Do P1 and P4 again

COL352 Problem Sheet 4

March 21, 2025

Problem 1. *Show that the following languages are not Context-free.*

1. $L_1 = \{a^m b^n \mid mn \text{ is the square of an integer }\}$ *is not context-free.* ab^{k^2}
2. $L_2 = \{x \in \{0,1\}^* \mid x \text{ is the binary representation of } 3^{n^2} \text{ for some } n \in \mathbb{N} \cup \{0\}\}$
3. $L_3 = \{a^m b^n \mid n \text{ is a multiple of } m\}$
4. $L_4 = \{x \in \{a,b\}^* \mid \# \text{ of } a\text{'s in } x \text{ is a multiple of } \# \text{ of } b\text{'s in } x\}$
5. $L_5 = \{xyx \mid x, y \in \{0,1\}^* \text{ and } |x| > 0,\ |y| > 0\}$

Problem 2. *The strict shuffle of two strings $x = x[1] \ldots x[n]$ and $y = y[1] \ldots y[n]$ of equal length is defined as $sshuffle(x, y) = x[1]y[1]x[2]y[2] \ldots x[n]y[n]$. The strict shuffle of two languages $L_1, L_2 \subseteq \Sigma^*$ is defined as $sshuffle(L_1, L_2) = \{sshuffle(x, y) \mid x \in L_1,\ y \in L_2,\ |x| = |y|\}$. Is the class of context-free languages closed under the sshuffle operation? Prove your answer.*

Problem 3. *We say that z is a shuffle of x and y if the characters in x and y can be interleaved, while maintaining their relative order within x and y, to get z. Formally, if $|x| = m$ and $|y| = n$, then $|z|$ must be $m + n$, and it should be possible to partition the set $\{1, 2, \ldots, m + n\}$ into two increasing sequences, $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_n$, such that $z[i_k] = x[k]$ and $z[j_k] = y[k]$ for all k. Given two languages $L_1, L_2 \subseteq \Sigma^*$, define*

$$shuffle(L_1, L_2) = \{z \in \Sigma^* \mid z \text{ is a shuffle of some } x \in L_1 \text{ and some } y \in L_2\}.$$

Is the class of context-free languages closed under the sshuffle operation? Prove your answer.

Problem 4. *Let $G = (V, \Sigma, R, S)$ be a grammar. Prove that for every $x \in L(G)$, there exists a parse tree of G with root S, yield x, and height at most $|V| \cdot (|x| + 1)$.*

Problem 5. *Let us say that an NPDA is a binary stack NPDA if the size of its set of stack symbols Γ is 2; assume $\Gamma = \{0, 1\}$ for concreteness. Prove that binary stack NPDAs and NPDAs are equivalent in terms of computation power.*

Problem 6. *An n-stack PDA is like a regular PDA except that it has n stacks instead of one.*

1. *Show that for all n, an n-stack PDA could be simulated by a 2-stack PDA.*
2. *Show that anything that can be computed by a turing machine can be computed by a 2-stack PDA.*

Problem 7. *Prove that each of the following functions is computable (You can assume that x, y are positive integers given in their binary representation, and you need the answer in binary representation too).*

1. $x - 1$ *(assuming $x > 0$)*
2. $x + y$
3. $x \times y$
4. x^y

Problem 8. *Show that every language in Problem 1 can be decided by a turing machine.*

1. $L_1 = \{a^m b^n \mid mn \text{ is the square of an integer }\}$ *is not context-free.*

Sol^n) Let $L = L_1 \cap \{a^* b\} = \{a^{n^2} b \mid n \in \mathbb{N} \cup \{0\})$

Claim:- L is NOT CFL

Advesary:- choose k

Me:- Take $a^{k^2} b$

Advesay choose $xuvwz = a^{k^2} b$ s.t. $|uw| > 0$

and $|uvw| \leq k$

Me:- if No b in uvw

$\Rightarrow$ for $i=2$ $\#b = 1$, $\#a < (k+1)^2$

if No a if uvw $\Rightarrow$ $b \in uvw$ $z = \varepsilon$

$\Rightarrow i=2 \Rightarrow a^{k^2} b^2$ notsquare

if

~~3.~~ $L_3 = \{a^m b^n \mid n \text{ is a multiple of } m\}$

$S = a^{k^2} b^{k^3}$

$k^2 + \mu(i-1) \mid k^3 + \lambda(i-1)$

$k^3 + k\mu(i-1)$

$k^2 + \mu(i-1) \mid (k\mu - \lambda)(i-1)$

$k^2 + \mu(i-1) > (k\mu - \lambda)(i-1)$

$$k^2 > (i-1)(k\mu - \lambda - \mu)$$

$$k^2 > k\mu - \lambda - \mu > k\mu - k$$

$$k > \mu - 1$$

$$a^m b^{mk}$$

$$w \cdot w \subseteq xyx$$

$$\underset{x}{a^k}\ \underset{y}{b^k}\ \underset{x}{a^{k+1}}$$

$$\underbrace{0}\,{}^n 1^n \quad 1 \quad 0^n 1^n$$

$$0^n \underbrace{1^n\ 1}\ 0^n\ 1^n$$

$$0^n\ 1^n\ \underbrace{1\ 0^n}\ 1^n$$

$$0^n\ 1^n\ 1\ \underbrace{0^n}\ 1^n$$

$$0^n\ 1^n\ 1\ \underbrace{0^n\ 1^n}$$

Shortest

Consider $x \in L$ with a parse tree
derivation $S \xrightarrow{h} x$
of height h

If $h > |V|(|x|+1)$

Consider the longest path in this parse tree
It has height $h \Rightarrow h-1$ nonterminals and
and 1 terminal.

$\#$ non-terminal $\geq |V|(|x|+1)$

$\Rightarrow \exists$ a nonterminal say V occuring atleast $|x|+1$ times.

Consider all the occurences of V

$$S \longrightarrow V_1 \;\underline{\ldots \ldots (1) \ldots}\; V_2 \;\underline{\ldots (1) \ldots}\; V_3 \;\underline{\ldots (1) \ldots}\; V_k \xrightarrow[|x| \geq 0]{} \Sigma$$

$k \geq |x|+1$

Claim:- In the derivation $V_i \longrightarrow x V_{i+1} y$ where $x, y \in \Sigma^*$
, $|xy| > 0$

Proof:- Assume $|xy| = 0$

$\Rightarrow V_i \longrightarrow V_{i+1}$ Contradiction to this being smallest parse tree.

$$|x| \geq k-1$$

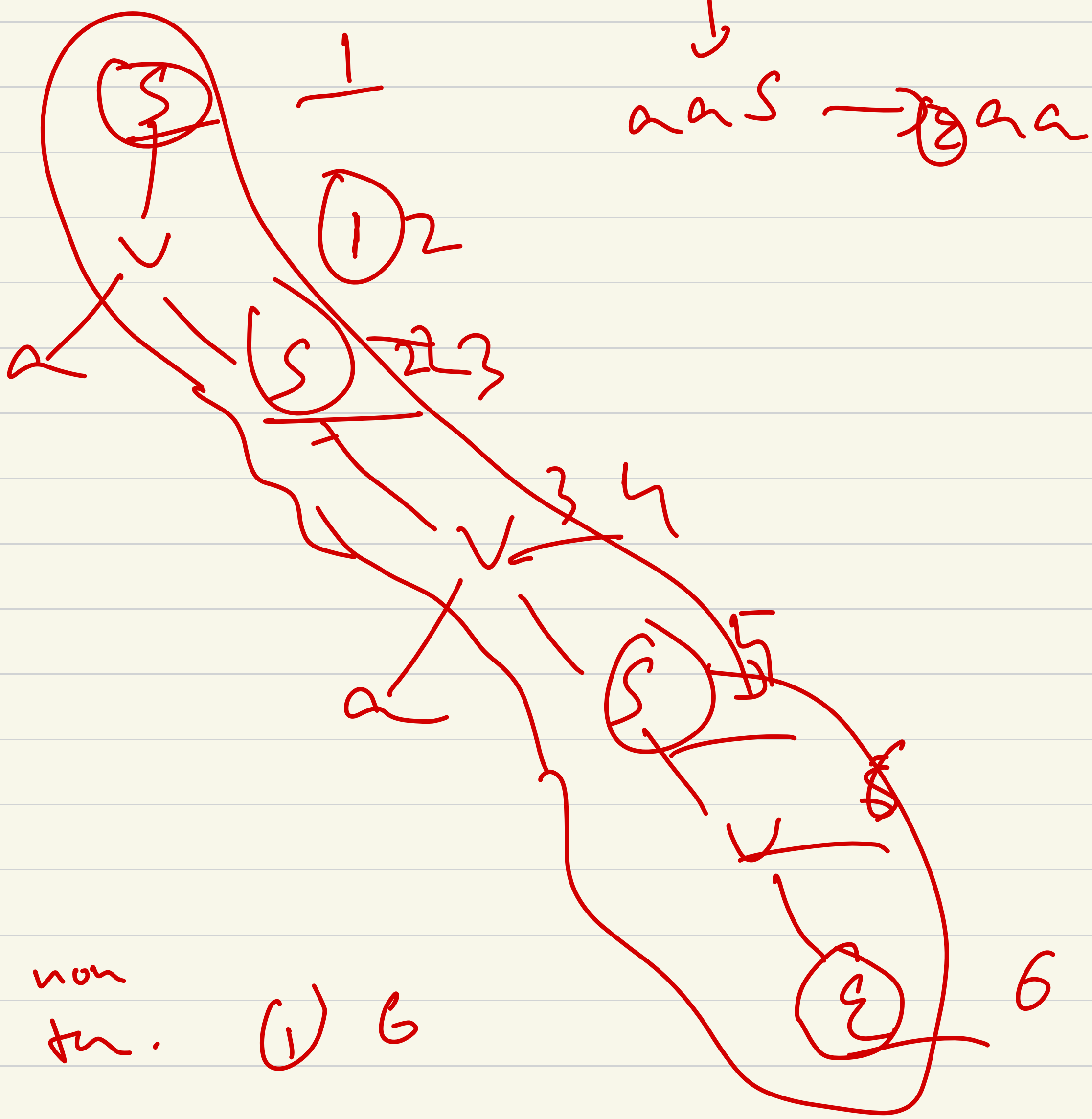
$$h \leq (k-1)|V| \leq |x| V$$

$S \to Y$

$Y \to aS \mid \varepsilon$

$S \to Y \to aS \to aV$

$\to aaS \to \cancel{\varepsilon}\,aa$



6 may non ter. (1) 6

Consider the smallest derivation of x in G.
In the parse tree consider the longest branch of non-terminal starting at S to a terminal (may be ε). This has height h.

Consider $x \in L(G)$ with a parse tree of it with smallest height h. .. (1)

i.e. $S \xrightarrow{h} x$

Consider the path from S to a character / ϵ in x in the parse tree of longest length $= h$.

On this path Non Terminals $= h$ and Terminal $= 1$ (maybe ϵ)

Since total non-terminals $= |V|$.

$\Rightarrow \exists\, P \in V$ s.t. it occurs $\lceil h/|V| \rceil$ no. of times on this path. Let $k = \left\lceil \frac{h}{|V|} \right\rceil$

Consider the k distinct occurence of P as $P_1 \cdots P_k$ and

$$S \xrightarrow{*} \alpha_1 P_1 \beta_1 \xrightarrow{*} \alpha_1 \alpha_2 P \beta_2 \beta_1 \cdots \rightarrow \alpha_1 \alpha_2 \cdots \alpha_i P_i \beta_i \cdots \beta_1$$

$$\cdots \rightarrow \alpha_1 \cdots \alpha_k P_k \beta_k \cdots \beta_1 \xrightarrow{*} x$$

Consider any 2 consecutive occurence of P. $(1 \le i \le k-1)$

$$\Rightarrow \alpha_1 \alpha_2 \cdots \alpha_i P_i \beta_i \beta_{i-1} \cdots \beta_1 \xrightarrow{*} \alpha_1 \alpha_2 \cdots \alpha_i \alpha_{i+1} P_{i+1} \beta_{i+1} \cdots \beta_1$$

Clearly we have $P_i \xrightarrow{*} \alpha_{i+1} P_{i+1} \beta_{i+1}$

Claim:- $|\alpha_{i+1} \beta_{i+1}| > 0$

Proof:- If not then $\alpha_{i+1} \beta_{i+1} = \epsilon \Rightarrow P_i \longrightarrow P_{i+1}$

$\Rightarrow$ Parse tree can be shortened. Contradiction to (1)

$\Rightarrow |\alpha_{i+1}\beta_{i+1}| \geq 1$

$\Rightarrow$ atleast 1 character gets added b/w 2 levels.

$\Rightarrow$ k levels $\Rightarrow$ k-1 char get added.

$\Rightarrow \quad |x| \geq k-1$

$\Rightarrow \quad (|x|+1) \geq \left\lceil \frac{n}{|x|} \right\rceil$

$\Rightarrow \quad |x|(|x|+1) \geq n$

H.P.

$$p+\lambda(i-1) \mid \lambda pq + \lambda\mu(i-1)$$

$$-p\mu$$

$$p+\lambda(i-1) \mid p(\lambda q-\mu)$$

$i = 2$