$$\begin{array}{l} \forall x\ [\exists y\ C(x,y) \rightarrow (S(x) \vee B(x) \vee L(x))] \\ \forall x\ [\exists y\ G(x,y) \rightarrow \exists z\ C(x,z)] \\ \forall x\ [\neg L(x)] \\ \forall x\ [J(x) \rightarrow \neg S(x)] \\ \hline \therefore \forall x\ [(\exists y\ G(x,y)) \rightarrow (J(x) \rightarrow B(x))] \end{array}$$

We will show that the above argument form is valid using rules of inference:

1. $\forall x\ [\exists y\ C(x,y) \rightarrow (S(x) \vee B(x) \vee L(x))]$ (Premise)
2. $\forall x\ [\exists y\ G(x,y) \rightarrow \exists z\ C(x,z)]$ (Premise)
3. $\forall x\ [\neg L(x)]$ (Premise)
4. $\forall x\ [J(x) \rightarrow \neg S(x)]$ (Premise)
5. $\exists y\ C(s,y) \rightarrow (S(s) \vee B(s) \vee L(s))$ for an arbitrary student $s$
(From (1) using Universal instantiation)
6. $\exists y\ G(s,y) \rightarrow \exists z\ C(s,z)$
(From (2) using Universal instantiation)
7. $\exists y\ G(s,y) \rightarrow (S(s) \vee B(s) \vee L(s))$
(From (5) and (6) using modus ponens)
8. $\forall x\ [\exists y\ G(x,y) \rightarrow (S(x) \vee B(x) \vee L(x))]$
(From (7) using Universal generalization)
9. $\forall x\ [(\forall y\ \neg G(x,y)) \vee S(x) \vee B(x) \vee L(x)]$
(From (8) using De Morgan's law for quantifiers and $p \rightarrow q \equiv p \vee q$)
10. $(\forall y\ \neg G(s,y)) \vee S(s) \vee B(s) \vee L(s)$ for an arbitrary student $s$
(From (9) using Universal generalization)
11. $\neg L(s)$
(From (3) using Universal generalization)
12. $(\forall y\ \neg G(s,y)) \vee S(s) \vee B(s)$
(Resolvent of (10) and (11))
13. $\forall x\ [\neg J(x) \vee \neg S(x)]$
(From (4) using $p \rightarrow q \equiv p \vee q$)
14. $\neg J(s) \vee \neg S(s)$
(From (13) using Universal generalization)

15. $(\forall y\ \neg G(s, y)) \vee \neg J(s) \vee B(s)$
    (Resolvent of (12) and (14))
16. $\forall x\ [(\forall y\ \neg G(x, y)) \vee \neg J(x) \vee B(x)]$
    (From (15) using Universal generalization)
17. $\forall x\ [(\exists y\ G(x, y)) \rightarrow (J(x) \rightarrow B(x))]$
    (From (16) using $p \rightarrow q \equiv p \vee q$)

3. (3 points) Prove or disprove: Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection and let $h : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined as $h(a, b, c) = f(f(a, b), c)$. Then $h$ is a bijection from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

**Solution:** We will prove that the given statement holds. To show that $h$ is a bijection, we need to show that $h$ is one-to-one and onto.

Claim 1: $h$ is a one-to-one function.

*Proof.* From the definition of one-to-one functions, we need to argue that for any inputs $(a, b, c), (a', b', c') \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, if $h(a, b, c) = h(a', b', c')$, then $a = a', b = b', c = c'$. Indeed, $h(a, b, c) = h(a', b', c')$ implies that $f(f(a, b), c) = f(f(a', b'), c')$. Since $f$ is one-to-one, this implies that $f(a, b) = f(a', b')$ and $c = c'$. Now using the fact that $f(a, b) = f(a', b')$ and that $f$ is one-to-one, we get that $a = a'$ and $b = b'$. So, we get that if $h(a, b, c) = h(a', b', c')$, then $a = a'$, $b = b'$, and $c = c'$. This completes the proof of the claim. $\square$

Claim 2: $h$ is onto.

*Proof.* Using the definition of onto functions, we need to argue that for any $r \in \mathbb{N}$, there exists $(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $h(a, b, c) = r$. Note that since $f$ is an onto function, there exists $(r', c') \in \mathbb{N} \times \mathbb{N}$ such that $f(r', c') = r$. Again, using the fact that $f$ in an onto function, there exists $(a', b') \in \mathbb{N} \times \mathbb{N}$ such that $f(a', b') = r'$. This means that $h(a', b', c') = f(f(a', b'), c') = f(r', c') = r$. This completes the proof of the claim. $\square$

From Claim 1 and Claim 2, we conclude that $h$ is a bijection.

4. (2 ½ points) Recall the definition of the big-O notation given in the lectures:

   Let $f(n)$ and $g(n)$ denote functions mapping positive integers to positive real numbers. The function $f(n)$ is said to be $O(g(n))$ if and only if there exists constants $C, n_0 > 0$ such that for all $n \geq n_0, f(n) \leq C \cdot g(n)$.

   Prove or disprove: For any functions $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ and $g : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ if $f(n)$ is $O(g(n))$, then $5^{f(n)}$ is $O(5^{g(n)})$.

   **Solution:** We will disprove the statement. Consider $f(n) = 2n$ and $g(n) = n$. For these functions we can show that $f(n) = O(g(n))$ since for all $n \geq 1$, $f(n) \leq 2 \cdot g(n)$. However, $5^{f(n)} = 5^{2n}$ and $5^{g(n)} = 5^n$. For any constant $c > 0$, if $c < 1$, then $5^{f(n)} > c \cdot 5^{g(n)}$ for all $n > 0$, otherwise we can show that for all $n \geq \lceil \log_5 c \rceil + 1$, $5^{f(n)} > c \cdot 5^{g(n)}$. This is because if $n \geq \lceil \log_5 c \rceil + 1$, then $5^n > c$, which further implies $5^{2n} > c \cdot 5^n$.