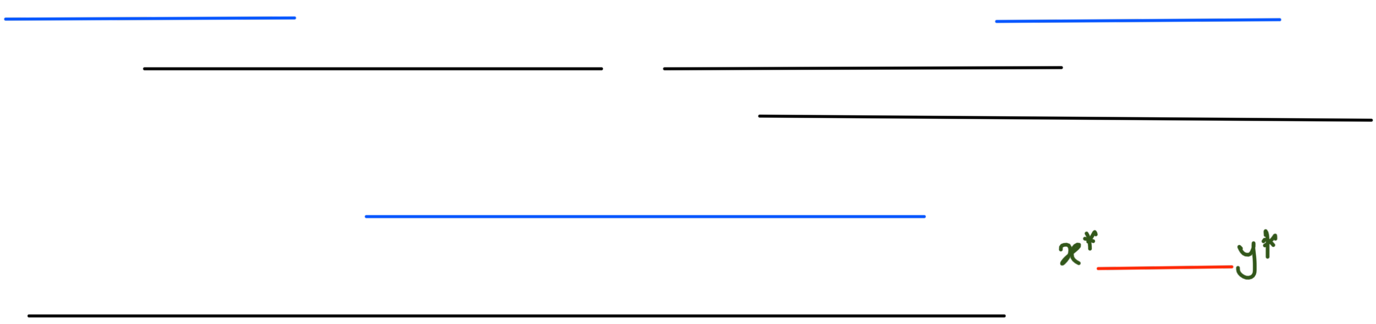
COL 351 Lecture 15 2023/02/13

Topic : Greedy Algorithms
(and how to design correct greedy algorithms)

Input: Set of intervals $[x_1, y_1], \ldots, [x_n, y_n]$

Output: Size of the largest subset of pairwise non-overlapping intervals.



Claim: Let $[x^*, y^*]$ be an interval which starts latest. Then there exists an opt. solution which includes $[x^*, y^*]$.

Proof: Any interval $[x, y]$ which intersects $[x^*, y^*]$ must satisfy $x \le x^*$, and $y \ge x^*$.

Suppose OPT is any optimum solution.

If $[x^*, y^*] \in$ OPT $\longrightarrow$ proved

Else: $[x^*, y^*]$ can intersect ≤ 1 interval in OPT.

(every interval in OPT that intersects $[x^*, y^*]$ must contain x^*. $\therefore \le 1$ such interval)

Include $[x^*, y^*]$ in OPT and remove the ≤ 1 intersecting interval to get a solution that is no worse than OPT. ∎

Algorithm

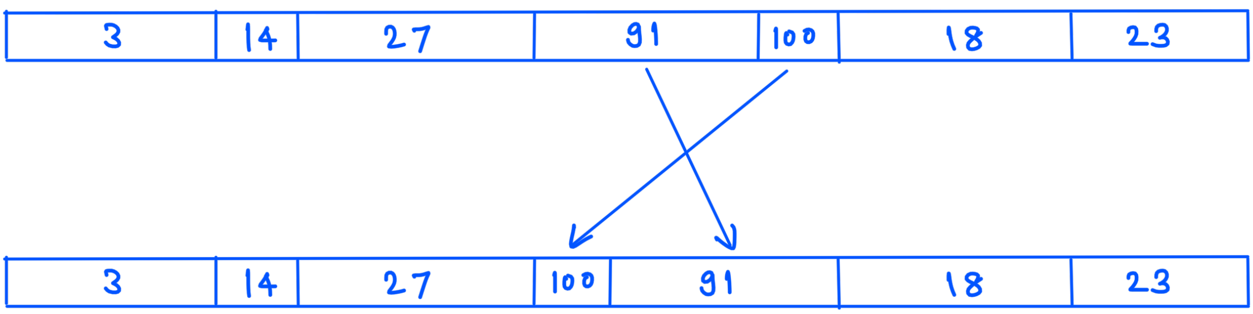
1. Include an interval I which starts latest.
2. Discard all intervals intersecting with I.
3. Recurse on the rest.

Input: Set of n jobs. p_i : processing time of i^{th} job.

Output: Order the jobs so that the total waiting time of jobs is minimised.

Waiting time of i : $W_i = \sum_{\substack{j \text{ scheduled} \\ \text{before } i}} p_j$

Minimise $\sum_i W_i$



$\Delta W_i = 0 \quad \forall i \neq 91, 100$

$\Delta W_{91} = p_{100}$

$\Delta W_{100} = -p_{91}$

$\therefore \Delta$ total waiting time $= p_{100} - p_{91}$

Swap profitable iff $p_{100} - p_{91} < 0$ ie $p_{100} < p_{91}$.

Claim : In every opt ordering, every pair of consecutive jobs should be such that the processing time of earlier job $\leq$ processing time of later job.

Proof: Suppose j is immediately after i, but $p_j < p_i$.
Exchange the position of i, j.

The waiting time of no other job changes.
$\Delta W_i = p_j \quad \Delta W_j = -p_i \quad \therefore \Delta$ total waiting time $= p_j - p_i$
$< 0 \longrightarrow$ contradiction

Greedy algorithm : Sort jobs in inc. order of processing time.

Input: n divisible objects ; W : capacity of knapsack
p_i : price of ith object w_i : weight of ith object.

Output : Pick fraction x_i of object i such that
total weight $\leq W$, total profit is max.

i.e. $x_1 \ldots x_n$ so that $\sum_i w_i x_i \leq W$
$x_i \in [0,1] \; \forall i$, max $\sum_i p_i x_i$

Suppose $x_1^* \ldots x_n^*$ is optimum solution. $\therefore \sum_i w_i x_i^* = W$
Suppose $x_i^* > 0$ and $x_j^* < 1$
Remove δ_i amount of i, add δ_j amount of j
$\delta_i \leq x_i^* \quad \delta_j \leq 1 - x_j^*$, $w_j \delta_j = w_i \delta_i$
Δ profit $= -p_i \delta_i + p_j \delta_j \quad \therefore$ Exchange profitable iff $p_j \delta_j > p_i \delta_i$.