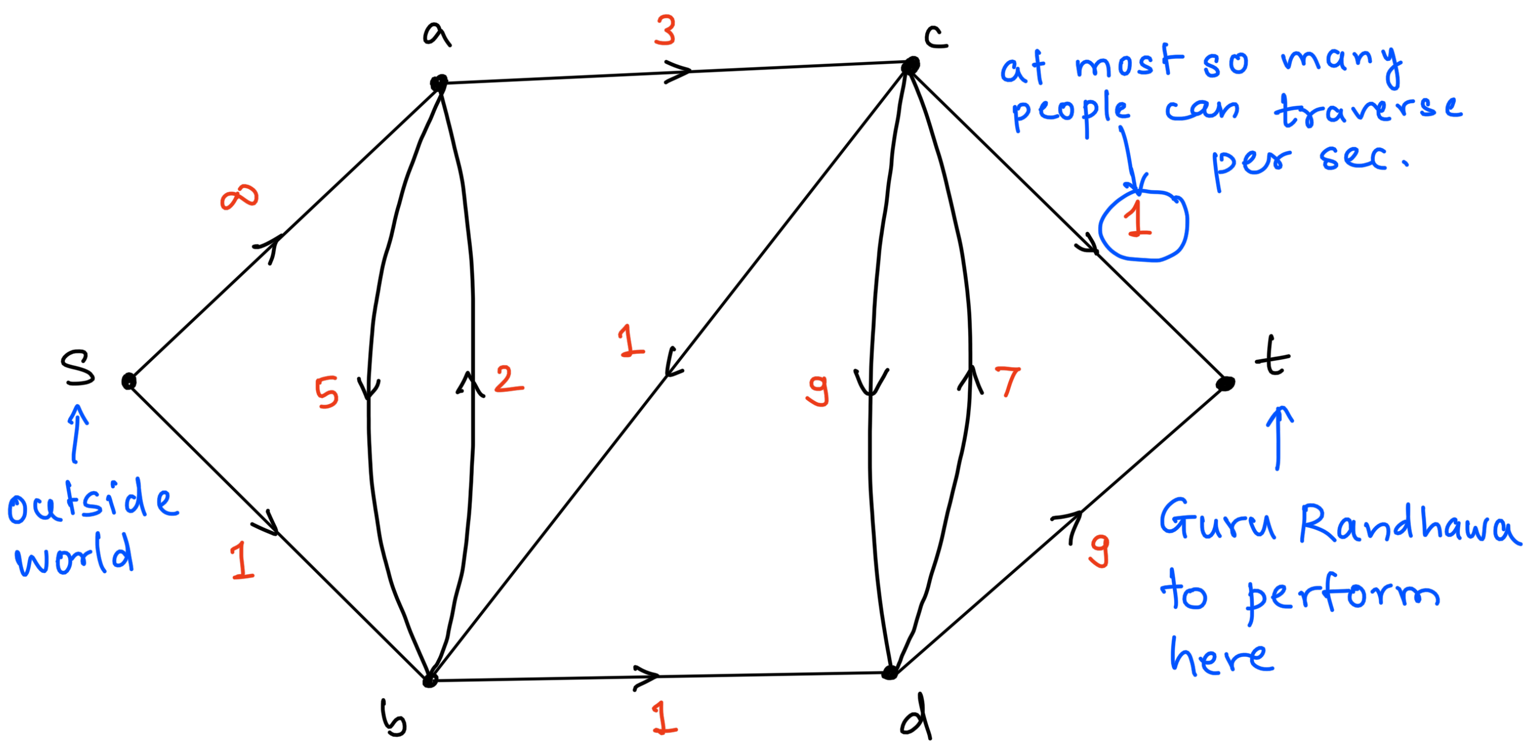
COL 351 Lecture 24 2023/03/13

Topic : Flows and Cuts



What is the max # people that can reach t froms per second ?

Definition: A network is a tuple (G, s, t, C), where

- $G = (V, E)$ is a directed graph (such that $(u,v) \in E$ iff $(v,u) \in E$.)
- $s, t \in V$
- $C : E \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a "capacity function".

Definition: A flow in a network (G, s, t, C), where $G = (V, E)$, is a function $f : E \to \mathbb{R} \cup \{-\infty, \infty\}$ such that:

1. $\forall (u,v) \in E.\ f(u,v) = -f(v,u)$ (skew-symmetry)
2. $\forall v \neq s,t :\ \sum_{u \text{ nbr of } v} f(v,u) = 0$ (conservation)
3. $\forall (u,v) \in E :\ f(u,v) \leq C(u,v).$ $\left(\therefore f(u,v) \geq -C(v,u) \right)$

Notation: $C(u,v) = 0$ and $f(u,v) = 0$ if $(u,v) \notin E$

Definition: Value of a flow f: $|f| = \sum_{v} f(s,v).$

$$\left(= \sum_{v} f(v,t) \right)$$

The Max-flow problem.

Input: Network (G, s, t, C)

Output: A flow of max possible value.

Q1: Does there exist a flow in every network?

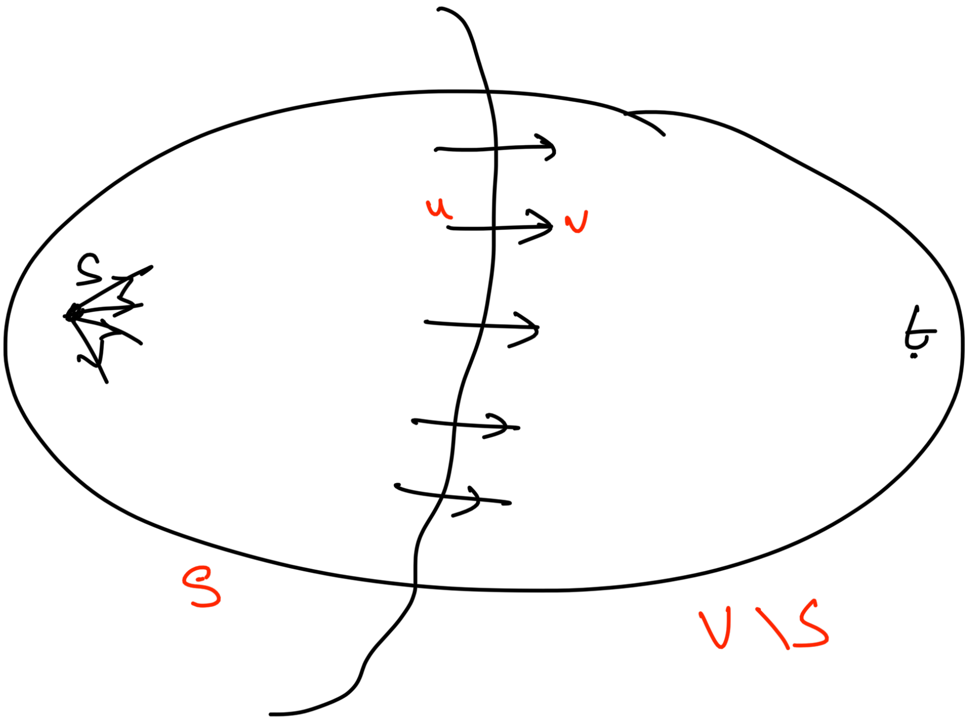
Yes; $f(u,v)=0 \ \forall \ u,v$ is a flow.

Q2: When does there exist a nonzero flow in a network?

Iff $\exists$ $s \to t$ path in which all edges have a +ve capacity. (If: obvious only if: ?)

Q2: When does there exist an infinite flow in a network?

Iff $\exists$ $s \to t$ path in which all edges have ∞ capacity (If: obvious only if: ?)



Intuition for proving upper bounds on flow values:

All flow emerging from s must go from S to V \ S.

Claim: Let (G, s, t, C) be a network and f be a flow in it. Let $S \subseteq V$ be such that $s \in S$, $t \notin S$.

Then $|f| = \sum_{u \in S} \sum_{v \notin S} f(u,v)$.

Proof: $|f| = \sum_{v} f(s,v)$ (def)

$$= \sum_{v} f(s,v) + \sum_{u \in S \setminus \{s\}} \underbrace{\sum_{v} f(u,v)}_{=0 \text{ by conservation}}$$

$$= \sum_{u \in S} \sum_{v} f(u,v)$$

$$= \underbrace{\sum_{u \in S} \sum_{v \in S} f(u,v)}_{0 \text{ by skew-symmetry}} + \sum_{u \in S} \sum_{v \notin S} f(u,v)$$

$$= \sum_{u \in S} \sum_{v \notin S} f(u,v)$$

Claim: Let (G, s, t, C) be a network and f be a flow in it. Let $S \subseteq V$ be such that $s \in S$, $t \notin S$.

Then $|f| \leq \sum_{u \in S} \sum_{v \notin S} C(u,v)$

Proof: $|f| = \sum_{u \in S} \sum_{v \notin S} f(u,v) \leq \sum_{u \in S} \sum_{v \notin S} C(u,v)$.