BPhO Computational Physics Challenge: Projectile Motion

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August 11, 2024

Extension Task: Modelling Projectile Motion with Diminishing Air Density with Altitude and Planetary Rotation along a Fixed Axis

Contents

1 Introduction

For this projectile motion model, we will only consider there to be 2 forces acting on the object:

1. Air Resistance
2. Gravitational Force

The vertical and horizontal positions will also be considered to be independent of each other, and can only be related through their expressions for time.

In basic projectile motion, the initial horizontal and vertical velocity can be expressed as:

$$v_x = ucos\theta$$

$$v_y = usin\theta$$

Integrating horizontal with respect to time results in an expression for the displacement of the projectile:

$$x = \int v_x \, dt = ucos\theta t + c$$

Where v_x and v_y are the horizontal and vertical velocities respectively.

An expression for the vertical position requires a double integration from the base downward acceleration of $g = -9.81\ ms^{-1}$:

$$v_y = \int g \, dt = gt + c$$

We know the initial velocity so $c = usin\theta$ and integrating again:

$$y = \int usin\theta + gt \, dt = usin\theta t + \frac{1}{2}gt^2 + c$$

Given that the initial horizontal and vertical displacements will always be set to 0 and h respectively, we can set our values for c for both equations, resulting in:

$$x = ucos\theta t$$
$$y = h + usin\theta t + \frac{1}{2}gt^2$$

Following along the basis of this derivation of a basic projectile motion model, we can achieve something similar with more complex equations, although the final differential equation derived will have to be solved with computational techniques.

2 Deriving the Air Density Equation

2.1 Gravitational Force

In the simplified projectile motion model, we assumed that g was a constant relative to the vertical height y. However this is far from the case:

$$g = \frac{GM_E}{r^2}$$

The distance between the 2 bodies r is the radius of the Earth D, in addition to the vertical altitude y so:

$$g(y) = \frac{GM_E}{(y+D)^2}$$

This can be used to provide more accurate results.

2.2 Density

Firstly, the pressure of the atmosphere, which greatly varies with height, will affect the air resistance acting on the projectile. We can derive an expression for the pressure in terms of the altitude as follows:

$$P = \rho g h$$

We can replace h in this case with the vertical altitude y and g with our function for $g(y)$:

$$P = \frac{\rho GM_E\, y}{(y+D)^2}$$

Assuming an infinitesimally small change in altitude, the density will remain constant relative to the altitude. We can therefore find how a change in altitude changes the pressure by differentiating our function for P in terms of y using the quotient rule:

$$u = \rho GM_E y \qquad v = (y+D)^2$$

$$\frac{d}{dx}[\frac{u(x)}{v(x)}] = \frac{u'v - v_y u}{v^2}$$

$$\frac{dP}{dy} = \frac{\rho GM_E(y+D)^2 - \rho GM_E y(2y+2D)}{[(y+D)^2]^2}$$

$$= \frac{\rho GM_E(y^2 + 2Dy + D^2 - 2y^2 - 2Dy)}{(y+D)^4}$$

$$= \frac{\rho GM_E(D^2 - y^2)}{(y+D)^4}$$

$$= \frac{\rho GM_E(y+D)(D-y)}{(y+D)^4}$$

$$\frac{dP}{dy} = \frac{\rho GM_E(D-y)}{(y+D)^3}$$

We can also relate density and pressure through the Ideal Gas Equation:

$$PV = nRT$$
$$PV = \frac{\rho V}{M} RT$$
$$P = \rho \frac{RT}{M}$$

Differentiating relative to y and treating ρ as a function of y results in:

$$\frac{dP}{dy} = \frac{d\rho}{dy} \frac{RT}{M}$$

Which we can equivalate to our previous value resulting in:

$$\frac{\rho\, GM_E(D-y)}{(y+D)^3} = \frac{d\rho}{dy} \frac{RT}{M}$$

This is a first order linear differential equation we can easily be solved mathematically:

$$\frac{MGM_E(D-y)}{RT(y+D)^3} = \frac{d\rho}{dy} \frac{1}{\rho}$$
$$\int \frac{MGM_E(D-y)}{RT(y+D)^3}\, dy = \int \frac{d\rho}{dy} \frac{1}{\rho}\, dy$$
$$ln(\rho) = \frac{MGM_E}{RT} \int \frac{(D-y)}{(y+D)^3}\, dy$$

We can solve this integral with a u-substitution:

$$\int \frac{(D-y)}{(y+D)^3}\, dy = -\int \frac{y-D}{(y+D)^3}$$

$$u = y + D, \qquad du = dy$$

$$-\int \frac{u-2D}{u^3} du = -\int \frac{1}{u^2} - \frac{2D}{u^3}$$

$$= \frac{1}{u} - \frac{D}{u^2} + c = \frac{u-D}{u^2} + c = \frac{y}{(y+D)^2} + c$$

Resulting in:

$$ln|\rho| = \frac{MGM_E\, y}{RT\, (y+D)^2} + \frac{MGM_E\, c}{RT}$$
$$\rho = B_\rho\, e^{|\frac{MGM_E\, y}{RT\, (y+D)^2}|}$$
$$\rho = B_\rho\, e^{-\frac{MGM_E\, y}{RT\, (y+D)^2}}$$

The constant B_ρ can be solved when $y = 0$ and is the air density at sea level. With the exponential being negative as this is logically what occurs.

2.3 Fluid Drag Force

The fluid drag force is defined as:

$$F_D = \frac{1}{2}\rho v^2 C_D A$$

This needs to be split into the x and y axis forces, and as the frictional force always opposes the velocity vector this results in:

$$F_D(x) = \frac{1}{2}\rho v\, v_x C_D A$$
$$F_D(y) = \frac{1}{2}\rho v\, v_y C_D A$$

Using the previously derived air density equation, this becomes:

$$F_D(x) = \frac{1}{2}B_\rho\, e^{-\dfrac{MGM_E\, y}{RT\,(y+D)^2}}\, v\, v_x C_D A$$
$$F_D(y) = \frac{1}{2}B_\rho\, e^{-\dfrac{MGM_E\, y}{RT\,(y+D)^2}}\, v\, v_y C_D A$$

2.4 Final Equations

Finally, we can formalise a second order differential equation for vertical position:

$$ma' = mg(y) - F_D$$

$$a' = g(y) - \frac{F_D}{m}$$

$$a' = \frac{GM_E}{(y + D^2)} - \frac{\frac{1}{2}B_\rho\, e^{-\dfrac{MGM_E\, y}{RT\,(y+D)^2}}\, \frac{dy}{dt}\, v\, C_D A}{m}$$

$$\frac{d^2y}{dt^2} = \frac{GM_E}{(y + D^2)} - \frac{\frac{1}{2}B_\rho\, e^{-\dfrac{MGM_E\, y}{RT\,(y+D)^2}}\, \frac{dy}{dt}\, v\, C_D A}{m}$$

In the horizontal component of motion, there is only one force acting against the projectile, the fluid drag force. Therefore:

$$\frac{d^2x}{dt^2} = -\frac{F_D}{m}$$

$$\frac{d^2x}{dt^2} = -\frac{\frac{1}{2}B_\rho\, e^{-\dfrac{MGM_E\, y(t)}{RT\,[y(t)+D]^2}}\, \frac{dx}{dt}\, v\, C_D A}{m}$$

Where

$$v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$$

Taylor series expansions could be used in this case as we can assume that given that the projectile will have a real position for every given time period, so the function can be infinitely differentiated to form an expansion in the form:

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 ...$$

The problem of course here being that the coefficients are unknown. However, we can easily find this with the initial conditions. Setting $t = 0$ for every derivative of the function $y(t)$ will unveil a new term. For example:

$$y(0) = a_0 + (a_1 \times 0) + (a_2 \times 0) + (a_3 \times 0) + (a_4 \times 0) = a_0 = h$$

$$\frac{dy}{dt}(0) = a_1 = v' = usin\theta$$

Thus we know that the first 2 coefficients and that the formula for the Taylor Expansion is:

$$\sum_{n=0}^{\infty} \frac{\frac{d^n y}{dt^n}(0)}{n!} t^n$$

So we can continue this as we know the second derivative. Setting $t = 0$ results in:

$$\frac{d^2y}{dt^2}(0) = \frac{GM_E}{(h + D^2)} - \frac{\frac{1}{2} B_\rho \, e^{-\frac{MGM_E \, h}{RT \, (h+D)^2}} \, u^2 \, sin\theta \, C_D A}{m}$$

Therefore giving the first 3 terms to be:

$$y(t) = h + usin\theta t + (\frac{GM_E}{(h + D^2)} - \frac{\frac{1}{2} B_\rho \, e^{-\frac{MGM_E \, h}{RT \, (h+D)^2}} \, u^2 \, sin\theta \, C_D A}{m}) \frac{t^2}{2}$$

This method has the problem of it being incredibly arduous and long to calculate, differentiating increasingly complex formulae. This in addition to the fact that a model for planet rotation will be added, we have decided to simply stick to a Verlet method, assuming a constant acceleration in small time interval steps.

3 Motion in a Rotating Frame of Reference

When considering the motion of a projectile in a rotating reference frame, such as A rotating planet, one must introduce fictional forces such that Newton's Laws of motion can be used. This is on account of the fact that a rotating reference frame's linear velocity constantly changes direction. Therefore any rotating reference frame is inherently non-inertial.

3.1 Modified Equations of Motion

Below is Newton's second law, modified for a rotating reference frame:

$$F' = F - m\frac{d\omega}{dt} \times r' - 2m\omega \times v' - m\omega \times (\omega \times r') = ma' \tag{1}$$

Where F' is the force in the rotating frame, ω is the angular velocity of the rotating frame with respect to the inertial frame, r' is the position vector of the object with respect to the rotating frame, v' is the object's velocity with respect to the rotating frame and a' is the object's velocity with respect to the rotating frame.
Now, consider a spherical planet which rotates (at a slow, constant angular velocity) fixed around the Z axis. For convenience we will place the origin at the centre of the planet. With these conditions in mind, we can simplify Equation 2. Since ω does not change, the term including its derivative will be 0. Furthermore, since the angular velocity of the planet is low, we can chose to ignore the $-m\omega \times (\omega \times r')$ term, as it is second order in ω.
We are now left with the equation below:

$$F - 2m\omega \times v' = ma'$$

The only fictional force that now remains is known as the Coriolis force.
Assuming no air resistance, the only 'real force' acting on the projectile is the force of gravitational attraction. Using Newton's Law of Gravitation yields :

$$F' = G\frac{Mm * \hat{r}}{||r^2||} - 2m\omega \times v' = ma' \tag{2}$$

Solving Equation 2 analytically is likely to be very arduous or even impossible. Therefore using a numerical method was deemed to be more advantageous.

3.2 Numerical Method

Consider a projectile of mass m , velocity v_0 , initial momentum p_0. Furthermore its position is described by the vector, r. At t=0, the projectile is at the surface of a planet which rotates about the Z (upwards) axis at a constant angular velocity of ω. Using Equation 2, we can work out the initial force, F_0 acting upon the projectile when it is still at the earth's surface:

$$F_0 = G\frac{Mm * \hat{r}}{||r^2||} - 2m\omega \times v'$$

Now assuming a small time interval, ΔT, we can calculate the new momentum of the projectile:

$$p_1 = p_0 + F_0 * \Delta T$$

Using this new momentum we can determine the projectile's new position:

$$r_1 = r + \frac{p_1 * \Delta T}{m}$$

Evidently, the new time is:

$$t_1 = t_0 + \Delta T$$

Now, we can calculate the forces currently acting on the projectile after the time interval of ΔT. We can repeat the above process repeatedly to determine the path of the projectile. This was done within the python simulation of projectile motion on a rotating planet produced by our team. A demonstration of this is given within our group's video submission and is available on our website.