COL 352 Introduction to Automata and Theory of Computation

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Lecture 33: Computational Complexity Theory (Part 2)

Time complexity and complexity classes

Let $t : \mathbb{N} \to \mathbb{N}$.

Definition

A language $L \subseteq \Sigma^*$ is said to be in class NTIME$(t(n))$ if there exists a non-deterministic Turing machine M such that $\forall x \in \Sigma^*$,

- each run of M halts on x in time $O(t(|x|))$, where $|x|$ indicates the length of x.
- if $x \in L$ then M accepts x on at least one run.
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A verifier for a language A is an algorithm V, where

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NP is the class of languages that have polynomial time verifiers. c is the "certificate" or "witness" or "proof" that $w \in A$.

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Idea: On input x, simulate the execution of M_x on x for $|x|^{1.5}$ steps using a Universal TM. If U outputs some bit $b \in \{0, 1\}$ in this time, then output the opposite answer (i.e., output $1 - b$). Else output 0.

Efficient simulation

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Theorem (Efficient universal Turing machine)

There exists a TM U such that for every $x, \alpha \in \{0,1\}^*, U(x,\alpha) = M_\alpha(x)$, where M_α denotes the TM represented by α. Moreover, if M_α halts on input x within T steps then $U(x,\alpha)$ halts within $CT \log T$ steps, where C is a number independent of $|x|$ and depending only on M_α's alphabet size, number of tapes, and number of states.

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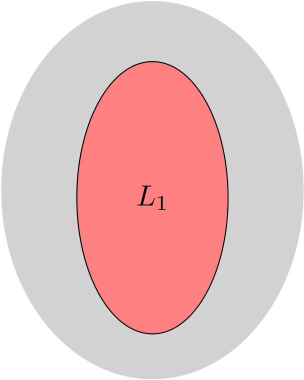

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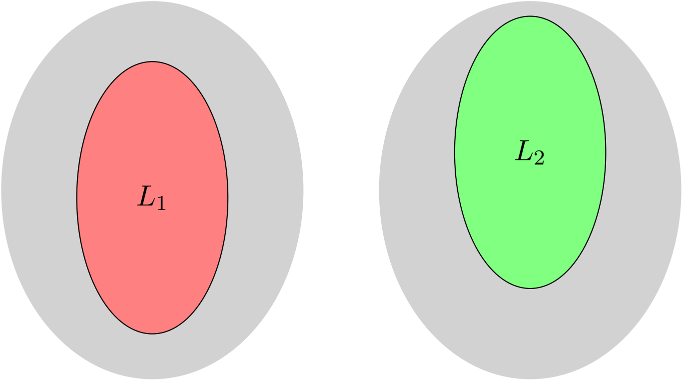

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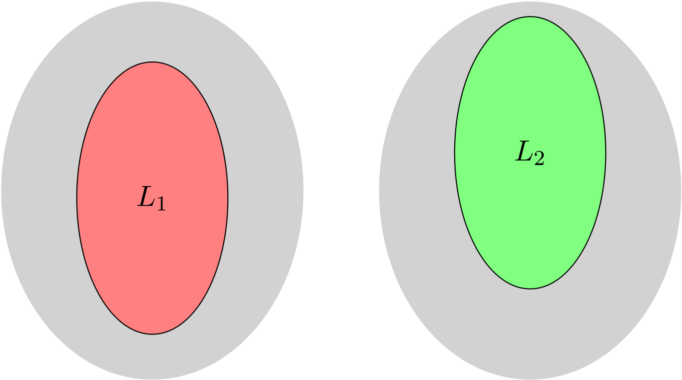

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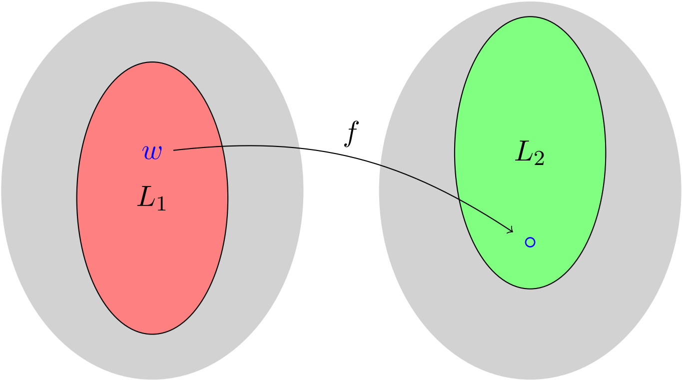

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