COL 352 Introduction to Automata and Theory of Computation

Nikhil Balaji

Bharti 420
Indian Institute of Technology, Delhi
nbalaji@cse.iitd.ac.in

April 5, 2023

Lecture 26: Reductions 1

Recap

Regular $\subsetneq$ Context-free $\subsetneq$ Decidable $\subsetneq$ Turing Recongizable

Undecidability

Recap

Regular $\subsetneq$ Context-free $\subsetneq$ Decidable $\subsetneq$ Turing Recongizable

Undecidability

$$A_M = \{\langle M, w\rangle \mid w \in L(M)\}$$

$$E_M = \{\langle M\rangle \mid L(M) = \varnothing\}$$

$$EQ_M = \{\langle M_1, M_2\rangle \mid L(M_1) = L(M_2)\}$$

Recap

Regular $\subsetneq$ Context-free $\subsetneq$ Decidable $\subsetneq$ Turing Recongizable

Undecidability

$$A_M = \{\langle M, w\rangle \mid w \in L(M)\}$$

$$E_M = \{\langle M\rangle \mid L(M) = \emptyset\}$$

$$EQ_M = \{\langle M_1, M_2\rangle \mid L(M_1) = L(M_2)\}$$

- A_{TM} is undecidable.

Recap

Regular $\subsetneq$ Context-free $\subsetneq$ Decidable $\subsetneq$ Turing Recongizable

Undecidability

$$A_M = \{\langle M, w\rangle \mid w \in L(M)\}$$

$$E_M = \{\langle M\rangle \mid L(M) = \varnothing\}$$

$$EQ_M = \{\langle M_1, M_2\rangle \mid L(M_1) = L(M_2)\}$$

- A_{TM} is undecidable.
- Universal Turing Machines.

Reducibility

Definition

A reduction from problem P_1 to problem P_2 is an algorithm to convert instances of a problem P_1 to instances of problem P_2 that have same answers. In this case we say that P_2 is as hard as P_1.

Reducibility

Definition

A reduction from problem P_1 to problem P_2 is an algorithm to convert instances of a problem P_1 to instances of problem P_2 that have same answers. In this case we say that P_2 is as hard as P_1.

Theorem

If there is a reduction from problem P_1 to problem P_2, then

- *If P_1 is undecidable then so is P_2.*
- *If P_1 is non-RE then so is P_2.*

Reducibility

Definition

A reduction from problem P_1 to problem P_2 is an algorithm to convert instances of a problem P_1 to instances of problem P_2 that have same answers. In this case we say that P_2 is as hard as P_1.

Theorem

If there is a reduction from problem P_1 to problem P_2, then

- *If P_1 is undecidable then so is P_2.*
- *If P_1 is non-RE then so is P_2.*

- If L is decidable then so is $\overline{L}$.

Reducibility

Definition

A reduction from problem P_1 to problem P_2 is an algorithm to convert instances of a problem P_1 to instances of problem P_2 that have same answers. In this case we say that P_2 is as hard as P_1.

Theorem

If there is a reduction from problem P_1 to problem P_2, then

- *If P_1 is undecidable then so is P_2.*
- *If P_1 is non-RE then so is P_2.*

- If L is decidable then so is $\overline{L}$.
- If L and $\overline{L}$ are Turing recognizable then they are both decidable.

Other undecidable problems and reducibility

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable.

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w).

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject,

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject, else do as per M on w.

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject, else do as per M on w.

$\mathcal{A}$ accepts (M, w) if M accepts w

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject, else do as per M on w.

$\mathcal{A}$ accepts (M, w) if M accepts w and rejects it if either M rejects w

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject, else do as per M on w.

$\mathcal{A}$ accepts (M, w) if M accepts w and rejects it if either M rejects w or M loops forever on w.

Other undecidable problems and reducibility

Reducing A_{TM} to another problem to prove undecidibility.

$$\text{Halt} = \{(M, w) \mid M \text{ halts on } w\}$$

We would like to show that Halt is undecidable.

Assume that Halt is decidable. Let $\mathcal{H}$ be the TM deciding Halt.

$\mathcal{A}$: Run $\mathcal{H}$ on (M, w). If it rejects then reject, else do as per M on w.

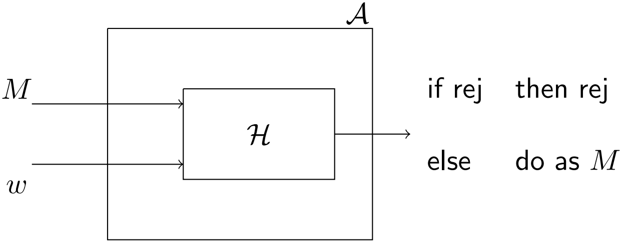
$\mathcal{A}$ accepts (M, w) if M accepts w and rejects it if either M rejects w or M loops forever on w.

$\mathcal{H}$ decides Halt if and only if $\mathcal{A}$ decides A_{TM}.

The halting problem

Lemma

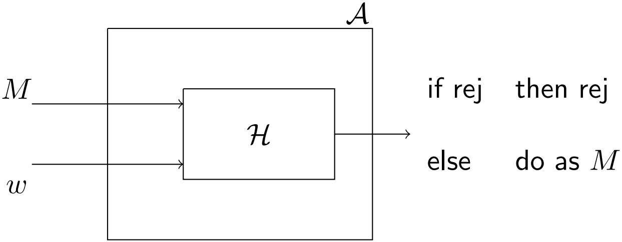
The halting problem, Halt = $\{(M, w) \mid M$ *halts on* $w\}$*, is undecidable.*



The halting problem

Lemma

The halting problem, Halt = $\{(M, w) \mid M$ halts on $w\}$, is undecidable.

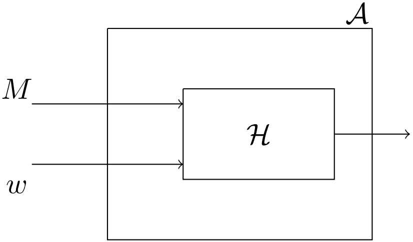


If Halt is decidable then $\mathcal{A}$ decides A_{TM}

The halting problem

Lemma

The halting problem, Halt = $\{(M, w) \mid M$ halts on $w\}$, is undecidable.



if rej then rej

else do as M

If Halt is decidable then $\mathcal{A}$ decides A_{TM}, which is a contradiction.

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M\rangle \mid L(M) = \varnothing\}$, is undecidable.

Assume for the sake of contradiction that it is decidable.

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}$, is undecidable.

Assume for the sake of contradiction that it is decidable. Let T be a machine that decides E_{TM}.

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M \rangle \mid L(M) = \emptyset\}$, is undecidable.

Assume for the sake of contradiction that it is decidable. Let T be a machine that decides E_{TM}.

Let $T'_{M,w}$ be as follows:

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M \rangle \mid L(M) = \emptyset\}$, is undecidable.

Assume for the sake of contradiction that it is decidable. Let T be a machine that decides E_{TM}.

Let $T'_{M,w}$ be as follows:

On input x

$\{$

if $w \neq x$ then reject

else do as per M

$\}$

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}$, is undecidable.

Assume for the sake of contradiction that it is decidable. Let T be a machine that decides E_{TM}.

Let $T'_{M,w}$ be as follows:

On input x

$\{$

if $w \neq x$ then reject

else do as per M

$\}$

$$L(T'_{M,w}) = \begin{cases} \{w\} & \text{if } M \text{ acc } w \\ \varnothing & \text{otherwise} \end{cases}$$

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M \rangle \mid L(M) = \varnothing\}$, is undecidable.

Assume for the sake of contradiction that it is decidable. Let T be a machine that decides E_{TM}.

Let $T'_{M,w}$ be as follows:

On input x

$\{$

if $w \neq x$ then reject

else do as per M

$\}$

$$L(T'_{M,w}) = \begin{cases} \{w\} & \text{if } M \text{ acc } w \\ \varnothing & \text{otherwise} \end{cases}$$

Let A be as follows:

Emptiness problem for TM

Lemma

The emptiness problem for TMs, $E_{TM} = \{\langle M\rangle \mid L(M) = \varnothing\}$, is undecidable.

Assume for the sake of contradiction that it is decidable. Let T be a machine that decides E_{TM}.

Let $T'_{M,w}$ be as follows:

On input x

{

if $w \neq x$ then reject

else do as per M

}

$$L(T'_{M,w}) = \begin{cases} \{w\} & \text{if } M \text{ acc } w \\ \varnothing & \text{otherwise} \end{cases}$$

Let A be as follows:

On input M, w

{

Create machine $T'_{M,w}$.

If T on $\langle T'_{M,w}\rangle$ rejects

then accept

else reject

}

This shows that if E_{TM} is decidable then A_{TM} is decidable.

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$*, is undecidable.*

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$, is undecidable.

Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$, is undecidable.

Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \varnothing$.

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$, is undecidable.

Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \emptyset$.

Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$, is undecidable.

Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \varnothing$.

Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$, i.e. to check whether $L(M_2) = \varnothing$ or not.

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$, is undecidable.

Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \varnothing$.

Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$, i.e. to check whether $L(M_2) = \varnothing$ or not.

This implies that if EQ_{TM} is decidable then E_{TM} is decidable.

Equality for TM

Lemma

The equality problem for TMs, $EQ_{TM} = \{(M_1, M_2) \mid L(M_1) = L(M_2)\}$, is undecidable.

Assume for the sake of contradiction that EQ_{TM} is decidable. Let M be the TM for it.

Let M_1 be a machine that rejects all strings. That is, $L(M_1) = \varnothing$.

Given a machine M_2 as an input, use M to check whether $L(M_2) = L(M_1)$, i.e. to check whether $L(M_2) = \varnothing$ or not.

This implies that if EQ_{TM} is decidable then E_{TM} is decidable.

But from the previous result we know that E_{TM} is undecidable.

Regularity checking

Lemma

REG$_{TM}$ = $\{\langle M \rangle \mid L(M)$ *is regular*$\}$ *is undecidable.*

Regularity checking

Lemma

$REG_{TM} = \{\langle M \rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R is a TM that decides REG_{TM}.

Regularity checking

Lemma

$REG_{TM} = \{\langle M\rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R is a TM that decides REG_{TM}.

Let $R'_{M,w}$ be s.t.

$$L(R'_{M,w}) = \begin{cases} \{0^n 1^n \mid n \geq 0\} & \text{if } M \text{ rej } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

Regularity checking

Lemma

$REG_{TM} = \{\langle M \rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R is a TM that decides REG_{TM}.

Let $R'_{M,w}$ be s.t.

$$L(R'_{M,w}) = \begin{cases} \{0^n1^n \mid n \geq 0\} & \text{if } M \text{ rej } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

If we get such an $R'_{M,w}$ we can design A as a follows.

Regularity checking

Lemma

$REG_{TM} = \{\langle M \rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R is a TM that decides REG_{TM}.

Let $R'_{M,w}$ be s.t.

$$L(R'_{M,w}) = \begin{cases} \{0^n 1^n \mid n \geq 0\} & \text{if } M \text{ rej } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

If we get such an $R'_{M,w}$ we can design A as a follows.

Regularity checking

Lemma

$REG_{TM} = \{\langle M\rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R is a TM that decides REG_{TM}.

Let $R'_{M,w}$ be s.t.

$$L(R'_{M,w}) = \begin{cases} \{0^n1^n \mid n \geq 0\} & \text{if } M \text{ rej } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

If we get such an $R'_{M,w}$ we can design A as a follows.

Let A be as follows:

Regularity checking

Lemma

$REG_{TM} = \{\langle M\rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R is a TM that decides REG_{TM}.

Let $R'_{M,w}$ be s.t.

$$L(R'_{M,w}) = \begin{cases} \{0^n1^n \mid n \geq 0\} & \text{if } M \text{ rej } w \\ \Sigma^* & \text{if } M \text{ acc } w \end{cases}$$

If we get such an $R'_{M,w}$ we can design A as a follows.

Let A be as follows:

On input M, w

$\{$

Create machine $R'_{M,w}$.

If R on $\langle R'_{M,w}\rangle$ accepts

then accept

else reject

$\}$

Regularity checking

Lemma

$REG_{TM} = \{\langle M \rangle \mid L(M)$ is regular$\}$ is undecidable.

Regularity checking

Lemma

$REG_{TM} = \{\langle M \rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R be a TM that decides REG_{TM}.

Let $R'_{M,w}$ be as follows:

Regularity checking

Lemma

$REG_{TM} = \{\langle M\rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R be a TM that decides REG_{TM}.

Let $R'_{M,w}$ be as follows:

On input x

$\{$

if $x = 0^n 1^n$

then accept

else run M on w and

if M acc w then acc

else rej

$\}$

Regularity checking

Lemma

REG_{TM} = $\{\langle M\rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R be a TM that decides REG_{TM}.

Let $R'_{M,w}$ be as follows:

On input x

$\{$

if $x = 0^n 1^n$

then accept

else run M on w and

if M acc w then acc

else rej

$\}$

Let A be as follows:

Regularity checking

Lemma

$REG_{TM} = \{\langle M\rangle \mid L(M)$ is regular$\}$ is undecidable.

Assume for the sake of contradiction that a TM R be a TM that decides REG_{TM}.

Let $R'_{M,w}$ be as follows:

On input x

{

if $x = 0^n1^n$

then accept

else run M on w and

if M acc w then acc
else rej

}

Let A be as follows:

On input M, w

{

Create machine $R'_{M,w}$.

If R on $\langle R'_{M,w}\rangle$ accepts

then accept

else reject

}