COL 352 Introduction to Automata and Theory of Computation

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Lecture 31: Beyond Undecidability

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- Oracle Turing machines!
- In addition to its ordinary read/write tape, there is a special one-way-infinite read-only input tape ("oracle tape") on which some infinite string ("oracle") is written.
- Machine can move its oracle tape head one cell in either direction in each step and make decisions based on the symbols written on the oracle tape.

Relative hardness

Definition

For $A, B \subseteq \Sigma^*$, we say that A is recursively enumerable (Turing recognizable) in B if there is an oracle TM M with oracle B such that $A = L(M)$. In addition, if M halts on all inputs, we write $A \leq_T B$ and say that A is recursive (decidable) in B or that A Turing reduces to B.

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- Given a TM M and input x, first ask the oracle whether M accepts x.
- Given M, x, query oracle whether M accepts x. If the answer is yes, then output "yes".
- If the answer is no, switch accept and reject states of M (M') and query the oracle whether M' accepts x. If the answer is yes, output "yes". If the answer is still no, output "no" (M should loop on x).

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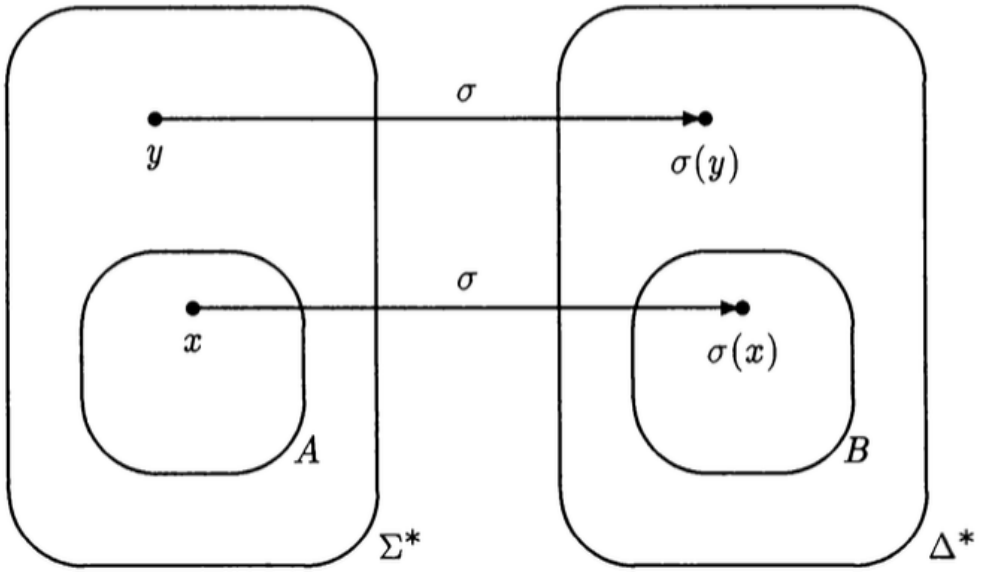
- On input M, x ask the oracle if M halts on x. If it answers no, output "no".
- If it answers "yes", run M on x and output the answer.

Many-one vs Turing reducibility



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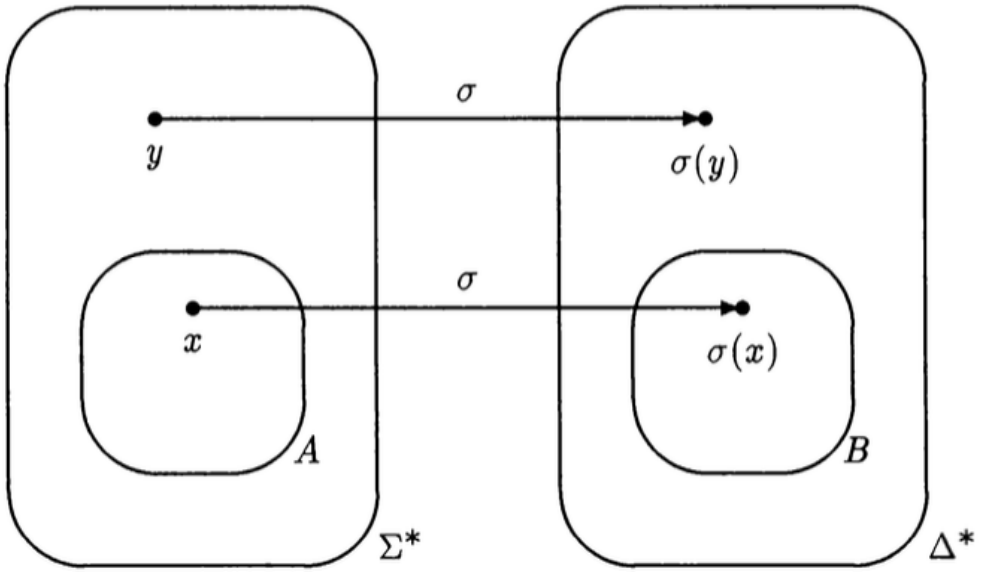

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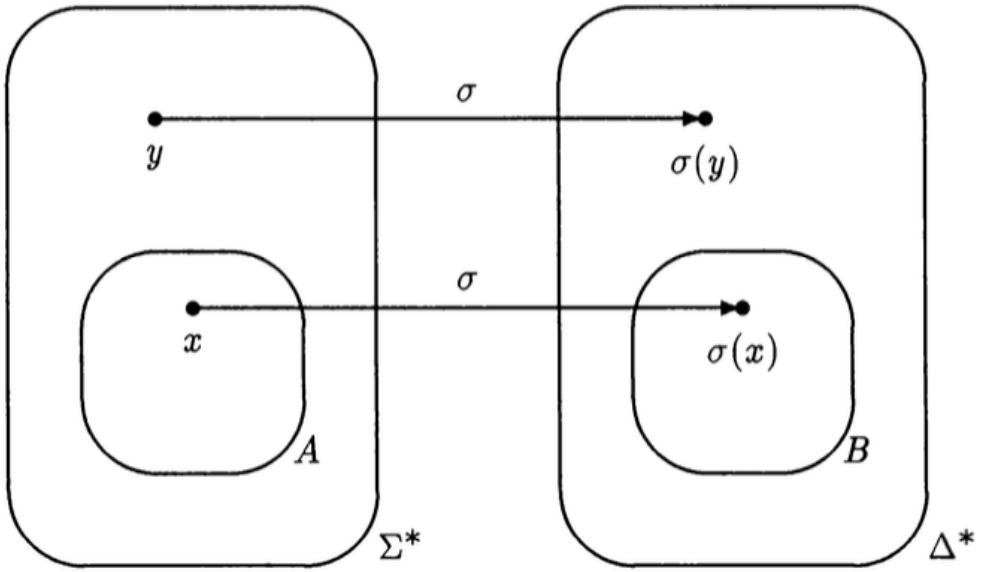

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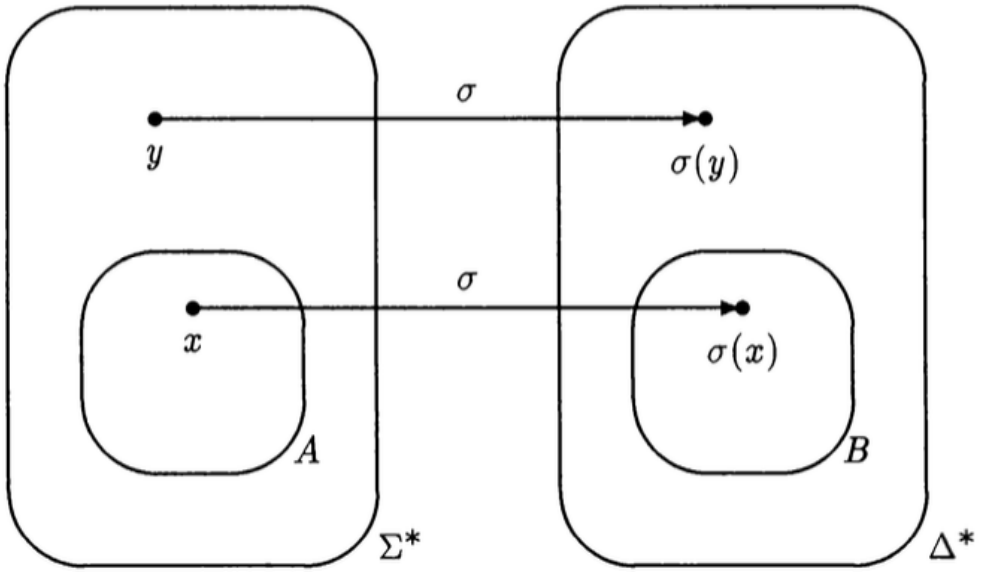

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- $\leq_T$ is transitive!

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A hierarchy of harder and harder problems!

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Theorem

1. *A set* A *is in* Σ_n^0 *if there exists a decidable* $(n+1)$*-ary predicate* R *such that*

$$A = \{x \mid \exists y_1, \forall y_2, \ldots Q y_n R(x, y_1, \ldots, y_n)\}$$

   where $Q = \exists$ *if* n *is odd and* $\forall$ *if* n *is even.*

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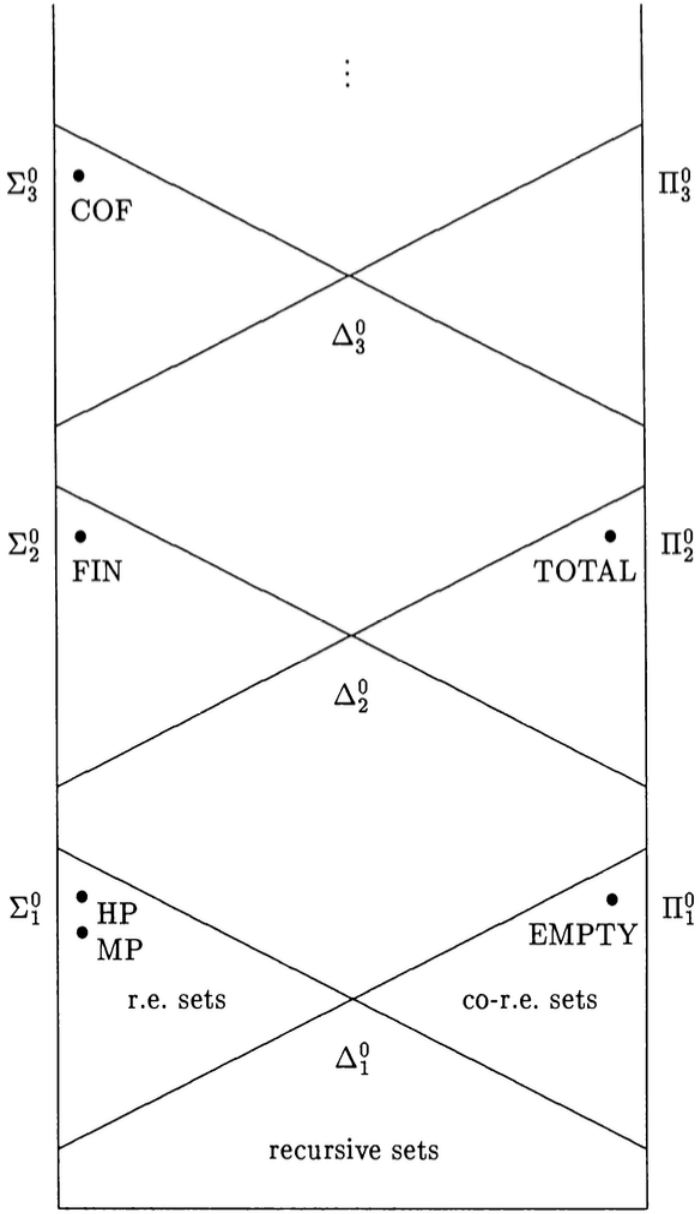

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- $COFIN$ is Σ_3^0-complete!

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- These classes do have natural complete problems!