

COL 351: Analysis and Design of Algorithms

Minor Exam

Timing: 9:10 am - 10:40 am

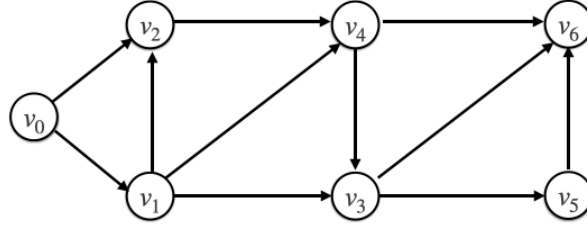
Important Guidelines:

1. This is a closed book exam. You cannot look at your notes or browse the internet.
2. Each solution must start on a new page.
3. You can directly reference an algorithm / theorem proved in the lectures.
4. Total marks are 42.
5. The scanned solution **must be** uploaded on Gradescope by 10:55 am sharp. The late submission deadline is 11:00 am, No submissions will be accepted later in any circumstance.

1 Short Questions [$3 \times 5 = 15$ marks]

- (a) Prove that edit-distance between two strings A, B of size respectively m, n is always bounded by $(m + n - 2|LCS(A, B)|)$, where $|LCS(A, B)|$ is the length of longest common subsequence of A and B (when these strings are viewed as sequences).
- (b) An edge (x, y) in an undirected graph G is said to be a *bridge* if each path from x to y in G passes through edge (x, y) .
Prove that an edge is a bridge if and only if it does not lie on any simple cycle of G .
- (c) Device an $O(m+n)$ time algorithm that verifies whether a given directed graph with n vertices and m edges is strongly connected (i.e. has exactly one SCC) using only BFS traversal.(Also provide short correctness argument).
- (d) Device the most efficient algorithm to verify whether a given directed graph with n vertices and m edges has a unique topological ordering. (Also provide a very short correctness proof of your algorithm).

- (e) Draw a DFS tree of the graph below with respect to source v_0 . Also provide start-time and finish-time of all the vertices with respect to your DFS tree. Compute a topological ordering of the graph using the finish-time of vertices with respect to the DFS tree compute by you.



2 Gemstones [6 marks]

You are given set of k precious gemstones and a function $F : \mathbb{N} \rightarrow \mathbb{N}$ where $F(i)$ denotes the market price of a box of i gemstones (for $1 \leq i \leq k$). Your task is to find the optimal way to partition k gemstones into smaller sets that on selling provides you maximum overall profit.

Design an $O(k^2)$ time algorithm to output the optimal partitioning for k gemstones in a 'list'. Also provide a short correctness argument.

3 Distances [(3 + 3 + 2 + 4) = 12 marks]

$G = (V, E, wt)$ is a directed, strongly-connected, weighted graph with n vertices and m edges, where wt is the edge-weight function. It is given that G has no cycle of negative weight. Let $s \in V$ be a vertex in G . Define a **new** weight function wt^* as follows:

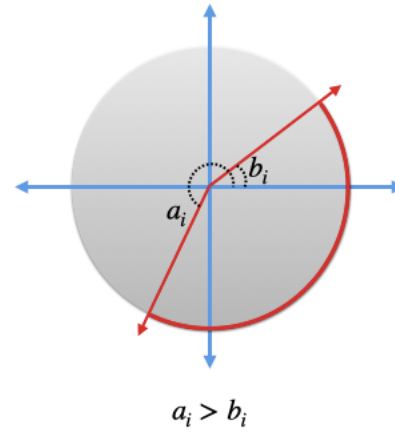
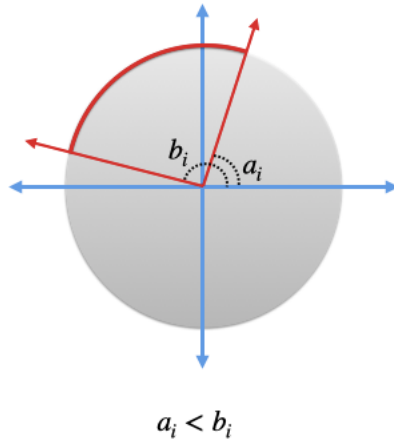
$$wt^*(x, y) := wt(x, y) + dist_G(s, x) - dist_G(s, y).$$

- Show that for each edge $(x, y) \in E$, $wt^*(x, y) \geq 0$. Further, the values $wt^*(e)$, for $e \in E$, are computable in $O(mn)$ total time.
- For a path P , define $wt(P) := \sum_{e \in P} wt(e)$ and $wt^*(P) := \sum_{e \in P} wt^*(e)$.
Prove that for any $P = (v_0, \dots, v_t)$, $wt^*(P) = wt(P) + dist_G(s, v_0) - dist_G(s, v_t)$.
- Use (b), to argue that for $x, y \in V$, a path Q is a shortest path from x to y according to weight function wt if and only if Q is a shortest path from x to y according to weight function wt^* .
- Present an $O(mn \log n)$ time algorithm to compute the distance between all the vertex-pairs in the input graph G . Also provide correctness argument.

4 Covering the circumference [6 + 3 = 9 marks]

Let C be a unit circle on $x - y$ plane centered at the origin. You are given n arcs on C numbered $1, \dots, n$. For $i \in [1, n]$, the arc i starts at an angle a_i from x -axis (in anticlockwise direction) and goes up to angle b_i (in anticlockwise direction).

Further, it is given to you that the union of all the arcs covers the entire circumference of the circle.



- (a) Device an $O(n \log n)$ time algorithm that given an arc (a_{i_0}, b_{i_0}) , finds a subset S of the minimum possible size such that $S \cup \{ (a_{i_0}, b_{i_0}) \}$ covers the circumference of the circle. (Also provide a correctness argument).
- (b) Use (a) to device an $O(n \log n)$ time algorithm to find a subset R of arcs satisfying:
 - union of arcs in R cover the circumference of the circle, and
 - $|R|$ is at most one larger than the minimum number of arcs needed to cover the circumference of the circle.

(Also provide a short correctness argument).