COL352 Problem Sheet 2

January 27, 2025

The problems marked as ** are relatively harder than the other problems

Problem 1. *Given a language L over alphabet Σ, define the language $cdr(L) = \{y \in \Sigma^* \mid ay \in L \text{ for some } a \in \Sigma\}$. Show that the regular languages are closed under cdr()*

Problem 2. *Conclude that a star-free regex always generates a finite language.*

Problem 3 (Efficiency of NFA). *Let $L_k = \{x \in \{0,1\}^* \mid |x| \geq k \text{ and the } k\text{'th character of } x \text{ from the end is a } 1\}$. Prove that every DFA that recognizes L_k has at least 2^k states. Also show that, on the other hand, there is an NFA with $k+1$ states that recognizes L_k.*

Problem 4. *A coNFA is like an NFA, except it accepts an input w if and only if every possible state it could end up in when reading w is an accept state. (By contrast, an NFA accepts w iff there exists an accept state it could end up in when reading w.) Show that the class of languages recognized by coNFAs is exactly the regular languages.*

Problem 5. *Construct the minimal DFA D that recognizes the language*

$$\{x \in \{0,1\}^* \mid x \text{ is the binary representation of a number coprime with } 6\}.$$

Prove its minimality by giving a string $z_{q,q'}$ for each pair of distinct states q, q' such that exactly one of $\delta(q, z_{q,q'})$ and $\delta(q', z_{q,q'})$ is an accepting state of D.

Problem 6. *Prove that for any infinite regular language L, there exist two infinite regular languages L_1, L_2 such that $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset^{**}$.*

Problem 7. *Construct a* ***minimal*** *DFA which accept the language $L = \{w \mid w \in \{a,b\}^* \text{ and } Na(w) \mod 3 = Nb(w) \mod 3\}$, where $Na(w)$ and $Nb(w)$ return the number of occurrences of a and b in w respectively.*

Problem 8. *Prove that the following languages are not regular.*

1. $\{xx \mid x \in \{0,1\}^*\}$
2. $\{x \in \{0,1\}^* \mid x = reverse(x)\}$
3. $\{0^{n_1}10^{n_2}1 \cdots 0^{n_k}1 \mid k, n_1, n_2, \cdots, n_k \in \mathbb{N} \cup \{0\} \text{ and } n_1, \ldots, n_k \text{ are distinct}\}$
4. $\{xyx \mid x, y \in \{0,1\}^* \text{ and } |x| > 0,\ |y| > 0\}$
5. $\{x \in \{0,1\}^* \mid x \text{ is the binary representation of } 3^{n^2}\text{, without leading 0's, for some } n \in \mathbb{N}\}$
6. $\{0^m1^n \mid m \neq n\}$ *(As a challenge, construct a clean proof using the pumping lemma only.)*
7. $\{x \in \{a,b,c\}^* \mid x \text{ contains an equal number of occurrences of } ab \text{ and } ba \text{ as substrings}\}$
8. $\{x \in \{0,1\}^* \mid x \text{ is the binary representation of } n!\text{, without leading 0's, for some } n \in \mathbb{N}\}$

Problem 9. *Design a context free grammar for the language $\{x \in \{0,1\}^* \mid \#0\text{'s in } x = \#1\text{'s in } x\}^{**}$.*

Problem 3 (Efficiency of NFA). *Let $L_k = \{x \in \{0,1\}^* \mid |x| \geq k$ and the k'th character of x from the end is a 1$\}$. Prove that every DFA that recognizes L_k has at least 2^k states. Also show that, on the other hand, there is an NFA with $k+1$ states that recognizes L_k.*

Sol^n) Consider $\alpha, \beta \in \Sigma^k$ s.t. $|\alpha| = |\beta| = k$ but $\alpha \neq \beta$.

let $\alpha = \alpha_1 \alpha_2 \cdots \alpha_k$ and $\beta = \beta_1 \cdots \beta_k$ with $\alpha_i, \beta_i \in \{0,1\}$

Since $\alpha \neq \beta \Rightarrow \exists i$ s.t. $\alpha_i \neq \beta_i$. Consider largest such i.

Since $\alpha_i \neq \beta_i \Rightarrow$ atleast one of α_i, β_i is 1.

WLOG $\alpha_i = 1$

Consider $\alpha_1 \alpha_2 \cdots \underbrace{\underset{=1}{\alpha_i} \cdots \alpha_k \, 0 \cdots 000}_{k} \in L$ and $\beta_1 \cdots \underbrace{\underset{=0}{\beta_i} \cdots \beta_k \, 0000}_{k} \notin L$

$\Rightarrow$ for $z = 0^{i-1}$, $\alpha z \in L$, $\beta z \notin L \Rightarrow \alpha \not\sim_L \beta$.

This is true for all α, β when $|\alpha| = k = |\beta|$

$\Rightarrow$ L has finite index of size $\geq 2^k$.

$\Rightarrow$ Minimum DFA of L has $\geq 2^k$ states by Myhill Nerode.

Problem 4. *A coNFA is like an NFA, except it accepts an input w if and only if every possible state it could end up in when reading w is an accept state. (By contrast, an NFA accepts w iff there exists an accept state it could end up in when reading w.) Show that the class of languages recognized by coNFAs is exactly the regular languages.*

Problem 8. *Prove that the following languages are not regular.*

1. $\{xx \mid x \in \{0,1\}^*\}$
2. $\{x \in \{0,1\}^* \mid x = reverse(x)\}$
3. $\{0^{n_1}10^{n_2}1 \cdots 0^{n_k}1 \mid k, n_1, n_2, \cdots, n_k \in \mathbb{N} \cup \{0\}$ *and* $n_1, \ldots, n_k$ *are distinct*$\}$
4. $\{xyx \mid x, y \in \{0,1\}^*$ *and* $|x| > 0,\ |y| > 0\}$
5. $\{x \in \{0,1\}^* \mid x$ *is the binary representation of* 3^{n^2}, *without leading 0's, for some* $n \in \mathbb{N}\}$
6. $\{0^m1^n \mid m \neq n\}$ *(As a challenge, construct a clean proof using the pumping lemma only.)*
7. $\{x \in \{a,b,c\}^* \mid x$ *contains an equal number of occurrences of ab and ba as substrings*$\}$
8. $\{x \in \{0,1\}^* \mid x$ *is the binary representation of* $n!$, *without leading 0's, for some* $n \in \mathbb{N}\}$

Sol^n) 1

A) Advesary chooses $k \geq 0$

I choose $x = 0^k$ $y = 1^k$ $z = 0^k1^k$

thus $xyz = 0^k1^k0^k1^k \in L$

let advesary choose $u = 1^p$ $v = 1^q$ $w = 1^r$

s.t $p+q+r = k$ and $q > 0$

Choose $i = 2$

$\Rightarrow xuv^2wz = 0^k1^{k+q}0^k1^k \notin L$ as $q > 0$

$\Rightarrow$ L is non regular.

B) Consider $i, j \in \mathbb{N}$ s.t $i \neq j$

$\Rightarrow 0^i \not\equiv_L 0^j$ as $0^i1^{2i}0^i \in L$ but $0^j0^{2i}0^i \notin L$

$\Rightarrow \infty$ index $\Rightarrow$

2 Similar.

3) $0^{n_1}10^{n_2}10^{n_3}1 \ldots 0^{n_k}1$

Trivial by Myhill Nerode.

Advesary chooses $k \geq 0$

I choose

$x = \varepsilon \quad y = 0^k, \; z = 1 0^{k-1} 1 0^{k-2} 1 \cdots 0^1 1 0^0 1$

let adversary choose $u = 0^p \quad v = 0^q \quad w = 0^r$ s.t.

$p + q + r = k$ and $q > 0$

$\Rightarrow$ for $i = 0$, we have

$x u v^0 w z = u w z$

$= 0^{k-q} 1 0^{k-1} 1 \cdots \quad 0^1 1 0^0 1$

is not unique.

④ $xyx, \; |y|, |x| > 0$

Advesary choose $k \geq 0$

we choose $x = 0^k 1^k \quad y = 0^k \quad z = \varepsilon$

$\Rightarrow 0^k 1^k 0^k \in L$

$u, v, w = 0^p, 0^q, 0^r$ s.t. $p + q + r = k, q > 0$

choose $i = 0$

$\Rightarrow 0^k 1^k \; 0^{k-q} \notin L$. L nonregular.

③ $x = 3^{n^2}$ in binary.

Advesary choose $k \geq 0$

let $s = (3^{k^2})_2 = xy$ with $|y| = k$

$\Rightarrow$ let adversary choose $y = uvw$

for $i = 2$

$r = |x u v^2 w| = |xy| + |v| \leq |xy| + |y|$

$\leq |s| + k$

Contradiction.

(6) $0^m 1^n, m \neq n$.

Adversary chooses k.

let $x = \varepsilon$, $y = 0^k$ $z = 1^{k+k!}$

Clearly $xyz = 0^k 1^{k+k!} \in L$ as $k \neq k+k!$ $\forall k \geq 0$

let adversary choose $u, v, w = 0^p, 0^q, 0^r$ s.t. $p+q+r = k$, $q > 0$

Consider xuv^iwz

$$= 0^{k+(i-1)q} 1^{k+k!}$$

Since $0 < q \leq k$ choose $i = 1 + \frac{k!}{q} \in \mathbb{N}$.

$\Rightarrow$ L is not regular.

(7) $\#ab = \#ba$ $\Sigma = \{a, b, c\}$

Consider $w_n = (abc)^n c$

Clearly $w_n \not\equiv_L w_m$ for $n \neq m$ because

$(abc)^n c (bac)^n \in L$ but $(abc)^m c (bac)^n \notin L$.

(8) Adversary choose $k \geq 0$

$\Rightarrow$ Take $s = (n!)_2 \in L$,

let $x = \varepsilon$, $|y| = k$ be such that $xyz = s$

$\Rightarrow$ adversary choose $u\,v\,w$ s.t. $|v| > 0$, $|u| + |v| + |w| = k$ $uvw = y$

Consider $i = 2$

$\Rightarrow |xuv^2wz| = |s| + |v|$

$\Rightarrow |s| < |s| + |v| \leq |s| + k$

Clearly $(xuv^2wz)_2 > n!$

$\Rightarrow (xuv^2wz)_2 \geq (n+1)!$

$\Rightarrow |xuv^2wz| \geq |s| + \log_2(n+1)$

Choose $n = 2^k - 1$

Contradiction.