COL 351 Lecture 31 2023/04/03

Topic : Hall's Theorem

Announcement: Quiz 4 April 13 19:30 – 20:30
(tentative)

Definition: A perfect matching is a matching in which all vertices are matched.

When does a bipartite graph $G = (L, R, E)$ (not) have a perfect matching ?

Necessary conditions ① $|L| = |R|$.

② $\forall$ subset L' of L, $|N(L')| \geq |L'|$

$N(L') = \{ v \in R \mid \exists u \in L' \text{ such that } \{u,v\} \in E \}$.

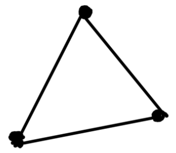
Are these sufficient?

Definition: A <u>vertex cover</u> of a graph $G = (V, E)$ is a subset C of V such that every edge of G has at least one endpoint in C.

Claim: If M is a matching and C is a vertex cover of the same graph G. Then $|M| \leq |C|$.

$$\therefore \max_{\text{matchings } M} |M| \leq \min_{\substack{\text{vertex covers} \\ C}} |C| \qquad \text{—— } (*)$$

n odd; consider cycle over n vertices



size of max matching $= \lfloor n/2 \rfloor$

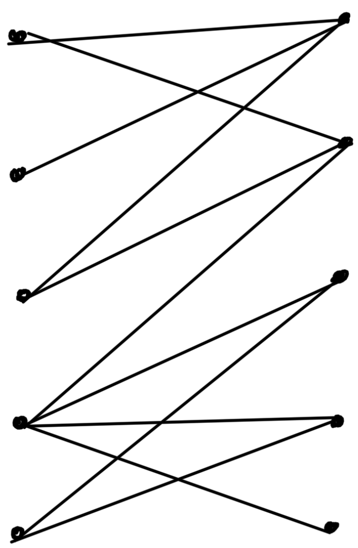
size of min vertex cover $= \lceil n/2 \rceil$

size of max matching : 1

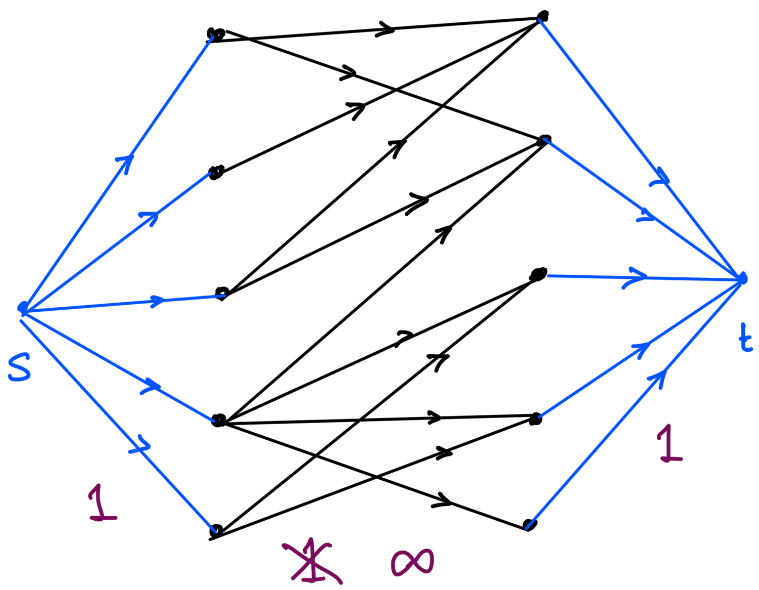
size of min vertex cover : 2

Can we claim equality in $(*)$ if a graph doesn't have odd cycles (ie is bipartite)?

G



$\mathcal{N}^{\infty}(G)$



Check: setting capacity of L to R edges to ∞ does not change the (existence of) bijection between matchings and integral flows.

Claim (*): Let S^* be a min cut in $\mathcal{N}^\infty(G)$. Then

$$S^* = \{s\} \cup L' \cup N(L'), \text{ where } L' = S^* \cap L.$$

Proof: Equivalent statement: $S^* \cap R = N(L')$.

If $N(L') \not\subseteq S^* \cap R$, then for $v \in N(L') \setminus (S^* \cap R)$, $\exists u \in L'$ such that $\{u, v\}$ is an edge of G. $\therefore C(S) = \infty$ because (u, v) is an ∞ capacity edge in $\mathcal{N}^\infty(G)$ that crosses C.

If $N(L') \subsetneq S^* \cap R$, then let

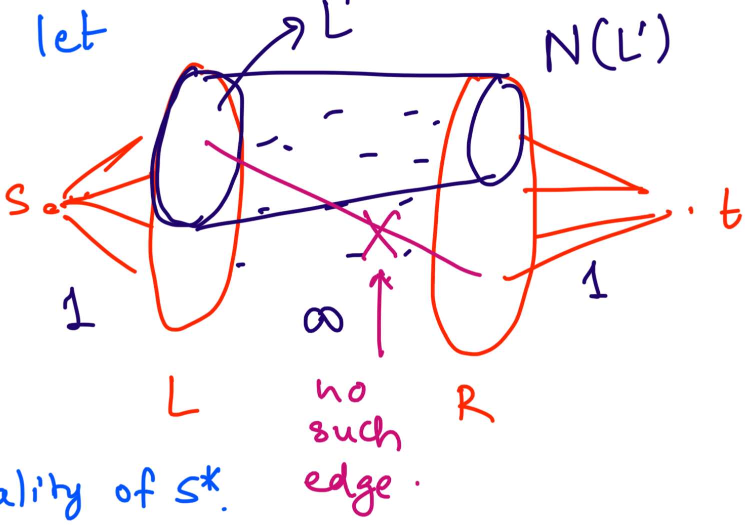
$S' = \{s\} \cup L' \cup N(L') \subsetneq S^*$.

$c(S') = |L \setminus L'| + |N(L')|$

$\quad < |L' \setminus L'| + |S^* \cap R|$

$\quad = c(S^*)$

$\rightarrow$ contradiction to minimality of S^*.



Hall's Theorem: Let $G = (L, R, E)$ be a bipartite graph with $|L| = |R|$. Then G does not have a perfect matching iff $\exists\, L' \subseteq L$ such that $|N(L')| < |L'|$.

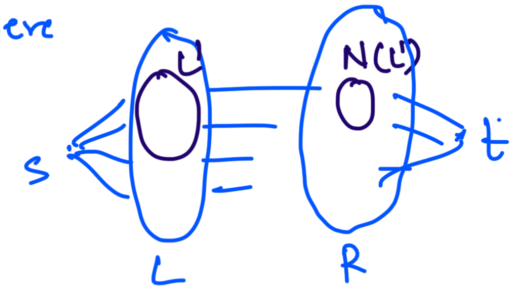
Proof: [If] $\rightarrow$ obvious.

[only if] Let $n = |L| = |R|$. G does not have a perfect matching.

$\therefore$ size of max matching $< n$. $\therefore$ maxflow in $\mathcal{N}^\infty(G) < n$.

$\therefore$ min cut in $\mathcal{N}^\infty(G) < n$. Let S^* be a min cut of $\mathcal{N}^\infty(G)$.

By claim (*), $S^* = \{s\} \cup L' \cup N(L')$, where $L' = S^* \cap L$. We know $C(S^*) < n$.



But $C(S^*) = |L \setminus L'| + |N(L')|$

$\therefore |L \setminus L'| + |N(L')| < n \;\therefore$

$\quad n - |L'| + |N(L')| < n \;\therefore |N(L')| < |L'|$.