COL 352 Introduction to Automata and Theory of Computation

Nikhil Balaji

Bharti 420
Indian Institute of Technology, Delhi
nbalaji@cse.iitd.ac.in

February 1, 2023

Lecture 10: Pumping Lemma: Examples

Pumping Lemma for regular languages

Pumping Lemma

$L \subseteq \Sigma^*$ is a regular language $\implies$
there exists $n \geq 1$ such that
for all strings $w \in L$ with $|w| \geq n$ we have that
there exists $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq n$ such that
for all $i \geq 0$ we have that
$xy^i z \in L$.

Pumping Lemma for regular languages

Pumping Lemma

$L \subseteq \Sigma^*$ is a regular language $\implies$
there exists $n \geq 1$ such that
for all strings $w \in L$ with $|w| \geq n$ we have that
there exists $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq n$ such that
for all $i \geq 0$ we have that
$xy^iz \in L$.

Contrapositive

If for all $n \geq 1$ such that
there exists a string $w \in L$ with $|w| \geq n$ such that
for all breakups $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq n$ we have that
there exists $i \geq 0$ such that
$xy^iz \in L$.
$\implies L \subseteq \Sigma^*$ is not a regular language

Applying Pumping Lemma: Example 1

Consider

$$L_{a,b} = \{a^n b^n \mid n \geq 0\}$$

Applying Pumping Lemma: Example 1

Consider

$$L_{a,b} = \{a^n b^n \mid n \geq 0\}$$

- For each n, we need a word. Let $w = a^n b^n$.

Applying Pumping Lemma: Example 1

Consider

$$L_{a,b} = \{a^n b^n \mid n \geq 0\}$$

- For each n, we need a word. Let $w = a^n b^n$.
- The first n characters of w are a^n. The breaks x and y are to be from within a^n.

Applying Pumping Lemma: Example 1

Consider

$$L_{a,b} = \{a^n b^n \mid n \geq 0\}$$

- For each n, we need a word. Let $w = a^n b^n$.
- The first n characters of w are a^n. The breaks x and y are to be from within a^n.
- Let $x = a^i, y = a^j$ where $i + j \leq n$ and $j \neq 0$.

$$w = a^i a^j a^{n-i-j} b^n$$

Applying Pumping Lemma: Example 1

Consider

$$L_{a,b} = \{a^n b^n \mid n \geq 0\}$$

- For each n, we need a word. Let $w = a^n b^n$.
- The first n characters of w are a^n. The breaks x and y are to be from within a^n.
- Let $x = a^i, y = a^j$ where $i + j \leq n$ and $j \neq 0$.

$$w = a^i a^j a^{n-i-j} b^n$$

- Choose $k = 0$ for each i, j. The corresponding word is

$$a^i (a^j)^0 a^{n-i-j} b^n = a^{n-j} b^n \notin L_{a,b}$$

Example 2

Consider

$$L = \{w \in \{a,b\}^* \mid w \text{ has equal number of a's and b's}\}$$

Example 2

Consider

$$L = \{w \in \{a,b\}^* \mid w \text{ has equal number of a's and b's}\}$$

Exercise: What is $L \cap a^*b^*$?

Example 3

Consider

$$PAL = \{w \cdot w^R \mid w \in \{a,b\}^*\}$$

Example 3

Consider

$$PAL = \{w \cdot w^R \mid w \in \{a, b\}^*\}$$

- For each n, let $w = a^n bba^n$.

Example 3

Consider

$$PAL = \{w \cdot w^R \mid w \in \{a,b\}^*\}$$

- For each n, let $w = a^n bba^n$.
- The first n characters of w are a^n

Example 3

Consider

$$PAL = \{w \cdot w^R \mid w \in \{a, b\}^*\}$$

- For each n, let $w = a^n bba^n$.
- The first n characters of w are a^n
- Let $x = a^i, y = a^j$ where $i + j \leq n$ and $j \neq 0$.

$$w = a^i a^j a^{n-i-j} bba^n$$

Example 3

Consider

$$PAL = \{w \cdot w^R \mid w \in \{a,b\}^*\}$$

- For each n, let $w = a^n bba^n$.
- The first n characters of w are a^n
- Let $x = a^i, y = a^j$ where $i + j \leq n$ and $j \neq 0$.

$$w = a^i a^j a^{n-i-j} bba^n$$

- Choose $k = 2$ for each i, j. The corresponding word is

$$a^i (a^j)^2 a^{n-i-j} bba^n = a^{n+j} bba^n \notin PAL$$

Food for thought

- If $L_1 \cup L_2$ is regular, are L_1 and L_2 regular?
- Is this regular: $\{a^n b^m \mid n \neq m\}$?
- Is this regular: $\{a^n b^m \mid n \geq m\}$?
- Is this regular: $\{a^n b^{n+1} \mid n \geq 0\}$?

Need for infinite memory

Feels like all non-regular languages needed to remember infinite memory.

In $\{a^n b^n \mid n \geq 0\}$ we need to remember the number of seen a's and count the b's to match.

Finite number of states cannot count unboundedly increasing number.

Example 4: Primes

$$L_{prime} = \{1^p \in |p \text{ prime}\}$$

- Pick $w = 1^p$ (p larger than pumping length).

Example 4: Primes

$$L_{prime} = \{1^p \in | p \text{ prime}\}$$

- Pick $w = 1^p$ (p larger than pumping length).
- $w = xyz$, $y \neq \epsilon$.
- For all $k \geq 0$ Let $x = 1^r, y = 1^s, z = 1^t$ where $r + s + t = p$

Example 4: Primes

$$L_{prime} = \{1^p \in | p \text{ prime}\}$$

- Pick $w = 1^p$ (p larger than pumping length).
- $w = xyz$, $y \neq \epsilon$.
- For all $k \geq 0$ Let $x = 1^r, y = 1^s, z = 1^t$ where $r + s + t = p$
- Consider $xy^kz = 1^r 1^{ks} 1^t = 1^{p+(k-1)s}$
- We need $p + (k-1)s$ to be prime for all $k \geq 0$ to satisfy pumping condition.

Example 4: Primes

$$L_{prime} = \{1^p \in | p \text{ prime}\}$$

- Pick $w = 1^p$ (p larger than pumping length).
- $w = xyz$, $y \neq \epsilon$.
- For all $k \geq 0$ Let $x = 1^r, y = 1^s, z = 1^t$ where $r + s + t = p$
- Consider $xy^kz = 1^r1^{ks}1^t = 1^{p+(k-1)s}$
- We need $p + (k-1)s$ to be prime for all $k \geq 0$ to satisfy pumping condition.
- Choose $k = p + 1$, we have $p + (k-1)s = p + ps = (s+1)p$.

Not a sufficient condition

Pumping lemma gives only a necessary condition for regularity, not a sufficient condition!

Not a sufficient condition

Pumping lemma gives only a necessary condition for regularity, not a sufficient condition!

Exercise: Find a language that is not regular, but which satisfies the pumping conditions.

Example 5

$$L = \{a^n b^m \mid n > m\}$$

Example 5

$$L = \{a^n b^m \mid n > m\}$$

- For each n, let $w = a^n b^n$

Example 5

$$L = \{a^n b^m \mid n > m\}$$

- For each n, let $w = a^n b^n$
- Any break-up of w has $x = a^i$ and $y = a^j$, where $i + j \leq n$ and $j \neq 0$

$$\overbrace{a^i}^{x}\,\overbrace{a^j}^{y}\,\overbrace{a^{n-j-i} b^n}^{z}$$

Example 5

$$L = \{a^n b^m \mid n > m\}$$

- For each n, let $w = a^n b^n$
- Any break-up of w has $x = a^i$ and $y = a^j$, where $i + j \leq n$ and $j \neq 0$

$$\overbrace{a^i}^{x}\ \overbrace{a^j}^{y}\ \overbrace{a^{n-j-i} b^n}^{z}$$

- if $k > 2$, we get $xy^k z = a^i a^{2j} a^{n-j-i} b^n \in L$, so NOT enough!

Example 5

$$L = \{a^n b^m \mid n > m\}$$

- For each n, let $w = a^n b^n$
- Any break-up of w has $x = a^i$ and $y = a^j$, where $i + j \le n$ and $j \ne 0$

$$\overbrace{a^i}^{x}\ \overbrace{a^j}^{y}\ \overbrace{a^{n-j-i}b^n}^{z}$$

- if $k > 2$, we get $xy^k z = a^i a^{2j} a^{n-j-i} b^n \in L$, so NOT enough!
- However, let $k = 0$ (pumping down).

Example 5

$$L = \{a^n b^m \mid n > m\}$$

- For each n, let $w = a^n b^n$
- Any break-up of w has $x = a^i$ and $y = a^j$, where $i + j \leq n$ and $j \neq 0$

$$\overbrace{a^i}^{x}\,\overbrace{a^j}^{y}\,\overbrace{a^{n-j-i}b^n}^{z}$$

- if $k > 2$, we get $xy^k z = a^i a^{2j} a^{n-j-i} b^n \in L$, so NOT enough!
- However, let $k = 0$ (pumping down). Then, $xy^k z = a^i a^{n-j-i} b^n \notin L$.

Homomorphisms: A tool to show regularity

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$h(xy) = h(x)h(y)$$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

We can derive the second condition from the first!

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

We can derive the second condition from the first!

$$|h(\varepsilon)| = |h(\varepsilon\varepsilon)|$$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

We can derive the second condition from the first!

$$\begin{aligned} |h(\varepsilon)| &= |h(\varepsilon\varepsilon)| \\ &= |h(\varepsilon)h(\varepsilon)| \end{aligned}$$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

We can derive the second condition from the first!

$$\begin{aligned} |h(\varepsilon)| &= |h(\varepsilon\varepsilon)| \\ &= |h(\varepsilon)h(\varepsilon)| \\ &= |h(\varepsilon)| + |h(\varepsilon)| \end{aligned}$$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

We can derive the second condition from the first!

$$\begin{aligned} |h(\varepsilon)| &= |h(\varepsilon\varepsilon)| \\ &= |h(\varepsilon)h(\varepsilon)| \\ &= |h(\varepsilon)| + |h(\varepsilon)| \\ \implies |h(\varepsilon)| &= 0 \end{aligned}$$

Homomorphisms: A tool to show regularity

Definition

A homomorphism is a map $h : \Sigma^* \to \Gamma^*$ such that for all $x, y \in \Sigma^*$

$$\begin{aligned} h(xy) &= h(x)h(y) \\ h(\varepsilon) &= \varepsilon \end{aligned}$$

We can derive the second condition from the first!

$$\begin{aligned} |h(\varepsilon)| &= |h(\varepsilon\varepsilon)| \\ &= |h(\varepsilon)h(\varepsilon)| \\ &= |h(\varepsilon)| + |h(\varepsilon)| \\ \implies |h(\varepsilon)| &= 0 \\ \implies h(\varepsilon) &= \varepsilon \end{aligned}$$

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$
- For $A \subseteq \Sigma^*$

$$h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$$

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$
- For $A \subseteq \Sigma^*$

$$h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$$

- Problem set 1: If A is regular then so is $h(A)$ (image of A).

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$
- For $A \subseteq \Sigma^*$

$$h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$$

- Problem set 1: If A is regular then so is $h(A)$ (image of A).
- For $B \subseteq \Gamma^*$,

$$h^{-1}(B) := \{x \in \Sigma^* \mid h(x) \in B\}$$

- Prove: If B is regular then so is $h^{-1}(B)$.

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$
- For $A \subseteq \Sigma^*$

$$h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$$

- Problem set 1: If A is regular then so is $h(A)$ (image of A).
- For $B \subseteq \Gamma^*$,

$$h^{-1}(B) := \{x \in \Sigma^* \mid h(x) \in B\}$$

- Prove: If B is regular then so is $h^{-1}(B)$.
- **Caveat:** This does not say that A is regular if $h(A)$ is. Example:

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$
- For $A \subseteq \Sigma^*$

$$h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$$

- Problem set 1: If A is regular then so is $h(A)$ (image of A).
- For $B \subseteq \Gamma^*$,

$$h^{-1}(B) := \{x \in \Sigma^* \mid h(x) \in B\}$$

- Prove: If B is regular then so is $h^{-1}(B)$.
- **Caveat:** This does not say that A is regular if $h(A)$ is. Example:

$$L_{a,b} = \{a^n b^n \mid n \in \mathbb{N}\}$$

Homomorphisms : Examples

- Given $\forall a \in \Sigma, h(a)$, we can uniquely determine $h(w)$ for every $w \in \Sigma^*$
- Any map $h : \Sigma \to \Gamma^*$ extends by induction uniquely to a map $h' : \Sigma^* \to \Gamma^*$
- For $A \subseteq \Sigma^*$

$$h(A) = \{h(x) \mid x \in A\} \subseteq \Gamma^*$$

- Problem set 1: If A is regular then so is $h(A)$ (image of A).
- For $B \subseteq \Gamma^*$,

$$h^{-1}(B) := \{x \in \Sigma^* \mid h(x) \in B\}$$

- Prove: If B is regular then so is $h^{-1}(B)$.
- **Caveat:** This does not say that A is regular if $h(A)$ is. Example:

$$L_{a,b} = \{a^n b^n \mid n \in \mathbb{N}\}$$

- What is $h(L_{a,b})$ under $h : \Sigma \to \Sigma$ given by $h(a) = h(b) = a$?

More non-regularity tricks

$$L = \{a^n b^m \mid n \geq m\}$$

More non-regularity tricks

$$L = \{a^n b^m \mid n \geq m\}$$

Consider

$$L^R = \{b^m a^n \mid n \geq m\}$$

- If L was regular so is L^R (reverse of L) - Prove!

More non-regularity tricks

$$L = \{a^n b^m \mid n \geq m\}$$

Consider

$$L^R = \{b^m a^n \mid n \geq m\}$$

- If L was regular so is L^R (reverse of L) - Prove!
- Apply $h : \Sigma \to \Sigma$ $(h(a) = b, h(b) = a$.

More non-regularity tricks

$$L = \{a^n b^m \mid n \geq m\}$$

Consider

$$L^R = \{b^m a^n \mid n \geq m\}$$

- If L was regular so is L^R (reverse of L) - Prove!
- Apply $h : \Sigma \to \Sigma$ $(h(a) = b, h(b) = a$.

$$L' = h(L^R) = \{a^m b^n \mid n \geq m\}$$

More non-regularity tricks

$$L = \{a^n b^m \mid n \geq m\}$$

Consider

$$L^R = \{b^m a^n \mid n \geq m\}$$

- If L was regular so is L^R (reverse of L) - Prove!
- Apply $h : \Sigma \to \Sigma$ $(h(a) = b, h(b) = a$.

$$L' = h(L^R) = \{a^m b^n \mid n \geq m\}$$

- What is $L \cap L'$?

More non-regularity tricks

$$L = \{a^n b^m \mid n \geq m\}$$

Consider

$$L^R = \{b^m a^n \mid n \geq m\}$$

- If L was regular so is L^R (reverse of L) - Prove!
- Apply $h : \Sigma \to \Sigma$ $(h(a) = b, h(b) = a$.

$$L' = h(L^R) = \{a^m b^n \mid n \geq m\}$$

- What is $L \cap L'$?

$$L \cap L' = \{a^n b^n \mid n \in \mathbb{N}\}$$

Converse of Pumping Lemma

Converse of Pumping Lemma

Example

Consider

$$L = \overbrace{\{ca^n b^n\}}^{L_1} \cup \overbrace{\{c^k w \mid k \neq 1, w \text{ starts with } a \text{ or } b\ \}}^{L_2}$$

Converse of Pumping Lemma

Example

Consider

$$L = \overbrace{\{ca^nb^n\}}^{L_1} \cup \overbrace{\{c^kw \mid k \neq 1, w \text{ starts with } a \text{ or } b\}}^{L_2}$$

- L_2 is regular (why?)

Converse of Pumping Lemma

Example

Consider

$$L = \overbrace{\{ca^n b^n\}}^{L_1} \cup \overbrace{\{c^k w \mid k \neq 1, w \text{ starts with } a \text{ or } b\ \}}^{L_2}$$

- L_2 is regular (why?)
- L_1 is not regular (why??)

Converse of Pumping Lemma

Example

Consider

$$L = \overbrace{\{ca^n b^n\}}^{L_1} \cup \overbrace{\{c^k w \mid k \neq 1, w \text{ starts with } a \text{ or } b\ \}}^{L_2}$$

- L_2 is regular (why?)
- L_1 is not regular (why??)