COL 352 Introduction to Automata and Theory of Computation

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February 21, 2023

Lecture 13: Myhill-Nerode Theorem

Recap of Module 1

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- DFA, NFA, Regular expressions and their equivalence.

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- Closure properties of regular languages.
- Non-regular languages and Pigeon Hole Principle.
- Pumping lemma and its applications.
- Myhill Nerode relation and characterization of regular languages.
- Algorithms for membership problem, emptiness problem and minimization problem.

Moving on

How do we add expressive power to DFA/NFA so that we can compute more functions?

2DFA: Two-way deterministic finite state automata

$\# \; w_1 \; w_2 \; \ldots \; \ldots \; \ldots \; \ldots \; \ldots \; \ldots \; w_n \; \$$

Input head moves left/right on this tape.

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It does not go to the left of $\#$.

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Can potentially get stuck in an infinite loop!

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- If the number of a seen is $0 \bmod 3$, move in the reverse direction now counting only $b \bmod 2$ ignoring a.
- If the number of b seen is not $0 \bmod 2$, then accept, else reject.

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Note: Needs only one start and accept state. Halts immediately after entering accept state.

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Exercise: Write down a formal definition of the example from previous slide.

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Fix input $x = a_1 \ldots x_n$, let $a_0 = \#, a_{n+1} = \$$. So $a_0 a_1 \ldots a_n a_{n+1} = \#x\$$.

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- Binary relation: $\xrightarrow[1]{x}$ defined as

  $\delta(p, a_i) = (q, L) \implies (p, i) \xrightarrow[1]{x} (q, i-1)$
- $\delta(p, a_i) = (q, R) \implies (p, i) \xrightarrow[1]{x} (q, i+1)$
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  $(p,i) \xrightarrow[n]{x} (q,j)$ and $(q,j) \xrightarrow[1]{x} (u,k) \implies (p,i) \xrightarrow[n+1]{x} (u,k)$.
- $$(p,i) \xrightarrow[*]{x} (q,j) \iff \exists n \ge 0 (p,i) \xrightarrow[n]{x} (q,j)$$

Acceptance by 2DFA

Definition

Let A be a 2DFA.

- A word w is said to be accepted by A if A reaches q_{acc} on w.

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Lemma

If L is regular then there is a 2DFA accepting L.

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If L is regular then there is a 2DFA accepting L.

This holds trivially.

2DFA: Two-way deterministic finite state automata

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Read the input and move right till \$ is encountered.

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2DFA: Two-way deterministic finite state automata

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$$L_{a,b} = \{a^n b^n \mid n \geq 0\}.$$

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We will say that two words, x, y are equivalent, i.e. $x \sim y$, if $T_x = T_y$.

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$T_x = T_y \Leftrightarrow x \equiv_A y$

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Moving on

How to we add expressive power to DFA/NFA so that we can compute more functions?