COL 352 Introduction to Automata and Theory of Computation

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Lecture 21:Turing Machines

So far..

- Regular languages and Finite Automata
  - DFA = NFA = 2-DFA = 2-NFA = Regex
  - All of the above capture regular languages
  - Non-regular languages exist - Pumping Lemma, Myhill Nerode
  - Algorithms for manipulating/reasoning about DFAs exist! (and are sometimes efficient)
- Context-free languages and Pushdown Automata
  - Nondeterministic Pushdown Automata = Context-free Grammars
  - Deterministic PDAs = DCFL
  - DCFL $\neq$ CFL
  - There exist languages which are not CFLs
  - Pumping Lemma for CFLs

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- What about all these machines with k pointers on the input tape?
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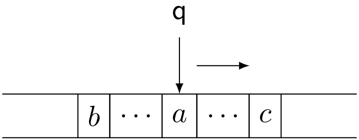
Is there a machine-independent notion of computation?

Turing machines

What is a Turing machine? (Informal description.)



- Read and write on the input tape. Head moves left/right.
- The tape is infinite.
- A special symbol $\&$ to indicate blank cells.
- Initially all cells blank except the part where the input is written.
- Special states for accepting and rejecting.

Formal definition

Definition

A Turing machine (TM) is given by $M = (Q, \Sigma, \Gamma, \delta, q_0, q_f, q_{rej})$

Q:	set of states	Σ:	input alphabet
q_0:	start state	Γ:	tape alphabet, $\Sigma \subseteq \Gamma$, $\& \in \Gamma$
q_{acc}:	accept state	q_{rej}:	reject state

$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$.

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Understanding δ

For a $q \in Q, a \in \Gamma$ if $\delta(q, a) = (p, b, L)$

then p is the new state of the machine,

b is the letter with which a gets overwritten,

the head moves to the left of the current position.

Configuration

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- We say that $u \cdot a \cdot q \cdot b \cdot v$ **yields** $u \cdot q' \cdot a \cdot c \cdot v$.
- We denote it by $u \cdot a \cdot q \cdot b \cdot v \mapsto u \cdot q' \cdot a \cdot c \cdot v$.

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A TM may not halt!

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and the machine is allowed to do either of the two.

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Turing decidable languages form a subclass of Turing recognizable languages.

Turing machine for a non-context free language

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On input string w:

1. Check if corresponding positions on either side of the $\#$ symbol are the same character. If not, or if no $\#$ is found, reject. Cross off symbols as they are checked.

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1. Check if corresponding positions on either side of the $\#$ symbol are the same character. If not, or if no $\#$ is found, reject. Cross off symbols as they are checked.
2. When all symbols to the left of the $\#$ have been crossed off, check for any remaining symbols to the right of the $\#$. If any symbols remain, reject; otherwise, accept.

Turing machine for a non-context free language

Example

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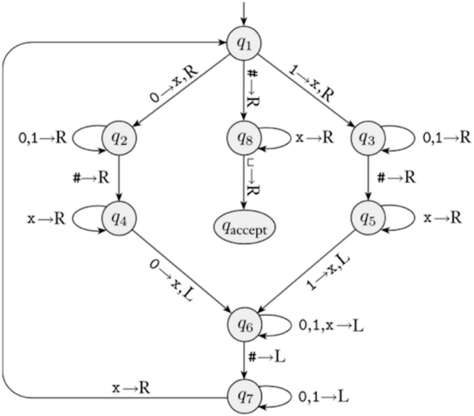

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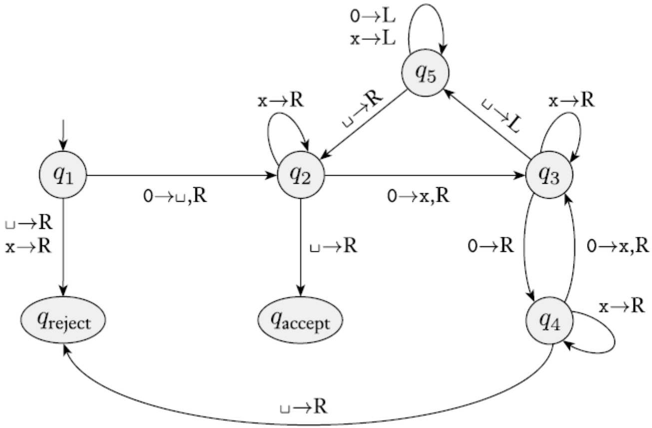

More examples

- $L_1 = \{0^{2^n} \mid n \in \mathbb{N}\}$
- $L_2 = \{a^i b^j c^k \mid i \times j = k$ and $i, j, k \geq 1\}$.
- $L_3 = \{0^n \mid n$ prime$\}$