COL 352 Introduction to Automata and Theory of Computation

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Lecture 9: Non-regularity: Pumping Lemma

Limitations of Finite Automata

$$L_{0,1} := \{0^n 1^n \mid n \geq 0\}$$

$$PAL := \{ww^R \mid w \in \Sigma^*\}$$

Generalise?

These arguments seem to be example specific.. can they be generalised?

Pumping Lemma

From the previous argument, we filter out a property of regular languages .

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Lemma

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$$\exists n \forall w \in L \left(|w| \geq n \implies \exists xyz.(xyz = w \wedge y \neq \epsilon \wedge |xy| \leq n \wedge (\forall k \quad xy^k z \in L)\right)$$

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- Exercise: Prove it holds for finite languages.
- What happens if $y = \epsilon$?

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- For any $k > 0$, $xy^k z$ takes A to q_f, hence accepting.
- Therefore, $\forall k > 0$, $xy^k z \in L$

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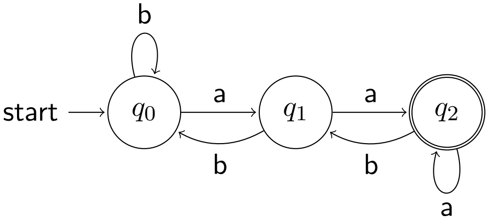
Example

Consider DFA A below.



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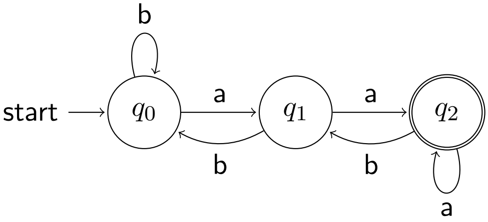

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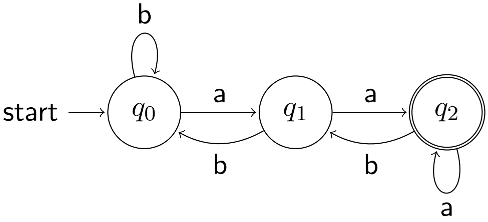

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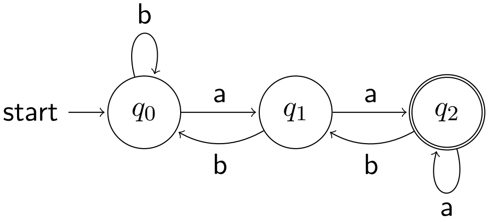

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The run is $q_0 \xrightarrow{a} q_1 \xrightarrow{b} q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_1 \xrightarrow{a} q_2$

q_1 is repeated in the first 4 states of the run.

Choose $x = a$, $y = ba$, $z = aba$

Therefore, $a(ba)^k aba \in L(A)$

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In words...

Theorem

If

- for every n
- there is a word $w \in L$ where $|w| \geq n$.
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How do we apply this lemma?

How to use the lemma?

Consider language L

- Let n be an arbitrary number (pumping length).
- (Cleverly) Find a representative string w of L of size $\geq n$.
- Try out all ways to break the string into xyz triplet satisfying that $|y| > 0$ and $|xy| \leq n$. There will be finitely many cases to consider.
- For every triplet show that for some i the string $xy^i z$ is not in L, and hence it yields contradiction with pumping lemma.

Applying Pumping Lemma: Example 1

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- Choose $k = 0$ for each i, j. The corresponding word is

$$a^i (a^j)^0 a^{n-i-j} b^n = a^{n-j} b^n \notin L_{a,b}$$

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Exercise: What is $L \cap a^* b^*$?