Date: 2021/03/15 Duration: 90 minutes

Read the instructions carefully.

Recall the definition of the shuffle operation from the first homework. Let $x, y, z \in \Sigma^*$. We said that z is a *shuffle* of x and y if the characters in x and y can be interleaved, while maintaining their relative order within x and y, to get z. Formally, if $|x| = m$ and $|y| = n$, then $|z|$ must be $m + n$, and it should be possible to partition the set $\{1, 2, \ldots, m + n\}$ into two increasing sequences, $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_n$, such that $z[i_k] = x[k]$ and $z[j_k] = y[k]$ for all k. Given two languages $L_1, L_2 \subseteq \Sigma^*$, we defined

$$\text{shuffle}(L_1, L_2) = \{z \in \Sigma^* \mid z \text{ is a shuffle of some } x \in L_1 \text{ and some } y \in L_2\}.$$

1. **[5 marks]** Prove that the class of context-free languages is not closed under the shuffle operation. (Note that in order to do this, you need to produce two context-free languages, L_1 and L_2, such that shuffle(L_1, L_2) is not context-free.)

2. **[5 marks]** Prove that the class of context-free languages is closed under shuffle with regular languages. That is, prove that if L_1 is a context-free language and L_2 is a regular language, then shuffle(L_1, L_2) is a context-free language. To do this, define a construction of an NPDA P recognizing shuffle(L_1, L_2) from an NPDA P_1 recognizing L_1 and a DFA D_2 recognizing L_2. Give a **short** proof of correctness (eg. by writing an inductive claim which will be obvious enough from your construction of P).