COL352 Problem Sheet 1

January 9, 2025

Problem 1. *Create a DFA that represents the language* $\{$*binary representation of* $n \in \mathbb{N} \mid n \mod 8$ *is either* 4 *or* $1\}$

Problem 2. *Create a regular expression over the alphabet* $\{0, 1\}$ *that represents the language mentioned in Problem 1.*

Problem 3. *Construct a DFA which accept the language* $L = \{w \mid w \in \{a, b\}^*$ *and* $Na(w) \mod 3 = Nb(w) \mod 3\}$, *where* $Na(w)$ *and* $Nb(w)$ *return the number of occurrences of* a *and* b *in* w *respectively.*

Problem 4. *Construct a DFA that recognizes the following language over the alphabet* $\{0, 1\}$.

$$\{x \mid \text{ 01 and 10 have equal number of occurrences as substrings in } x\}$$

Problem 5. *Let* $C_n = \{x \mid x$ *is a binary number that is a multiple of* $n\}$. *Show that for each* $n \geq 1$, *the language* C_n *is regular. Provide a general construction for* C_i *for* $i \geq 0$.

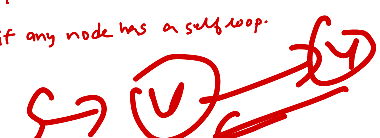
Problem 6. *You already know from the lectures that regular languages are closed under complementation. Given DFAs* D_1 *and* D_2 *that recognize languages* L_1 *(over* Σ_1*) and* L_2 *(over* Σ_2*) respectively, construct an automaton* D *recognizing the following languages, if you believe that the class of regular languages is closed under the following operations. Provide a counterexample otherwise.*

1. ***Difference:*** $L_1 \setminus L_2 := \{x \mid x \in L_1 \text{ and } x \notin L_2\}$
2. ***Star:*** $L_1^* := \{w_1 w_2 \dots w_n \mid w_i \in L_1 \text{ for } n \geq 0, \text{ and every } 1 \leq i \leq n\}$

Problem 7. *Design an efficient algorithm that takes input the description of a DFA* D *and determines if the resulting language* $L(D)$ *is*

for (v : adj[u])
{ dfs(v, u);
u=
}

1. *Empty* → No path from q_0 to q_i. $q_i \in F$.
2. *Infinite* → From q_0 to $q \in F$, if any node has a self loop.
3. Σ^* → Check $\neg L$ is empty.



Problem 8. *Given two DFAs,* D_1 *and* D_2*, design an efficient algorithm to determine if* $L(D_1) = L(D_2)$.

Problem 9. $L_1 = (0|1)^*0(0|1)^*1(0|1)^*$, $L_2 = (0|1)^*01(0|1)^*$. *Show that the two languages are equal.*

Problem 10. *Let* L_1 *be a regular language and* L_2 *be any language (not necessarily regular) over the same alphabet* Σ. *Prove that the language* $L = \{x \in \Sigma^* \mid x \cdot y \in L_1$ *for some* $y \in L_2\}$ *is regular by defining a DFA for* L *starting from a DFA for* L_1 *and the language* L_2.

Problem 11. $\phi : \Sigma^* \to \Gamma^*$ *is called a homomorphism over strings if for all* $x, y \in \Sigma^*$, $\phi(xy) = \phi(x)\phi(y)$. *Show that if* L *is a regular language, then* $\phi(L) := \{y \in \Gamma^* \mid y = \phi(x), x \in L\}$ *where* ϕ *is a homomorphism as defined above is also regular.*

Problem 12. *Prove that the class of regular languages is closed under inverse homomorphisms. That is, prove that if* $L \subseteq \Gamma^*$ *is a regular language and* $\phi : \Sigma^* \to \Gamma^*$ *is a string homomorphism, then* $\phi^{-1}(L) = \{x \in \Sigma^* \mid f(x) \in L\}$ *is regular.*

$q_i \xrightarrow{R} q_{accept}$

$R \cap L_2 = \phi$?

GNFA for $\phi^{-1}(L)$

$(q, a, q') \in \Delta$

if $\delta(q, \phi(a)) = q'$

Problem 1. *Create a DFA that represents the language* $\{$*binary representation of* $n \in \mathbb{N} \mid n \mod 8$ *is either* 4 *or* $1\}$

sol^n) Define $D = (Q, \Sigma, \delta, q_0, F)$

as $Q = \{ q_i \mid 0 \leq i < 8 \}$

$\Sigma = \{0, 1\}$

$F = \{q_4, q_1\}$

and δ as

$$\delta(q_i, 0) = q_{(2i \bmod 8)}, \quad \delta(q_i, 1) = q_{(2i+1) \bmod 8}$$

Problem 2. *Create a regular expression over the alphabet* $\{0, 1\}$ *that represents the language mentioned in Problem 1.*

sol^n) $(1 \mid 100 \mid \Sigma^* 001 \mid \Sigma^* 100)$

Problem 3. *Construct a DFA which accept the language* $L = \{w \mid w \in \{a, b\}^*$ *and* $Na(w) \mod 3 = Nb(w) \mod 3\}$, *where* $Na(w)$ *and* $Nb(w)$ *return the number of occurrences of* a *and* b *in* w *respectively.*

sol^n) $D = (Q, \{a, b\}, \delta, q_0, F)$ as

$Q = \{q_0, q_1, q_2\}$. $\quad F = \{q_0\}$

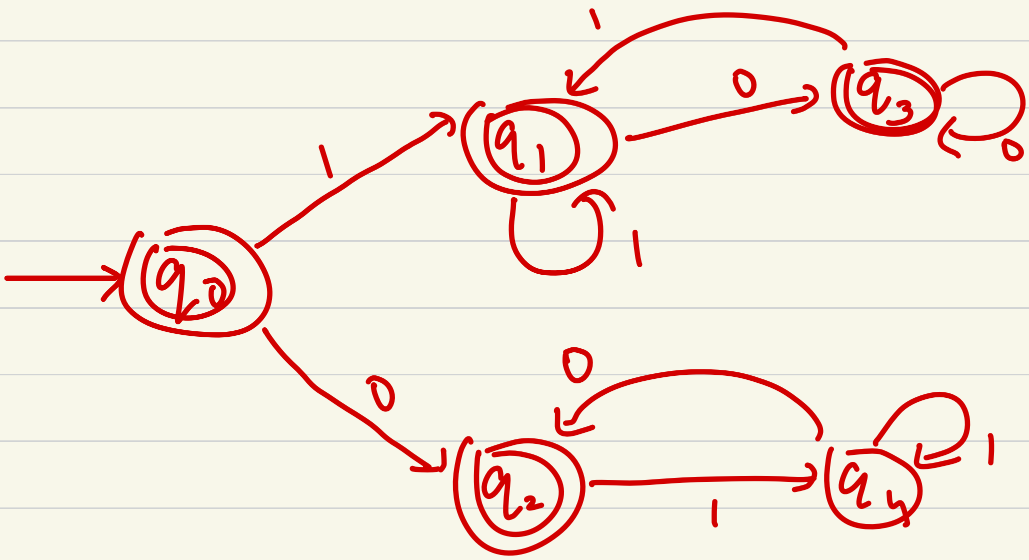
and

$$\delta(q_i, a) = q_{(i+1) \bmod 3}$$

$$\delta(q_i, b) = q_{(i-1) \bmod 3}$$

Problem 4. *Construct a DFA that recognizes the following language over the alphabet* $\{0, 1\}$.

$$\{x \mid \text{01 and 10 have equal number of occurrences as substrings in } x\}$$



1. ***Difference:*** $L_1 \setminus L_2 := \{x \mid x \in L_1 \text{ and } x \notin L_2\}$

Sol^n) Claim:- If L is regular with alphabets Σ_1 then L is regular with alphabet $\Sigma_1 \cup \Sigma_2$.

Let $M = (Q, \Sigma_1, \delta, q_0, F)$ recognize L over Σ_1

then $M' = (Q', \Sigma_1 \cup \Sigma_2, \delta', q_0, F)$ as

$Q' = Q \cup \{q_{reject}\}$

$$\delta'(q,a) = \begin{cases} \delta(q,a) & \text{if } q \in Q,\ a \in \Sigma_1 \\ q_{reject} & \text{if } q \in Q,\ a \notin \Sigma_1,\ a \in \Sigma_2 \\ q_{reject} & \text{if } q = q_{reject},\ a \in \Sigma_1 \cup \Sigma_2 \end{cases}$$

$\Rightarrow$ Clearly M' also recognizes L.

Original Problem:- Let M_1 and M_2 recognize L_1 and L_2 over $\Sigma_1 \cup \Sigma_2$

as $M_1 = (Q_1, \Sigma_1 \cup \Sigma_2, \delta_1, q_0^1, F_1)$ and

$M_2 = (Q_2, \Sigma_1 \cup \Sigma_2, \delta_2, q_0^2, F_2)$.

Define M to recognize $L_1 \setminus L_2$ as

$(Q, \Sigma_1 \cup \Sigma_2, \delta, q_0, F)$ to be

$Q = Q_1 \times Q_2$

$q_0 = q_0^1 \times q_0^2$

$F = F_1 \times (Q_2 \setminus F_2)$

$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$

Problem 10. *Let L_1 be a regular language and L_2 be any language (not necessarily regular) over the same alphabet Σ. Prove that the language $L = \{x \in \Sigma^* \mid x \cdot y \in L_1 \text{ for some } y \in L_2\}$ is regular by defining a DFA for L starting from a DFA for L_1 and the language L_2.*

Sol^n) Let $D_1 = (Q, \Sigma, \delta, q_0, F)$ be

Define a NFA for L

Problem 9. $L_1 = (0|1)^*0(0|1)^*1(0|1)^*$, $L_2 = (0|1)^*01(0|1)^*$. *Show that the two languages are equal.*

Sol^n) Let $w \in L_2 \Rightarrow w = (0|1)^n 01 (0|1)^m$

$= (0|1)^n 0 (0|1)^0 1 (0|1)^m \in L_1$

Since w was arbitrary, $L_2 \subseteq L_1$.

Now to prove all $w \in L_1$ are in L_2.

Let $w = (0|1)^a 0 (0|1)^b 1 (0|1)^c \in L_1$

We induct on b.

Base Case :- $b = 0$, trivially in L_2.

Inductive Hypothesis :- Assume true for $0, 1, \ldots, b$.

For $b+1$, if $(0|1)^{b+1} = 0(0|1)^b$, then

$w = (0|1)^a 0 (0|1)^{b+1} 1 (0|1)^c$

$= (0|1)^a 00 (0|1)^b 1 (0|1)^c$

$= (0|1)^{a+1} 0 (0|1)^b 1 (0|1)^c \in L_2$ (Inductive hypothesis)

if $(0|1)^{b+1} = 1(0|1)^b$

$\Rightarrow w = (0|1)^a 0 (0|1)^{b+1} 1 (0|1)^c$

$= (0|1)^a 01 (0|1)^b 1 (0|1)^c$

$= (0|1)^a 0 (0|1)^0 1 (0|1)^{b+c+1} \in L_2$

Problem 10. *Let L_1 be a regular language and L_2 be any language (not necessarily regular) over the same alphabet Σ. Prove that the language $L = \{x \in \Sigma^* \mid x \cdot y \in L_1 \text{ for some } y \in L_2\}$ is regular by defining a DFA for L starting from a DFA for L_1 and the language L_2.*

Sol^n Claim:- Question is WRONG.

Counter Example:-

let L_2 = undecidable language with $\varepsilon \in L_2$.

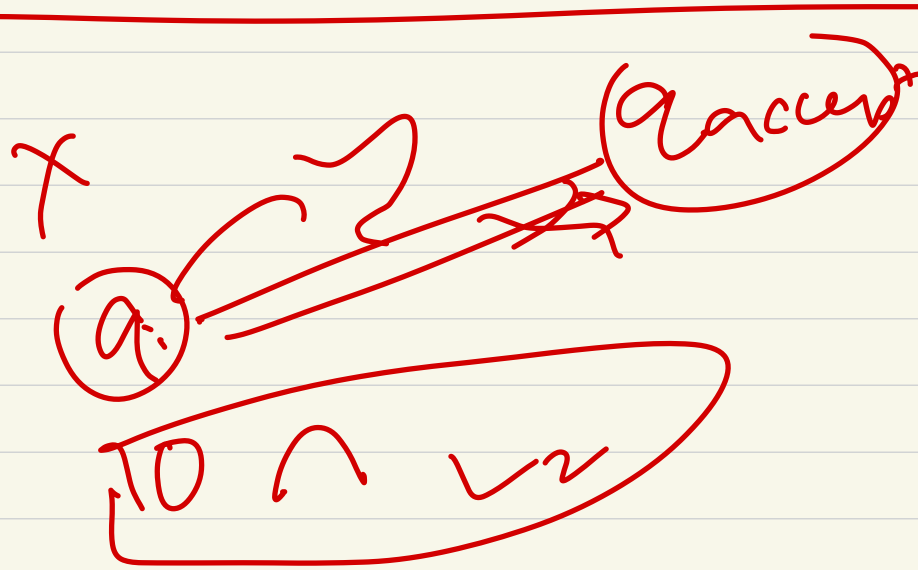
Claim:- $L_3 \cap L_2 = \phi$? is also a undecidable if L_3 is regular.

Proof:- Assume not. Let M decide $L_1 \cap L_2 = \phi$?
Then design D to decide L_2 as given w to test $w \in L_2$?, check $\{w\} \cap L_2 = \phi$? This decides L_2.

Let L_1 be the following GNFA for Regular Expression R.

$$q_0 \xrightarrow{R} q_{accept}$$

Then in GNFA,



L_2 = undec

$L_1 = \{a^n b^m \mid n \geq m\}$ $L_1 \cup L_2 = a^* b^*$
$L_2 = \{a^n b^m \mid n \leq m\}$

Problem 13. *Prove that the class of* ***non-regular*** *languages is* ***not*** *closed under the Union operation.* ***Hint:*** *use the closure properties mentioned in the lectures.*

Problem 14. *Let L be an arbitrary regular language. Prove that the following languages are regular.*

1. $\{x \mid x \cdot reverse(x) \in L\}$
2. $\{x \mid x \cdot reverse(x) \cdot x \in L\}$.
3. $\{x \mid xxx \in L\}$

Problem 15. *Let L be a regular language with DFA D. We define $Pre(L) = \{x \mid x$ is a prefix of some $y \in L\}$. Show that $Pre(L)$ is regular by constructing a DFA.*

Problem 16. *Let A be any language. Define DROP-OUT(A) to be the language containing all strings that can be obtained by removing one symbol from a string in A. Thus, DROP-OUT(A) $= \{xz \mid xyz \in A$ where $x, z \in \Sigma^*, y \in \Sigma\}$. Show that the class of regular languages is closed under the DROP-OUT operation.*

Problem 15. *Let L be a regular language with DFA D. We define $Pre(L) = \{x \mid x$ is a prefix of some $y \in L\}$. Show that $Pre(L)$ is regular by constructing a DFA.*

Sol^n) let $D = (Q, \Sigma, \delta, q_0, F)$, then define $E = (Q', \Sigma, \delta', q_0', F')$ to accept pre(L) such that

$Q' = Q$

$q_0' = q_0$ Use ε transition

$\delta' = \delta$

$F' = \{q \mid \exists q' \in F$ such that q' is reachable from $q\}$.

Problem 16. *Let A be any language. Define DROP-OUT(A) to be the language containing all strings that can be obtained by removing one symbol from a string in A. Thus, $DROP\text{-}OUT(A) = \{xz \mid xyz \in A$ where $x, z \in \Sigma^*, y \in \Sigma\}$. Show that the class of regular languages is closed under the DROP-OUT operation.*

Sol^n) Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A.

Create a NFA $N = (Q_1 \cup Q_2, \Sigma, \Delta, Q_0, F_0)$ recognizing dropout(A)

as $Q_1 = \{q_1 \mid q \in Q\}$, $Q_2 = \{q_2 \mid q \in Q\}$

$Q_0 = \{q_0\}$, $F_0 = \{q_2 \mid q \in F\}$

and

$\Delta(q_{i,1}, a) = \{q_{j,1} \mid \delta(q_i, a) = q_j\}$ $a \in \Sigma$

and $\Delta(q_{i,2}, a) = \{q_{j,2} \mid \delta(q_i, a) = q_j\}$ $a \in \Sigma$

and $\Delta(q_{i,1}, \varepsilon) = \{q_{j,2} \mid \exists a \in \Sigma$ s.t $\delta(q_i, a) = q_j\}$

Problem 14. *Let L be an arbitrary regular language. Prove that the following languages are regular.*

1. $\{x \mid x \cdot reverse(x) \in L\}$
2. $\{x \mid x \cdot reverse(x) \cdot x \in L\}$.
3. $\{x \mid xxx \in L\}$

Sol^n) Let $D = (Q, \Sigma, \delta, q_0, F)$ be the DFA recognizing L.

① Create a NFA $N = (Q', \Sigma, \Delta, Q_0, F_0)$ as

$Q' = Q \times Q$

$Q_0 = \{q_0\} \times F$

$F_0 = \{(q, q) \mid q \in Q\}$

$\Delta((q_1,q_2),a) = \{ (\delta(q_1,a), q_3) \mid \delta(q_3,a) = q_2 \}$

② q_j $\underbrace{abc}$ $\underbrace{cba}$ q_i $\underbrace{abc}$ q_j

$Q' = Q \times Q$

$Q_0 = \{q_0\} \times Q$

$F_0 = Q \times F$

$\Delta((q_1,q_2),a) = \{ (\delta(q_1,a), q_3) \mid \delta(q_3,a) = q_2 \}$

Define a language L_i such that

$Q' = Q \times Q$

$Q_0 = (q_0, q_i)$

$F_0 = Q \times F$

$\Delta((q_1,q_2),a) = \{ (\delta(q_1,a), q_3) \mid \delta(q_3,a) = q_2 \}$

$\{x \mid rev(x) \cdot x \in L\} = \bigcup L_i$

$\bigcup_{i,j} (L_{0i} \cap Rev(L_{ij}) \cap L_{jf})$

③ q_0 abc q_i abc q_j abc

Define language L_{ij} with a DFA as

$Q' = Q \times Q \times Q$

$q_0' = (q_0, q_i, q_j)$

$F' = \{(q_i, q_j, q) \mid q \in F\}$

$\delta((q_1,q_2,q_3),a) = (\delta(q_1,a), \delta(q_2,a), \delta(q_3,a))$

Then $\{xxx \in L\} = \bigcup_{i,j} L_{ij}$

$0 \underset{x}{\text{————}} i \quad \underbrace{\qquad\qquad\qquad}$

$$L_{0i} \cap \mathrm{Rev}(L_{if})$$