Connections between
(1) quantum mean values,
(2) low-degree polynomials, and
(3) quantum query complexity

Sergey Bravyi, David Gosset, Ramis Movassagh. *Classical algorithms for quantum mean values*. arxiv:1909.11485 (2019)

Mean value problem

What is μ? Mean value: $\mu = \langle 0^n | U^\dagger O U | 0^n \rangle$

$O = \underbrace{O_1 \otimes \cdots \otimes O_n}_{\text{Hermitian}}$ U constant-depth circuit

MVP: Output $\tilde{\mu}$ s.t. $|\tilde{\mu} - \mu| \leq \delta$

Applications to variational algorithms

VQE: $H = \sum_i P_i$

Can classical algorithms solve MVP?

Main theorem

$|\mu|$ in 2^{n^ε} time for $\varepsilon < 1$

How to prove the main theorem

$\varepsilon < 1$

There is a 2^{n^ϵ} time algorithm to estimate $|\mu|$ up to δ error.

$|\tilde{\mu} - \mu| \leq \delta$

Quantum query complexity

$\downarrow$

Low-degree polynomial

$\downarrow$

Estimate $|\mu| \leftarrow \mu^2$

From quantum query complexity to LDP approximating delta function

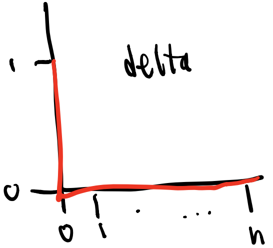
Quantum query complexity

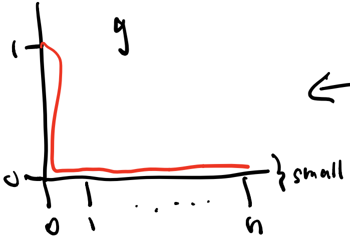
What is the minimum # of queries to $O: |i,y\rangle \mapsto |i, y \oplus x_i\rangle$ $\quad x = x_1 \cdots x_n$

for a quantum algorithm to solve some function on x?

e.g. Grover's Algorithm: $U_{\sqrt{n}} O_{\sqrt{n}} U_{\sqrt{n}-1} \cdots O_1 U_1 |0^n\rangle$

Goal: Obtain a **low-degree** polynomial g that "looks like" the delta function for inputs $0, 1, \ldots, n$





Problem: high-degree

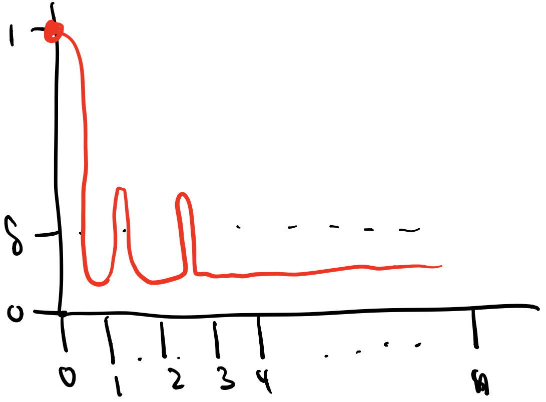
degree n

goal: $\sqrt{n}$

Query complexity.

(LDP Theorem): LDP approximating delta function

Theorem: Let $0 < \delta < 1/2$. There is a univariate polynomial $g : \mathbb{R} \to \mathbb{R}$ (with coefficients computable in polynomial time) with degree $\boxed{L = O(\sqrt{n}\lg(\delta^{-1}))}$, $\boxed{g(0) = 1}$, and $\boxed{|g(c)| \leq \delta}$ for all $c = 1, 2, ..., n$



LDP Theorem proof (part 1): Query complexity

Goal: $L = O(\sqrt{n \lg(\delta^{-1})})$, $g(0) = 1$, $|g(c)| \leq \delta$ for all $c = 1, \ldots, n$

Query complexity of one-sided error OR? (A)

$$A(x) = \begin{cases} 0 & \text{if } x = 0^n \\ 1 \text{ w/p} \geq 1-\delta & \text{if } x \neq 0^n \end{cases} \iff 0 \text{ w/p} \leq \delta \text{ if } x \neq 0^n$$

[Buhrman, et al. '99] One-sided error OR algorithm using few queries

for $\delta > 0$ quantum algo $T = O(\sqrt{n \lg(1/\delta)})$

$$A(x) = \begin{cases} 0 & \text{if } x = 0^n \\ 1 \text{ w/p} \geq 1-\delta & \text{if } x \neq 0^n \end{cases} \iff 0 \text{ w/p} \leq \delta \text{ if } x \neq 0^n$$

LDP Theorem proof (part 2): Low query complexity to low-degree polynomial

Goal: $L = O(\sqrt{n \lg(\delta^{-1})})$, $g(0) = 1$, $|g(c)| \le \delta$ for all $c = 1, \ldots, n$

[Beals, et al. '98] For any quantum algorithm making T queries, there is a degree $2T$ polynomial $p : \mathbb{R}^n \to \mathbb{R}$ s.t. $p(x)$ is the probability of outputting 0 on input x.

$p(x)$
prob. outputting 0.

Proof: $U_T O_T U_{T-1} \cdots O_1 U_1 |0^n\rangle$ s.t. $O|i, y, \vec{z}\rangle = |i, y \oplus x_i, \vec{z}\rangle$

- U_i is independent of $x \Rightarrow$ does not factor into p
- Each query to O increases amplitudes' degrees by 1:

Amplitude of $|i, 0, \vec{z}\rangle$ in $O\left(\sum_{i,y,z} \alpha_{i,y,z} |i, y, \vec{z}\rangle\right)$?

$(1 - x_i)\alpha_{i,0,\vec{z}} + x_i \alpha_{i,1,\vec{z}}$

$\Rightarrow$ degree T amplitudes

sum up amplitudes $|i, 0, z\rangle$ for all i, z

2T degree

Recap: The LDP approximating delta function

Goal: $L = O(\sqrt{n \lg(\delta^{-1})})$, $g(0) = 1$, $|g(c)| \le \delta$ for all $c = 1, ..., n$

Not obvious: $p(x)$ only depends on $|x|$, so we actually have a g s.t. $\underbrace{p(x) = g(|x|)}$

g has degree $2T$

$g(0) = 1$

$c = 1, \dots, n$ $\quad |g(c)| \le \delta$

$\deg(g) \le 2T = O(\sqrt{n \lg(1/\delta)})$

From LDP approximating delta function to mean value approximation

Theorem: There is a 2^{n^ε} $(\varepsilon < 1)$ time classical algorithm to δ-approximate $|\langle 0^n | U | 0^n \rangle|^2$

Proof (part 1):

- Define $H = \sum_{j=1}^{n} U |1\rangle\langle 1|_j U^\dagger$ w/ eigenvectors $U|x\rangle$, eigenvalue $|x|$ $\forall x \in \{0,1\}^n$

- (Spectral decomposition) $= \sum_{x \in \{0,1\}^n} |x| \, U|x\rangle\langle x| U^\dagger$

Claim: $g(H)$ is δ-close $U|0^n\rangle\langle 0^n|U^\dagger$ (spectral norm)

$$\| g(H) - U|0^n\rangle\langle 0^n|U^\dagger \|$$

$$\| \sum_x g(|x|) U|x\rangle\langle x|U^\dagger - U|0^n\rangle\langle 0^n|U^\dagger \| \le \| \underbrace{\sum_{x \neq 0^n} \delta \, U|x\rangle\langle x|U^\dagger}_{} + \cancel{U|0^n\rangle\langle 0^n|U^\dagger} - \cancel{U|0^n\rangle\langle 0^n|U^\dagger} \|$$

$$\le \delta \Rightarrow | \langle 0^n | g(H) | 0^n \rangle - \langle 0^n | U |0^n\rangle\langle 0^n| U^\dagger |0^n\rangle | \le \delta$$

How to efficiently estimate the mean value

Recall g is degree $L = O(\sqrt{n \lg(1/\delta)})$

$$(\|X\|_1 + \cdots + \|X\|_n)^r$$

$$\langle 0^n | g(H) | 0^n \rangle = \sum_{r=0}^{L} c_r \langle 0^n | H^r | 0^n \rangle$$

$\downarrow$ definition of H

$$= \sum_{r=0}^{L} c_r \underbrace{\langle 0^n | U \left(\sum_{j=1}^{n} \|X\|_j \right)^r U^\dagger | 0^n \rangle}$$

$$\|X\|_1 \cdot \|X\|_2 = \|X\|_{12}$$

n^r terms of the form $\boxed{\langle 0^n | U \, \| \cdots \| X \| \cdots \|_S \, U^\dagger | 0^n \rangle}$

$S \subseteq [n]$

$|S| \leq r$

depth of U

$$\underbrace{U \, \| \cdots \| X \| \cdots \|_S \, U^\dagger}$$

$2^d \cdot |S|$ qubits $\leq 2^d \cdot r$ qubits

$2^{O(2^d \cdot r)}$ time to calculate

What is the time complexity of mean value estimation?

Need to compute $\langle 0^n | g(H) | 0^n \rangle = \underbrace{\sum_{r=0}^{L}} \underbrace{\sum_{\substack{S \subseteq [n] \\ |S| \le r}}}_{n^r \le n^L} \underbrace{\overset{e_r}{\langle 0^n | U (|1\cdots1\rangle\langle 1\cdots1|_S) U^\dagger | 0^n \rangle}}_{2^{O(2^d \cdot r)} \le 2^{O(2^d \cdot L)}}$

$\boxed{2^{n^\varepsilon} \quad \varepsilon \approx 1/2}$

in time: $L \cdot n^L \cdot 2^{O(2^d \cdot L)}$

$O(1) \Rightarrow 2^d = O(1)$

$L \cdot n^{O(\sqrt{n \lg(1/\delta)})} \cdot 2^{O(2^d \cdot \sqrt{n \lg(1/\delta)})}$

$2^{\tilde{O}(\sqrt{n \lg(1/\delta)})}$

$L = O(\sqrt{n \lg(1/\delta)})$

Conclusion and Open Problem

1) Quantum query complexity: Compute OR using few queries and low one-sided error (δ)

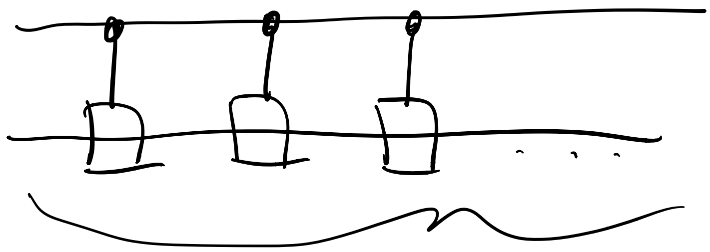
2) Low-degree polynomial approximating delta function: $g(0)=1 \quad |g(c)|\le\delta \quad \forall c=1,\dots,n$

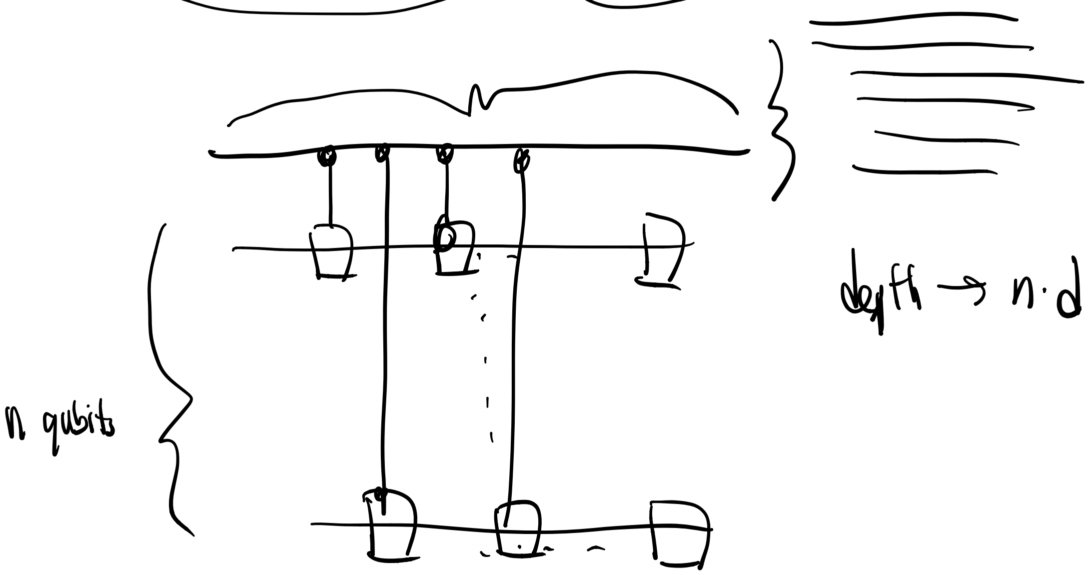
3) Estimate μ^2 in subexponential time: using $g(H)=\sum_x g(|x|)\, U|x\rangle\langle x|U^\dagger$

Open Problem: Can we decide $\mathrm{sign}(\mu)$ in subexponential time?

$\langle 0^n | U | 0^n \rangle$

$n=1$





depth $\to n \cdot d$