COL 352 Introduction to Automata and Theory of Computation

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Lecture 25: Undecidability

Recap

- Turing machines - definition (single tape, deterministic), examples.
- Languages - Turing recognizable vs Turing decidable
- Robustness: TMs are extemely robust
- Variants: k-tapes, doubly infinite tapes, Enumerators, NTMs, Queue machines, 2 stacks, counter machines,...

Recap

- Turing machines - definition (single tape, deterministic), examples.
- Languages - Turing recognizable vs Turing decidable
- Robustness: TMs are extemely robust
- Variants: k-tapes, doubly infinite tapes, Enumerators, NTMs, Queue machines, 2 stacks, counter machines,...
- Today: undecidability.

Turing recognizability vs Decidability

Definition

A language L is said to be Turing recognizable if there is a Turing machine M such that $\forall w \in L$, M has at least one accepting run on w. For words not in L

- the machine may run forever
- or may reach q_{rej}

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A language L is said to be Turing decidable if there is a Turing machine M such that for all $w \in \Sigma^*$, M halts on w and

- if $w \in L$, M has an accepting run on w.
- if $w \notin L$, all runs of M on w are rejecting runs.

Hierarchy of languages and devices

Regular $\subsetneq$ Context-free $\subsetneq$ Decidable $\subsetneq$ Turing Recongizable

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- Reading exercise: Theorems 4.2-4.9 in Sipser's book.

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- Important: Encoding programs as data.

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Any encoding of TMs will have a null character, say 010101. Then for any string $\alpha \in \{0,1\}^*$, suppose it represents machine M then all strings of the form $(010101)^*\alpha$ also represent the same machine M.

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If $|x| \neq |x'|$ then $\phi(x) \neq \phi(x')$. If $|x| = |x'|$, then bin$(1x) \neq$ bin$(1x')$ as long as $x \neq x'$.

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Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and $2^{\mathbb{N}}$ (set of all subsets of $\mathbb{N}$).

Proof.

Suppose for the sake of contradiction that there is a bijection, say f, between set of all subsets of $\mathbb{N}$.

	0	1	2	3	...
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The inverted diagonal set does not belong to any of the existing sets! □

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- Therefore, set of all languages is uncountable.
- However, the set of all TMs is countable. ($\{0,1\}^*$ is countable.)
- There must be a language which is not Turing recognizable.

A decision problem about TMs

$$A_{TM} = \{(M, w) \mid M \text{ accepts } w\}$$

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Is A_{TM} decidable?

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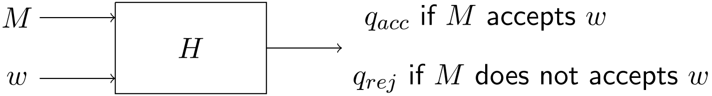
Assume that there exists M such that M decides A_{TM}.



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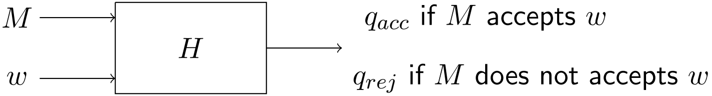

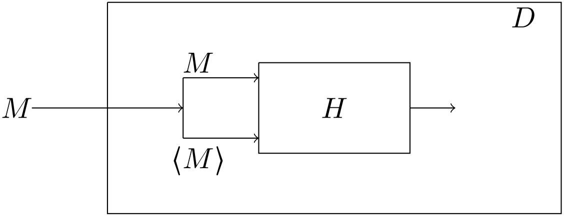

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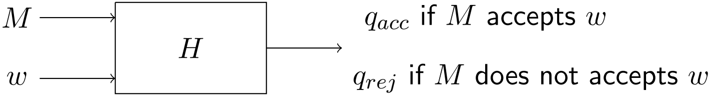

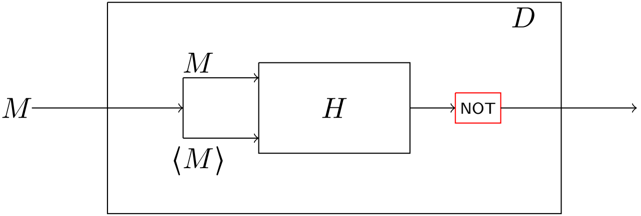

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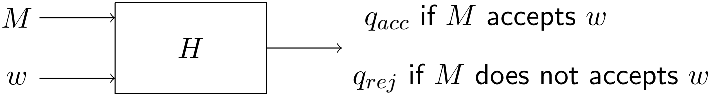

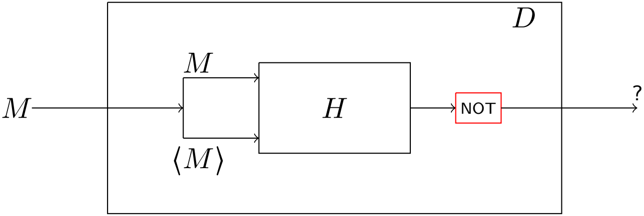

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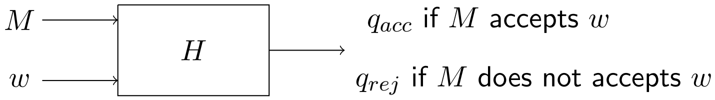

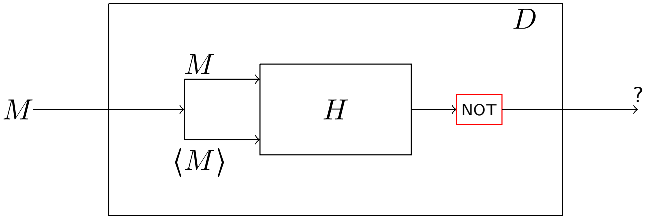

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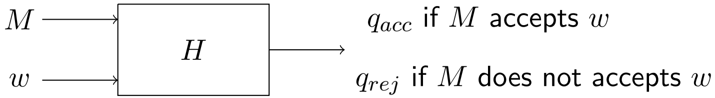

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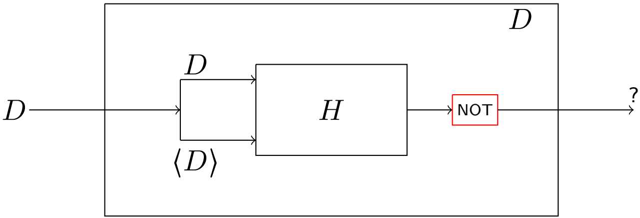

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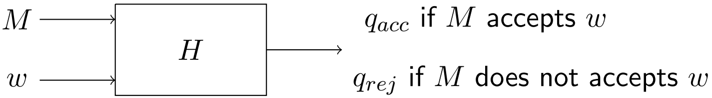

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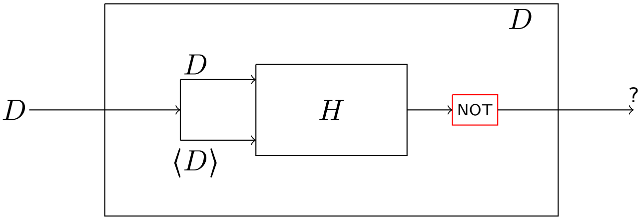

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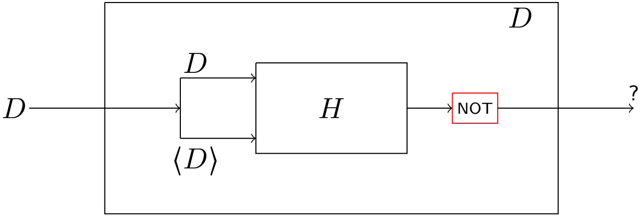

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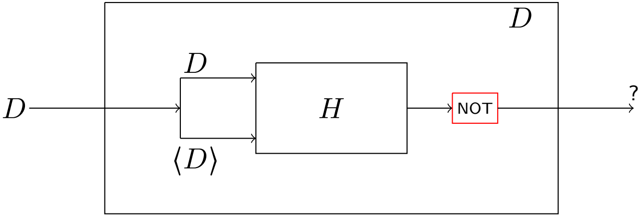

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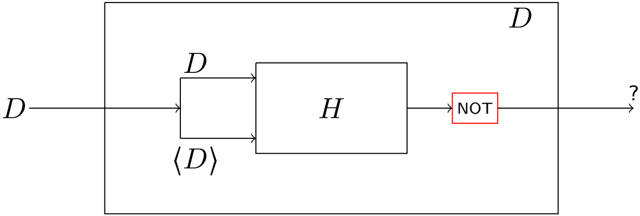

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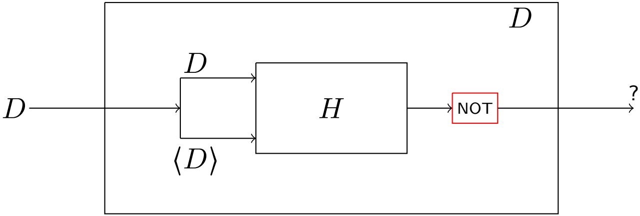

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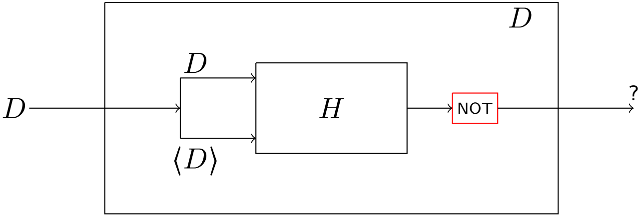

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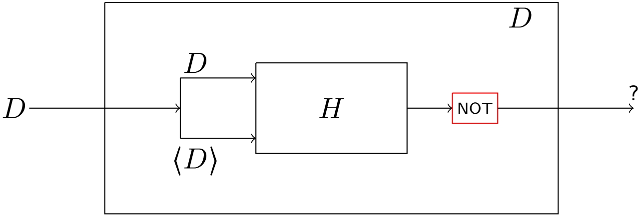

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Diagonalization inside the proof

Behaviour of the machines.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$		$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$		$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$		$\times\ \ldots$	$\checkmark\ldots\times\checkmark\ \ldots$

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Behaviour of the machines.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	✓		✓	✓ $\ldots$	$\ldots\ldots$
M_2	✓	$\times$		$\times$ $\ldots$	✓ $\ldots \times$ ✓ $\ldots$
M_3	$\times$	$\times$	✓	$\ldots \times$	✓ $\ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of H.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$	$\times$	$\checkmark$	$\checkmark\ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$	$\times$	$\times\ \ldots$	$\checkmark\ldots\times\checkmark\ \ldots$
M_3	$\times$	$\times$	$\checkmark$	$\ldots\times$	$\checkmark\ldots\ldots$
$\vdots$					
$\vdots$					

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\checkmark$	$\times$	$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\times$	$\times$	$\times$ $\ldots$	$\checkmark \ldots \times \checkmark$ $\ldots$
M_3	$\times$	$\times$	$\checkmark$	$\ldots \times$	$\checkmark \ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of D.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\ldots\ldots$	$\ldots\ldots$
M_1	$\cancel{\checkmark}$ $\times$	$\times$	$\checkmark$	$\checkmark \ldots$	$\ldots\ldots$
M_2	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\times$	$\times \ldots$	$\checkmark \ldots \times \checkmark \ldots$
M_3	$\times$	$\times$	$\cancel{\checkmark}$ $\times$	$\ldots \times$	$\checkmark \ldots\ldots$
$\vdots$					
$\vdots$					

Diagonalization inside the proof

Behaviour of D on itself.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\dots \langle D \rangle \dots$	$\dots\dots$
M_1	$\cancel{\checkmark}$ $\times$	$\times$	$\checkmark$	$\checkmark \dots$	$\dots\dots$
M_2	$\checkmark$	$\cancel{\times}$ $\checkmark$	$\times$	$\times \dots$	$\checkmark \dots \times \checkmark \dots$
M_3	$\times$	$\times$	$\cancel{\checkmark}$ $\times$	$\dots \times$	$\checkmark \dots\dots$
$\vdots$					
$\vdots$					
D				$\dots$ $?$ $\dots$	$\dots\dots$

Back to Comparing decidability and recognizability

Theorem

A language L is Turing decidable if and only if L and $\overline{L}$ are both Turing recognizable.

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Corollary

$\overline{A_{TM}}$ is not Turing recognizable.

Universal Turing machines

Definition

A Turing machine is called a Universal Turing machine if it can given the description of any Turing machine M and an input w, simulate the machine M on w.

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