


Connections between

- (1) quantum mean values,
- (2) low-degree polynomials, and
- (3) quantum query complexity



# Mean value problem

What is  $\mu$ ? Mean value:  $\mu = \langle 0^n | U^\dagger O U | 0^n \rangle$

$$O = \underbrace{O_1 \otimes \dots \otimes O_n}_{\text{Hermitian}}$$

$U$  constant-depth circuit

MVP: Output  $\tilde{\mu}$  st.  
 $|\tilde{\mu} - \mu| \leq \delta$

Applications to variational algorithms

$$\text{VQE: } H = \sum_i P_i$$

Can classical algorithms solve MVP?

Main theorem

$|\mu|$  in  $2^{n^\epsilon}$  time for  $\epsilon < 1$

# How to prove the main theorem

$$\epsilon < 1$$

There is a  $2^{n^\epsilon}$  time algorithm to estimate  $|\mu|$  up to  $\delta$  error.

$$|\tilde{\mu} - \mu| \leq \delta$$

Quantum query complexity



Low-degree polynomial



Estimate  $|\mu| \leftarrow \mu^2$

# From quantum query complexity to LDP approximating delta function

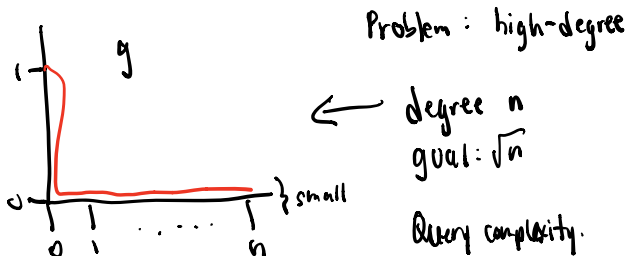
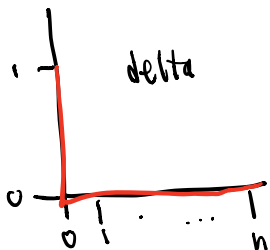
Quantum query complexity

What is the minimum # of queries to  $O: |i, y\rangle \mapsto |i, y \oplus x_i\rangle$   
for a quantum algorithm to solve some function on  $x$ ?

$$x = x_1 \cdots x_n$$

e.g. Grover's Algorithm:  $U_{\sqrt{n}} O_{\sqrt{n}} U_{\sqrt{n}-1} \cdots O_1 U_1 |0^n\rangle$

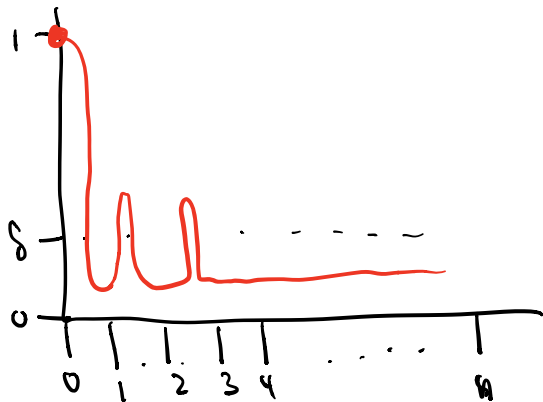
Goal: Obtain a **low-degree** polynomial  $g$  that “looks like” the delta function for inputs  $0, 1, \dots, n$



# (LDP Theorem): LDP approximating delta function

**Theorem:** Let  $0 < \delta < 1/2$ . There is a univariate polynomial  $g : \mathbb{R} \rightarrow \mathbb{R}$  (with coefficients computable in polynomial time)

with degree  $L = O(\sqrt{n \lg(\delta^{-1})})$ ,  $g(0) = 1$ , and  $|g(c)| \leq \delta$  for all  $c = 1, 2, \dots, n$



# LDP Theorem proof (part 1): Query complexity

Goal:  $L = O(\sqrt{n \lg(\delta^{-1})})$ ,  $g(0) = 1$ ,  $|g(c)| \leq \delta$  for all  $c = 1, \dots, n$

Query complexity of one-sided error OR? (A)

$$A(x) = \begin{cases} 0 & \text{if } x = 0^n \\ 1 & \text{w/p } \geq 1-\delta \text{ if } x \neq 0^n \end{cases} \iff 0 \text{ w/p } \leq \delta \text{ if } x \neq 0^n$$

[Buhrman, et al. '99] One-sided error OR algorithm using few queries

for  $\delta > 0$  quantum algo  $T = O(\sqrt{n \lg(1/\delta)})$

$$A(x) = \begin{cases} 0 & \text{if } x = 0^n \\ 1 & \text{w/p } \geq 1-\delta \text{ if } x \neq 0^n \end{cases} \iff 0 \text{ w/p } \leq \delta \text{ if } x \neq 0^n$$

# LDP Theorem proof (part 2): Low query complexity to low-degree polynomial

Goal:  $L = O(\sqrt{n \lg(\delta^{-1})})$ ,  $g(0) = 1$ ,  $|g(c)| \leq \delta$  for all  $c = 1, \dots, n$

[Beals, et al. '98] For any quantum algorithm making  $T$  queries, there is a degree  $2T$  polynomial

$p : \mathbb{R}^n \rightarrow \mathbb{R}$  s.t.  $p(x)$  is the probability of outputting 0 on input  $x$ .

Proof:  $U_T \Theta_T U_{T-1} \dots \Theta_1 U_1 |0^n\rangle$  s.t.  $\Theta |i, y, \vec{z}\rangle = |i, y \oplus x_i, \vec{z}\rangle$

-  $U_i$  is independent of  $x \Rightarrow$  does not factor into  $p$

- Each query to  $\Theta$  increases amplitudes' degrees by 1:

Amplitude of  $|i, 0, \vec{z}\rangle$  in  $\Theta \left( \sum_{i, y, z} \gamma_{i, y, z} |i, y, \vec{z}\rangle \right)$ ?

$$(1 - x_i) \gamma_{i, 0, \vec{z}} + x_i \gamma_{i, 1, \vec{z}}$$

$\Rightarrow$  degree  $T$  amplitudes

Sum up  $|i, 0, \vec{z}\rangle$  for all  $i, z$

2T degree

$p(x)$   
prob. outputting 0.



# Recap: The LDP approximating delta function

Goal:  $L = O(\sqrt{n \lg(\delta^{-1})})$ ,  $g(0) = 1$ ,  $|g(c)| \leq \delta$  for all  $c = 1, \dots, n$

Not obvious:  $p(x)$  only depends on  $|x|$ , so we actually have a  $g$  s.t.  $\underbrace{p(x) = g(|x|)}$

$g$  has degree  $2T$

$$g(0) = 1$$

$$c = 1, \dots, n \quad |g(c)| \leq \delta$$

$$\deg(g) \leq 2T = O(\sqrt{n \lg(1/\delta)})$$



# From LDP approximating delta function to mean value approximation

Theorem: There is a  $2^{n^\xi}$  ( $\xi < 1$ ) time classical algorithm to  $\delta$ -approximate  $|\langle 0^n | U | 0^n \rangle|^2$

Proof (part 1):

- Define  $H = \sum_{j=1}^n U |X_j| U^\dagger$  w/ eigenvectors  $U|x\rangle$ , eigenvalue  $|x|$   $\forall x \in \{0,1\}^n$

- (Spectral decomposition)  $= \sum_{x \in \{0,1\}^n} |x| U|x\rangle\langle x| U^\dagger$

Claim:  $g(H)$  is  $\delta$ -close  $U|0^n\rangle\langle 0^n|U^\dagger$  (spectral norm)

$$\|g(H) - U|0^n\rangle\langle 0^n|U^\dagger\|$$

$$\left\| \sum_x g(|x|) U|x\rangle\langle x| U^\dagger - U|0^n\rangle\langle 0^n|U^\dagger \right\| \leq \left\| \underbrace{\sum_{x \neq 0^n} \delta U|x\rangle\langle x| U^\dagger}_{\text{cancel}} + \cancel{U|0^n\rangle\langle 0^n|U^\dagger - U|0^n\rangle\langle 0^n|U^\dagger} \right\|$$

$$\leq \delta \Rightarrow |\langle 0^n | g(H) | 0^n \rangle - \langle 0^n | U | 0^n \rangle \langle 0^n | U^\dagger | 0^n \rangle| \leq \delta$$

# How to efficiently estimate the mean value

Recall  $g$  is degree  $L = O(\sqrt{n \lg(1/\delta)})$

$$\langle 0^n | g(H) | 0^n \rangle = \sum_{r=0}^L c_r \langle 0^n | H^r | 0^n \rangle$$

$$= \sum_{r=0}^L c_r \langle 0^n | U \left( \sum_{j=1}^n \|X_j\| \right)^r U^\dagger | 0^n \rangle$$

↓ definition of  $H$

$$(\|X_1\|_1 + \dots + \|X_n\|_1)^r$$

$$\|X\|_1 \circ \|X\|_2 = \|X\|_{12}$$

$n^r$  terms of the form

$$\langle 0^n | U \|1 \dots \|X_1 \dots \|X_S U^\dagger | 0^n \rangle$$

$$S \subseteq [n] \\ |S| \leq r$$

depth of  $U$

$$U \|1 \dots \|X_1 \dots \|X_S U^\dagger$$

$$2^d \cdot |S| \text{ qubits} \leq 2^d \cdot r \text{ qubits}$$

$$2^{O(2^d \cdot r)}$$

time to calculate

# What is the time complexity of mean value estimation?

Need to compute  $\langle 0^n | g(H) | 0^n \rangle = \sum_{r=0}^L \sum_{\substack{S \subseteq [n] \\ |S| \leq r}}^{e_r} \langle 0^n | U (|1 \dots 1 X 1 \dots 1\rangle_S) U^\dagger | 0^n \rangle$

$n^r \leq n^L$        $2^{O(2^d \cdot r)} \leq 2^{O(2^d \cdot L)}$

$2^{n^2} \quad \epsilon \approx 1/2$

in time:  $L \cdot n^L \cdot 2^{O(2^d \cdot L)}$

$O(1) \Rightarrow 2^d = O(1)$

$$L = O(\sqrt{n \lg(1/\delta)})$$

$$L \cdot n^{O(\sqrt{n \lg(1/\delta)})} \cdot 2^{O(2^d \cdot \sqrt{n \lg(1/\delta)})}$$

$\rightarrow 2^{\tilde{O}(\sqrt{n \lg(1/\delta)})}$

# Conclusion and Open Problem

(8)

1) Quantum query complexity: Compute OR using few queries and low one-sided error

2) Low-degree polynomial approximating delta function:  $g(0)=1$   $|g(c)| \leq \delta \quad \forall c=1, \dots, n$

3) Estimate  $n^2$  in subexponential time: using  $g(H) = \sum_x g(|x|) U|x\rangle\langle x| U^\dagger$

Open Problem: Can we decide  $\text{sign}(n)$  in subexponential time?