COL 352 Introduction to Automata and Theory of Computation

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Lecture 7: Pattern Matching and Regular Expressions

Recap

- DFA = NFA = ε-NFA.
- All of them recognize (compute/decide) exactly regular languages
- Regular languages are closed under union, intersection, complement, concatenation, Kleene star, ...

A number of widely different and equiexpressive formalisms precisely capture the same class of languages:

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- Certain Algebraic connection (acceptability via finite semi-group): Eilenberg'76
- Regular expressions (Kleene'50s).
- Rational languages.

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Kleene star

For a language L, its Kleene closure, denoted L^* is the set of all strings obtained by taking any number of strings from L with possible repetitions and concatenating all of them.

$$L^* = \cup_{i \geq 0} L^i$$

$$L^0 = \{\epsilon\}, L^i = L \circ L^{i-1}$$

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$$L(\alpha) = \{x \in \Sigma^* \mid x \text{ matches } \alpha\}$$

Atomic Patterns

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6. x matches $\alpha + \beta$ if $L(\alpha + \beta) = L(\alpha) \cup L(\beta)$
7. x matches $\alpha \cap \beta$ if $L(\alpha \cap \beta) = L(\alpha) \cap L(\beta)$
8. x matches $\alpha\beta$ if $x = yz$ where $L(\alpha\beta) = L(\alpha)L(\beta)$
9. x matches $\overline{\alpha}$ if $L(\overline{\alpha}) = \overline{L(\alpha)} = \Sigma^* \setminus L(\alpha)$

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9. x matches $\overline{\alpha}$ if $L(\overline{\alpha}) = \overline{L(\alpha)} = \Sigma^* \setminus L(\alpha)$
10. x matches α^* if x can be expressed as zero or more of strings that match α, i.e., $L(\alpha^*) = L(\alpha)^*$
11. x matches α^+ if x can be expressed as one or more of strings that match α, i.e., $L(\alpha^+) = L(\alpha)^+$

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Express each of these languages as a pattern.

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- Can you get rid of complementation?

Regular expressions

For a regular expression E we write $L(E)$ for its language. The set of valid regular expressions RegEx can be defined recursively as the following:

	Syntax	Semantics
Empty String	ϵ	$L(\epsilon) = \{\epsilon\}$
Empty Set	$\varnothing$	$L(\varnothing) = \varnothing$
Single Letter	a	$L(a) = \{a\}$
Union	$E + F$	$L(E + F) = L(E) \cup L(F)$
Concatenation	$E.F$	$L(E.F) = L(E) \circ L(F)$
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Precedence rules: $* > \circ > +$

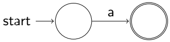
Language defined by regular expression

Lemma

The language defined by any regular expression is regular.

Example

$$(a + b)^*$$



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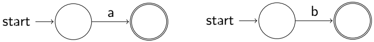

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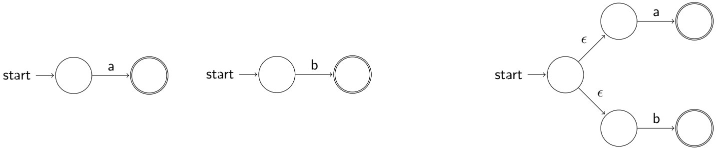

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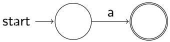

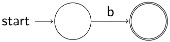

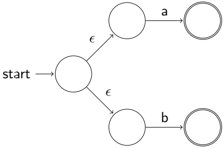

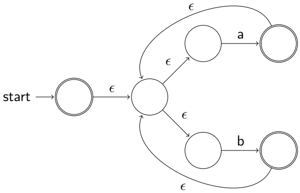

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What about the converse?

Example

- $(aaa)^* + (aaaaa)^*$
- $(11+0)^*(00+1)^*$

A few axioms we can use for simplification

Associativity

$$\alpha + (\beta + \gamma) \equiv (\alpha + \beta) + \gamma$$
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Identity	$\alpha + \emptyset \equiv \alpha$
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Subset order	$\beta + \alpha\gamma \leq \gamma \implies \alpha^*\beta \leq \gamma$
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where

$$\begin{aligned} \alpha \le \beta &\iff L(\alpha) \subseteq L(\beta) \\ &\iff L(\alpha + \beta) = L(\beta) \\ &\iff \alpha + \beta = \beta \end{aligned}$$

A few consequences that follow

Exercise!

$$
\begin{aligned}
(\alpha\beta)^*\alpha &\equiv \alpha(\beta\alpha)^* \\
(\alpha^*\beta)^*\alpha^* &\equiv (\alpha+\beta)^* \\
\alpha^*(\beta\alpha^*)^* &\equiv (\alpha+\beta)^* \\
(\epsilon+\alpha)^* &\equiv \alpha^* \\
\alpha\alpha^* &\equiv \alpha^*\alpha
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$$(1+01+001)^*(\varepsilon+0+00) \quad \equiv \quad ((\varepsilon+0+00)1)^*(\varepsilon+0+00)$$

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- $(11 + 0)^*(00 + 1)^*$
- $(1 + 01 + 001)^*(\varepsilon + 0 + 00)$

$$\begin{aligned}(1 + 01 + 001)^*(\varepsilon + 0 + 00) &\equiv ((\varepsilon + 0 + 00)1)^*(\varepsilon + 0 + 00)\\ &\equiv ((\varepsilon + 0)(\varepsilon + 0)1)^*(\varepsilon + 0)(\varepsilon + 0)\end{aligned}$$

Example

- $(aaa)^* + (aaaaa)^*$
- $(11 + 0)^*(00 + 1)^*$
- $(1 + 01 + 001)^*(\varepsilon + 0 + 00)$

$$
\begin{aligned}
(1 + 01 + 001)^*(\varepsilon + 0 + 00) &\equiv ((\varepsilon + 0 + 00)1)^*(\varepsilon + 0 + 00) \\
&\equiv ((\varepsilon + 0)(\varepsilon + 0)1)^*(\varepsilon + 0)(\varepsilon + 0)
\end{aligned}
$$

$(1 + 01 + 001)^*(\varepsilon + 0 + 00)$ = all strings over $\{0, 1\}$ with no substring of more than two adjacent 0's.

DFA to regular expression

Lemma

Any regular language can be specified by a regular expression

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Want: Given any DFA, convert it into a regular expression.

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Given any DFA A, we can obtain a regular expression, say R_A, such that $L(A) = L(R_A)$.

Computing with labelled graphs

Lemma

Any regular language can be specified by a regular expression

Want: Given any DFA, convert it into a regular expression.

Lemma

Given any DFA A, we can obtain a regular expression, say R_A, such that $L(A) = L(R_A)$.