EXTRA CHALLENGES 2DBI00

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Some students may appreciate some extra challenging questions. Most of the questions in this document are no material for the exam, but they are nice problems with matrices that may be important in practical applications. Start with (or only do) the questions that you find appealing!

You can send your answers to the lecturer to get feedback.

(1) **(Class 1)** A more abstract linear algebra question: let $A(p)$ be the polynomial with $(A(p))(x) = x\,p'(x)$ (so the operation of differentiation followed by multiplication with x), which is a linear operator in p. Give the matrix representing this operation, going from all polynomials of degree ≤ 3 to itself, with the standard basis $\{\,1, x, x^2, x^3\,\}$.

(2) A more abstract linear algebra question: let $\mathrm{Int}(p)$ be the polynomial with $(\mathrm{Int}(p))(x) = \int_0^x p(t)\,dt$ (so the operation of integration), which is a linear operator in p. Give the matrix representing this operation, going from the space of all polynomials of degree ≤ 2 to all polynomials of degree ≤ 3. Take the standard basis $\{\,1, x, x^2, \dots\,\}$ in each case. The matrix is nonsquare (4×3).

(3) If you already know about eigenvalues: what are all eigenvalues of the 4×4-matrix representing differentiation (see slides)? Why is this natural?
What is the only eigenvector? Why is this natural?

(4) **(Class 2)** Give the full details for question **(10)** of the homework of Class 2.
In particular, consider the vectors $\mathbf{e}_1$ and $\mathbf{e}$ (the vector consisting of all ones).

(5) Prove the Cauchy–Schwarz inequality for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$: $|\mathbf{x}^T\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$.
Hint: Consider $0 \le \|\mathbf{x} + \mu\,\mathbf{y}\|^2 = \|\mathbf{y}\|^2\,\mu^2 + 2\,\mathbf{x}^T\mathbf{y}\,\mu + \|\mathbf{x}\|^2$ as a polynomial in μ.
What can we know about the discriminant in view of the $0 \le \cdots$?
Background: the desired result is also a natural consequence of $\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos(\mathbf{x}, \mathbf{y})$, but this usually is done the other way: since we know by Cauchy–Schwarz that $\frac{|\mathbf{x}^T\mathbf{y}|}{\|\mathbf{x}\|\,\|\mathbf{y}\|} \le 1$, we can define $\cos(\mathbf{x}, \mathbf{y})$ to be equal to this quantity.

(6) **(Class 4)** Show that the number of flops to solve an $n \times n$ linear system is equal to $\frac{2}{3}n^3$, ignoring lower-order terms (with n^2, n, and constants).
Hints and answers, e.g.:
https://math.stackexchange.com/questions/1161410/floating-point-arithmetic-operations-when-
$\cdots$ row-reducing-matrices

(7) Assume we do not perform row permutations. Explain that row reduction is of the form $L_{n-1} \cdots L_1\, A = U$, so $A = L_1^{-1} \cdots L_{n-1}^{-1} U$. Explain by a 3×3 example that $L_1^{-1} \cdots L_{n-1}^{-1}$ is in fact a new lower triangular matrix L. This gives the *LU decomposition* of A.

(8) Suppose A is a 5×5 symmetric tridiagonal matrix (e.g. with "stencil" $[-1,\ 2,\ -1]$).
Let $A = LU$ be the LU decomposition (e.g.: Matlab `[L,U,P] = lu(A)`). Explain:

- L and U are bidiagonal (lower or upper?)
- L^{-1} and U^{-1} are not banded (but triangular)
- is $A^{-1} = U^{-1}L^{-1}$ banded (sparse) or full?
- is A^{-1} symmetric?

(9) **(Class 7)** The following matrix is quite famous. We will encounter it in Class 12 on least squares.

Let $A = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}$. Prove that $\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$.

($\prod$ denotes a product, similar to $\sum$ for a sum.)

Hint: first try $\det \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = (y-x)(z-x)(z-y)$. Think of elementary matrices.

See https://en.wikipedia.org/wiki/Vandermonde_matrix for tips and answers, but try to solve it yourself!

Background: this question nearly "drove me nuts" when I was a first-year student in "19??"! However, I tried to do it "with bare hands", without elementary matrices.

(10) Let $E = \begin{bmatrix} A & B \\ O & D \end{bmatrix}$ be a *block matrix*, which means that the A, B, C are matrices; O is the zero matrix as usual. Suppose for convenience that $A, B, D \in \mathbb{R}^{2\times 2}$, but the result holds for general n as well. Show that $\det(E) = \det(A)\,\det(D)$.

Background: this means that for this special 2×2 block matrix (with here 2×2 blocks), the same formula holds as when A, B, D were numbers.

Now consider the more general case $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with A nonsingular.

Prove that $\det(E) = \det(A)\,\det(D - CA^{-1}B)$.

Note that in general this is not equal to $\det(AD - BC)$.

Hint: start with $\begin{bmatrix} I & O \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ O & D - CA^{-1}B \end{bmatrix}$

See also `https://en.wikipedia.org/wiki/Determinant#Block_matrices`.

Conclude that $\det(E) = \det(AD - ACA^{-1}B)$. This is very close to the known formula $\det = ad - bc$ for a 2×2-matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with scalars (numbers) instead of matrices as elements. Since in the block case the items are matrices, that do not commute in general, we cannot state that $\det(AD - ACA^{-1}B) = \det(AD - CB)$, nor will it be equal to $\det(AD - BC)$.

Similarly, also show that $\det(E) = \det(A - BC^{-1}D)$ when C is nonsingular.

(11) Work out all details of the slide that explains the relation between

$\det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$ and $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ for the volume of a parallelepiped.

(12) **(Class 8)** If $A\mathbf{x} = \mathbf{b}$ is a linear system where all entries in A and $\mathbf{b}$ are integers, and $\det(A) \neq 0$, explain:

- $\det(A)$ is an integer
- $\mathbf{x}$ consists of rational numbers (fractions)
- How about the eigenvalues of A with integer elements? Are they guaranteed to be rational?

(13) Let A be the tridiagonal matrix $\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$ of size n.

Show that the eigenvalues of A are $2 - 2\cos(\frac{k\pi}{n+1})$, where $k = 1, \ldots, n$.

Hint: try eigenvectors of the form $\begin{bmatrix} \sin(\frac{k\pi}{n+1}) \\ \sin(\frac{2k\pi}{n+1}) \\ \vdots \\ \sin(\frac{nk\pi}{n+1}) \end{bmatrix}$ for $k = 1, \ldots, n$.

Use Calculus formulas for $\sin(a+h)$ and $\sin(a-h)$ to show that

$$-\sin(a-h) + 2\sin(a) - 2\sin(a+h) = (2 - 2\cos(h))\sin(a).$$

Analyze the first and last row separately.

Background: this tridiagonal matrix is important in applied math: it is a discretization of minus the second-derivative operator. For general $n \times n$ matrices, there is no expression for the eigenvalues, but this is a relatively "easy" matrix. We can conclude that all eigenvalues of this matrix are in the interval [0,4].

(14) (If you know how to compute with complex numbers.)

Compute the eigenvalues and eigenvectors of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Hint: try on your computer to get a hint!

Background: this is a rotation over $\frac{\pi}{2}$. Therefore, it is natural that the matrix has no real eigenvectors, as these would stay "in place". However, according to the main theorem of algebra, a 2×2 matrix has 2 eigenvalues, which are complex in this case.

(15) **(Class 9)** Use the first question of Class 8 in this document to answer this question. Consider a cyclic cluster of n nodes: every node is (only) connected to its 2 neighbors. Show that this gives the matrix A of 2 questions ago, and that the second smallest eigenvalue is $\lambda_2 = 2 - 2\cos(\frac{\pi}{n+1})$. Using Calculus, show that this means $\lambda_2 = \mathcal{O}(\frac{1}{n^2})$. This means that this network is very suitable to cluster: the value is small.

(16) Now consider a cluster of 10 nodes, in which every node is connected to all other nodes. By a calculation on your computer, show that this means that $\lambda_2 = 10$ (and it is multiple). This means that this network is very unsuitable to cluster: the value is large.

(17) **(Class 13)** Explain:

- The eigenvalues of an orthogonal projection in $\mathbb{R}^2$ are 0 and 1
- The singular values of an orthogonal projection in $\mathbb{R}^2$ are 0 and 1
- The eigenvalues of an oblique projection in $\mathbb{R}^2$ are 0 and 1
- The singular values of an oblique projection in $\mathbb{R}^2$ are 0 and α; what is α in terms of the vectors $\mathbf{u}$ and $\mathbf{v}$?
- The eigenvalues of a reflection in $\mathbb{R}^2$ are -1 and 1
- The singular values of a reflection in $\mathbb{R}^2$ are 1 and 1
- The eigenvalues of a rotation in $\mathbb{R}^2$ are not real
- The singular values of a rotation in $\mathbb{R}^2$ are 1 and 1

(18) **(All classes)** Study the extra chapters of the book by Eldén, as indicated in the document `background-info-2DBI00`.

(19) Search online for a challenging question on your favorite topic, and submit it to the lecturer!