COL 351 Lecture 41 2023/04/26

Topic : Directed Hamiltonian Cycle is NP-complete

3SATtoDHC (φ):

1. $n \leftarrow$ # vars in φ (say $x_1, \ldots, x_n$ are the vars)

   $m \leftarrow$ # clauses in φ (say $C_1, \ldots, C_m$ are the clauses)

2. $G \leftarrow$ backbone graph $B(n, m)$

3. For each clause C_j in φ

   Add vertex c_j to G

   For each variable x_i

      If x_i is a literal in C_j

         Add edges $(l_i, c_j), (c_j, r_i)$ to G

      If $\overline{x_i}$ is a literal in C_j

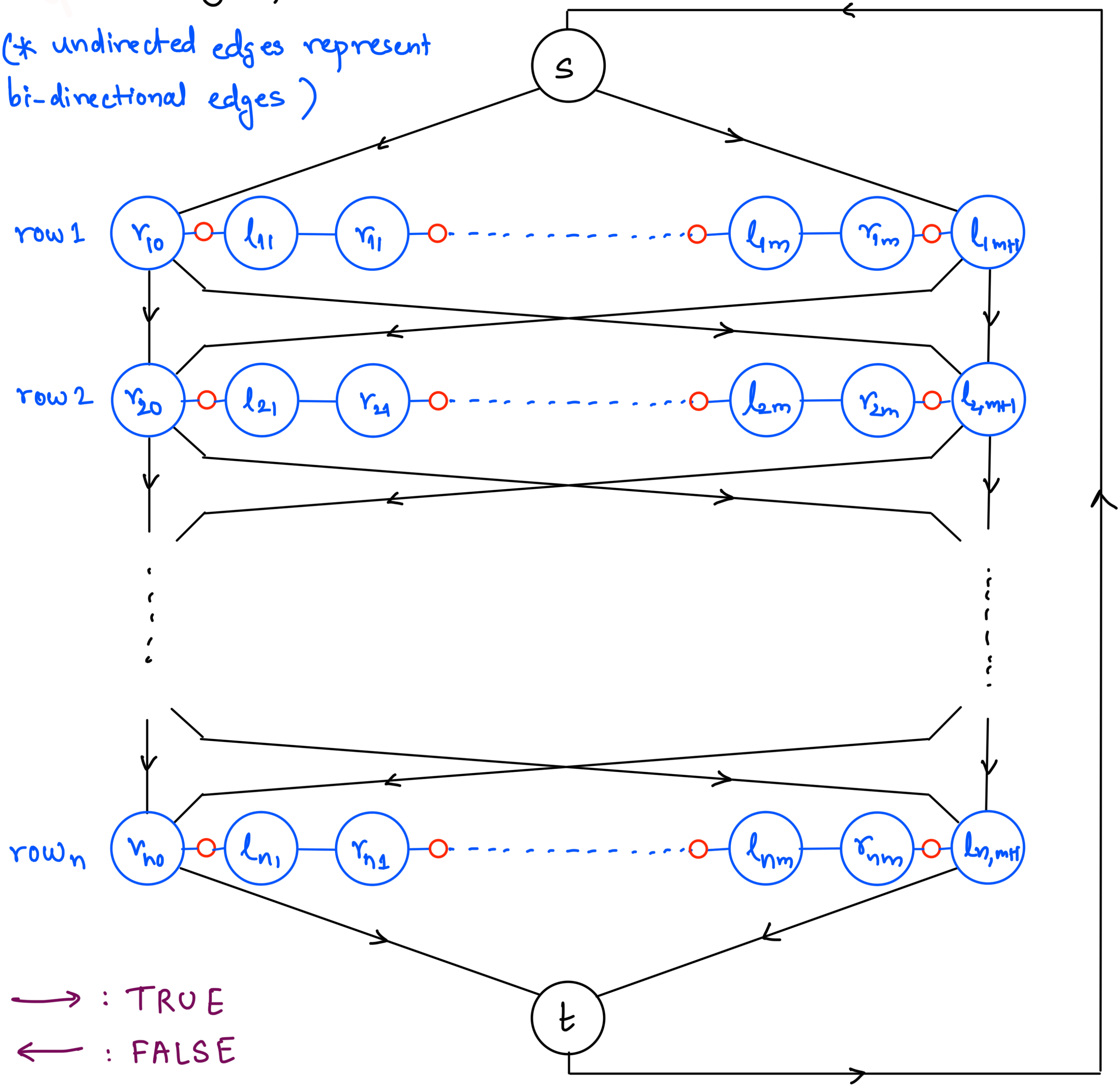
         Add edges $(r_i, c_j), (c_j, l_i)$ to G

4. Return G.

Note: $O(mn)$ running time.

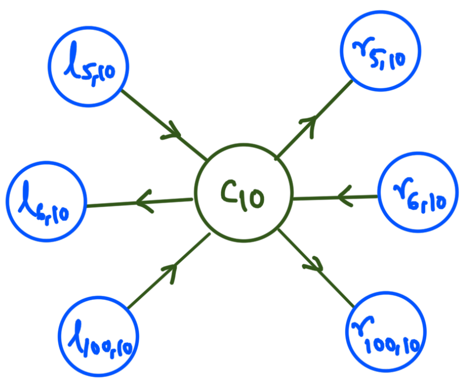
Backbone graph : $B(n, m)$

(* undirected edges represent bi-directional edges)



Additions to backbone graph: 1 vertex and ≤ 6 edges per clause.

eg. $C_{10} : x_5 \vee \overline{x_6} \vee x_{100}$



Claim: φ satisfiable $\Rightarrow$ G has a directed Hamiltonian cycle.

Proof: Let $(x_1^*, \ldots, x_n^*)$ be a satisfying assignment of φ.

Let T be the DHC of the backbone graph corresponding to $(x_1^*, \ldots, x_n^*)$ (ie traverse row i from left to right if x_i^* true, right to left if x_i^* false).

For each clause C_j, identify one true literal in C_j, say x_i or $\overline{x_i}$

If x_i is a true literal in C_j, insert C_j in T between l_{ij} and r_{ij}.

If $\overline{x_i}$ is a true literal in C_j, insert C_j in T between r_{ij} and l_{ij}.

This makes T a directed Hamiltonian cycle of G.

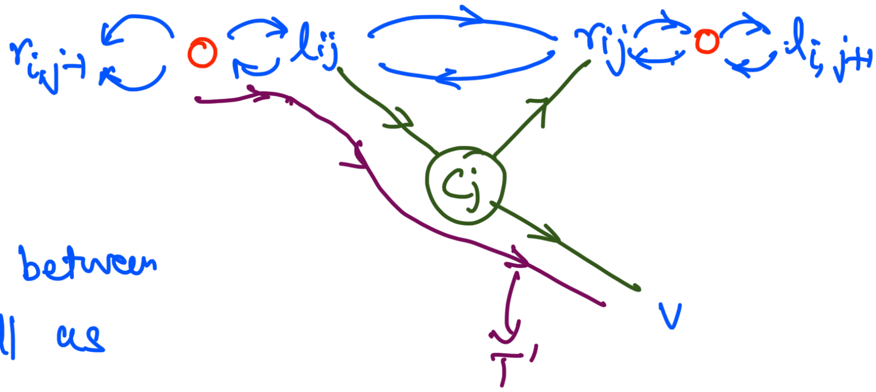
Claim: Let T' be a directed Hamiltonian cycle of G. For every j and i the following is true.

If T' visits l_{ij} before C_j, then it visits r_{ij} after C_j, and

if T' visits r_{ij} before C_j, then it visits l_{ij} after C_j.

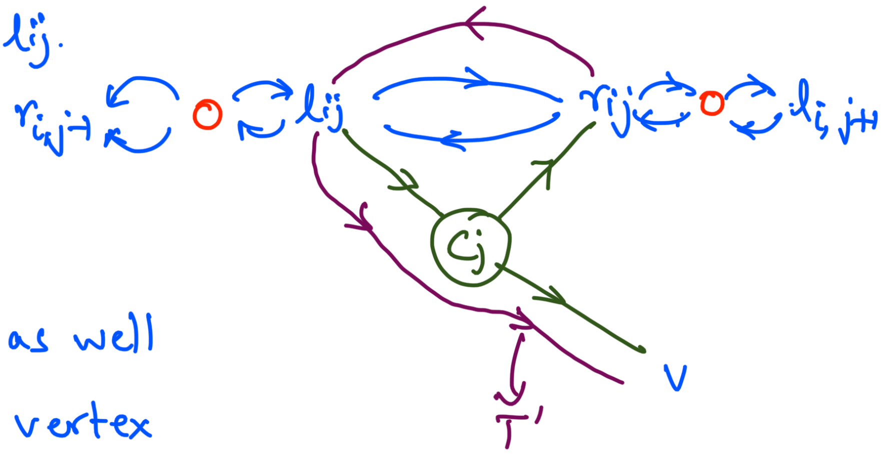
Proof: Suppose T' visits l_{ij} before C_j, and some $v \neq r_{ij}$ after C_j

Case 1: T' visits the red vertex between r_{ij-1} and l_{ij} before l_{ij}.



T' must visit the red vertex between r_{ij} and $l_{i,j+1}$ before as well as after it visits r_{ij} $\rightarrow$ contradiction.

Case 2: T' visits r_{ij} before l_{ij}.



T' must visit $r_{i,j-1}$ before as well as after it visits the red vertex between $r_{i,j-1}$ and $l_{i,j}$ $\rightarrow$ contradiction.

A symmetric argument gets a contradiction if T' visits r_{ij} before C_j, and some $v \neq l_{ij}$ after C_j.