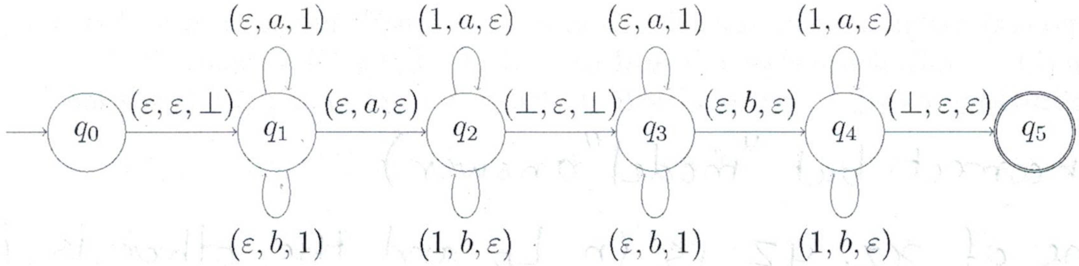
Name: ______________________ Entry number: ______________________

COL352-Holi2024 Quiz 2 Duration: 1 hour

1. Consider the NPDA P over the input alphabet $\{a, b\}$ and stack alphabet $\{1, \perp\}$ shown in the following diagram[1].



Recall the definition of the relation $\vdash^*$ between instantaneous descriptions[2]. Let $L' = \{x \in \{a, b\}^* \mid (q_0, x, \varepsilon) \vdash^* (q_3, \varepsilon, \perp)\}$. Write down a context-free grammar that recognizes each of the following languages. Specify the initial non-terminal clearly. Proof of correctness is not required.

1. (3 points) L', with at most 5 production rules.

$A \to aAa \mid aAb \mid bAa \mid bAb \mid a$

Initial nonterminal : A

2. (3 points) $\mathcal{L}(P)$, with at most 11 production rules.

$S \to AB$

$A \to aAa \mid aAb \mid bAa \mid bAb \mid a$

$B \to aBa \mid aBb \mid bBa \mid bBb \mid b$.

Initial nonterminal : S

(4 points) For every $n \in \mathbb{N} \cup \{0\}$, write down a string $w_n \in \{a, b\}^{2n}$ such that the string $a^n \cdot w_n \cdot b^n$ is rejected by P. Give a short proof (ideally, at most 4 sentences).

$\mathcal{L}(P) = \{x_1 x_2 \mid |x_1|, |x_2|$ are both odd, and the middle characters of x_1, x_2 are a, b respectively$\}$.

If $|x_1 x_2| = 4n$ and $|x_1|, |x_2|$ are both odd, then their middle characters are at distance $2n$.

For $x_1 x_2$ to be rejected by P, we need the middle character of x_1 to be b or the middle character of x_2 to be a.

Thus, if $x_1 x_2 = a^n w_n b^n$, then $w_n = b^n a^n$.

[1] Label (B, a, C) means B is popped off the stack, a is read from the input, and then C is pushed on the stack.

[2] An instantaneous description (q, y, α) denotes that the NPDA is in state q, it is yet to read y from its input, and its stack content read from top to bottom is α.

2. (10 points) Call a pair (L_0, L_1) of disjoint languages over a common alphabet Σ *DFA-separable* if there exists a DFA that rejects every string in L_0 and accepts every string in L_1. Our goal is to come up with a Myhill-Nerode-like result for DFA-separability.

   (a) (2 points) Define an equivalence relation $=_{(L_0,L_1)}$ on Σ^* in such a way that $=_{(L_0,L_1)}$ has a finite number of equivalence classes **if and only if** (L_0, L_1) is DFA-separable.
   Definition. $x =_{(L_0,L_1)} y$ *if (and only if) there does not exist* $z \in \Sigma^*$ *such that*

   (incorrect but "model" answer)

   one of $x \cdot z$, yz is in L_0 and the other is in L_1.

   (b) (8 points) Prove that if (L_0, L_1) is DFA-separable, then $=_{(L_0,L_1)}$, as defined above, has finitely many equivalence classes. (The proof of the converse is not required, but if the converse doesn't hold, you get a 0 points.)

   (incorrect but "model" answer)

   Suppose DFA D separates L_0 and L_1.

   We claim that $=_D$ refines $=_{L_1,L_2}$, i.e.

   $\forall x, y : x =_D y \Rightarrow x =_{L_1,L_2} y$. — $(*)$

   But $=_D$ has finitely many equivalence classes, so $=_{L_1,L_2}$ also has finitely many equivalence classes.

   Proof of $(*)$: If $x =_D y$, then $\forall z$; D accepts both xz, yz, or D rejects both xz, yz.

   In the former case, none of xz, yz is in L_0, and in the later case, none of xz, yz is in L_1.

   Thus, $\nexists z$ such that one of xz, yz is in L_0 and the other is in L_1

   $\therefore x =_{L_1,L_2} y$.