COL 351 Lecture 42 2023/04/27

Topic : Leftover from last class

Context: Prove that DHC is $\mathbb{NP}$-hard.

Claim: 3SAT $\leq_{Karp}$ DHC

Reduction: Formula φ with n vars m clauses $\longrightarrow$ Graph G = Backbone graph of size $\Theta(mn)$ + m extra vertices encoding clauses

We proved: φ satisfiable $\Rightarrow$ G has a directed Hamiltonian cycle

We also proved: Let T' be a directed Hamiltonian cycle of G.
$\forall j$ and i :

If T' visits l_{ij} before c_j, then it visits r_{ij} after c_j, and
if T' visits r_{ij} before c_j, then it visits l_{ij} after c_j.

Claim: G has a directed Hamiltonian cycle $\Rightarrow \varphi$ is satisfiable.

Proof: Let T' be a directed Hamiltonian cycle of G.
Bypassing the vertices c_j in T', we get a directed Hamiltonian cycle, say T, of the backbone graph $B(n, m)$.

T is of the following form:

1. Start from s.
2. For $i = 1$ to n, traverse row i from left to right or right to left.
3. Visit t and return to s.

Let x_i^* be True if T traverses row i from left to right
Let x_i^* be False if T traverses row i from right to left.
We claim $(x_1^*, \ldots, x_n^*)$ satisfies φ.
Consider an arbitrary clause c_j in φ.
T' visits c_j between l_{ij} and r_{ij} for some i.
Case 1: T' visits l_{ij} followed by c_j followed by r_{ij}
$\therefore$ (l_{ij}, c_j) and (c_j, r_{ij}) are edges of G.
$\therefore$ x_i is a literal in c_j
T traverses l_{ij} before r_{ij} $\therefore$ T traverses row i from left to right
$\therefore$ $x_i^* =$ true $\therefore$ c_j is satisfied by $(x_1^*, \ldots, x_n^*)$.
Case 2: T' visits r_{ij} followed by c_j followed by l_{ij}
Symmetric argument implies c_j is satisfied by $(x_1^* \ldots x_n^*)$
$\therefore$ $(x_1^*, \ldots, x_n^*)$ satisfies φ.

A summary of Karp reductions done in class / tutorials to prove NP-hardness