COL 352 Introduction to Automata and Theory of Computation

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Lecture 34: Computational Complexity Theory (Part 3)

Recap

- Time complexity

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- TIME(n), NTIME(n)

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- Today: NP-completeness

Polynomial time reductions and NP-hardness

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Theorem

3-SAT is polynomial time reducible to k-Clique.

$\phi = (a_1 \vee b_1 \vee c_1) \wedge \ldots (a_k \vee b_k \vee c_k)$

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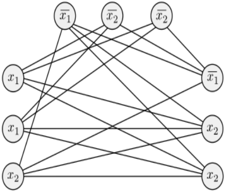

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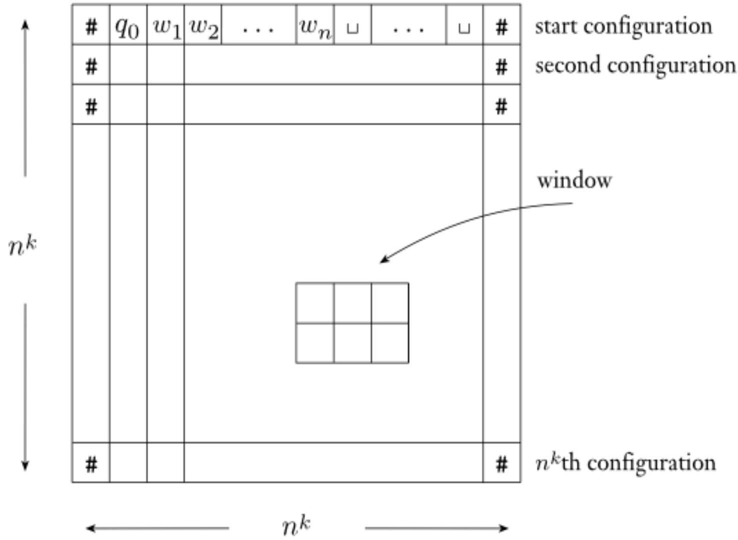
Theorem ([Cook-Levin, 1970])

SAT is NP-complete. If L is NP-complete and $L \in P$ then, P = NP.

Proof sketch



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- Let $Q :=$ States, $\Gamma :=$ Tape alphabet. $C = Q \cup \Gamma \cup \{\#\}$
- Variables: $\forall i, j \in [n^k], k \in C, \quad x_{i,j,s}$

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