COL 352 Introduction to Automata and Theory of Computation

Nikhil Balaji

Bharti 420
Indian Institute of Technology, Delhi
nbalaji@cse.iitd.ac.in

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Module 1: Finite Automata

Formal Languages and Problems

- An alphabet $\Sigma = \{a, b, c\}$ is a finite set of letters.
- A language is a set of strings over some alphabet.
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- CONNECTED = $\{w \in \{0, 1\}^* | G_w$ is connected $\}$

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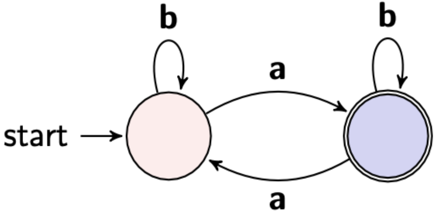

Finite Automata

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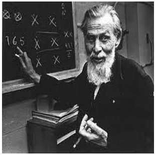

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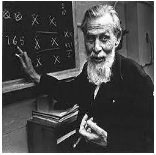

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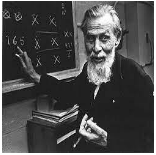

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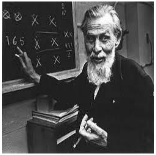

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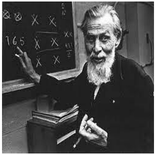

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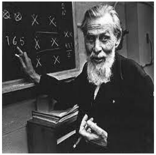

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- Nice theoretical properties
- Applications in digital circuit/protocol verification, compilers, pattern recognition, etc

Finite Automata you use daily

Exercise: Design Automata for these!

Finite state automata

Example 1

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Idea: Start scanning from left, if you see an ’a’ check if the next character is also ’a’. If yes, accept, else reset. If you reach end of string, reject.

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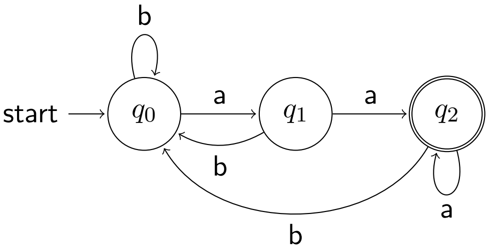

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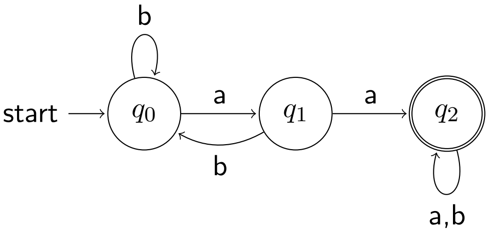

Finite state automata

Example 3

Input: $w \in \{a, b\}^*$

Check: does w have odd number of a's?

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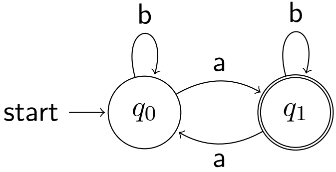

Exercise: Design an automaton to check if w has an even number of a's in every block of length 4 in w.

Deterministic Finite State Automata

An automaton has

- (Finite set of) States
- (Finite) Alphabet
- Initial state
- Accepting/final state
- (Finite) Set of transitions

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$\forall q \in Q, \forall a \in \Sigma$, $|\delta(q, a)| \leq 1$.

Run of an automaton

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Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA. A run of A on word $w = a_1 \dots a_n$ is a sequence of states $q_0, \dots, q_n$ such that $q_i = \delta(q_{i-1}, a_i)$ for all $1 \le i \le n$.

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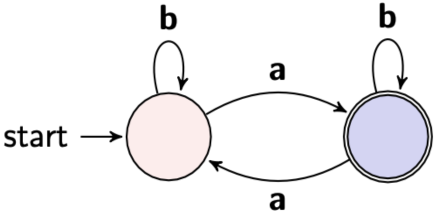

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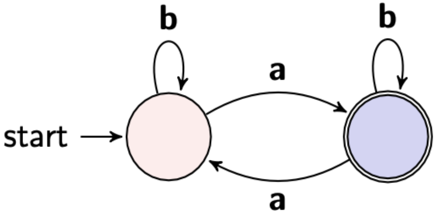

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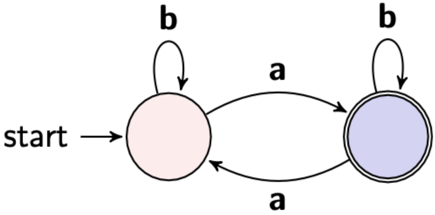

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Run gives the sequence of states: $q_0\ q_1\ q_0\ q_1\ q_0$.

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$$L(A) = \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\}$$

Regular languages

REG

A language is said to be a **regular** if it is accepted by some DFA.
L is a regular language if there exists some DFA A such that $L(A) = L$

Examples

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L &= \{w \in \{a,b\}^* \mid w \text{ ends with aa}\} \\
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L_3 &= \{w \in \{0,1\}^* \mid w \text{ encodes a number in binary divisible by } 3\}
\end{aligned}
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Can we solve all problems using computers?

Theorem (Turing (1936))

There are some problems for which it is impossible to write a program solving it correctly on all inputs.

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- Hence the set of all languages over $\{0, 1\}$ is the power-set of the set of all strings.
- By Cantor's theorem (for any set $|A| < |2^A|$), it must be the case that for some languages there is no recognizing program.

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Example 4

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Idea 2: If you read a 0 at a state q then go to state $2q \pmod 3$, else go to state $2q+1 \pmod 3$

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