

COL 352 Introduction to Automata and Theory of Computation

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Lecture 32: Computational Complexity Theory (Part 1)

Effective computation

Turing machines with resource constraints.

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Resources for computation.

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Many different ways exist. ...

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$$O(n) + \frac{n}{2}O(n) = O(n^2)$$

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- 4 Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
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$$L \in \text{TIME}(n \log n)$$

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- 3 Scan across the 1s on tape 1 until the end of the input. For each 1 read on tape 1, cross off a 0 on tape 2. If all 0s are crossed off before all the 1s are read, reject.
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$$L \in \text{TIME}(n)$$

Exercise: There is no single-tape TM solving L in $O(n)$ time.

Relationships between models

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Proof idea: DFS or BFS.

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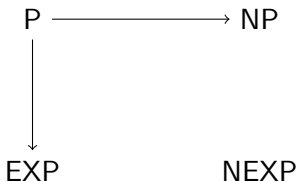
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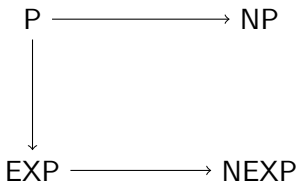
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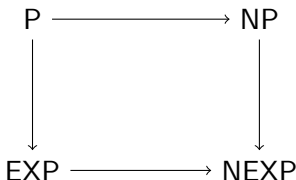
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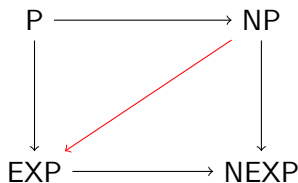
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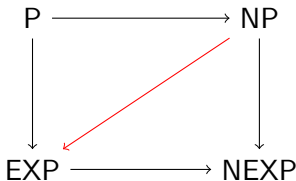
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NP is the class of languages that have polynomial time verifiers. c is the “certificate” or “witness” or “proof” that $w \in A$.