Quiz4B (COL 351)

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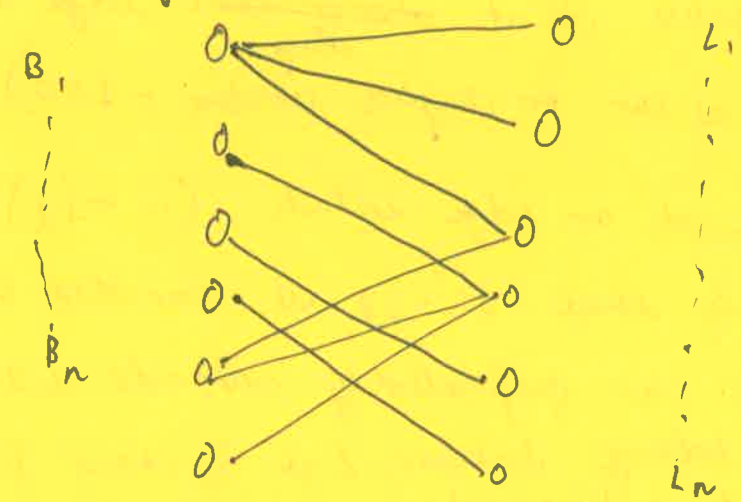
You are given a set of n borrowers $B_1, \ldots, B_n$ and n lenders $L_1, \ldots, L_n$. For each pair of borrower B_i and lender A_j, you are given the amount of money borrowed by B_i from the lender A_j (this amount could be 0 also). You are also given that the following condition holds: for each borrower $B_i, i = 1, \ldots, n$, the total amount of money borrowed by B_i (from all the lenders) is Rs 100. Similarly, for each lender $L_j, j = 1, \ldots, n$, the total amount of money lent by L_j (to all the borrowers) is Rs 100. We want to find a permutation $\sigma_1, \sigma_2, \ldots, \sigma_n$ of $1, 2, \ldots, n$ such that borrower B_i has borrowed non-zero amount of money from lender L_{σ_i} for each $i = 1, \ldots, n$.

Example: Suppose there are 3 lenders and 3 borrowers. B_1 has borrowed Rs 100, Rs 0, Rs 0 from L_1, L_2, L_3 respectively. B_2 has borrowed Rs 0, Rs 60, Rs 40 from L_1, L_2, L_3 respectively. B_3 has borrowed Rs 0, Rs 40, Rs 60 from L_1, L_2, L_3 respectively. Then one valid permutation is $(1, 3, 2)$ because B_1 has borrowed non-zero amount from L_1, B_2 has borrowed non-zero amount from L_3, and B_3 has borrowed non-zero amount from L_2.

Show that this problem can be formulated as a bipartite matching problem. Then prove that such a permutation always exists.

We will formulate the bipartite graph as follows :-



Draw edge for every B_i & lender L_j such that B_i borrowed non zero amount from L_j

$\exists e \in G$, ~~st~~ $B_i - L_j$ s.t B_i borrows non zero from L_j.

Note that this graph is bipartite as each edge exists from ~~one~~ vertex from $B_1 \ldots B_n$ & one from $L_1 \ldots L_n$.

Now selecting / maximum matching of this bipartite graph we claim will lead to a permutation of vertices $1, 2 \ldots n$.

For the permutation :- each L_j matched to B_i, ~~on left~~ in our maximum matching $\Leftrightarrow$ $\sigma_i = j$. (as each edge connects no 2 same vertex)

There will exist atmost n edges in the matching, we prove that there will be exactly n edges in the matching

~~Each edge is connected to a~~

Each vertex on left & right is connected to atmost one edge. So we get a permutation

We will show ~~that~~ by contradiction that it is not possible for any vertex on the left side to be unmatched.

Consider B_i s.t no edge exists from B_i to L_j in the maximum matching, there will exist some L_v which maps to no values.

Consider set of edges from B_i to $L_{j_1}, L_{j_2} \dots L_{j_k}$ in the bipartite graph.

Now, as B_i is not matched, $\Rightarrow j_1, j_2 \dots j_k$ are matched to some other vertices on the left side. (otherwise we could select it).

Sum of edges $(i-j_1) \dots (i-j_k)$ is 100.

Since $L_{j_1}, L_{j_2} \dots L_{j_k}$ are matched to other vertices, it is not possible that ~~all of~~ any the edge weights ~~are~~ is 100. (as sum on right vertex = 100).

$\therefore$ there will exist an edge which $(i - j_l)$ which has the property that $i' - j_l$ is connected in the matching, we can repeatedly cascade until we get a matching between L_v & some B_t & hence get a permutation.

Consider a greedy algorithm (similar to the one discussed in class) which repeatedly picks edges with the minimum weight possible.

It will