COL 352 Introduction to Automata and Theory of Computation

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Lecture 5: Nondeterminism: Subset Construction

Power of NFAs

Lemma

Let A be an NFA. Then $L(A)$ is a regular language.

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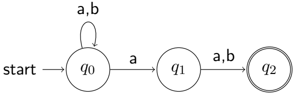

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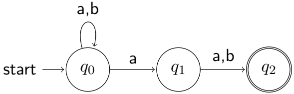

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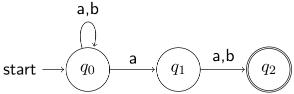

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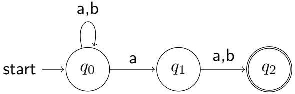

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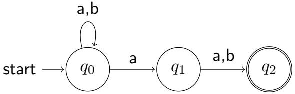

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Let $A = (Q, \Sigma, q_0, F, \delta)$. We will construct a DFA $A' = (Q', \Sigma, q_0', F', \Delta)$ such that $L(A') = L(A)$.

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$Q' = 2^Q$,

$q'_0 = \{q_0\}$,

$F' = \{S \subseteq Q \mid S \cap F \neq \varnothing\}$.

For each $S \subseteq Q, \Delta(S, a) = \bigcup_{p \in S} \delta(p, a)$.

$\square$

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From Lemma, we have for all $w \in \Sigma^*, \hat{\delta}(q_0, w) = \hat{\Delta}(\{q_0\}, w)$. Hence,for all $w, \hat{\delta}(q_0, w) \cap F \neq \varnothing$ iff $\hat{\Delta}(\{q_0\}, w) \in F'$.

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Two questions

- Does this blowup really occur when only considering reachable states?
- On examples where it does not occur can we have a subset construction that is efficient?

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$$L = \{x \in \{a\}^* \mid |x| \text{ is divisible by } 3 \text{ or } 5\}$$

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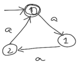

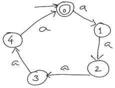

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