Q1 Well-behaved numbers

4 Points

Call a complex number x *well-behaved* if there exists a natural number d and there exist $d+1$ integers, $a_0, a_1, \ldots, a_d$, not all zero, such that $\sum_{i=0}^{d} a_i x^i = 0$. Let $W \subseteq \mathbb{C}$ denote the set of all well-behaved complex numbers.

(To get some intuition, observe that every rational number is well-behaved, and so are some irrational real numbers like $2^{1/4}$, and some non-real complex numbers like $-(1/2) + (\sqrt{3}/2)\iota$, etc.)

Is W countable? Prove your answer. You may use the fact that a nonzero polynomial of degree d has at most d roots.

Claim: W is countable.

Proof:

Consider the equation

$$a_o + a_1 x + a_2 x^2 + ... a_d x^d = 0$$

Let A_d denote the set of roots of the above equation. Notice that the above equation is a non-zero polynomial of degree d, so it must have at most d roots. Let us denote the set of roots of the above equation by A_d. Then we have, $|A_d| \leq d$.

Every well-behaved complex number must satisfy at least one such equation by definition. Therefore, we may write the set of well behaved complex numbers as,

$$W = \bigcup_{n=1}^{\infty} A_d$$

Now we know by the result proved in class, a countably infinite union of countable sets is itself countable. Here, A_d are finite for all d and since degrees are natural numbers (which are countable), thus it is a countably infinite union.

Hence, W is countable. ■

Q2 Non-conflicting transportation

6 Points

Call two walks in a graph $\mathit{conflicting}$ if there exists an edge in the graph traversed by both of them (possibly in opposite directions). Given a connected graph G and a subset T of its vertices with $|T|$ even, we call a set R of $|T|/2$ paths a $\mathit{transportation\ of} T\ \mathit{in}\, G$ if the set of endpoints of paths in R is exactly the set T. We call such a transportation R a $\mathit{non\text{-}conflicting\ transportation\ of} T\ \mathit{in}\, G$ if no two (different) paths in R are conflicting.

Prove or disprove the following statement. For every connected graph G and every even-sized subset T of its vertices, there exists a non-conflicting transportation of T in G.

Claim: Consider the statement,
$P(n)$: For every connected graph G of size m and every even-sized subset T of its vertices of size n, there exists a non-conflicting transportation of T in G.
We will attempt to prove using induction on n.

Base Case: Let $n = 2$. In that case, only one path exists from the two vertices to one another. No two different paths conflict and the claim is hence true.

Let $P(n)$ be true for some even n. Then, consider $P(n + 2)$.
Consider any two connected vertices v1, v2 (i.e. edge exists between v1 and v2) and consider the remaining n vertices.

By induction hypothesis there exist n/2 non conflicting paths amongst the other n vertices. Thus we simply append the path v1v2 to the set of non conflicting paths and we are done.

1.

The set of polynomials with integer coefficients can be injectively mapped to the set of finite length sequences of integers, which is a countable set. Therefore, the set of polynomials with integer coefficients is countable.

Each such polynomial p has a finite set of roots, say R_p. The set of well-behaved numbers is, by definition, given by W = union over all p of R_p. Thus, W is the union of a countable collection of finite (and therefore, countable) sets. Therefore, W is countable.

2.

Solution 1

Define a set A as,

A = { {w_1, . . . , w_{T/2}} | w_1, . . . , w_{T/2} are walks in G such that the set of their endpoints are exactly the set T }.

Since graph G is connected, A is a non-empty set.

Define the set B as,

B = { len(w_1) + . . . + len(w_{T/2}) | {w_1, . . . , w_{T/2}} belongs to A }.

Similarly, B is a non-empty subset of natural numbers.

By WOP, B has a minimum element say m. Let W = { w_1, . . . w_{T/2} } be the set of walks in A with total length m.

Claim: w_1, . . . w_{T/2} all are paths in G and no two of them are conflicting.

Proof. Suppose w_c is not a path. Then w_c = v_1, . . . v_k such that v_i = v_j for some I < j.

Now we can replace w_c by w’_c = v_1, . . . v_i, v_{j+1}, . . . v_k in W to get another member W’ of A. Clearly len(w’_c) < len(w_c), so the total length of walks in W’ is less than m, which is a contradiction.

Now suppose there exists a conflicting edge between two walks w_i and w_j and let that edge be (u, v).

w_i = a_1, a_2, . . . u, v, . . . a_p ; w_j = b_1, b_2, . . . , b_l, u, v, b_{l+2}, . . . b_q.

Replace w_i and w_j by the walks w’_i = a_1, a_2, . . . u, b_l, . . . , b_2, b_1 and

w’_j = a_p, . . . , v, b_{l+2}, . . . b_q to get another member W’ of A. len(w’_i) + len(w’_j) = len(w_i) + len(w_j) - 2, so the total length of walks in W’ is less than m, which is a contradiction.

Solution 2

Since the graph G is connected, it has a spanning tree, say H. Since the edges in H are a subset of G, if we prove the claim for H, it directly follows for the graph G.

So, we prove that there exists a non-conflicting transportation in a tree H for any subset T of vertices where |T| is even.

Proof by induction of number of vertices in H,

Base case: |V| = 2, then we can pick the single edge between these 2 vertices if T=V, and the empty set of paths If T is empty.

Induction Hypothesis: Assume there exists a non-conflicting transportation #vertices < n.

Inductive step: Consider a tree H with n vertices, and a subset T of vertices. Let a be a leaf node of this tree which is connected to vertex b.

Case 1: a does not belong to T. In this case we can simply remove a from the graph and get the result using IH.

Case 2: Both a, b belong to T. In this case we include the edge (a, b) to the transportation, remove the vertex a from graph and a,b from T. Now we can complete the argument using IH.

Case 3: a belongs to T, b does not belong to T. In this case we remove the vertex a from the graph, and replace a by b to T. From IH we get a non-conflicting transportation. Consider a path which has an end point as b, extend this path by appending edge (a, b) to it. This creates a non-conflicting transportation for original tree.