COL 352 Introduction to Automata and Theory of Computation

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Lecture 30: Rice's Theorem (Part 2)

Rice's theorem

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Let P be a non-trivial property of Turing recognizable languages. Let $\mathcal{L}_P = \{M \mid L(M) \in P\}$. Then $\mathcal{L}_P$ is undecidable.

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We now learn how to apply Rice's theorem

$\{M \mid L(M) \text{ contains } \langle M \rangle\}$.

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Since P is non-trivial, there exists a TM M_1 s.t. $L(M_1)$ has Property P.

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Run M on w

if M halts on w, then run M_1 on x

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[1] We will remove this assumption later.

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Conclude undecidibility of $\mathcal{L}_P$.

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- Given an input M, x for HALT, construct M_x that halts on every input (final state is useful!) if and only if M halts on x.

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