COL 352 Introduction to Automata and Theory of Computation

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Lecture 22:Turing Machines: Variants, CT Thesis

Recap

- Deterministic single-tape Turing Machines

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Definition

A Turing machine (TM) is given by $M = (Q, \Sigma, \Gamma, \delta, q_0, q_f, q_{rej})$

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q_0:	start state	Γ:	tape alphabet, $\Sigma \subseteq \Gamma$, $\& \in \Gamma$
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- L is Turing recognizable $\implies \exists M \forall w \in L$, ($M$ has an accepting run on w).
- L is Turing decidable $\implies \exists M(\forall w \in L,\ M$ has an accepting run on w) and ($\forall w \notin L, M$ has a rejecting run on w).

A hierarchy of languages

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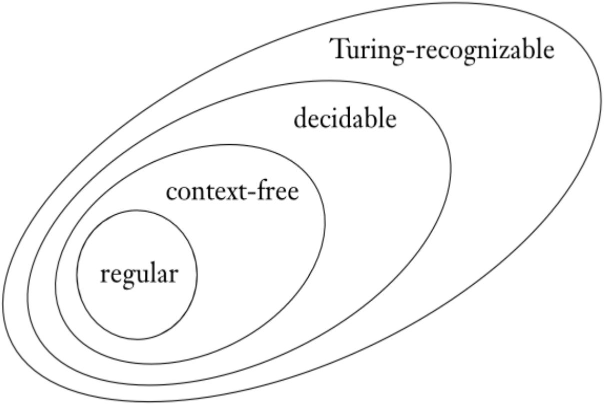

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Turing decidable languages are closed under the following operations:

1. *Union*
2. *Intersection*
3. *Complement*
4. *Concatenation*
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- Variants of TM with multiple tapes or with nondeterminism abound.
- Original model of a TM and its variants all have the same computation power, i.e., they recognize the same class of languages.
- Hence, robustness of TM definition is measured by the invariance of its computation power to certain changes in design features of the machine.

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- Sketch of proof:
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Exercise: What about $\delta : Q \times \Gamma \to Q \times \Gamma \times \{R, S\}$

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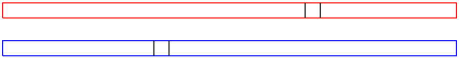

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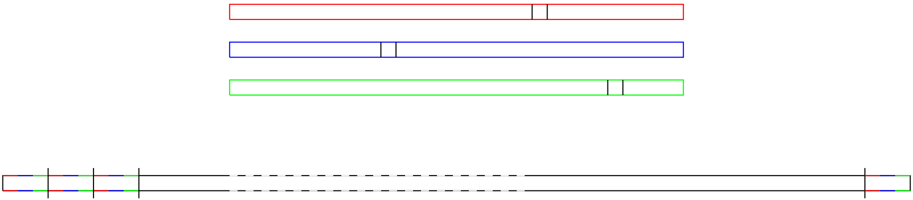

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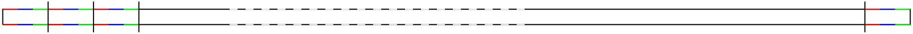

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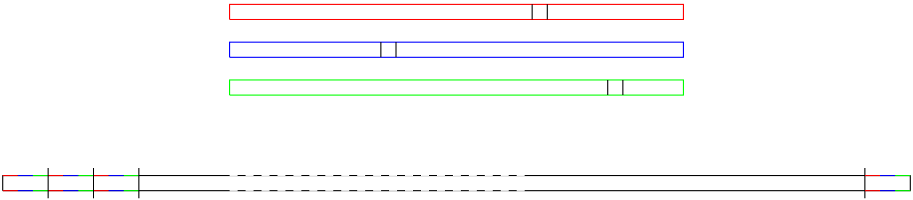

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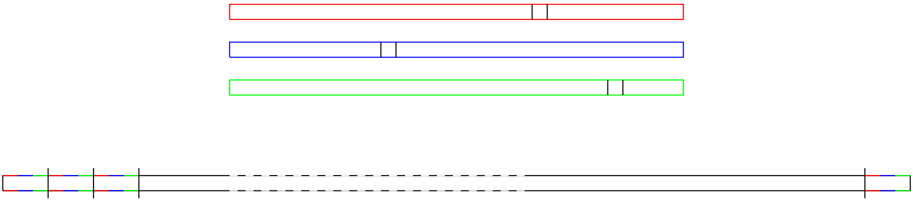

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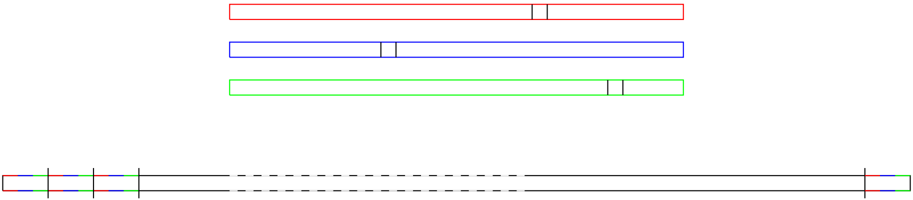

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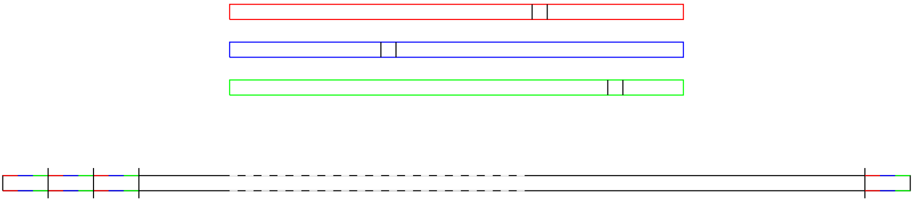

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$\overline{\Gamma}$ symbols used to denote tape head positions.

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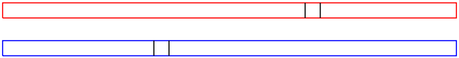

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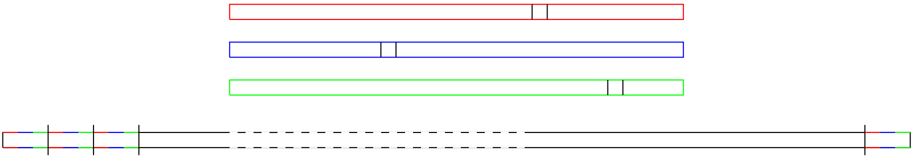

To simulate 1 step of M, M' works follows:

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- uses δ to determine the next state,
- sweeps the input left to right again to update marked symbols.

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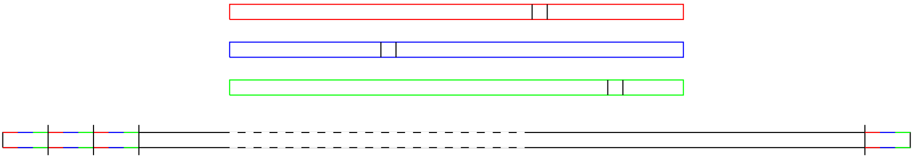

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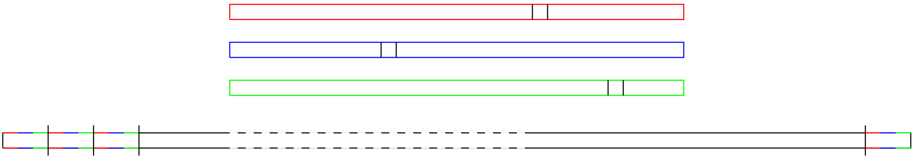

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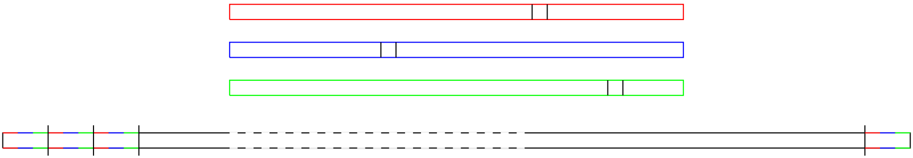

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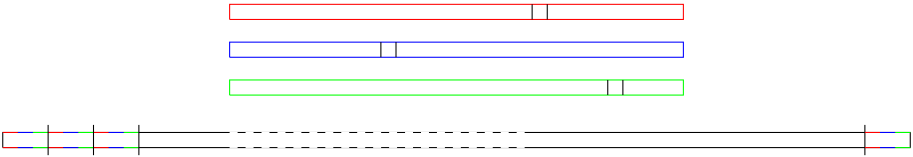

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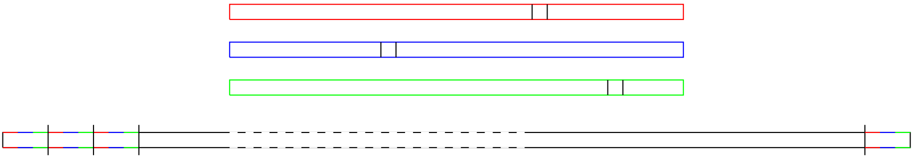

To simulate 1 step of M, M' works follows:

- reads the tape left to right once, remembering the marked symbols in its states,
- uses δ to determine the next state,
- sweeps the input left to right again

k-tape Turing machines

Theorem

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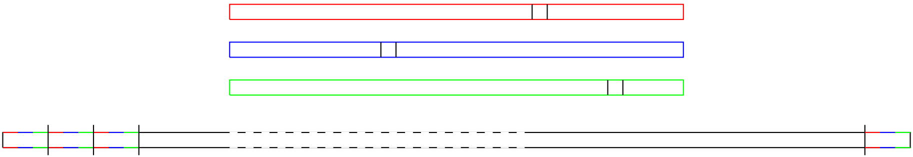
Proof sketch:



To simulate 1 step of M, M' works follows:

reads the tape left to right once, remembering the marked symbols in its states,

uses δ to determine the next state,

sweeps the input left to right again to update marked symbols.