COL 352 Introduction to Automata and Theory of Computation

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Lecture 17: Context-Free Grammars

Pushdown Automata

Definition

A non-deterministic pushdown automaton (NPDA)
$A = (Q, \Sigma, \Gamma, \delta, q_0, \perp, F)$, where

Q:	set of states	Σ:	input alphabet
Γ:	stack alphabet	q_0:	start state
$\perp$:	start symbol	F:	set of final states

$$\delta \subseteq Q \times \Sigma \times \Gamma \times Q \times \Gamma^*.$$

Understanding δ

For $q \in Q, a \in \Sigma$ and $X \in \Gamma$, if $\delta(q, a, X) = (p, \gamma)$,
then p is the new state and γ replaces X in the stack.

if $\gamma = \epsilon$ then X is popped.
if $\gamma = X$ then X stays unchanges on the top of the stack.
if $\gamma = \gamma_1\gamma_2 \dots \gamma_k$ then X is replaced by γ_k
and $\gamma_1\gamma_2 \dots \gamma_{k-1}$ are pushed on top of that.

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T is a set of terminal symbols or the alphabet,

P is a set of (rewriting rules) productions, $P \subseteq V \times (V \cup T)^*$,

$S_0 \in V$, a start ("sentence") symbol.

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For all $\alpha \in (V \cup T)^*$, we say that $\alpha \Rightarrow^0 \alpha$.

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For all $\alpha, \beta, \gamma \in (V \cup T)^*$,

if $\alpha \Rightarrow^{k-1} \beta$ and $\beta \Rightarrow \gamma$ then $\alpha \Rightarrow^k \gamma$.

For all $\alpha, \beta \in (V \cup T)^*$, we say that $\alpha \Rightarrow^* \beta$, if $\exists k \geq 0$ s.t. $\alpha \Rightarrow^k \beta$.

Language of Arithmetic expressions

Consider the language of arithmetic expressions over binary numbers.

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Words extending at many places

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- In regular languages, words are extended at the end depending on the finite information collected on the word so far.
- In CFLs, words are extended at unboundedly many points, which gives CFLs more power.
- To understand the above intuition, we view the words in derivations as tree.

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- Let's review the Tree terminology:
  - A tree is a directed acyclic graph (DAG) where every node has at most incoming edge.
  - Edge relationship as parent-child relationship
  - Every node has at most one parent, and zero or more children
  - We assume an implicit order on children ("from left-to-right")
  - There is a distinguished root node with no parent, while all other nodes have a unique parent
  - There are some nodes with no children called leaves—other nodes are called interior nodes
  - Ancestor and descendent relationships are closure of parent and child relationships, resp

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  - Each leaf is either a variable, terminal, or ϵ. However, if a leaf is ϵ it is the only child of its parent.
  - If an interior node is labeled A and has children labeled $X_1, X_2, ..., X_k$ from left-to-right, then

    $$A \to X_1 X_2 \dots X_k$$

    is a production is P. Only time X_i can be ϵ is when it is the only child of its parent, i.e. corresponding to the production $A \to \epsilon$.
- **Exercise:** Give parse tree representation of examples seen so far.

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where $a \in T$, $A, B, C \in V$, neither B nor C is the start variable, i.e. start variabe does not appear on the right of any rule. Moreover, epsilon does not appear on the right of any rule except as $S \to \epsilon$.

Lemma

Any context-free grammar G can be converted into another context-free grammar G'

Chomsky normal form

Definition

A context-free grammar is said to be in Chomsky normal form if every rule is of the form

$$A \to BC$$

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Lemma

Any context-free grammar G can be converted into another context-free grammar G' such that $L(G) = L(G')$ and G' is in the Chomsky normal form.

Chomsky normal form

Step 1: Add a new start symbol

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$S_0 \to S$

Chomsky normal form

Step 1: Add a new start symbol

$S_0 \rightarrow S$

Step 2: Remove ϵ rules.

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If $R \to A$ is a rule then delete the rule and add $R \to \epsilon$ unless $R \to \epsilon$ was already removed.

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If $A \to B$ is a rule and if $B \to u$ appears

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If $u_i \in T$, moreover replace each u_i with a variable U_i and add $U_i \to u_i$.

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- Indeed, this looks like $\{a^n b^n \mid n \geq 0\}$ which we proved cant be accepted by any NFA.
- Remember, pumping lemma for regular languages was a property of regular languages. What can we do for CFLs?

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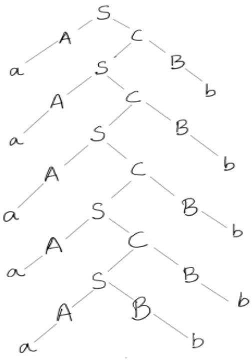

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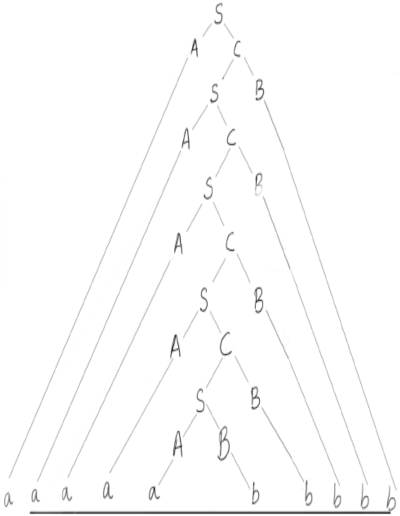

Pumping Lemma for CFLs