COL 351 Quiz 6B

KushagraGupta

TOTAL POINTS

10 / 10

QUESTION 1

1 Q1 **10 / 10**

✓ **+ 2 pts** *We first show that this problem is in NP. The verifier takes as input a graph G and a subset S of size k vertices and checks if S is an independent set or not*

✓ **+ 2 pts** *We now reduce the independent set problem to the LargeIndSet problem. Let (G, k) be an input to the independent set problem.*
We map it to an input (G′, k′) of the LargeIndSet problem. Let n be the number of vertices in G. We obtain G′ as follows: we add a set W of n new vertices to G and there are no edges incident with any vertex in W.

✓ **+ 1 pts** *The parameter $k' = k + n$.*

✓ **+ 1 pts** *Note that if n' is the number of vertices in G', then*
$n' = 2n$ and so, $k' \geq n'/2$

✓ **+ 2 pts** *Argue that G has an independent set of size k iff G′ has a large independent set of size k′.*
Suppose G has an independent S of size k. Then $S \cup W$ is an independent set in G′ of size $n + k = k'$

✓ **+ 2 pts** *Conversely, suppose G′ has an independent set S of size k′. Now at most n of the vertices in S can belong to W . The remaining vertices $S \setminus W$ belong to G and form an independent set in G. Thus, we have an independent set of size at least $k' - n = k$ in G*

+ 0 pts No submission / Incorrect

Quiz 6B (COL 351)

Name KUSHAGRA GUPTA Entry No. 2021CS50592

Give precise arguments. You can use the fact that the following problems are NP-complete: 3-Satisfiability, Clique, Vertex Cover, Independent Set, Subset Sum.

Given an undirected graph G on n vertices, we say that a subset S of vertices in G is a large independent set if S is an independent set AND $|S| \geq n/2$. Prove that the following problem, called LargeIndSet, is NP-complete: given a graph G and a value $k \geq n/2$, does G have a large independent set of size at least k? Recall that a subset S of vertices is said to be an independent set if there is no edge between any pair of vertices in S.

We will first show that Large Ind Set is in NP.
Given any subset S which is a solution / Large Ind Set,
we ~~also~~ can check in polynomial time that $|S| \geq n/2$
AND $|S|$ is independent (check every pair of vertices in $O(|S|^2)$ time)

We will now show that

$$IS \leq_p \text{Large Ind Set}.$$

$\boxed{G, k}$ (G, k) & Decision version. G', k'

(any general instance of IS (G,k) may be converted to a specific instance of Large Ind Set, i.e it is reducible to Large Ind set).

If $k \leq n/2$,
Construct G' by adding ~~2~~ $(n-2k)$ disconnected independent vertices to G, and choose $k' = n-k$. where $n=|G|$.
else if $k \geq n/2$,
Choose $G' = G$ and $k' = k$.

$G' = G \cup S'$ where S' has $n-2k$ disconnected vertices

Note that this conversion is done in polynomial time (adding $n-2k$ vertices → polynomial)

We will now show that corresponding to every solved instance of $IS(G, k)$, $\exists$ solved instance of Large Ind Set (G', k')

~~i.e~~ & vice-versa i.e

$$IS(G, k) \longrightarrow \text{Large } IS(G', k') \quad -(1)$$
$$\text{Large } IS(G', k') \longrightarrow IS(G, k). \quad -(2)$$

(also note that if $k \geq n/2$, Large $IS(G', k')$ can exist as $\frac{|G'|}{2} \leq k'$
if $k < n/2$, " as $\frac{|G'|}{2} = \frac{2n-2k}{2} \leq k' = n-k$.)

(1) Now, if there exists a solution to $IS(G, k)$

↳ if $k \geq n/2$ → it exists in Large $IS(G', k')$ as $\frac{|G'|}{2} \leq k'$
$|G'| = n$

if $k < n/2$ →

On adding $n-2k$ disconnected vertices to G, we now have $k + n - 2k = n-k$ disconnected vertices in G' (disconnected = independent) (independent).

Since $k' = n-k$, we would be able to find a sol^n for Large $IS(G', k')$ as $k' = n-k = \frac{|G'|}{2}$
$k' \geq \frac{|G'|}{2}$

✓

(2) Now, if $\exists$ a sol^n to Large $IS(G', k')$.

↳ if $k \geq n/2$ same argument as above ($G' = G$ & $k' = k$)
Corresponding to sol^n for Large $IS(G', k')$
if $k < n/2$ $\exists$ a sol^n to $IS(G, k)$

Since we added only $n-2k$ independent vertices & $k' = n-k$, $\therefore$ there must exist k independent vertices in the original graph G, & $k' \geq |G'|/2$
Corresponding to those k vertices, we have a sol^n to $IS(G, k)$.

$\therefore$ $IS(G, k)$ reducible to Large Ind Set (G, k') ~~reducible~~ in polynomial time