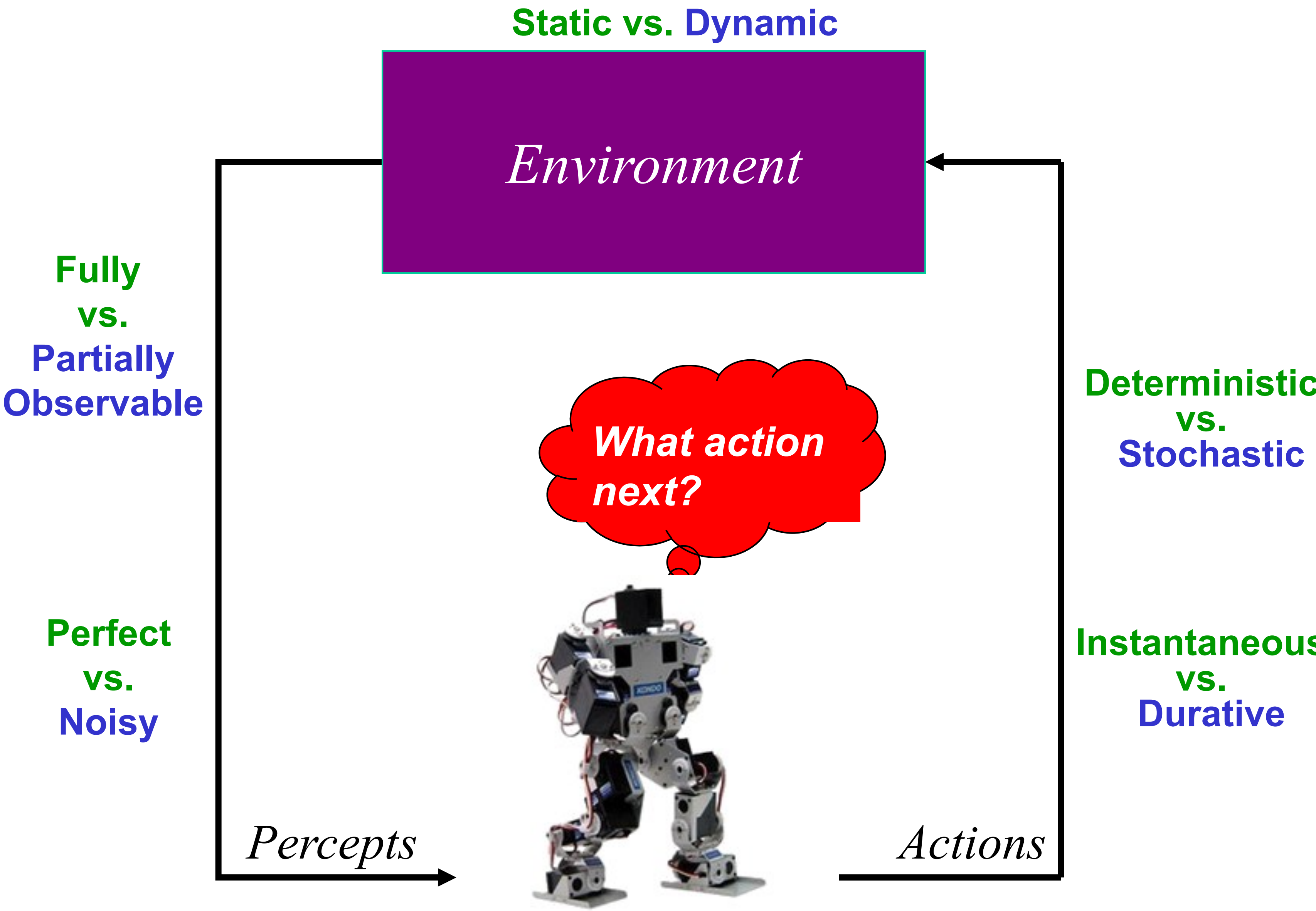
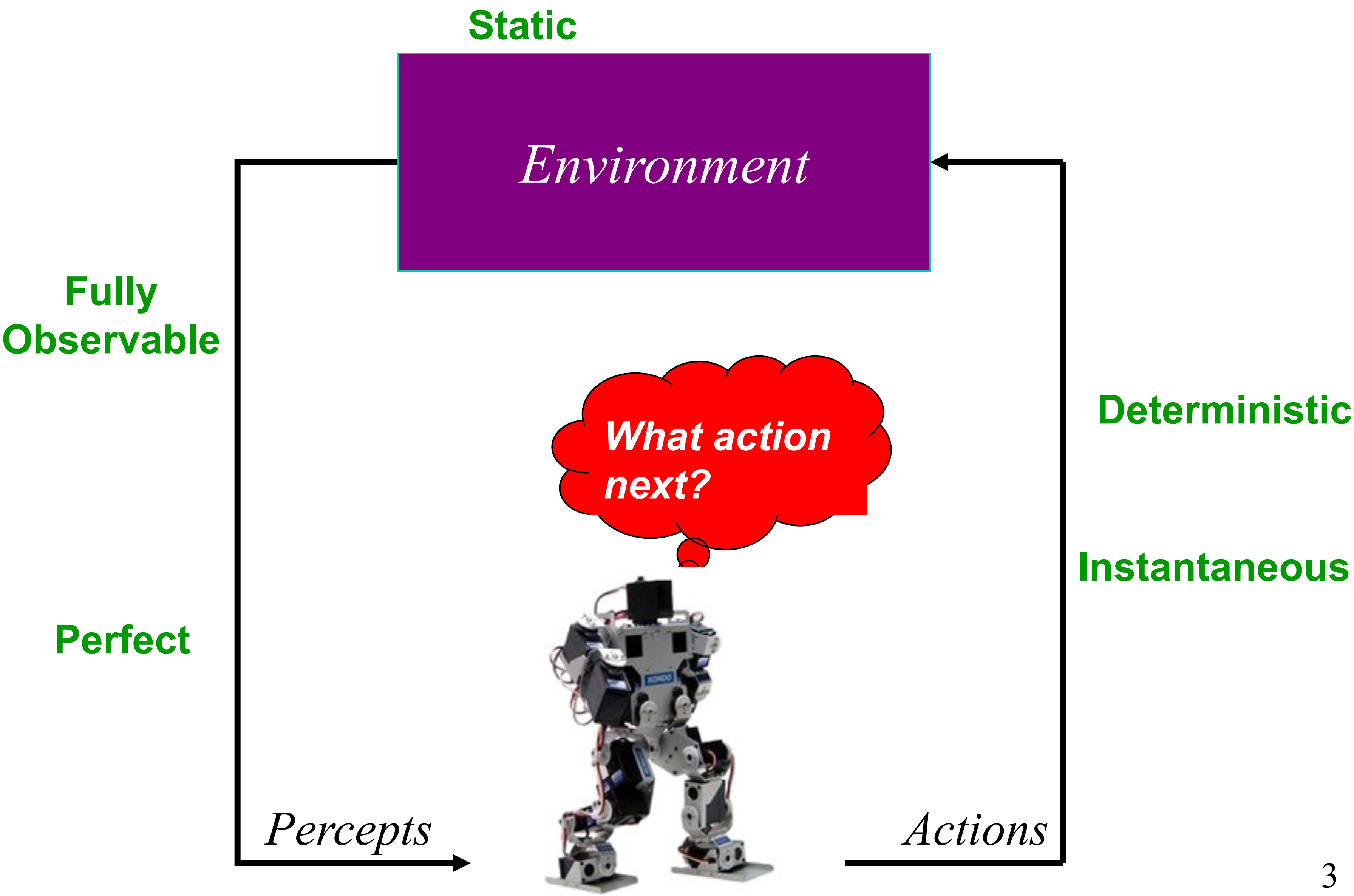
# Markov Decision Processes
## Chapter 17

Mausam

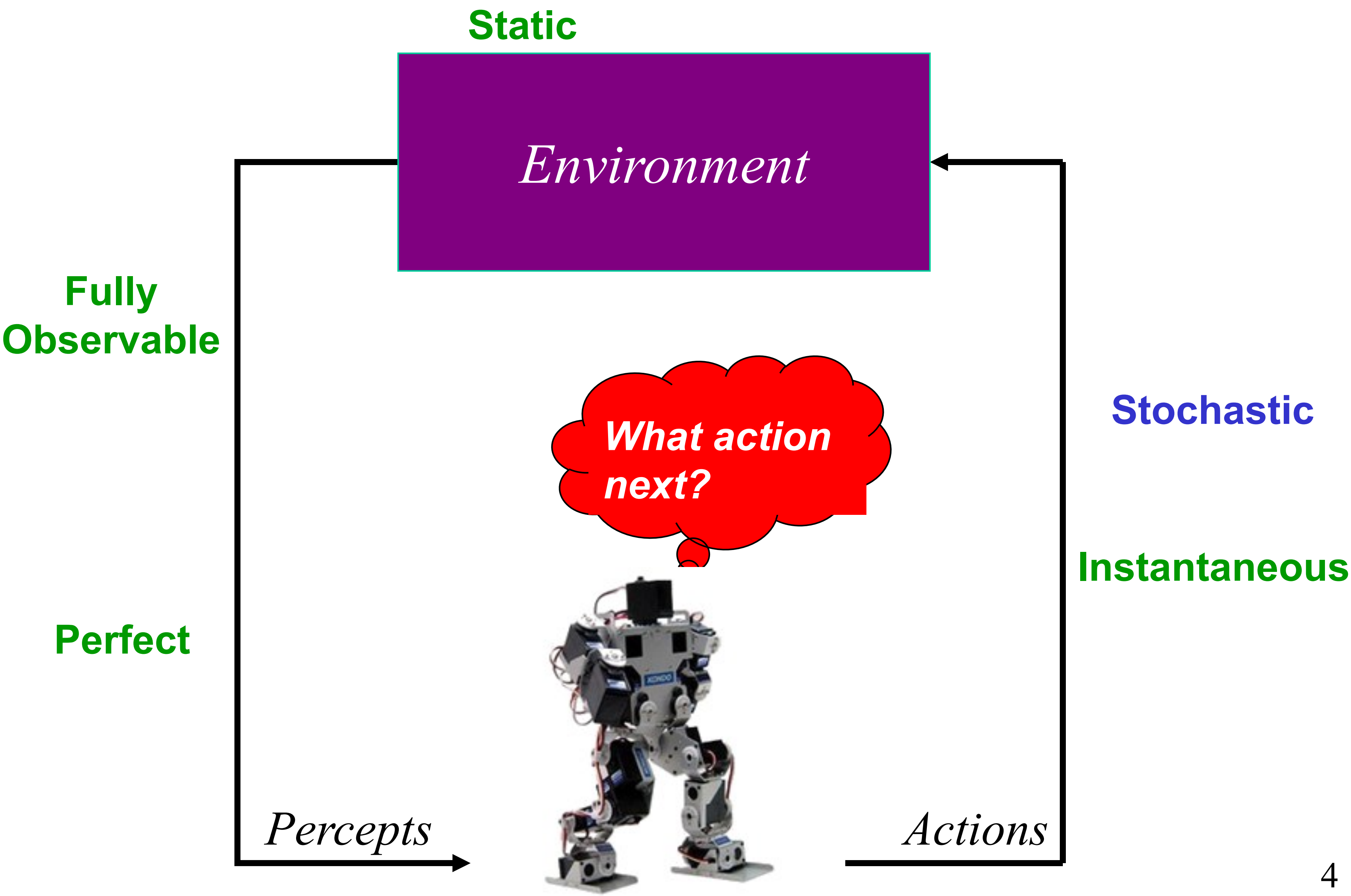
# Planning Agent



# Search Algorithms



# Stochastic Planning: MDPs



# MDP vs. Decision Theory

- Decision theory - episodic

- MDP -- sequential

# Markov Decision Process (MDP)

- $\mathcal{S}$: A set of states
- $\mathcal{A}$: A set of actions
- $\mathcal{T}$(s,a,s'): transition model
- $\mathcal{C}$(s,a,s'): cost model
- $\mathcal{G}$: set of goals
- $s_0$: start state
- $\gamma$: discount factor
- $\mathcal{R}$(s,a,s'): reward model

factored

Factored MDP

absorbing/
non-absorbing

# Objective of an MDP

- Find a policy $\pi: \mathcal{S} \rightarrow \mathcal{A}$

- which optimizes
  - minimizes $\begin{pmatrix} \text{discounted} \\ \text{or} \\ \text{undiscount.} \end{pmatrix}$ expected cost to reach a goal
  - maximizes expected reward
  - maximizes expected (reward-cost)

- given a ____ horizon
  - finite
  - infinite
  - indefinite

- assuming full observability

# Role of Discount Factor ($\gamma$)

- Keep the total reward/total cost finite
  - useful for infinite horizon problems

- Intuition (economics):
  - Money today is worth more than money tomorrow.

- Total reward: $r_1 + \gamma r_2 + \gamma^2 r_3 + \ldots$
- Total cost: $c_1 + \gamma c_2 + \gamma^2 c_3 + \ldots$

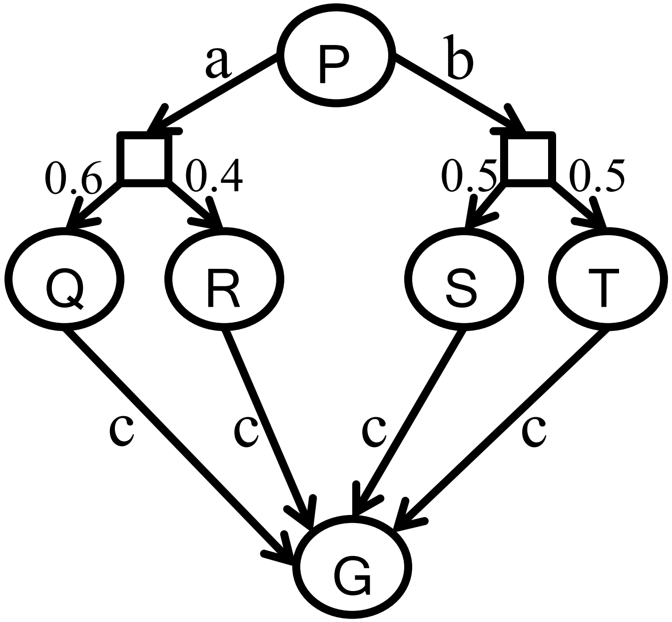
# Examples of MDPs

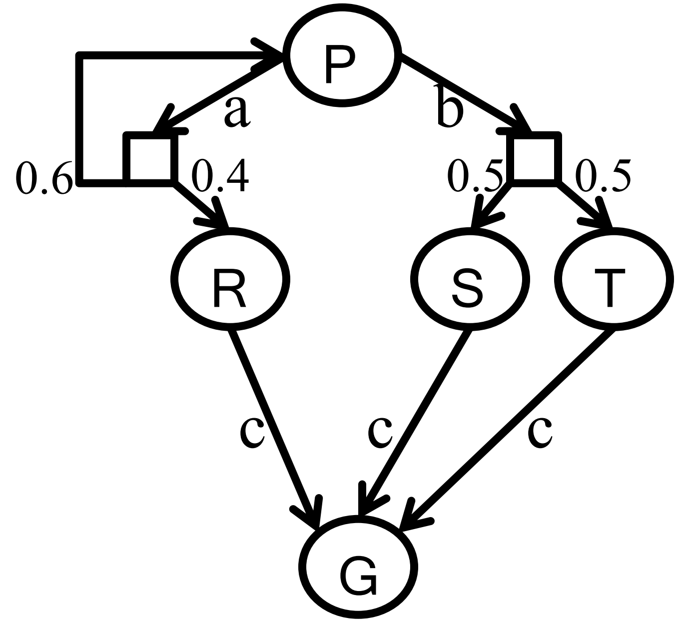
- Goal-directed, Indefinite Horizon, Cost Minimization MDP
  - $<\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{C}, \mathcal{G}, s_0>$
  - Most often studied in planning, graph theory communities
- Infinite Horizon, Discounted Reward Maximization MDP
  - $<\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{R}, \gamma>$
  - Most often studied in machine learning, economics, operations research communities

**most popular**

- Oversubscription Planning: Non absorbing goals, Reward Max. MDP
  - $<\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{G}, \mathcal{R}, s_0>$
  - Relatively recent model

# Acyclic vs. Cyclic MDPs




C(a) = 5, C(b) = 10, C(c) =1

Expectimin works

- V(Q/R/S/T) = 1
- V(P) = 6 – action a

Expectimin doesn't work

- infinite loop
- V(R/S/T) = 1
- Q(P,b) = 11
- Q(P,a) = ????
- suppose I decide to take a in P
- Q(P,a) = 5+ 0.4*1 + 0.6Q(P,a)
- ➔ = 13.5

# Brute force Algorithm

- Go over all policies $\pi$
  - How many? $|A|^{|S|}$ finite
- Evaluate each policy how to evaluate?
  - $V^{\pi}(s) \leftarrow$ expected cost of reaching goal from s
- Choose the best
  - We know that best exists (SSP optimality principle)
  - $V^{\pi*}(s) \leq V^{\pi}(s)$

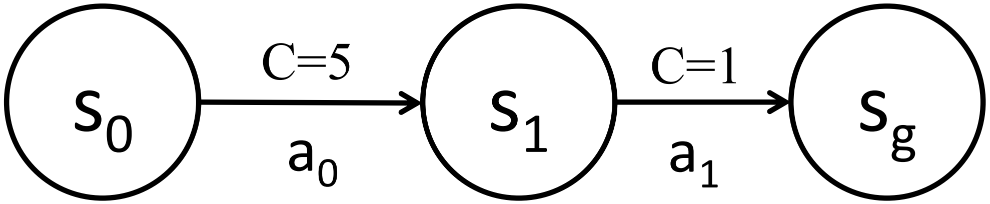
# Policy Evaluation

- Given a policy $\pi$: compute $V^\pi$
  - $V^\pi$ : cost of reaching goal while following $\pi$

# Deterministic MDPs

- Policy Graph for $\pi$

  $\pi(s_0) = a_0;\ \pi(s_1) = a_1$

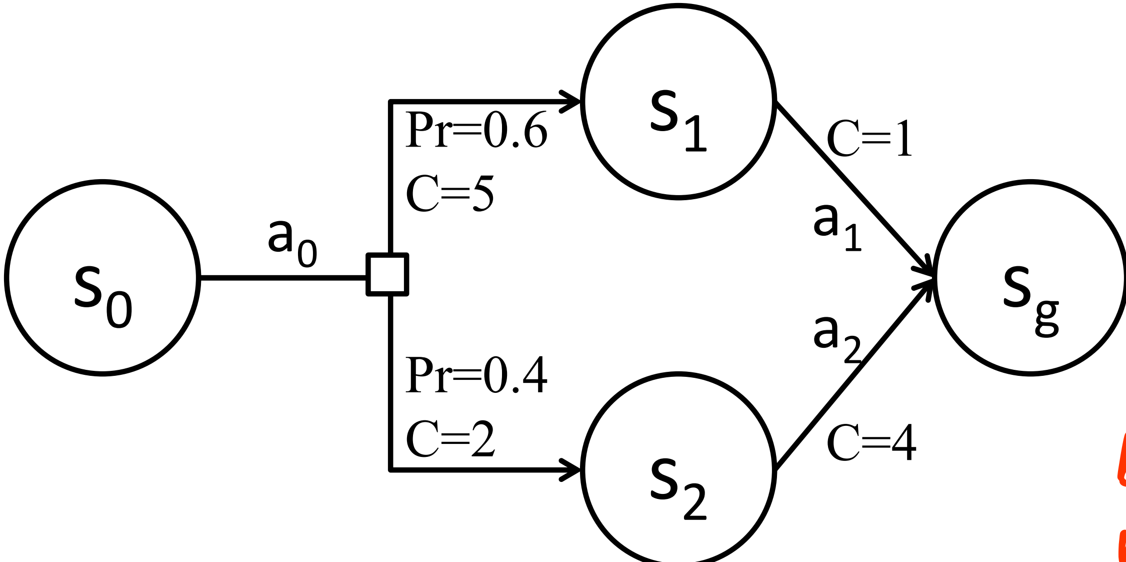



- $V^{\pi}(s_1) = 1$
- $V^{\pi}(s_0) = 6$

*add costs on **path** to goal*

# Acyclic MDPs

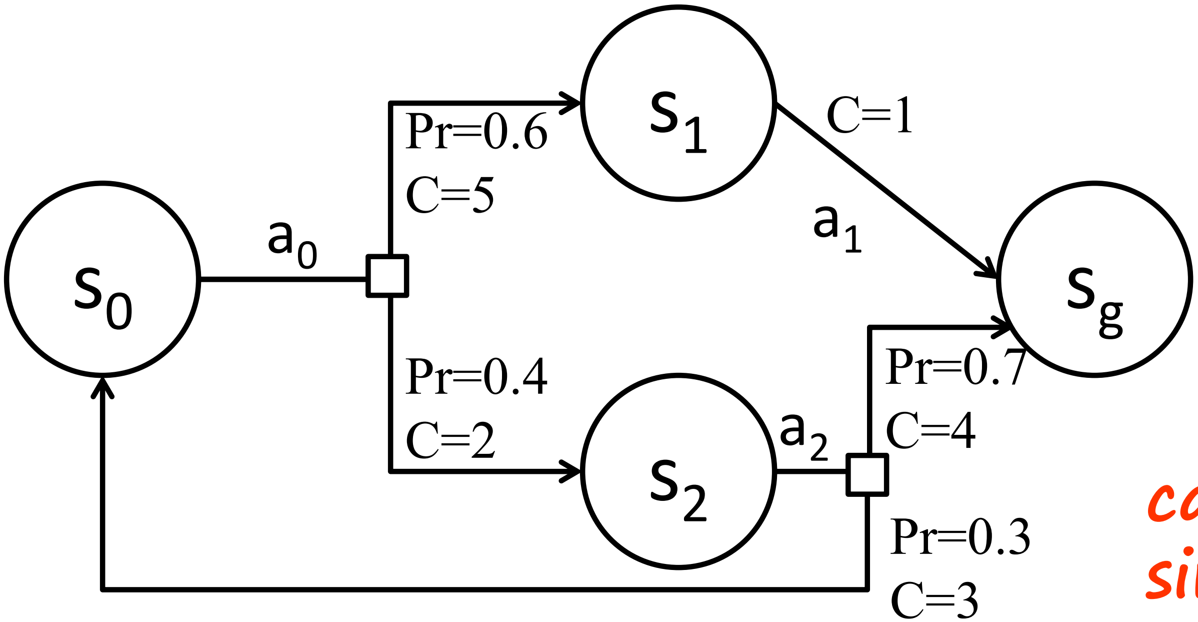
- Policy Graph for $\pi$




- $V^\pi(s_1) = 1$
- $V^\pi(s_2) = 4$
- $V^\pi(s_0) = 0.6(5+1) + 0.4(2+4) = 6$

backward pass in reverse topological order

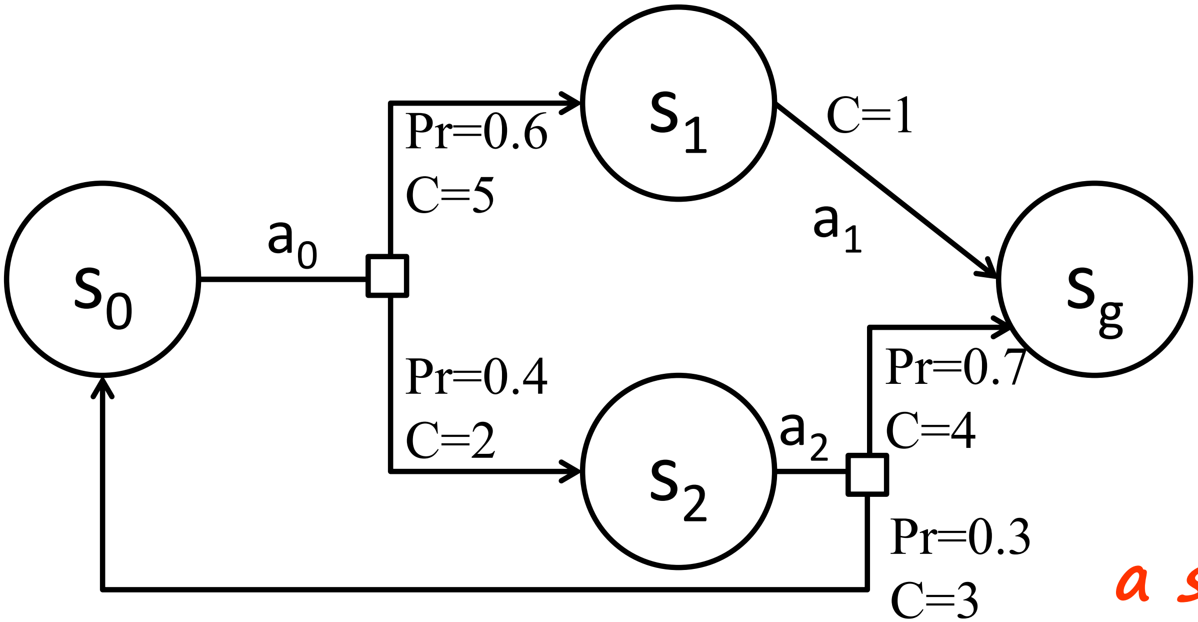
# General MDPs can be cyclic!




cannot do a simple single pass

- $V^{\pi}(s_1) = 1$
- $V^{\pi}(s_2) = ??$ (depends on $V^{\pi}(s_0)$)
- $V^{\pi}(s_0) = ??$ (depends on $V^{\pi}(s_2)$)

# General SSPs can be cyclic!




a simple system of linear equations

- $V^{\pi}(g) = 0$
- $V^{\pi}(s_1) = 1 + V^{\pi}(s_g) = 1$
- $V^{\pi}(s_2) = 0.7(4 + V^{\pi}(s_g)) + 0.3(3 + V^{\pi}(s_0))$
- $V^{\pi}(s_0) = 0.6(5 + V^{\pi}(s_1)) + 0.4(2 + V^{\pi}(s_2))$

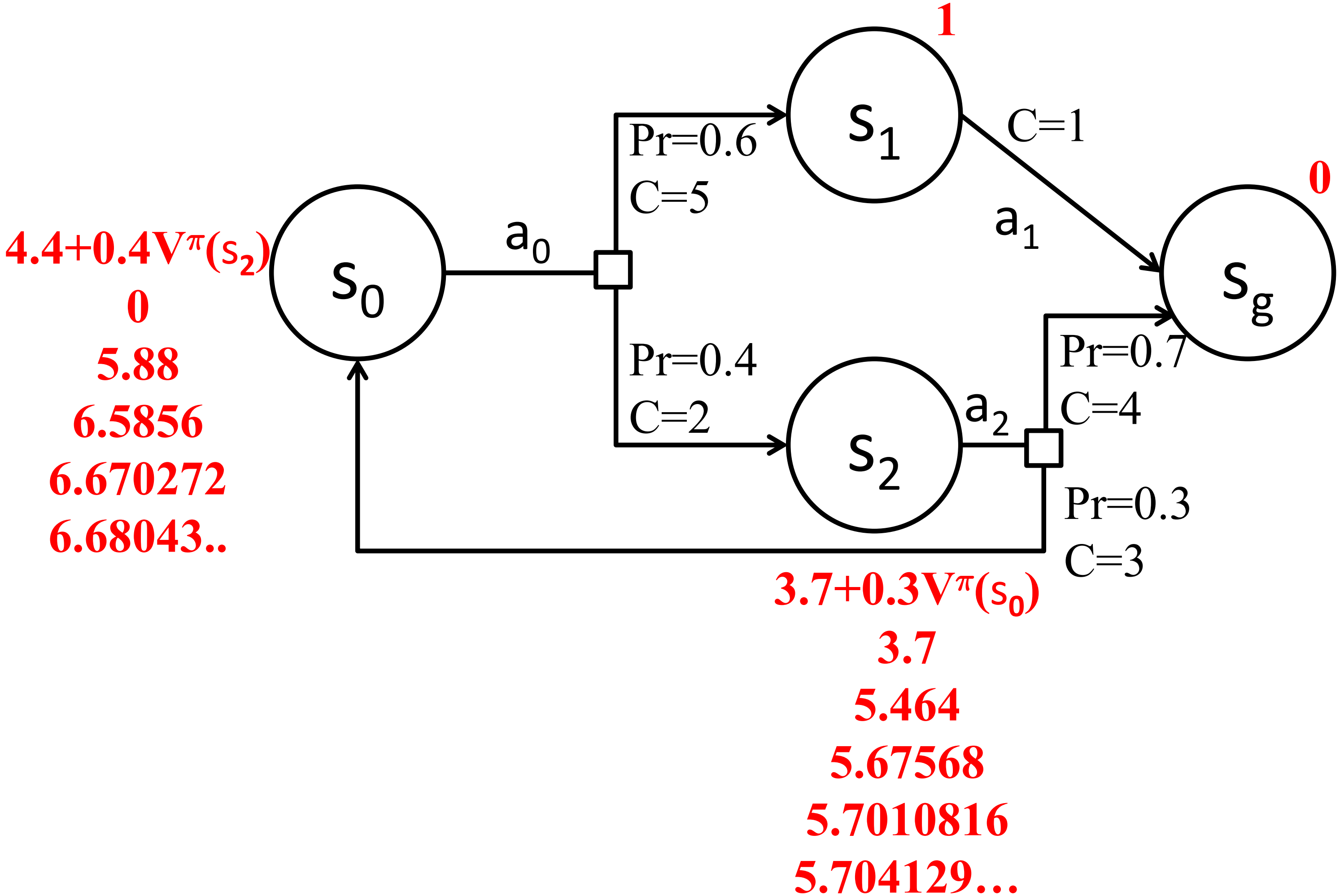
# Policy Evaluation (Approach 1)

- Solving the System of Linear Equations

$$
\begin{aligned}
V^{\pi}(s) &= 0 \qquad \text{if } s \in \mathcal{G} \\
&=
\end{aligned}
$$

- $|S|$ variables.
- $O(|S|^3)$ running time

# Iterative Policy Evaluation



# Policy Evaluation (Approach 2)

$$V^\pi(s) = \sum_{s' \in \mathcal{S}} \mathcal{T}(s, \pi(s), s') \left[\mathcal{C}(s, \pi(s), s') + V^\pi(s')\right]$$

*iterative refinement*

$$V_n^\pi(s) \leftarrow \sum_{s' \in \mathcal{S}} \mathcal{T}(s, \pi(s), s') \left[\mathcal{C}(s, \pi(s), s') + V_{n-1}^\pi(s')\right]$$

# Iterative Policy Evaluation

```
//Assumption: π is proper
initialize V_0^π arbitrarily for each state
```

iteration n

ϵ-consistency

termination condition

## Convergence & Optimality

For a proper policy $\pi$

Iterative policy evaluation
converges to the true value of the policy, i.e.

$$\lim_{n\rightarrow\infty} V_n^\pi = V^\pi$$

irrespective of the initialization $V_0$

# Policy Evaluation → Value Iteration (Bellman Equations for $MDP_1$)

- $<\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{C}, \mathcal{G}, s_0>$
- Define V*(s) {optimal cost} as the minimum expected cost to reach a goal from this state.
- V* should satisfy the following equation:

$$
\begin{aligned}
V^*(s) &= 0 \qquad \text{if } s \in \mathcal{G} \\
&=
\end{aligned}
$$

**Q*(s,a)**

$V^*(s) = \min_a Q^*(s,a)$

# Bellman Equations for $MDP_2$

- $<\mathcal{S}, \mathcal{A}, \mathcal{T}, \mathcal{R}, s_0, \gamma>$
- Define **V*(s)** {optimal **value**} as the **maximum** expected **discounted reward** from this state.
- V* should satisfy the following equation:

$$\boxed{V^*(s) \quad = \quad \max_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \mathcal{T}(s,a,s') \left[ \mathcal{R}(s,a,s') + \gamma V^*(s') \right]}$$

# Fixed Point Computation in VI

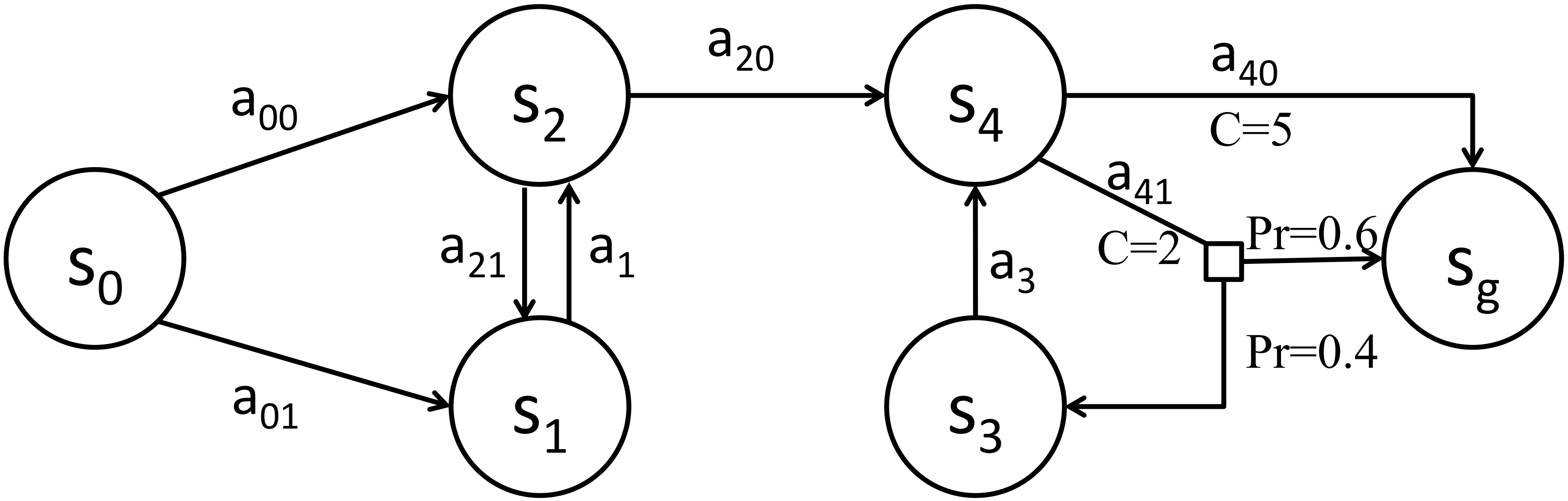
$$V^*(s) = \min_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \mathcal{T}(s,a,s') \left[ \mathcal{C}(s,a,s') + V^*(s') \right]$$

iterative refinement

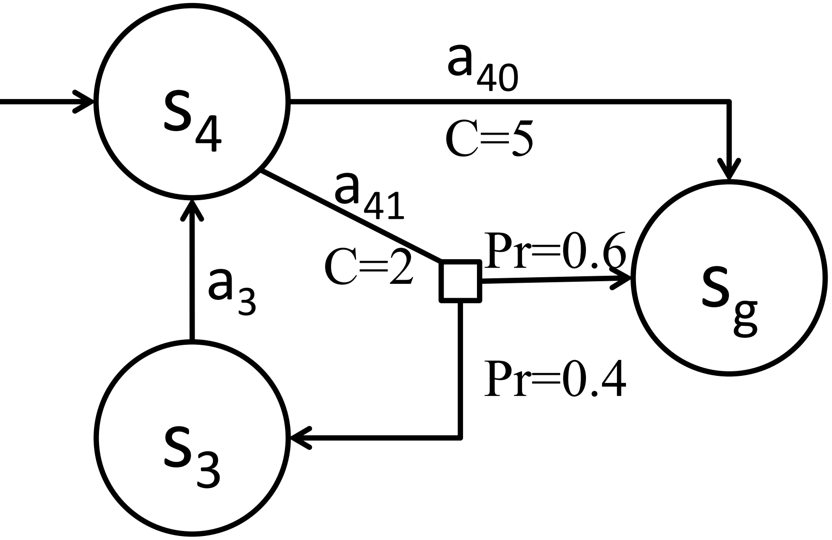
$$V_n(s) \leftarrow \min_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \mathcal{T}(s,a,s') \left[ \mathcal{C}(s,a,s') + V_{n-1}(s') \right]$$

non-linear

# Example

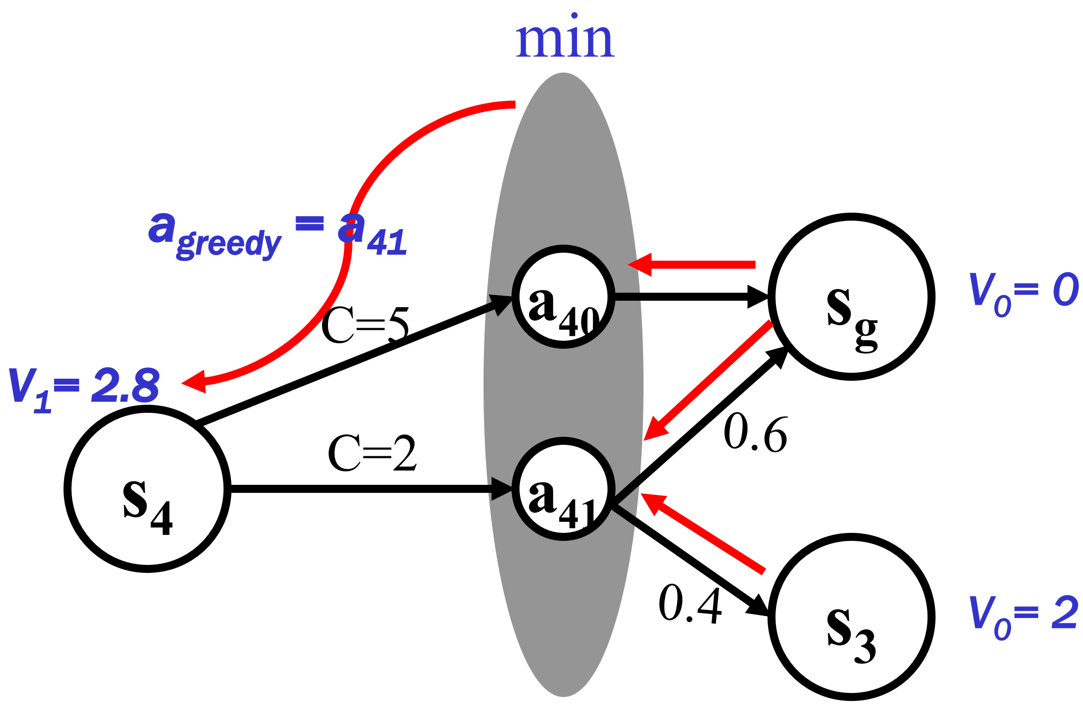


# Bellman Backup




$Q_1(s_4,a_{40}) = 5 + 0$

$Q_1(s_4,a_{41}) = 2 + 0.6 \times 0$
$+ 0.4 \times 2$
$= 2.8$



# Value Iteration [Bellman 57]

No restriction on initial value function

iteration n

```
initialize V_0 arbitrarily for each state
n ← 0
repeat
    n ← n + 1
    foreach s ∈ S do
        compute V_n(s) using Bellman backup at s
        compute residual_n(s) = |V_n(s) − V_{n−1}(s)|
    end
until max_{s∈S} residual_n(s) < ε;
return greedy policy: π^{V_n}(s) = argmin_{a∈A} Σ_{s'∈S} T(s,a,s') [C(s,a,s') + V_n(s')]
```

initialize $V_0$ arbitrarily for each state

$n \leftarrow 0$

**repeat**

  $n \leftarrow n + 1$

  **foreach** $s \in \mathcal{S}$ **do**

    compute $V_n(s)$ using Bellman backup at $s$

    compute residual$_n(s) = |V_n(s) - V_{n-1}(s)|$

  **end**

**until** $\max_{s \in \mathcal{S}}$ residual$_n(s) < \epsilon$;

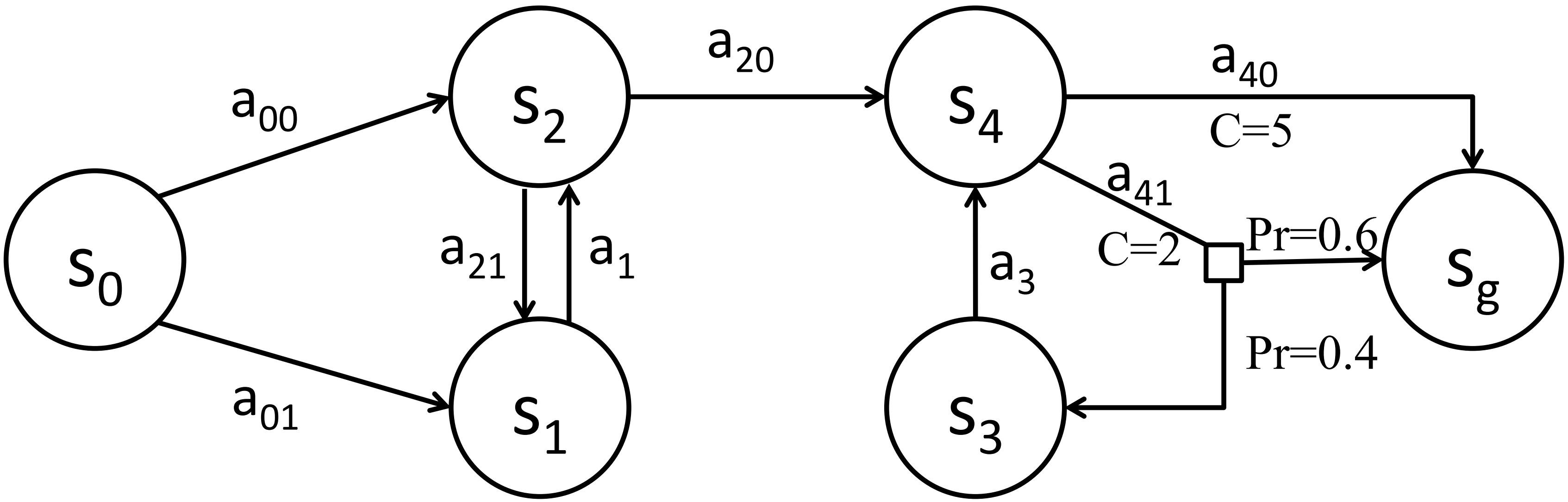
return greedy policy: $\pi^{V_n}(s) = \text{argmin}_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \mathcal{T}(s,a,s') \left[\mathcal{C}(s,a,s') + V_n(s')\right]$

ϵ-consistency

termination condition

# Example

(all actions cost 1 unless otherwise stated)




| n | $V_n(s_0)$ | $V_n(s_1)$ | $V_n(s_2)$ | $V_n(s_3)$ | $V_n(s_4)$ |
|---|---|---|---|---|---|
| 0 | 3 | 3 | 2 | 2 | 1 |
| 1 | 3 | 3 | 2 | 2 | 2.8 |
| 2 | 3 | 3 | 3.8 | 3.8 | 2.8 |
| 3 | 4 | 4.8 | 3.8 | 3.8 | 3.52 |
| 4 | 4.8 | 4.8 | 4.52 | 4.52 | 3.52 |
| 5 | 5.52 | 5.52 | 4.52 | 4.52 | 3.808 |
| 20 | 5.99921 | 5.99921 | 4.99969 | 4.99969 | 3.99969 |

# Comments

- Decision-theoretic Algorithm
- Dynamic Programming
- Fixed Point Computation
- Probabilistic version of Bellman-Ford Algorithm
  - for shortest path computation
  - $MDP_1$ : Stochastic Shortest Path Problem

- Time Complexity
  - one iteration: $O(|\mathcal{S}|^2|\mathcal{A}|)$
  - number of iterations: $\text{poly}(|\mathcal{S}|, |\mathcal{A}|, 1/\epsilon, 1/(1-\gamma))$
- Space Complexity: $O(|\mathcal{S}|)$

# Changing the Search Space

- Value Iteration
  - Search in value space
  - Compute the resulting policy

- Policy Iteration
  - Search in policy space
  - Compute the resulting value

# Policy iteration [Howard'60]

- assign an arbitrary assignment of $\pi_0$ to each state.
- repeat
  - Policy Evaluation: compute $V_{n+1}$: the evaluation of $\pi_n$
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmin}_{a\in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

**costly: $O(n^3)$**

**approximate by value iteration using fixed policy**

**Modified Policy Iteration**

# Advantage

- searching in a finite (policy) space as opposed to uncountably infinite (value) space ⇒ convergence in fewer number of iterations.
- all other properties follow!

# Modified Policy iteration

- assign an arbitrary assignment of $\pi_0$ to each state.

- repeat
  - Policy Evaluation: compute $V_{n+1}$ the *approx.* evaluation of $\pi_n$
  - Policy Improvement: for all states s
    - compute $\pi_{n+1}(s)$: $\text{argmin}_{a\in Ap(s)} Q_{n+1}(s,a)$
- until $\pi_{n+1} = \pi_n$

# Advantage

- probably the most competitive synchronous dynamic programming algorithm.

# VI → Asynchronous VI

- Is backing up *all* states in an iteration essential?
  - No!
- States may be backed up
  - as many times
  - in any order
- If no state gets starved
  - convergence properties still hold!!

# Applications

- Stochastic Games
- Robotics: navigation, helicopter manuevers…
- Finance: options, investments
- Communication Networks
- Medicine: Radiation planning for cancer
- Controlling workflows
- Optimize bidding decisions in auctions
- Traffic flow optimization
- Aircraft queueing for landing; airline meal provisioning
- Optimizing software on mobiles
- Forest firefighting
- …

# Extensions

- Heuristic Search + Dynamic Programming
  - AO*, LAO*, RTDP, …

- Hierarchical MDPs
  - hierarchy of sub-tasks, actions to scale better
- Reinforcement Learning
  - learning the probability and rewards
  - acting while learning – connections to psychology
- Partially Observable Markov Decision Processes
  - noisy sensors; partially observable environment
  - popular in robotics