HOMEWORK ASSIGNMENTS 2DBI00

This homework forms, together with the slides, the heart of the course. The group assignments and the online homework are also important, but these homework assignments are even more important. Unless explicitly stated otherwise, do the exercises by hand, and not using computer programs. The main reason for this is that you have to be able to do it by hand on the exam. For many questions you can check your answer using your laptop (and Python, Julia, Matlab, R, C++, Wolfram Alpha, ...). Answers to these questions will usually be made available the evening after the instruction class.

You do not have to hand in your homework. **You are encouraged to work together with classmates, but make sure that you understand the material yourself.**

Class 1: Matrices and vectors

(1) Form groups of 4–6, and register your group on Canvas **before the deadline**.

(2) Compute A^T, $A+B$, $A-B$, AB and BA for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$.

(3) **(a)** Compute $A\mathbf{x}$, $\mathbf{x}^T A$ and $\mathbf{x}^T A\mathbf{x}$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) What can you conclude about the sizes (= dimensions) of these 3 ?

(4) **(a)** Compute $\mathbf{x}^T\mathbf{x}$ and $\mathbf{x}\,\mathbf{x}^T$ for $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Which of these 2 have you already seen in the Calculus course? What did it mean?

(5) **(a)** Show that if A and B are 2×2 upper triangular matrices, then AB is also upper triangular.
(b) Does the opposite also hold: if AB is upper triangular, are A and B upper triangular?

(6) **(a)** If A and B are 2×2 diagonal matrices, show that AB is also a diagonal matrix.
(b) Is this also true when A and B are $n \times n$ diagonal matrices?

(7) When A is an $n \times n$ matrix, show that $B = A + A^T$ is a symmetric matrix.

(8) **(a)** Find 2×2-matrices $A, B \neq O$ such that $AB = O$. (Try matrices with many 0's and some 1's.) Is this also possible when A and B are real numbers (1×1 matrices)?

(b) Find a 2×2-matrix A with $A \neq O$ and $A^2 = O$.
Is this also possible when A is a real number (so a 1×1 matrix)?

(c) Find a 2×2-matrix A with $A \neq I$ and $A \neq -I$ such that $A^2 = I$.
Is this also possible when A is a real number (so a 1×1 matrix)?

(9) When A is an $n \times n$ matrix with $A \neq O$, is it possible to find an $\alpha \neq 0$ such that $\alpha A = O$?

(10) When A is a symmetric $n \times n$ matrix, show that $B = \alpha A$ is also a symmetric matrix.

(11) When A is an $n \times n$ matrix, show that $A^T A$ is a symmetric matrix.
NB: this exercise is important! It teaches you how to manipulate with symbols in a linear algebra way. If you write this out in matrix elements it becomes very messy, but there is a very elegant proof without this.

(12) When A and B are symmetric 2×2 matrices, is AB also a symmetric matrix ?
Prove, or give a counterexample.

(13) Do exercises **(2)** and **(3)** using your computer, for instance making use of Python, Julia, Matlab, R, C++, or Wolfram Alpha.

(14) Optional: past exam questions on this class

Class 2: Vector and matrix norms

(1) If you have not done so already, **SEE QUESTION (1) OF CLASS 1 !**

(2) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalize $\mathbf{x}$ such that: **(a)** $\|\mathbf{x}\|_1 = 1$; **(b)** $\|\mathbf{x}\|_2 = 1$; **(c)** $\|\mathbf{x}\|_\infty = 1$.

(3) Let $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Determine $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$. Which is the largest and the smallest? Will this be the case for any vector? (See also exercise **(9)** on "equivalent" norms.)

(4) Let $A = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix}$. Determine $\|A\|_1$, $\|A\|_\infty$, and $\|A\|_F$. Which norm is the largest? Which is the smallest? Will this be the case for any matrix? Why?

Background: Later in this course we will learn how to compute the most important $\|A\|_2$. This quantity is related to the important "SVD".

(5) For A and $\mathbf{x}$ mentioned in the previous 2 exercises, compute $A\mathbf{x}$ and check that indeed $\|A\mathbf{x}\|_1 \le \|A\|_1 \cdot \|\mathbf{x}\|_1$ and $\|A\mathbf{x}\|_\infty \le \|A\|_\infty \cdot \|\mathbf{x}\|_\infty$.

(6) Let $\mathbf{x} \in \mathbb{R}^2$. What can you conclude if it is given that $\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_1$?

(7) Explain why $\|A\|_1 = \|A^T\|_\infty$ for every matrix $A \in \mathbb{R}^{m\times n}$.
Does it also hold that $\|A\|_F = \|A^T\|_F$?

(8) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. For each of the properties below, determine whether they are true or false. When they are true, give a simple proof to show why. When they are false, give a counterexample (an example in which the statement is incorrect), and give a correct statement instead.

(a) $\|\alpha\mathbf{x}\|_1 = \alpha\,\|\mathbf{x}\|_1$ **(b)** $\|-\mathbf{x}\|_2 = \|\mathbf{x}\|_2$

(c) $\|\mathbf{x}+\mathbf{y}\|_\infty = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$ **(d)** $\|A\mathbf{x}\|_1 = A\mathbf{x}$

(9) This is a bit challenging but important exercise. It turns out that all norms in $\mathbb{R}^n$ are *equivalent*, which means that one norm can be bounded from below and above by another. Let $\mathbf{x} \in \mathbb{R}^2$. For each part below, find the maximal $\beta > 0$ (lower bound) and the minimal $\gamma > 0$ (upper bound) number so that the inequalities are valid for all $\mathbf{x}$ (β and γ may be different for each part).

(a) $\beta\,\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le \gamma\,\|\mathbf{x}\|_\infty$ **(b)** $\beta\,\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \gamma\,\|\mathbf{x}\|_\infty$

(c) $\beta\,\|\mathbf{x}\|_1 \le \|\mathbf{x}\|_2 \le \gamma\,\|\mathbf{x}\|_1$

(10) What do you think will happen in the previous question if the vector is $\in \mathbb{R}^n$ (instead of $\in \mathbb{R}^2$)? You don't have to prove this. **Hint:** the change is quite "logical".

(11) Do the following quantities define a vector norm for vectors $[x,\, y]^T \in \mathbb{R}^2$?
(a) $2x + y$ **(b)** $|2x + y|$ **(c)** $2\,|x| + |y|$ **(d)** $x^2 + |y|$ **(e)** $\max(2\,|x|, |y|)$

(12) Do exercises **(3)** and **(4)** using your computer.

(13) This is a good time to do the group assignment on movie tastes (recommender systems) together with your group!

(14) Optional: past exam questions on this class

Class 3: Google PageRank and Markov chains

(1) We have learned that the solution to $A\mathbf{x} = \mathbf{x}$ is an eigenvector corresponding to an eigenvalue $\lambda = 1$. In class 4 we will learn about *linear systems*. A linear system is of the form $C\mathbf{x} = \mathbf{b}$, where C and $\mathbf{b}$ are given and we have to solve for $\mathbf{x}$.

Explain why the PageRank vector that satisfies $A\mathbf{x} = \mathbf{x}$ can also be viewed as a solution to a certain linear system $C\mathbf{x} = \mathbf{b}$. What are C and $\mathbf{b}$ in this case?

Background: we will treat the important topic of linear systems in more detail in the next class. You can use the answer to this question for the next 2 questions.

(2) Consider the smallest possible internet: 2 sites, with a link from site 1 to site 2. Determine the connectivity matrix G, then $\widehat{G}$ by making sure all columns have 1-norm equal to 1. Since one column of G has only zeros, fill this column with $\frac{1}{2}$'s in $\widehat{G}$ (with equal probability to site 1 and 2). Subsequently, let $A = 0.85 \cdot \widehat{G} + 0.15 \cdot \frac{1}{2} \cdot \text{ones}(2)$, where ones(2) is the 2×2 matrix with all elements equal to 1. Compute **by hand** and a **simple calculator** the eigenvector corresponding to eigenvalue $\lambda = 1$. (If you don't have a calculator, you may use the computer, but only use elementary operations; don't use an eigenvalue routine.) What are the PageRanks of the 2 sites? Is this as expected?

(3) Now consider the same internet: 2 sites, but now with a link from site 1 to site 2 and vice versa. Determine G, $\widehat{G}$, and A similar to the previous question. Compute **by hand** the eigenvector corresponding to eigenvalue $\lambda = 1$. What are the PageRanks of the 2 sites? Is this as expected?

(4) Why does it make sense for Google PageRank that the diagonal of G contains only zeros?

(5) For the connectivity matrix $G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, compute the PageRank on the computer.

Is the result as expected?

(6) Consider the connectivity matrix $G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. What is special about site 4?

What do you expect that the PageRank will look like?
Now compute the PageRank on the computer. Is the result as expected?

(7) In question **(5)**, how many different paths are there from site 3 to site 2 in exactly 3 steps?
In question **(6)**, how many different paths are there from site 1 to site 4 in ≤ 3 steps?
Use both powers of G (on the computer) and counting by hand using a picture of the graph.

(8) If $\widehat{G}$ is a column-stochastic matrix, and $\frac{1}{n} \cdot \text{ones}(n)$ is a column-stochastic matrix as well, show that $A = 0.85 \cdot \widehat{G} + 0.15 \cdot \frac{1}{n} \cdot \text{ones}(n)$ is also a column-stochastic matrix.

(9) Explain that if a connectivity matrix G is symmetric, one may also view the graph as undirected, which means that the edges "have no arrows".

(10) Player X and Y start both with €2. Each turn X gets €1 from Y with probability p, or instead Y gets €1 from X with probability $q = 1 - p$. The game ends when X or Y has €4.

- **(a)** Sketch the Markov chain and give the column-stochastic matrix A.
- **(b)** Compute the eigenvalues using your computer for *any* $0 < p < 1$ (e.g. $p = 0.5$). Does this matrix satisfy the Perron–Frobenius theorem? Can you explain this?
- **(c)** Suppose $p = 0.5$. What are the probabilities after 10 turns that: X has won; Y has won; the game is undecided? Do the same for $p = 0.6$.
- **(d)** Suppose $p = 0.51$. What are the probabilities that X will win; Y will win the game?

(11) This is a good time to do the group assignment on the Google PageRank type ranking together with your group!

Class 4: Linear systems Do all exercises by hand, except for **(7)** and **(8)**.

(1) Give the reduced row echelon form (rref) of the matrices:

(a) $\begin{bmatrix} 4 & 16 & -44 \\ -1 & 0 & -1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & -2 & -6 \\ -4 & 6 & 10 \\ -4 & 8 & 8 \end{bmatrix}$ **(c)** $\begin{bmatrix} 2 & 8 & -8 & -12 & -30 \\ 5 & 22 & -8 & -62 & -55 \\ -3 & -16 & -9 & 76 & 11 \end{bmatrix}$

(2) Determine by means of an augmented matrix the solution to the systems of equations:

(a) $\begin{aligned} x - 3y &= -8 \\ 3x - 8y &= -21 \end{aligned}$ **(b)** $\begin{bmatrix} 3 & 3 & -3 \\ -1 & 1 & -7 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ -3 \end{bmatrix}$ **(c)** $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 3 \\ 3 & -3 & 3 & 0 & -18 \\ -3 & -2 & -2 & 2 & 13 \\ 3 & 0 & 3 & -3 & -24 \end{array}\right]$

(3) Consider the following systems of linear equations with parameter a in the variables x_1, x_2, x_3, and x_4:

$$\begin{aligned} 2x_1 + x_2 + 5x_3 + 8x_4 &= 10 \\ x_2 + 3x_3 + 4x_4 &= 4 \\ x_1 + x_2 + 4x_3 + 6x_4 &= a \end{aligned}$$

(a) For which value(s) of a does this system have a solution?

(b) Give a parameter representation of all solutions if $a = 7$.

(4) Consider the following system of linear equations with parameter c and variables x_1, x_2, and x_3:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + cx_2 + 3x_3 &= 5 \\ 3x_1 + 6x_2 + cx_3 &= 9 \end{aligned}$$

(a) Check if the system has a solution for $c = 9$.

(b) Give all solutions of the system for $c = 4$.

(c) For which values of c does the system have exactly 1 solution?

(5) This question is **important** for your understanding.
Let $A \in \mathbb{R}^{3\times 3}$ and $\mathbf{x}, \mathbf{y}, \mathbf{b} \in \mathbb{R}^3$. Suppose that $A\mathbf{x} = \mathbf{b}$ and also $A\mathbf{y} = \mathbf{b}$.
Does it follow that $\mathbf{x} = \mathbf{y}$?
If so, why? **If not**, under which condition does it hold?

(6) Fruit is important for hard-working students; let's buy some apples (a), bananas (b), cherries (c) and dates (d) (NL: dadels). Total cost of one of each type: €9; $a + 2b + 3c + 4d =$ €22. $4a + 3b + 2c + 2d =$ €25. $a + c$ are €3 more expensive than $b + d$.
What does each item cost?

(7) Compute **1(c)** using your computer, e.g., by using a command of the form `rref([1 2; 3 4])` in Matlab or `rref {{1,2},{3,4}}` in Wolfram Alpha.
(In Python this is also possible, but less standard. It may require some extra functionality of the form:
`from sympy import Matrix,` `A = Matrix([[1., 2., 3.], [2., 4., 5.]]),` `A.rref()`
The first output is the rref-form, while the second is the columns that contain pivots.)

(8) Compute **2(a)** and **2(b)** using your computer. Do you get the right solutions?
(The **(b)** part is tricky since it has ∞ many solutions. Depending on the program you use, it may or may not work. Wolfram Alpha seems to do fine, how about Python or Matlab?)

(9) Can you think of examples where one needs to solve linear systems in your field of study?

(10) Solve the parametrized linear systems on the final slides of Class 4.

(11) Optional: past exam questions on this class

Class 5: Linear algebra on your computer

(1) Let $A, B \in \mathbb{R}^{n\times n}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. In this question, assume that A and B are **full** (also called dense; not sparse). The number of flops to compute $\mathbf{x}^T\mathbf{y}$ is $2n-1$; ignoring the less important term we can say: $\approx 2n$. In the same way, give the numbers of flops for:

(a) $A\mathbf{x}+\mathbf{y}$	**(c)** $\alpha A\mathbf{x}$	**(e)** $\mathbf{y}^T A\mathbf{x}$
(b) $(A+B)\,\mathbf{x}$	**(d)** $\alpha AB\mathbf{x}$	**(f)** $\alpha\,\mathbf{y}^T\mathbf{x}$

Important! Parts **(b)**–**(f)** can be computed in several ways. Show that you have chosen the fastest way, by comparing the numbers of flops.

(2) Repeat **(a)**–**(e)** of the previous exercise, but now for the practically important case that A and B are tridiagonal (so bandwidth $\ell = 1$).

(3) Let x, y, z be machine numbers (which means that we consider them as exact, and do not look at rounding errors from storing a real number x on the computer). Let $z \neq 0$. What is the maximal relative error in $\frac{xy}{z}$? Give a rounding error analysis with δ's.

(4) Let x and y be machine numbers. We are going to see that on a computer, $x^2 - y^2$ may be different from $(x+y)(x-y)$!

(a) Carry out a rounding error analysis with δ's, and show that the rounding errors in $x^2 - y^2$ may be larger than $(x+y)(x-y)$ when x and y are large and $x \approx \pm y$.

(b) Now on the computer try a command of the form (and compare `a` and `b`)

```
x = pi*1e14; y = x-0.1; a = x^2-y^2; b = (x+y)*(x-y); (a-b)/b
```

`^` means "to the power"; in Python, this should be replaced by `**`.

(5) We now look at the famous math identity $\sin^2(x) + \cos^2(x) = 1$, which is not necessarily true on a computer!

(a) Using an analysis with δ's, give the maximal rounding error.

(b) Now try a command such as `x = 0.17; sin(x)^2 + cos(x)^2 - 1` on the computer.

(6) Cancellation: suppose $x = 0.001$ has a relative error of at most 5%, and $y = 0.00099$ has a relative error of at most 1%, what is the maximal relative error in $x - y$?

(7) Consider Table 1. Explain the following answers given by a computer, using standard IEEE double precision arithmetic. Here, `eps` $= 2^{-52} \approx 2.2 \cdot 10^{-16}$.
`==` tests whether the left-hand side and right-hand side are equal.

Table 1

Command	Answer
`1+eps == 1`	`false`
`1+eps/2 == 1`	`true`
`2^-1000`	`9.3326e-302`
`2^-1100`	`0`
`2^1000`	`1.0715e+301`
`2^1100`	`Inf`

Table 2

Command	Answer
`2+eps == 2`	??
`sqrt(2)+eps == sqrt(2)`	??
`1+eps-1`	??
`1e16 +1 -1e16`	??
`1/2 == 0.5`	??
`0.6/0.2 == 3`	??

(8) Now consider Table 2. First try to predict what answers your computer will give. Then try them on the computer to see if you are right. (For `eps` $\approx 2.2 \cdot 10^{-16}$: see slides.)

(9) Optional: past exam questions on this class

Class 6: Inverse matrices

(1) Determine whether the following matrices are nonsingular. If so, determine the inverse matrix.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 6 & 5 & -4 \\ -2 & -1 & 2 \\ 4 & 4 & -2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 6 & 5 & -4 \\ -2 & -1 & 2 \\ 4 & 4 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix}$

(e) $\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$ **(f)** $\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 4 & 1 \end{bmatrix}$ **(g)** $2\,I_{1000}$

(2) Explain what you see when you look at the answers to parts **1(d)**, **1(e)**, **1(f)**.

(3) In view of question **(1)**, what is the inverse of the $n \times n$ matrices

(a) $\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$ (all $a_i \neq 0$) **(b)** $\alpha\, I_n$ $(\alpha \neq 0)$ **(c)** Π (permutation matrix) ?

(4) Let A be a nonsingular $n \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^n$ an n-vector, and consider the linear system $A\mathbf{x} = \mathbf{b}$. One can solve the system and compute $\mathbf{x} = A^{-1}\mathbf{b}$ in two ways.
Explain which two, and which method is the most efficient in which situation.

(5) Now carry out both methods of question **(4)** on $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}$

(6) If A is a square nonsingular matrix, then what would $(A^{-1})^{-1}$ be ?
Check your hypothesis on $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(7) Let A be nonsingular. Find the inverse of A^2, by which you also show that A^2 is also nonsingular.

(8) Let A be symmetric and nonsingular. Is the inverse A^{-1} also symmetric?
Show, or give a counterexample.

(9) Which is easier: computing $B = A^{-1}$, or checking if a given B is the inverse of A:
(a) by hand? **(b)** by computer?

(10) This question is **very important** for your understanding of the material.
Let $A \in \mathbb{R}^{n\times n}$. Explain why the following 5 statements are equivalent:
A is nonsingular $\iff$ A has n pivots $\iff$ $\mathrm{rref}(A) = I_n$
$\iff$ $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for every $\mathbf{b} \in \mathbb{R}^n$
$\iff$ $A\mathbf{x} = \mathbf{0}$ has only $\mathbf{x} = \mathbf{0}$ as solution.
(*Background*: in the next classes we will come back with more equivalent statements.)

(11) Now answer **(1)(c)** on your computer.

(12) **Challenge:** By row reduction on the general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, deduce the formula for the inverse $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Tip: Start with $[A \,|\, I]$ and try to arrive at $[I \,|\, A^{-1}]$. While solving this question, do not worry whether quantities in the denominator are 0, check this only at the end. Check your answer on **1(a)**.
This formula may be useful in several situations, so it might be an idea learning it by heart.

(13) Optional: past exam questions on this class

Class 7: Determinants

(1) What is the determinant of the 3×3 diagonal matrix $\text{diag}(\alpha, \beta, \gamma)$?
And of the general $n \times n$ diagonal matrix $\text{diag}(a_1, \ldots, a_n)$?

(2) What is the determinant of the 3×3 upper triangular matrix $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$?
And of the general $n \times n$ upper triangular matrix with $(a_1, \ldots, a_n)$ on the diagonal ?

(3) What is the area of the parallelogram spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$? (So the other points are the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$.) Can you determine this easily by another method?

(4) What is the volume of the parallelepiped spanned by $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$?

(5) When does the 2×2 matrix $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right]$ have a unique solution?
Give a constraint in terms of the coefficients. What is notable about the condition?

(6) When is the 2×2 matrix $\begin{bmatrix} 1 & 3 \\ 2 & \alpha \end{bmatrix}$ singular? Why is this natural?

(7) For which values of α is: **(a)** $\begin{bmatrix} \alpha+2 & 1 \\ \alpha+2 & -\alpha-2 \end{bmatrix}$ singular? **(b)** $\begin{bmatrix} \alpha+3 & 3 & 1 \\ -8 & \alpha-6 & -1 \\ 4 & 0 & \alpha-3 \end{bmatrix}$ singular?

(8) Let A, B, and C be $n \times n$ matrices, and I the $n \times n$ identity matrix. For each of the properties below, determine whether they are **true or false**. When they are true, give a **simple proof** to show why. When they are false, give a **counterexample** (an example in which the statement is incorrect), and give a **correct statement** instead.

- **(a)** $\det(AB) = \det(BA)$
- **(b)** $\det(-I) = -1$
- **(c)** $\det(A) = 0 \Longrightarrow A = O$
- **(d)** $\det(ABC) = \det(A)\det(B)\det(C)$
- **(e)** $\det(A^2) > 0$
- **(f)** $\det(AB^{-1}) = \frac{\det(A)}{\det(B)}$

(9) Show that for 1×1 matrices we have $\det(A+B) = \det(A) + \det(B)$, but show by giving a 2×2 example that this generally does **not** hold for matrices of larger size.

(10) Compute **by hand** the determinant of $\begin{bmatrix} 1 & & & & 2 \\ & 3 & & 4 & \\ & & 5 & & \\ & 4 & & 3 & \\ 2 & & & & 1 \end{bmatrix}$. What does the answer mean?

(11) Compute **by computer** the determinant of $\begin{bmatrix} F & F & R & R & \\ F & & R & & R \\ F & F & R & R & \\ F & & R & R & \\ F & & R & & R \end{bmatrix}$ where $F = 1$ and $R = 2$.

(Can you recognize the initials of a popular soccer club in The Netherlands?)

Can you explain the result? (**Hint:** the answer is the volume of a 5-dimensional object. What do you see when you look at the points defined by the 5 columns ?)

(12) **Important for your understanding**:
Explain briefly to yourself the reason why we can efficiently check on $\det(A) = 0$ to see if a matrix is singular. Use the notions of "elementary matrices", "pivots", and "rref".

(13) Optional: past exam questions on this class

Class 8: Eigenvalues and eigenvectors

(1) What are the eigenvalues and eigenvectors of: **(a)** $\begin{bmatrix} 2 & 1 \\ & -3 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & -3 & -7 \\ & 2 & -1 \\ & & 3 \end{bmatrix}$

(2) What are the eigenvalues of the general $n \times n$ upper triangular matrix with $(a_1, \dots, a_n)$ on the diagonal? Of the eigenvectors, 1 is very easy; which one? (See **(1)(b)**)

(3) What are the eigenvalues and eigenvectors of: **(a)** $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 4 & -3 & 1 \\ -1 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$

Hint: for **(b)** one of the eigenvalues is $\lambda = 3$.

(4) Show that the matrix $\begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$ has no real eigenvalues.

Background: We will see later that this is a rotation matrix over $\frac{\pi}{6}$ (= 30 degrees). A rotation leaves no vector in place, so it is natural that there is no real eigenvalue/vector.

(5) It can be shown that the eigenvalues of a symmetric matrix are real, that is, cannot be complex. Show this for a 2×2 matrix: prove that the eigenvalues of $A = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$ must be real. **Hint:** look at $\det(A - \lambda I) = 0$ and use the quadratic formula ("abc-formule").

(6) Can a 3×3 (real) matrix have: **(a)** 3 real eigenvalues? **(b)** 2 real eigenvalues and 1 complex? **(c)** 1 real eigenvalue and 2 complex? **(d)** 3 complex eigenvalues?
NB: by "complex", we mean "non-real". **Hint:** consider the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$, which is of degree 3, and use your Calculus expertise!

(7) Suppose $(\lambda, \mathbf{x})$ is an eigenpair of a A. Show the following:

- **(a)** $(\lambda - \alpha, \mathbf{x})$ is an eigenpair of $A - \alpha I$
- **(b)** $(\alpha\lambda, \mathbf{x})$ is an eigenpair of αA
- **(c)** $(\lambda^2, \mathbf{x})$ is an eigenpair of A^2
- **(d)** $(\lambda^{-1}, \mathbf{x})$ is an eigenpair of A^{-1} (when A is nonsingular)

(8) Illustrate all parts of the previous question by considering $A = \begin{bmatrix} 2 & 1 \\ & -3 \end{bmatrix}$, $\alpha = 3$, $\lambda = 2$, and the corresponding eigenvector $\mathbf{x}$.

(9) Show: the eigenvalues of A^T are the same as the eigenvalues of A.
Hint: Look at $\det(A - \lambda I)$ and $\det(A^T - \lambda I)$.

(10) Show that a **row**-stochastic matrix (all row sums are 1) has an eigenvalue $\lambda = 1$.
Hint: find an "easy" vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$. (By the previous question, this is also true for a column-stochastic matrix; this gives a link with Google PageRank.)

(11) Let A, $B \in \mathbb{R}^{n \times n}$ with B nonsingular. Show that the eigenvalues of AB are exactly those of BA. So although AB is usually $\neq BA$, their eigenvalues are the same.
Hint: Consider $\det(B)\det(AB - \lambda I)\det(B)^{-1}$.

(12) Let A, B, C be $n \times n$ matrices with B nonsingular. Let $C = BAB^{-1}$, i.e., C is similar to A.

(a) Show that the eigenvalues of C are the same as the eigenvalues of A.

(b) Show that $\mathbf{x}$ is an eigenvector of A corresponding to eigenvalue λ if and only if $B\mathbf{x}$ is an eigenvector of C.

(c) Show that $A = \begin{bmatrix} -2 & 2 & 3 \\ 2 & -1 & 6 \\ 3 & 6 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 1 & 1 \\ 4 & -1 & 4 \\ 9 & 9 & 1 \end{bmatrix}$ are similar.

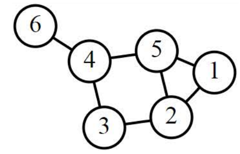
Hint: Try to find a suitable diagonal matrix B.

(d) Now conclude that the eigenvalues of C are real, without actually computing them.

(13) Compute **(3)(b)** on your computer.

(14) Optional: past exam questions on this class

Class 9: Clustering via the Fiedler vector; eigenvalue location



(1) Compute using the computer 2 clusters according to the Fiedler vector of this graph. Do you think the result is logical?

(2) Explain why every Laplacian matrix has an eigenvalue $\lambda = 0$. What is the corresponding eigenvector?

(3) Explain why the fact that a Laplacian matrix is of the form $B^T B$ (see slides) implies that all eigenvalues are ≥ 0.

(4) Explain why Gershgorin's Theorem, applied to a Laplacian matrix, implies that all eigenvalues are ≥ 0.

(5) What can you say using Gershgorin's Theorem about the **maximum** eigenvalue of a Laplacian matrix? And therefore about **all** eigenvalues?

(6) Compute **by hand** the Fiedler clustering of the undirected graph with 2 nodes and 1 edge. Do you think the result is logical?

(7) Compute **by hand** the Fiedler clustering of the undirected graph with 3 nodes and 2 edges: one from node 1 to node 2, and one from node 2 to node 3. Do you think the result is logical?

(8) Compute **by computer** the result for the previous question, but now the edge connecting nodes 1 and 2 has double weight.

(9) Consider $\begin{bmatrix} 3 & -1 & \\ -1 & 4 & -1 \\ & -1 & 5 \end{bmatrix}$. Why are the eigenvalues of this matrix real? Using Gershgorin, what is the smallest set in which the eigenvalues are guaranteed to be located?

(10) The same questions as the last exercise, but now for $\begin{bmatrix} 3 & -0.1 & \\ -0.1 & 4 & -0.1 \\ & -0.1 & 5 \end{bmatrix}$.

(11) Using $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$, what info do we get about the eigenvalues of A in **(9)**?

(12) Show that $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ is positive definite in 2 different ways:

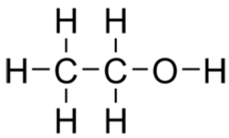
(a) by Calculus techniques: show that $\mathbf{x}^T A \mathbf{x} > 0$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

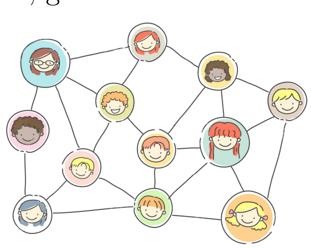
(b) by computing the eigenvalues.

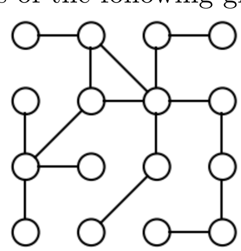
(13) Show in an easy way that the **column** version of Gershgorin's theorem is also true: all eigenvalues are contained in $\bigcup_i \{\, |x - a_{ii}| \leq \sum_{j \neq i} |a_{ji}| \,\}$.

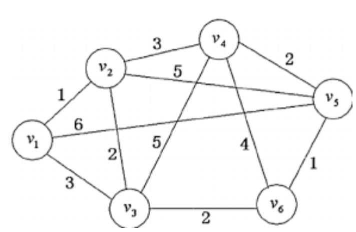
(14) Is $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ **(a)** SPD? **(b)** SPSD? **(c)** Weakly diagonally dominant?

(15) Using your computer, give 2 Fiedler clusters of the following graphs:









(16) Which are SPD? **(a)** $\begin{bmatrix} 3 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 4 \\ 1 & 4 & 5 \end{bmatrix}$ **(c)** Matrix of **(9)** **(d)** Matrix of **(10)**

(17) This is a good time to do the group assignment on Fiedler vector based clustering together with your group!

(18) Optional: past exam questions on this class

Class 10: Linear dependence, orthogonality, bases, and rank

(1) Are these vectors independent? If not, give a linear combination that gives $\mathbf{0}$.

(a) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\-5\end{bmatrix}$ **(c)** $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ **(d)** $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

(e) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ **(f)** $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ **(g)** $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}$

(2) **(a)** Determine (by hand) the rank of $A = \begin{bmatrix}3 & 4 & -1\\-2 & 3 & 1\\-9 & 5 & 4\end{bmatrix}$.

(b) What can you conclude about A being singular or nonsingular ?
(c) What can you conclude about $\det(A)$?
(d) Are the columns of A independent?
(e) Do the columns of A form a basis for $\mathbb{R}^3$?

(3) Now change only the -9 in position $(3,1)$ to -8 and answer the same questions.

(4) What are the ranks of $A = \begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2\\3 & 6.0001\end{bmatrix}$?

Can you explain from this that determining the rank of a matrix may be a tricky subject in practical situations?

(5) Determine if the columns of the following matrices:

- form an **orthonormal** basis,
- form an **orthogonal** basis (but not orthonormal),
- form a **basis** (but not orthogonal, let alone orthonormal),
- or do **not** form a basis at all (are linearly dependent).

(a) $\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$ **(b)** $\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ **(c)** $\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\-\frac{1}{2} & \frac{1}{2}\end{bmatrix}$ **(d)** $\begin{bmatrix}\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2}\\-\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2}\end{bmatrix}$

(e) $\begin{bmatrix}1 & 1 & 1\\1 & 0 & -1\\-1 & 1 & 0\end{bmatrix}$ **(f)** $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$ **(g)** $\begin{bmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\0 & 1 & 0\\-\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$ **(h)** $\begin{bmatrix}\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2}\\0 & 1 & 0\\-\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2}\end{bmatrix}$

(6) Which of the matrices in the previous question are orthogonal?

(7) Given is that the columns of the matrix $\begin{bmatrix}1 & 0 & 0\\\alpha & 2 & 0\\\beta & \gamma & 3\end{bmatrix}$ form an orthogonal basis.

Give all options for α, β, and γ such that this is possible.

(8) Construct a 4×4 matrix with only ± 1 as elements so that the columns are orthogonal.

(9) Is this also possible for a 3×3 matrix? Explain your answer.

(10) Consider $A = \begin{bmatrix}\alpha & \alpha\\\alpha & \alpha\end{bmatrix}$, where $\alpha \in \mathbb{R}$. What is:
(a) the maximal and minimal possible rank of A ? (by varying α)
(b) the "generic" rank of A ? This means: almost always, except for exceptional cases.

(11) Without looking at the slides, try to deduce why for an orthogonal $n \times n$ matrix Q:
(a) Q has orthonormal columns; **(b)** Q has orthonormal rows; **(c)** $\|Q\mathbf{x}\| = \|\mathbf{x}\|$;
(d) $\angle(Q\mathbf{x}, Q\mathbf{y}) = \angle(\mathbf{x}, \mathbf{y})$; **(e)** $\det(Q) = \pm 1$.

(12) Check **(2)(a)**, **(3)(a)**, and **(5)(g,h)** on your computer.

(13) Finish the online tests, any overdue homework, and the group work as much as you can.

(14) Optional: past exam questions on this class

Class 11: Singular Value Decomposition (SVD)

Do all exercises by hand, except for **(10)** and parts of **(1)** and **(7)**.

(1) Determine the singular values of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Give the exact (algebraic) answer.
What is $\|A\|$ $(= \|A\|_2)$? Now check this using the computer.

(2) The singular values of A are the square roots of the eigenvalues of A^TA.
To be able to take a square root, the eigenvalues of A^TA should be real and ≥ 0.

(a) Why are we certain that these are real?

(b) And why that they are ≥ 0 ? **Hint:** Rayleigh quotient.

(3) Compute by hand the SVD $A = U\Sigma V^T$ of **(a)** $\begin{bmatrix} & 2 \\ -3 & \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & & \\ & 4 & \\ & & -5 \end{bmatrix}$ **(c)** $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(4) $\|A\|_2$ is as least as large as $\max_\lambda |\lambda|$, the absolute value of the maximal eigenvalue (of A, not of A^TA). Show this by taking the 2-norm of the equation $A\mathbf{x} = \lambda\mathbf{x}$.
Hint: We have seen this before in the class on Gershgorin.

(5) Consider the $m \times n$ matrix A, where $m \geq n$. Suppose we know the SVD $A = U\Sigma V^T$. Explain how we can easily determine the SVD of A^T from this. Give a formula, and also draw a picture such as the one on the slide with the yellow color ("colored areas").

(6) Explain using formulas why from the SVD $A = U\Sigma V^T$ of A, we cannot easily get the SVD of A^2.

(7) Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$ and its SVD $A = U\Sigma V^T$.

(a) What do you expect that:
- $\mathbf{u}_1$, the first column of U (the first left singular vector),
- and $\mathbf{v}_1$, the first column of V (the first right singular vector)

will be approximately? Think of the "dominant" column and row, and take care that they have to have norm 1.

(b) And what do you think that
- $\mathbf{v}_2$, the second column of V (the second singular vector)

will be approximately?

(c) Now compute the SVD on the computer. Is it as expected?

(8) Considering the position of U and V in $A = U\Sigma V^T$, how can you remember that the left singular vectors U give information about the rows, and that the right singular vectors V give information about the columns ?

(9) From $A\mathbf{v} = \sigma\mathbf{u}$ and $A^T\mathbf{u} = \sigma\mathbf{v}$ it follows that $A^TA\mathbf{v} = \sigma^2\mathbf{v}$. Deduce this.
This gives a nice relation between the singular values of A and the eigenvalues of A^TA.

(10) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Using your computer, compute **(a)** both eigenvalues of A^TA;
(b) both singular values of A; **(c)** $\|A\|$. Explain what you see.

(11) We know that $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ (as usual, the norm is the 2-norm).
Show using the SVD that $\|A\| = \|U\Sigma V^T\| = \|\Sigma\| = \sigma_1$.
Hint: use the properties of orthogonal matrices of the previous class.

(12) This is a good time to do the group assignment on the term–document (or product–customer, or keyword–tweet) matrix together with your group!

(13) Optional: past exam questions on this class

Class 12: Least squares and fitting

(1) (Important question!) Without looking at the slides, show that $A^T A$ is symmetric positive semidefinite (SPSD).

(2) Suppose we have the inconsistent (contradictory) equations $x = 8$ and $x = 10$. Solve the least squares problem $\min_x (x-8)^2 + (x-10)^2$ in the "Calculus way" (set derivative to 0). This method is always possible but gets trickier and trickier for more variables.

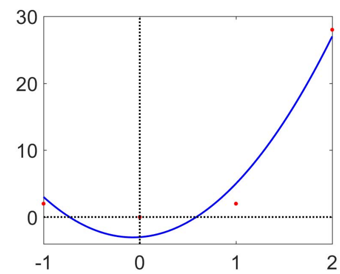
(3) Now solve the above problem in a linear algebra way: set up $Ax \approx \mathbf{b}$, and determine $A \in \mathbb{R}^{2\times 1}$ and $\mathbf{b} \in \mathbb{R}^2$ (so A and $\mathbf{b}$ are of the same size here). Then solve x via the normal equations $A^T A x = A^T \mathbf{b}$. Is the result natural?

(NB: since here x is a number rather than a vector, we prefer a non-boldface letter.)

(4) Determine **by hand** the best straight line (of the form $y(t) = x_1 + x_2\, t$) fitting the points $(0,0)$, $(1,1)$, and $(2,1)$ by an $A\mathbf{x} \approx \mathbf{b}$ approach, where $A \in \mathbb{R}^{3\times 2}$ and $\mathbf{b} \in \mathbb{R}^3$.
Draw a picture to see if your answer makes sense, and compute the sum of squares.

(5) What is the best fit in **(4)** if one considers a degree-0 (contant) polynomial $y(t) = x_1$?

(6) You are an expert in computer graphics, and your boss asks you to determine a suitable curve (blue line) that approximately fits the 4 red points. She states that she prefers a simple polynomial of a low degree.



(a) Determine **by hand** the best **quadratic** polynomial fitting these points $(-1,2)$, $(0,0)$, $(1,2)$, and $(2,28)$ by a least squares approach $A\mathbf{x} \approx \mathbf{b}$, where $A \in \mathbb{R}^{4\times 3}$, $\mathbf{b} \in \mathbb{R}^4$, and $\mathbf{x} \in \mathbb{R}^3$, to give a curve of the form $y(t) = x_1 + x_2\, t + x_3\, t^2$.

(b) Without doing any computations, explain what will happen if we take a **cubic** polynomial ($y(t) = x_1 + x_2\, t + x_3\, t^2 + x_4\, t^3$) instead.

(7) Check your answer to the previous question **by computer**, using "fitting" or solve the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ for appropriate A and $\mathbf{b}$.

(8) **(Important for your understanding and to improve your technical skills!)**
Let A be $m \times n$ with $m > n$. Show that the following statements are equivalent:
(a) A has full rank n
(b) The columns of A are linearly independent
(c) There does not exist a nonzero $\mathbf{v}$ with $A\mathbf{v} = \mathbf{0}$
(d) There does not exist a nonzero $\mathbf{v}$ with $A^T A\mathbf{v} = \mathbf{0}$
(e) $A^T A$ is nonsingular
This means that the least squares problem $A^T A\mathbf{x} = A^T\mathbf{b}$ has a unique solution if and only if A is of full rank n.

(9) The best fitting quadratic polynomial through the points $(-1,1)$, $(0,0)$, $(1,1)$, and $(2,4)$ evidently is: ...... (fill the dots!). Now, we consider the "perturbed" points $(-1,\ 1.1)$, $(0,\ 0.1)$, $(1,\ 0.9)$, and $(2,\ 4.1)$. (Here, "perturbed" means: "changed a little bit".)
Determine **by computer** the 3 coefficients of the best fitting quadratic polynomial.
Is the answer close to that to the first part of this question?

(10) First show that the determinant of $\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix}$ is $(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)$. Conclude that this matrix is nonsingular if the t_i's are distinct. Then show: if a quadratic polynomial $at^2 + bt + c$ is zero in three different points, then it must be the zero polynomial!

(11) Optional: past exam questions on this class

Class 13: Rotations, projections, reflections

(1) **(a)** Show, using the definition, that $P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a projection.

(b) Again using the definition, is it also an orthogonal projection?

(2) The same questions for $P = \begin{bmatrix} \frac{4}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{3}{7} \end{bmatrix}$.

(3) Give the 2×2 matrices representing a rotation of $\frac{\pi}{2}$, π, $\frac{3}{2}\pi$, $-\frac{\pi}{2}$, $\frac{2}{3}\pi$, and $-\frac{2}{3}\pi$.
Which of these is also a reflection in a line? And the reflection in the origin?

(4) Let $\mathbf{v} \in \mathbb{R}^2$ be a vector with $\|\mathbf{v}\| = 1$, and let $R = I - 2\,\mathbf{v}\mathbf{v}^T$ be the corresponding reflection across the line orthogonal to $\mathbf{v}$. What do you expect that R^2 is (two times reflecting is ...)? Now check this algebraically: $(I - 2\,\mathbf{v}\mathbf{v}^T)(I - 2\,\mathbf{v}\mathbf{v}^T) = \ldots$.

(5) Multiply the rotation matrices $\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ (over angle α), and similar over angle β.
Use the Calculus formulas $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \ldots$, and a similar one for $\cos(\alpha+\beta)$, to show that the product of these 2 matrices represents the rotation over angle $\alpha + \beta$.

(6) What is the inverse of the 2×2 rotation matrix $\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$?

(7) Give the 2×2 matrix for the orthogonal projection onto the line $y = 4x$.
Test your matrix on two well-chosen vectors (on the axis of projection, and orthogonal to the axis of projection) to see if it gives the desired result.

(8) Give the 2×2 matrix for the reflection in the line $y = 4x$.

(9) Give the 2×2 matrix for the oblique projection onto the x-axis along the line $y = -2x$.
Test your matrix on the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: see if it gives the desired result.

(10) True or false? Show using algebra arguments.
(a) If Q is a 2×2 rotation, then so is Q^T.
(b) If Q is a 2×2 rotation, then so is Q^2.
(c) If Q is a 2×2 rotation, then $Q^2 = Q$.
(d) The 2×2 identity matrix I_2 is a projection.
(e) The 2×2 zero matrix O_2 is a projection.
(f) If P is a 2×2 projection onto a line, then so is P^T.
(g) If P is an $n \times n$ projection, then $P^3 = P$.
(h) If P is an $n \times n$ projection, then $P^{99} = P$.
(i) Every 2×2 projection matrix is nonsingular.
(j) Every 2×2 projection matrix is singular.
(k) If P is a 2×2 orthogonal projection matrix onto a line, then p_{11} can be < 0.

(11) Show that a 2×2 rotation matrix Q has $\det(Q) = 1$. Why is this natural?

(12) Show that $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$ is a reflection. Across which axis?

(13) Give the 3×3 matrix of a rotation of angle α around the z-axis.

(14) Although this question also holds in the $\mathbb{R}^n$, we consider it in $\mathbb{R}^2$.
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and let P be an orthogonal projection in the $\mathbb{R}^2$.
(a) Show Pythagoras' theorem: if $\mathbf{x} \perp \mathbf{y}$, then $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.
Hint: start with $(\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) = \ldots$. Also make a sketch.
(b) Show that $P\mathbf{x} \perp (I-P)\mathbf{x}$. *Hint:* start with $\mathbf{x}^T(I-P)^T P\mathbf{x} = \ldots$.
(c) Use (a) and (b) to show: $\|\mathbf{x}\|^2 = \|P\mathbf{x}\|^2 + \|(I-P)\mathbf{x}\|^2$. Also sketch a figure.
(d) Conclude that $\|P\mathbf{x}\| \le \|\mathbf{x}\|$. Explain why this logical.

(15) Optional: past exam questions on this class

Class 14: Splines and truncated SVD

(1) [Splines] Suppose we have a cubic spline on the intervals $[0, 1]$, $[1, 2]$, and $[2, 3]$. List the "dofs" (degrees of freedom, number of variables that can/should be chosen), and the number of requirements.

(2) We are interested in a cubic spline fitting the points $(-1, 1)$, $(1, 0)$, and $(2, 4)$.

(a) Write down the requirements for the 2 cubic polynomials p (on the interval $[-1, 1]$) and q (on the interval $[1, 2]$). Add the 2 extra degrees of freedom for $p'(-1)$ and $q'(2)$.

(b) Formulate a linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ contains the coefficients $x_1, \ldots, x_8$.

(c) If time allows, solve $\mathbf{x}$ on the computer, and plot the spline; take $p'(-1) = 1$ and $q(2) = -1$. (This part **(c)** is not important for the exam.)

(3) [Truncated SVD] Draw a picture with "boxes" of the sizes of the following matrices in the (truncated) SVDs, for A with size 10×5:

(a) Draw $A = U\Sigma V^T$ with Σ of size 10×5.

(b) Draw $A = U\Sigma V^T$ with Σ of size 5×5. Explain the difference with **(a)**.

(c) Draw the sizes for the truncated SVD (TSVD) $A \approx A_k = U_k \Sigma_k V_k^T$ for $k = 2$.

It is not necessary to give any SVD, just to draw a picture of the sizes.

(4) Suppose $A \approx A_k = U_k \Sigma_k V_k^T$ is a truncated SVD (TSVD) of $A \in \mathbb{R}^{m\times n}$.
Assume $k < \min(m, n)$, so that it is a real truncation.

(a) Are the columns of U_k linearly independent ? The columns of V_k ?

(b) Are the columns of U_k orthogonal, orthonormal ? The columns of V_k ?

(c) Are the columns of U_k a basis for $\mathbb{R}^m$? The columns of V_k a basis for $\mathbb{R}^n$?

(d) Is U_k orthogonal? V_k orthogonal?

(5) What is the best rank-1 approximation of the matrices:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(6) What is the best rank-1, and the best rank-2 approximation of $\begin{bmatrix} 1 & & \\ & -2 & \\ & & 3 \end{bmatrix}$?

(7) Suppose you have an image of 200×320 pixels, and every pixel is stored as a real number.

(a) How many bytes are needed to store this image?
(Recall: 1 number takes 64 bits = 8 bytes.)

(b) Suppose we approximate A by A_{30}, the TSVD with $k = 30$.
How many bytes are needed now? What are the savings?

(8) Finish all electronic homework tests. This takes time, start early!

(9) Finish all homework.

(10) Ask any questions that you have to the instructors.

(11) Past exams!

GOOD LUCK ON THE EXAM !

See next page for some review / exam training

Review / exam training

For all questions, give a 2×2 matrix, not equal to a multiple of I or O.

(1) Give a matrix such that $A^2 = A$. What is the name of this type of matrix?

(2) Give a matrix such that $A^{-1} = A$, or in other words $A^2 = I$. What type of matrix is an example of this?

(3) Give a matrix such that $A^{-1} = A^T$, or in other words $A^T A = I$. What is the name of this type of matrix?

(4) Give a matrix such that $A^T = A$. What is the name of this type of matrix?

(5) Give a matrix such that $A^T = -A$. (What is the name of this type of matrix?)

(6) Give a matrix such that $A^2 = O$. (What is the name of this type of matrix?)

(7) Give a matrix such that $A^2 = -I$. What type of matrix is an example of this?

(8) Give a matrix such that $A^3 = I$. What type of matrix is an example of this?

(9) Give a matrix such that $A^3 = -I$. What type of matrix is an example of this?

(10) Give a matrix such that $A^3 = A$.

(11) Give 2 properties of a matrix of the form $B = A^T A$.

(12) Name 3 classes (topics) where we have seen a matrix of the form $B = A^T A$.

(13) If we know the eigenvalues and eigenvectors of A, what can we say about those of:
(a) A^2 **(b)** A^{-1} **(c)** A^T **(d)** SAS^{-1}

(14) The determinant satisfies $\det(AB) = \det(A)\,\det(B)$.
Can you give examples of functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy:
(a) $f(xy) = f(x) \cdot f(y)$ **(b)** $f(xy) = f(x) + f(y)$ **(c)** $f(x + y) = f(x) \cdot f(y)$

(15) Explain to yourself: ε, `eps`, overflow, underflow, cancellation.

(16) Explain to yourself the relations between the notions of:
(non)singular, determinant, inverse matrix, rref, eigenvalues, singular values, independent columns, independent rows, rank, pivots, solution to homogeneous linear system, solution to inhomogeneous linear system, basis.

(17) Assign the following words to either "Square only" or "Square and nonsquare":
determinant, eigenvalue, linear system, least squares problem, matrix norm, SVD, pivot, rank, orthogonal matrix, projection, reflection, rotation, Laplacian, stochastic matrix.