Code Swendsen-Wang Dynamics

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An important open question about Markov chains for preparing quantum Gibbs states is proving rapid mixing. However, rapid mixing at low temperatures was only proved for Gibbs states with no thermally stable phases such as the 2D toric code. Inspired by Swendsen-Wang dynamics, in this work we give a simple, new Markov chain for preparing Gibbs states of commuting Hamiltonians. We prove rapid mixing of this chain for a class of quantum code Hamiltonians with thermally stable phases, including the 4D toric code, at any temperature. We conjecture its efficiency for all code Hamiltonians away from first-order phase transition points.

Contents

1 Introduction

Recently, there have been tremendous advancements in algorithms for quantum Gibbs sampling—the task of preparing quantum Gibbs states. The main open problem is establishing rapid mixing of these chains, thereby establishing efficiency of the Markov chain.

At high temperatures, recently proposed algorithms based on (quasi-)local dynamics do rapidly mix to the Gibbs state [KB16; BK19; YL23; BLMT24; RFA25]. However, as the temperature decreases, many systems cross a phase transition at a critical temperature, below which local dynamics mix only after exponentially long times. This slowdown can be attributed to the emergence of thermally stable phases, or modes of the Gibbs state, separated by *energy barriers* which local dynamics cannot traverse (except with exponentially small probabilities) [AFH09; GKZ24; RPBK24].

In this paper, we introduce a quantum Markov chain for Gibbs states of commuting Hamiltonians that *do* exhibit energy barriers. Furthermore, we show that it mixes quickly for code Hamiltonians of codes with an approximate "graphic" or "cographic" representation, such as certain good LDPC codes and 2D and 4D toric codes. This gives a simpler, discrete alternative to the 2D toric code Markov chain in [DLLZ24] and is the first Markov chain that rapidly mixes to the 4D toric code from any initial configuration.

Our starting point is the Swendsen-Wang (SW) dynamics for the 2D Ising model. The Ising model has two ground states: one with all $+1$-spins and one with all -1-spins. When one runs local (Glauber) dynamics on this model at low temperatures, one is met with a severe bottleneck: it takes an exponential amount of time to traverse from predominantly $+1$ spins to predominantly -1 spins and vice-versa [BCK+99]. In 1987, Swendsen and Wang [SW87] introduced a Markov chain for the 2D Ising model which departed significantly from local updates and instead updated *clusters*. Clusters are connected subgraphs of the 2D lattice and capture the correlations between the spins. Every spin configuration σ defines a subgraph $E(\sigma)$ of the 2D lattice given by those edges of the lattice on which the spins align, defining clusters. The key idea of the Swendsen-Wang algorithm is to alternate between updates of spin configurations and updates of cluster configurations:

- *Cluster formation:* Given a spin configuration σ, clusters are formed by iid. removing edges in $E(\sigma)$ with probability $e^{-2\beta}$. The connected components of the resulting sublattice are the clusters.

- *Cluster update:* Given a cluster configuration, the new spin configuration is sampled by flipping an unbiased coin for every cluster and aligning the spins in each cluster accordingly.

Remarkably, this Markov chain mixes to the Gibbs distribution of the Ising model *and* faces no obstruction to crossing energy barriers. At low temperatures, clusters grow rapidly. Since all spins within each cluster are resampled in every step of the algorithm (cluster update step), once large clusters emerge, the barrier between the modes of the distribution centered at the all $+1$-spins configuration and the all -1-spins configuration is traversable in a *single* step. Consequently, the SW algorithm mixes rapidly for Ising models on any graph [GJ17; Ull14].

The focus of this paper is the formulation of a natural generalization of the SW dynamics to commuting Pauli Hamiltonians. In what follows, we will principally describe this chain—the *code Swendsen-Wang (SW)* dynamics—for quantum CSS codes. The goal of the algorithm is then to prepare the Gibbs state $\rho_\beta \propto e^{-\beta H}$ for the Hamiltonian

$$H = -\sum_{A \in \text{X checks}} X_A - \sum_{A \in \text{Z checks}} Z_A, \tag{1}$$

where X_A denotes the Pauli-X operator supported on the qubits in A, and likewise for Z_A. The key observation of the quantum chain is that a measurement of the stabilizers projects the system into an eigenstate of H. What remains is to switch into an eigenstate whose eigenvalue is distributed according to the correct (Gibbs) distribution. To achieve this, in each step of the quantum code SW chain, we measure X and Z stabilizers and apply complementary Z and X errors drawn from two classical Markov chains whose stationary distributions are Gibbs distributions

$$\pi(x) \propto e^{-\beta H(x)} \tag{2}$$

of the classical X and Z code, each with energy function

$$H(x) = -\sum_{A \in \text{checks}} (-1)^{\sum_{i \in A} x_i}. \tag{3}$$

Because the two sectors evolve independently through the course of the quantum Markov chain, the mixing time of the quantum chain is determined by the slower of the two classical chains. It thus remains to find classical Markov chains for the classical code Hamiltonians that mix rapidly for systems of interest. This is achieved by a generalization of the Ising SW dynamics to binary linear codes.

(Classical) code SW dynamics Like the SW dynamics, code SW dynamics iterates updates of error configurations and (the code analogue of) cluster configurations.

- *Cluster formation:* Given an error configuration x, let $E(x)$ be the checks satisfied by x. Remove checks from $E(x)$ iid. with probability $e^{-2\beta}$, resulting in $E(x)'$.
- *Cluster update:* Sample a new error configuration x' by sampling a codeword from the parity check matrix defined by $E(x)'$, i.e., sample x' uniformly s.t. $\langle x', A\rangle = 0$ for all $A \in E(x)'$.

We show that the stationary distribution of this chain is given by Eq. (2). In fact, the SW dynamics for the Ising model is a special case of the code SW dynamics where the interactions are restricted to be pairwise: in Eq. (3), let $H(x)$ be the Ising model Hamiltonian, where each *check* is a pairwise check between adjacent vertices in the interaction graph. Then in the cluster formation step of the original SW dynamics, a *cluster* is now just one *dimension* of the (repetition) codespace defined by $E(x)'$, and the cluster update step outputs a random linear combinations of these dimensions.

To prove polynomial mixing of this dynamics, we extend the approach of Guo and Jerrum [GJ17] and Ullrich [Ull14] for rapid mixing of the original SW dynamics on the Ising model. Their approach for solving this longstanding open problem relies on a series of three closely related Markov chains, propagating the proof of rapid mixing from the last of these chains to the first:

1. *SW dynamics on the RC model*: Consider the original SW chain where we focus our attention not on the spin configurations but on the *edge* configurations. The stationary distribution of this chain is a distribution over subgraphs called the *random cluster* (RC) model with probability distribution $\phi(S) \propto p^{|S|}(1-p)^{|E \setminus S|}2^{k(S)}$ for $S \subset E$ and $p = 1-e^{-2\beta}$, where $k(S)$ is the number of connected components of the subgraph S.
2. *Single-bond dynamics on the RC model*: The Markov chain which picks a random edge and resamples it according to the Metropolis rule for ϕ. This can be seen as a "slowed down" version of SW dynamics which updates a single, random edge rather than all edges.

3. *JS chain on the even-subgraph model*: The Jerrum-Sinclair (JS) chain efficiently samples from the distribution supported on even subgraphs $B \subset E$ (subgraphs where every vertex has even degree) proportional to $\xi(B) \propto p^{|B|}(1-p)^{|E \setminus B|}$. This chain works by expanding the state space to include subgraphs with two defects (vertices with odd degree).

Ullrich [Ull13] showed that rapid mixing for single-bond dynamics implies rapid mixing for the SW dynamics on the RC model and therefore also the Ising model. Guo and Jerrum [GJ17] gave an argument in which they "lift" the canonical paths from the rapid mixing proof of the JS chain [JS93] to show rapid mixing of the single-bond dynamics.

We can directly follow this proof for linear codes which have a *graphic* form [Oxl11] to show rapid mixing in this case. This covers the 2D toric code (which has no energy barrier) and certain LDPC codes without phase transitions (including some good LDPC codes). However, the 4D toric does not have such a graphic form and the analogue of the JS chain acts on more complex, high-dimensional objects for which the canonical path argument fundamentally breaks down. To show rapid mixing in this case, we use a different coupling that couples the RC model to the distribution of syndromes of the Gibbs state. The syndromes are themselves codewords of another code with a *dual* parity check matrix. For the 4D toric code this dual parity check is sufficiently *close* to being graphic and we can again lift the JS paths to show rapid mixing.

Altogether, using the correspondence between the quantum and classical chains, and the series of classical Markov chains just described, we can prove that our Gibbs sampling algorithm is efficient at all temperatures for commuting Pauli Hamiltonians with positive coefficients whose classical checks are close to being graphic or cographic (i.e., the dual is close to graphic). In fact, we conjecture that it is efficient at all temperatures for all classical and quantum stabilizer codes other than around first-order phase transition points.

Discussion and related work Sampling from commuting Hamiltonians, and in particular from the 2D toric code, has already been investigated in recent, as well as older work. Already in 2009, Alicki, Fannes, and Horodecki [AFH09] showed that a local Davies generator for the 2D toric code thermalizes quickly in a time upper bounded by $e^{O(\beta)}$, a scaling that matches the results for Glauber dynamics of the 1D Ising model [LPW09]. Improving the scaling in β, recently, Ding et al. [DLLZ24] showed that a nonlocal Davies generator for the 2D toric code on n qubits thermalizes in time at most $\min\{e^{O(\beta)}, \mathsf{poly}(n)\}$, leading to an efficient, nonlocal Gibbs sampler at arbitrary low temperatures. With an entirely different algorithmic approach based on a reduction to classical Gibbs sampling for 2-local Hamiltonians—that is different from our approach—Hwang and Jiang [HJ25] showed that the Gibbs states of certain commuting Hamiltonians, including the 2D toric code can be prepared efficiently at arbitrary temperatures. However, none of these works have been able to generalize their results to the 4D toric code.

Recently, Bergamaschi, Gheissari, and Liu [BGL25] showed that certain quasi-local block dynamics mix quickly *within* a logical sector of the 4D toric code, but require exponential time to traverse its energy barrier. When applied to a specific input state—the maximally mixed state on the code space—this algorithm can be used to efficiently prepare the full Gibbs state of the 4D toric code. In contrast, the code SW algorithm rapidly converges to the global Gibbs distribution from any initial state.

The code SW algorithm is also closely related to the study of approximation algorithms for the Tutte polynomial of binary matroids [JS09; Oxl11]. The mixing-time proof of Guo and Jerrum [GJ17] implies approximation algorithms for graphic as well as cographic matroids using duality properties of the Tutte polynomial. But while this algorithm just used the original SW algorithm for graphs, our algorithm shows how to explicitly sample from the random cluster and Gibbs distributions associated with a matroid. Our mixing-time bounds further imply extensions of those approximation results to matroids that are close to being graphic or cographic.

2 Technical overview

In this section, we give a more detailed overview of the key ideas in our proof of rapid mixing for the code SW dynamics. We will then give the full proofs of correctness and rapid mixing in the subsequent sections, Section 3 and Section 4, respectively.

Consider the series of three Markov chains used in the proof of rapid mixing of the original SW dynamics for the Ising model described above. What changes when we generalize the dynamics to arbitrary linear codes via the code SW dynamics? Suppose we have a parity check matrix $h \in \{0,1\}^{c\times n}$ and run the SW dynamics on h as defined in the introduction. In the case of the code SW dynamics, we can define three related Markov chains on the space of subsets of the *checks* indexed by $E := [c]$ as follows:

1. *Code SW dynamics on the RC model*: Consider the code SW chain where we focus our attention not on the spin configurations but on the *check* configurations. The stationary distribution of this chain is a *random cluster* model on subsets of checks $S \subset E$ distributed as $\phi(S) \propto p^{|S|}(1-p)^{|E\setminus S|}2^{k(S)}$ with $p = 1 - e^{-2\beta}$. Here, $k(S)$ is the dimension of the code defined by the checks in S.

2. *Single-check dynamics on the RC model*: The Markov chain which picks a random check and resamples it according to the Metropolis rule for ϕ.

3. *"Code JS" chain on the even cover model*: Consider the chain that attempts to sample from the distribution $\xi(B) \propto p^{|B|}(1-p)^{|E\setminus B|}$ supported on *even covers* $B \subset E$, that is, subsets of checks where every variable is incident to an even number of checks. This chain works by expanding the state space to include covers with the minimal number of defects needed to traverse between any two even covers.

Ullrich's proof for the Ising model [Ull14] directly extends to our setting and implies that the code SW dynamics on the RC model mixes faster than the single-check dynamics. Furthermore, if there is a good set of canonical paths for the even-cover model, then, by Guo and Jerrum's proof [GJ17], the single-check dynamics mixes rapidly. Thus, to assess when this proof strategy works we must examine the rapid mixing of the code JS chain.

The original JS chain on a graph with edges E works as follows: Given any state $B \subset E$ with at most two defects, the chain randomly flips single edges while maintaining the invariant of at most *two* defects in the intermediate subgraphs. Because every pair of even subgraphs can be connected via disjoint loops, canonical paths can be defined by unwinding and creating loops with at most two defects. Crucially, the number of subgraphs with only two defects is polynomially related to the number of even subgraphs, so this Markov chain is rapidly mixing.

Unfortunately, when considering even covers of arbitrary codes, the higher-dimensional connectivity poses a fundamental obstacle: the state space may need to be expanded by an exponential factor. As an example, consider the 4D toric code, where bits lie on faces of the 4D torus and checks are associated to edges, i.e., each edge checks its four incident faces. In this setting, the even covers are unions of 3D volumes and to traverse from one even cover to another using single-check flips we must unwind these volumes. In doing so, we necessarily pass through states with an extensive number of defects which means that the state space must expand by an exponential factor, thereby invalidating the proof strategy.

Our insight is that although there is a coupling between the RC model and the even-cover model (chains 2 and 3 above), which we call the *primal* view, there is yet another *dual* view which couples the RC model to the *syndrome distribution* of the Gibbs state. The cluster formation step of the SW dynamics defines this coupling: In this step, we sample a set of violated checks (syndromes) according to the Gibbs distribution, then add more violated checks independently with probability $e^{-2\beta}$ and output the set of *satisfied* checks. This is a sample from the RC model.

The syndromes form another linear subspace, captured by a dual parity check matrix. We can now define an analogous code JS chain on the syndrome distribution, replacing the code JS dynamics on the even covers.

For the 4D toric code, in this dual view, it turns out that even covers can be unwound efficiently, giving a good set of canonical paths. To see this, consider the edge-vertex incidence matrix g of the 4D torus and let h be the (edges by faces) parity check matrix for the 4D toric code. Then $g^T h = 0$ because g^T and h are boundary maps from $1 \to 0$- and $2 \to 1$-dimensional objects, respectively, and the boundary of a boundary is zero. But this condition also implies that $\mathrm{col}(h)$, or the set of syndromes of the code, is contained in the set of all even subgraphs of the 4D torus. As long as the difference between the dimension of the syndrome space $\mathrm{col}(h)$ and the dimension of the even-subgraph space $\ker(g^T)$ is not too big, which is the case for the 4D toric code, we obtain a good set of canonical paths.

We can then lift these canonical paths on the syndrome distribution to a flow on the RC model, analogously to Guo and Jerrum [GJ17]. This flow then implies rapid mixing of the single-check dynamics [JS93]. Then, the code SW dynamics also mix rapidly by an analogous argument to Ullrich [Ull14].

Theorem 1. *The code SW algorithm for the 4D toric code mixes rapidly at any temperature.*

In fact, we are able to show rapid mixing of the code SW chain for a larger class of codes, using the notion of a *graphic* or *cographic* parity check matrix. A parity check matrix is graphic if the linear dependencies of its rows are captured by a graph, i.e., $\ker(h^T) = \ker(g^T)$ for the edge-vertex incidence matrix g of a graph. It is cographic if there is a graphic (column-)generator matrix for the orthogonal complement $\mathrm{col}(h)^\perp$ of the column space $\mathrm{col}(h)$ of h. We can relax this notion to parity checks that are *close* to being graphic or cographic. We say that h is Δ-(co)graphic if at most Δ columns need to be added to h (its dual) to make it graphic.

Theorem 2 (Rapid mixing for Δ-graphic or Δ-cographic codes)**.** *Given a parity check matrix h, the code SW algorithm for the Gibbs distribution of h mixes in time $2^\Delta \cdot \mathsf{poly}(n)$ at any temperature if h is Δ-graphic or Δ-cographic.*

Examples of quantum codes that satisfy the conditions of Theorem 2 are:

- the 2D toric code. The graph defined by their parity check is a cycle graph.
- codes that arise from any chain complex on a two-dimensional surface, i.e. parity check h which has the property that $g^T h = 0$ for the edge-vertex incidence matrix g of a graph, e.g., the X and Z checks of the 4D toric code.
- codes whose parity checks have no linear dependencies such as certain good LDPC codes [HGL25; PRBK24]. The graph defined by their parity check is a line graph.

3 Quantum Markov chain for code Hamiltonians

In this section, we describe a reduction for sampling Gibbs states of Hamiltonians with positive-coefficient and commuting Pauli terms (stabilizer Hamiltonians) to sampling from the classical Gibbs distribution of a corresponding linear code. We show that the mixing time of the quantum Markov chain is upper bounded by that of a classical Markov chain for the classical Gibbs distribution.

3.1 Preliminaries: Codes and their Gibbs states

3.1.1 Quantum and classical codes

The *linear code* defined by the parity-check matrix $h \in \{0,1\}^{c\times n}$ is given by $C_h = \ker(h)$. Let $P_1, \ldots, P_c \in \mathcal{P}_n$ be a set of commuting Pauli operators in the n-qubit Pauli group $\mathcal{P}_n$, i.e., $[P_i, P_j] = 0$ for all i, j, and let $\mathcal{S} = \langle P_1, \ldots, P_c\rangle \le \mathcal{P}_n$ be the stabilizer group generated by $P_1, \ldots, P_c$ with $\dim(\mathcal{S}) = n - k$. The *stabilizer code* associated with $\mathcal{S}$ is given by

$$C_{\mathcal{S}} = \{|\psi\rangle \in (\mathbb{C}_2)^{\otimes n} : S\,|\psi\rangle = |\psi\rangle\,, \forall S \in \mathcal{S}\}. \tag{4}$$

A convenient representation of Pauli operators is the *stabilizer matrix representation.* Let X, Y, Z be single-qubit Pauli operators and X_i be the Pauli-X operator acting on qubit i. Let $X(x) = \prod_{i\in[n]} X_i^{x_i}$ and likewise for $Z(z)$ with $x, z \in \{0,1\}^n$. We can thus represent every Pauli operator by a string $p = (x, z) \in \{0,1\}^{2n}$ as $P(p) = i^{x\cdot z} X(x)Z(z)$. For example, for a single qubit, $P(0,0) = \mathbb{1}$, $P(1,0) = X$, $P(0,1) = Z$ and $P(1,1) = Y$. The (multiplicative) Pauli group $\mathcal{P}_n$ without phases is therefore equivalent to the (additive) group $\mathbb{Z}_2^{2n}$ in the sense that for $x, y \in \{0,1\}^{2n}$, $P(x+y) \propto P(x)Q(y)$. The commutativity of Pauli operators is captured by the symplectic form ω defined as follows

$$[P(x), P(y)] = 0 \Leftrightarrow \omega(x,y) := x^T \omega y = 0, \quad \omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \tag{5}$$

Given a stabilizer $\mathcal{S}$ generated by Pauli operators $P_1 = P(g_1), \ldots, P_c = P(g_c)$ for $g_i \in \{0,1\}^{2n}$, we can therefore represent $\mathcal{S}$ by the *stabilizer matrix* g with rows g_i as $\mathcal{S} = \mathcal{P}(g) := \langle P(g_1), \ldots, P(g_c)\rangle$. In fact, by linearity, only the row space of g matters and we have $\mathcal{S} = \mathcal{P}(\mathrm{row}(g))$. Let $h = g\omega$. The commutant of $\mathcal{S}$ is given by $\mathcal{C} = \mathcal{P}(\ker(h))$ and contains both the stabilizer itself and the logical operators $\mathcal{L} = \mathcal{C}/\mathcal{S}$ of the code $C_{\mathcal{S}}$.

We define the logical $\overline{|0\rangle}$ state in terms of the code space projector $\overline{|0\rangle\langle 0|} = 2^{-(n-k)}\sum_{S\in\mathcal{S}} S$. Letting $k = \dim(C_{\mathcal{S}})$, we can index a complete set of logical Pauli operators $\overline{P(l)}$ with X and Z parts $\overline{X(l_x)}$ and $\overline{Z(l_z)}$, respectively, such that $\mathcal{L} = \langle \overline{P(l)}, l \in \{0,1\}^{2k}\rangle$. The *syndrome* of $C_{\mathcal{S}}$ is the column range $S_h = \mathrm{col}(h)$. It is the set of possible measurement outcomes when measuring the Pauli operators corresponding to the rows of h. For every $s \in S$, let $h^{-1}s$ be the set of preimages of s under h. We can associate a representative Pauli error $e_s = \min(h^{-1}s)$ given by the lexicographically first preimage and denote $E(s) = P(e_s)$. It will be important that for any $e \in h^{-1}s$, $e_s + e \in \ker(h)$.

3.1.2 Gibbs states of quantum and classical codes

Given a classical code with parity-check matrix $h \in \{0,1\}^{c\times n}$, we associate the energy function

$$H(x) = 2|hx|, \tag{6}$$

where $|x| = \sum_i x_i$ is the Hamming weight of x. The corresponding Gibbs distribution is the probability distribution

$$\pi(x) \propto e^{-\beta H(x)}. \tag{7}$$

Likewise, given a quantum stabilizer code with stabilizer matrix $g \in \{0,1\}^{c\times 2n}$, we associate the Hamiltonian

$$H = -\sum_{i\in[c]} P(g_i), \tag{8}$$

with Gibbs state

$$\rho_\beta \propto e^{-\beta H}. \tag{9}$$

3.2 Lifting classical code sampling to quantum code sampling

We can now give a Markov chain for Gibbs states of quantum stabilizer codes that lifts sampling from a Markov chain for the Gibbs distribution of error configurations of a certain classical code to a Markov chain for the quantum Gibbs state preparation. We formulate it here for general stabilizer codes. The formulation for CSS codes (with two underlying chains) given in the introduction directly follows from it, observing that in this case, the Pauli-X and Z terms decouple and therefore the Gibbs distribution of errors is a product distribution over Z and X errors, which we can sample from independently. In the more general case, we have to consider X and Z-part of the errors jointly.

To this end, consider a stabilizer code with dimension k defined by the stabilizer matrix $g \in \{0,1\}^{c\times 2n}$. The states

$$\overline{|l,s\rangle} = E(s)\overline{X(l)}\,\overline{|0\rangle} \tag{10}$$

for syndrome $s \in S_g$ and $l \in \{0,1\}^k$ form an orthonormal basis in which

$$H = \sum_{s\in S_g} 2|s|\Pi(g,s) - c\mathbb{1}, \tag{11}$$

is diagonal, with $\Pi(g,s) = \sum_{l\in\{0,1\}^k} \overline{|l,s\rangle\langle l,s|}$. Accordingly, the corresponding Gibbs state is given by

$$e^{-\beta H} \propto \sum_{s\in S_g} e^{-2\beta|s|}\Pi(g,s) \tag{12}$$

Let π be the Gibbs distribution associated with $h \coloneqq g\omega$ at inverse temperature β. We will lift a classical Markov chain Q with stationary distribution π to a quantum Markov chain with fixed point ρ_β.

Algorithm 1 Quantum code Gibbs sampling

Input: Stabilizer matrix g, classical Markov chain Q, initial state $|\phi_0\rangle$, initial error configuration x_0

1: Let $|\phi\rangle \leftarrow |\phi_0\rangle, x \leftarrow x_0$, cutoff time T.
2: Iterate T times:
3: i. Measure the stabilizers $\mathcal{S}$ on $|\phi\rangle$, yielding syndrome $s \in S_g$, and post-measurement state $|\phi_s\rangle$.
ii. Sample $y \leftarrow Q(x,\cdot)$, and apply $P(y)E(s)$, letting $|\phi\rangle \leftarrow P(y)E(s)\,|\phi_s\rangle$, $x \leftarrow y$.

Output: $|\phi\rangle$

Theorem 3. *The outputs $|\phi\rangle$ sampled from Algorithm 1 satisfy*

$$\lim_{T\to\infty} \mathop{\mathbb{E}}_{\phi} |\phi\rangle\langle\phi| \propto e^{-\beta H}, \tag{13}$$

for any initial state $|\phi_0\rangle$.

Proof. To show that Algorithm 1 converges to the correct distribution, observe that as $T\to\infty$, $Q^T(x_0,\cdot) \to \pi$ for any initial state $x_0 \in \{0,1\}^{2n}$. Therefore, to show correctness, we can assume that $y \leftarrow \pi$ is sampled from π.

We begin by observing that the post-measurement state is given by an erroneous logical state $E(s)\overline{|\varphi\rangle}$ with $\overline{|\varphi\rangle} \in \mathrm{span}_{l\in\{0,1\}^k}(\overline{|l,0\rangle})$ for some $s \in S_g \subset \{0,1\}^c$.

Now, observe that a sample $y \leftarrow \pi$ can be decomposed as $y = d + e + f$ in terms of a logical Pauli, a representative error, and a stabilizer represented by $d, e, f \in \{0,1\}^{2n}$, respectively. By basic stabilizer calculus, the string y has the property that

$$P(y)\overline{|0\rangle} \propto \overline{|l, s\rangle}, \tag{14}$$

up to a global phase for a syndrome $s = he$ and a logical $l \in \{0,1\}^{2k}$ such that $P(d) = \overline{P(l)}$. In particular, observe that the distribution π only depends on the syndrome of the error e and therefore l and f are uniformly random.

Applying $E(s)$ moves the state to the logical subspace, i.e., the 0-syndrome subspace. Applying $P(y)$ then maps to a syndrome subspace sampled according to the Gibbs distribution and twirls the logical state, giving

$$\begin{aligned} \rho &\propto \sum_{s \in S_g} e^{-2\beta|s|} E(s) \left[\mathbb{E}_{l \in \{0,1\}^{2k}} \overline{P(l)}\, \overline{|\varphi\rangle\langle\varphi|}\, \overline{P(l)} \right] E(s) && (15)\\ &= \frac{1}{2^k} \sum_{s \in S_g} e^{-2\beta|s|} E(s) \Pi(g,0) E(s) && (16)\\ &= \frac{1}{2^k} \sum_{s \in S_g} e^{-2\beta|s|} \Pi(g,s) \propto e^{-\beta H} && (17) \end{aligned}$$

Here, we have used that twirling any state over the Pauli group yields a maximally mixed state and therefore

$$\mathbb{E}_{l \in \{0,1\}^{2k}} \overline{P(l)}\, \overline{|\varphi\rangle\langle\varphi|}\, \overline{P(l)} = \frac{1}{\operatorname{tr}[\Pi(g,0)]} \Pi(g,0) = \frac{1}{2^k} \Pi(g,0) \tag{18}$$

for any logical state $\overline{|\varphi\rangle}$.

□

We can now show that the mixing time of the classical chains determines the mixing time of the quantum chain. More precisely, for an ergodic (classical) Markov chain on X with transition matrix P and stationary distribution π, we define the mixing time

$$\tau(P) := \min_t \left[\max_{x \in X} \sum_y |P^t(x,y) - \pi(y)| \leq e^{-1} \right]. \tag{19}$$

We furthermore define the mixing time of Algorithm 1 with output $|\phi\rangle$ after time T as

$$\tau_q := \min_T \left[\max_{|\phi_0\rangle \in (\mathbb{C}^2)^n} \left\| \mathbb{E}_\phi |\phi\rangle\langle\phi| - \rho_\beta \right\|_{\mathrm{tr}} \leq e^{-1} \right]. \tag{20}$$

Lemma 4 (Coupling of quantum and classical chains)**.** *The mixing times of Q and Algorithm 1 satisfy $\tau_q \leq \tau(Q)$.*

Proof. Observe that after time t the state of the algorithm is given by

$$\begin{aligned} \mathbb{E}_{y \leftarrow Q^t} |\phi\rangle\langle\phi| &= \sum_y Q^t(x_0, y) \sum_{s \in S_g, l, l' \in \{0,1\}^{2k}} P(y) E(s) \Pi(s,l) |\phi_0\rangle\langle\phi_0| \Pi(s,l') E(s) P(y) && (21)\\ &= \sum_y Q^t(x_0,y) P(y) \overline{|\phi_0\rangle\langle\phi_0|} P(y). && (22) \end{aligned}$$

where $\overline{|\phi_0\rangle\langle\phi_0|} = \left[\sum_{l,l'\in\{0,1\}^{2k}} \Pi(0,l)\,|\phi_0\rangle\langle\phi_0|\,\Pi(0,l')\right]$ is the projection of the initial state to the code space. This gives us

$$\left\|\mathbb{E}_{y\leftarrow Q^t}|\phi\rangle\langle\phi| - \rho_\beta\right\| = \left\|\sum_y \left(Q^t(x_0,y) - \frac{1}{Z}e^{-2\beta|hy|}\right) P(y)\overline{|\phi_0\rangle\langle\phi_0|}P(y)\right\| \tag{23}$$

$$\leq \sum_y |Q^t(x_0,y) - \frac{1}{Z}e^{-2\beta|hy|}| \leq 1/e, \tag{24}$$

where $Z = \sum_y e^{-2\beta|hy|}$ and we used that $\Pi(s,l)\Pi(s',l) = \delta(s,s')$, i.e., the terms in the sum with distinct syndrome are orthogonal. □

3.3 Quantum Markov chains beyond code Hamiltonians

We note that Algorithm 1 is not restricted to stabilizer Hamiltonians, but will also be correct for *signed* stabilizer Hamiltonians. Such Hamiltonians are parameterized by a stabilizer matrix $g \in \{0,1\}^{c\times 2n}$ as well as a vector $t \in \{0,1\}^c$ as

$$H = -\sum_{i\in[c]} (-1)^{t_i} P(g_i). \tag{25}$$

For this signed case—analogous to antiferromagnetic Ising Hamiltonians—it is well known, however, that sampling is intractable. In fact, an efficient sampler would be able to solve MAXCUT, an NP-complete problem. Therefore, we also do not expect an efficient quantum sampler for this case.

We can also conceive of analogous dynamics for Hamiltonians which are sums of arbitrary—potentially non-commuting—Pauli operators. Importantly, the distribution π is still well-defined in this case and the energy function just corresponds to the number of Hamiltonian terms an "error" anticommutes with. However, in this case, syndrome extraction cannot be done for all terms simultaneously and the measurement will always project into the joint eigenspace of a subset of the Hamiltonian terms, giving a non-convergent chain. We believe it is an interesting future direction to explore the dynamics that arise from Algorithm 1 for different measurement protocols of the Hamiltonian terms, however. Some examples that could be interesting are: (i) sequentially measure random maximal subsets of commuting terms and (ii) weakly measure the Hamiltonian terms.

4 Gibbs sampling from classical codes

In this section, we formally define the (classical) code SW chain for arbitrary classical linear codes with parity check matrix $h \in \{0,1\}^{c\times n}$. We show that it mixes to the correct Gibbs distribution. Finally, we define a notion of parity check matrices (Δ-graphic and Δ-cographic) which when satisfied, lead to rapid mixing for the code SW chain.

4.1 Recap: Swendsen-Wang dynamics for the Ising model

Let $G = (V,E)$ be a graph. We consider spins on the vertices with configuration space $\Sigma = \{-1,+1\}^V$. The (ferromagnetic) Ising model on Σ is described by the energy function

$$H(\sigma) = -\sum_{(u,v)\in E} \sigma_u\sigma_v. \tag{26}$$

We can thus think of every edge (u,v) as a constraint which is satisfied if $\sigma(u) = \sigma(v)$.

It is convenient to denote the configurations satisfying a subset $A \subset E$ of the edges by

$$\Sigma(A) \coloneqq \{\sigma \in \Sigma : \sigma(u) = \sigma(v),\ \{u, v\} \in A\}, \quad A \subset E, \tag{27}$$

and the set of edges satisfied by a given configuration $\sigma \in \Sigma$ by

$$E(\sigma) \coloneqq \{\{u, v\} \in E :\ \sigma(u) = \sigma(v)\}. \tag{28}$$

We can then rewrite the energy function as $H(\sigma) = -2|E(\sigma)| + |V|$. The Gibbs distribution π of the Ising model at inverse temperature β is then

$$\pi(\sigma) \propto e^{2\beta|E(\sigma)|}. \tag{29}$$

The Swendsen-Wang (SW) algorithm now iterates the following two steps, starting from a spin configuration $\sigma \in \Sigma$.

1. *Cluster formation:* For each $e \in E(\sigma)$ include it in A with probability $p = 1 - e^{-2\beta}$, giving $A \subset E$

2. *Cluster update:* For every connected component $C \subset V$ of A let $\tau(C) = +1$ with probability $1/2$ and -1 otherwise, giving a new state $\tau \in \Sigma$.

These steps are motivated from a coupling of the Ising model on spins to the so-called random-cluster (RC) model on subsets of edges with probability distribution

$$\phi(A) \propto \left(\frac{p}{1-p}\right)^{|A|} 2^{k(A)} \mathbb{1}(A \subset E(\sigma)), \tag{30}$$

where we denote the number of connected components of the subgraph defined by $A \subset E$ by $k(A)$, and we write the indicator of an event e as $\mathbb{1}(e)$. ϕ and π are coupled by the Fortuin-Kasteleyn (FK) [FK72] measure

$$\mu(\sigma, A) \propto \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(A \subset E(\sigma)). \tag{31}$$

in the sense that its marginal distributions satisfy

$$\sum_{A \subset E} \mu(\sigma, A) = \pi(\sigma), \quad \sum_{\sigma \in \Sigma} \mu(\sigma, A) = \phi(A), \tag{32}$$

with $p = 1 - e^{-2\beta}$. Crucially, the Swendsen-Wang transition probabilities of step 1 and 2, $P(\sigma, A)$ and $P(A, \tau)$, respectively, are precisely given by the conditional distributions[1]

$$P(\sigma, A) = \mu(A|\sigma) = 2^{-k(A)} \mathbb{1}(\sigma \in \Sigma(A)) \tag{33}$$

$$P(A, \tau) = \mu(\tau|A) = \left(\frac{p}{1-p}\right)^{|A|} (1-p)^{|E(\sigma)|} \mathbb{1}(A \subset E(\tau)). \tag{34}$$

From this, we can immediately see that the SW dynamics has the distribution π as its unique fixed point, since its transition matrix $P(\sigma, \tau) = \sum_A P(\sigma, A) P(A, \tau)$ satisfies detailed balance

$$\frac{P(\sigma, \tau)}{P(\tau, \sigma)} = \frac{\sum_A \mu(A|\sigma)\mu(\tau|A)}{\sum_A \mu(A|\tau)\mu(\sigma|A)} = \frac{\pi(\tau)}{\pi(\sigma)}, \tag{35}$$

where we have simply used the law of conditional probabilities $\mu(A|\sigma) = \mu(\sigma, A)/\pi(\sigma)$ and likewise for $\mu(\sigma|A)$. To complete the argument, observe that the chain is aperiodic and ergodic since after step 1 not a single cluster survives with finite probability, and therefore, in step 2, an arbitrary state can be reached.

[1] Note that $\sigma \in \Sigma(A) \Leftrightarrow A \subset E(\sigma)$.

4.2 Swendsen-Wang algorithm for codes

To generalize the SW dynamics to classical codes, we observe that the Ising model on a graph $G = (V, E)$ defines a particular code. To see this, notice that every edge of the graph defines a constraint on the ground states, and therefore the code C_h defined by the edge-vertex incidence matrix $h \in \{0,1\}^{|E|\times|V|}$ of G is the ground space of h. For a connected graph, $C_h = \mathrm{span}(|0^n\rangle, |1^n\rangle)$, and thus h defines a repetition code. The SW algorithm then generalizes to an arbitrary linear code with parity check h whose rows define hyperedges.

More precisely, in the following we consider a code with parity check $h \in \{0,1\}^{c\times n}$ and the state spaces of the spin configurations $X = \{0,1\}^v$ and the set of check indices $E = [c]$. The correspondent of a subset of edges in the Ising model is now simply a subset of checks. For $A \subset E$, the associated parity check matrix is h_A, which is the row-submatrix of h indexed by A. We denote the number of "connected components" of the checks h_A by $k(A) = \dim(\ker(h_A))$, and for $x \in \{0,1\}^n$, we let $\mathrm{supp}(x) = \{i \in [n] : x_i = 1\}$. Therefore, the set of checks satisfied by a configuration x is given by

$$E(x) \coloneqq E \setminus \mathrm{supp}(hx), \tag{36}$$

and the set of configurations satisfying a set of checks $A \subset E$ is given by

$$X(A) \coloneqq \ker(h_A). \tag{37}$$

Our goal is to sample from the Gibbs distribution $\pi(x) \propto e^{2\beta|hx|}$, see Eq. (7). The code Swendsen-Wang (SW) algorithm achieves this as follows.

Algorithm 2 The code SW algorithm

Input: Parity check h, inverse temperature β, initial state x, cutoff time T.

1: Let $x \leftarrow x_0, A \leftarrow \emptyset$.

2: Iterate T times:

i. *Cluster formation:* For every $e \in E(x)$, include it in A with probability $p = 1 - e^{-2\beta}$, giving $A \subset E$.

ii. *Cluster update:* Pick $y \in \ker(h_A)$ uniformly at random and let $x \leftarrow y, A \leftarrow \emptyset$.

Output: x

To see that the code SW algorithm converges to its unique fixed point π, we observe that it, too, works by coupling to the random cluster model on codes, defined for $A \subset E$ as

$$\phi(A) \propto \left(\frac{p}{1-p}\right)^{|A|} 2^{k(A)}, \tag{38}$$

via the generalized FK measure

$$\mu(x, A) \propto \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(A \subset E(x)), \tag{39}$$

at $p = 1 - e^{-2\beta}$.

Lemma 5 (Correctness of code SW dynamics)**.** *The code SW algorithm (Algorithm 2 converges to π as $T \to \infty$ for any initial state x_0.*

Proof. Clearly, the chain is aperiodic and ergodic, since after a single iteration of the algorithm any state can be reached since $\Pr(A = \emptyset) > 0$ for $\beta < \infty$. To show that it satisfies detailed balance, we use Eq. (35) and all that remains is to show that the transition probabilities of step 1. and 2. in Algorithm 2 are

$$P(x, A) = \mu(A|x), \qquad P(A, y) = \mu(y|A). \tag{40}$$

To see this, we simply observe that μ is a valid coupling, i.e.,

$$\sum_{x \in X} \mu(x, A) \propto \sum_{x} \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(x \in \ker(h_A)) = \left(\frac{p}{1-p}\right)^{|A|} 2^{k(A)} \propto \phi(A), \tag{41}$$

and

$$\sum_{A \subset E} \mu(x, A) = \sum_{k=0}^{|E(x)|} \left(\frac{p}{1-p}\right)^{k} \binom{|E(x)|}{k} \tag{42}$$

$$= (1-p)^{-|E(x)|} \sum_{k=0}^{|E(x)|} p^k (1-p)^{|E(x)|-k} \binom{|E(x)|}{k} \propto \pi(x), \tag{43}$$

where we have simply used the binomial formula and the fact that $|E(x)| = |E| - |hx|$.

It remains to consider the case of zero temperature. In this case, we can see that the algorithm converges rapidly: In the cluster-formation step all previously satisfied checks are kept. In the cluster-update step, if there is an unsatisfied check prior to the update, this check will be satisfied with probability 1/2 after the update. Therefore the number of unsatisfied checks converges to zero exponentially fast. Once no unsatisfied checks remain, the cluster update step is a sampler from the zero-temperature Gibbs state. □

4.3 Mixing time of the code Swendsen-Wang algorithm

We are now ready to prove that the code SW algorithm is efficient for the 4D toric code.

Theorem 6 (Fast mixing of SW)**.** *The code SW algorithm (Algorithm 2) mixes in time $O(n^5)$ for the 2D and in time $O(n^9)$ 4D toric code at any temperature.*

To show rapid mixing, we apply the proof strategy of Ullrich [Ull14] and Guo and Jerrum [GJ17]. To apply it, however, we need to make some crucial amendments.

For the proof, we will first switch to the RC view of the SW algorithm. To this end, observe that if we halt the algorithm after the cluster formation step (1.), the sampled subset of checks will be distributed accoridng to the RC model ϕ (as $T \to \infty$). To prove convergence of the SW algorithm it is sufficient to prove convergence after either of the steps.

The proof idea of Guo and Jerrum [GJ17] and Ullrich [Ull14] is based on a series of reductions: from the RC model to the so-called even-subgraph model, and finally to the so-called worm-distribution [JS93; PS01].

4.3.1 Comparison to single-check updates

We start with a comparison to the standard (lazy) single-check-update Metropolis dynamics in which with probability 1/2 no change is made and otherwise, an update $B = A \oplus e$ is proposed for uniformly random $e \in E$. This proposal is accepted with probability

$$\Pr(\text{Accept}) = \min\left\{1, \frac{\phi(B)}{\phi(A)}\right\}, \tag{44}$$

and rejected otherwise. Let $P_{\mathrm{Metropolis}}$ the corresponding transition matrix, and likewise P_{SW} be the transition matrix of the SW process for the RC model.

Lemma 7 (SW is faster than Metropolis dynamics)**.** *The SW dynamics of the RC model is faster than Metropolis dynamics (up to a constant), i.e.,*

$$\tau(P_{Metropolis}) \geq \frac{1}{2}\tau(P_{SW}). \tag{45}$$

The proof of Lemma 7 is an adaptation of the result of Ullrich [Ull14] to our setting, which we give in Appendix A.

4.3.2 Fast mixing of single-check dynamics

In this section, we show that the single-check dynamics for the RC model of a parity check matrix h mixes rapidly if h has an approximate "graphic" or "co-graphic" representation.

Definition 8. *Let $h \in \{0,1\}^{c\times n}$ be a parity check matrix. We say that h is*

- *Δ-graphic if there exists an edge-vertex incidence matrix $g \in \{0,1\}^{c\times n'}$ of a graph on $n' \leq O(n)$ vertices such that $\ker(g^T) \supset \ker(h^T)$ and $\dim(\ker(g^T)) - \dim(\ker(h^T)) \leq \Delta$.*
- *Δ-cographic if a column-generator matrix $h^\perp$ of $\mathrm{col}(h)^\perp$ is Δ-graphic.*

In the first case, we call the edge-vertex incidence matrix g the primal coupling *to h. In the second case, we call the edge-vertex incidence matrix g that is the primal coupling to $h^\perp$ the* dual coupling *to h.*

Which codes admit primal or dual couplings?

Claim 9. *The parity check matrix h of any linear code with independent checks is 0-graphic.*

Proof. Let g be the edge-vertex incidence matrix of a line graph. Then $\ker(g^T)$ and $\ker(h^T)$ are both 0-dimensional. □

Claim 10. *The X and Z parity check matrices h of the 2D toric code are 0-graphic*

Proof. Consider the edge-vertex incidence matrix $g \in \{0,1\}^{c\times c}$ for a cycle graph. Then $\ker(g^T)$ is spanned by the all 1's vector, which matches exactly with the even covers of the 2D toric code. □

Claim 11. *The X and Z parity check matrices h of the 4D toric code are 4-cographic.*

Proof. Let g be the edge-vertex incidence matrix of the 4D torus. Then $g^T h = 0$ because g^T and h are boundary maps of a chain complex. Therefore, $\ker(g^T) \supset \mathrm{col}(h) = \ker((h^\perp)^T)$. On the other hand, a set of generators for the even subgraphs of the 4D lattice are the trivial loops (boundaries of faces) and nontrivial loops (crossing the boundary of one of the dimensions). There are four generators for the latter, corresponding to each dimension, so $\dim(\ker(g^T)) - \dim(col(h)) = 4$. □

To show rapid mixing of the single-check dynamics of the RC model for these models, we will use the method of *flow congestion.* To introduce it, we first introduce the concept of a path in the state space. Let the state space of our Markov chain with transitions P be Ω, and the *graph* of P be defined by the edges $\mathcal{E}(P) := \{(U,V) \in \Omega\times\Omega : P(U,V) > 0\}$. A path

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{L(\gamma)}) \subset \Omega^*, \tag{46}$$

of length $L(\gamma) \in \mathbb{N}$ is a sequence of states $\gamma_i \in \Omega$ which are connected by transitions of the Markov chain, i.e., $(\gamma_i, \gamma_{i+1}) \in \mathcal{E}(P)$. Let $\Gamma(I, F) \coloneqq \{\gamma \in \Omega^* : \gamma_0 = I, \gamma_{-1} = F\}$ be the set of paths from I to F and $\Gamma = \bigcup_{I,F} \Gamma(I, F)$. A *flow* with respect to a probability distribution p on Ω is a function $f : \Omega^* \to [0, 1]$ assigning a weight $f(\gamma)$ to every path such that

$$\sum_{\gamma \in \Gamma(I,F)} f(\gamma) = p(I)p(F) \tag{47}$$

for all $I \neq F \subset E$.

Good flows are such that they have low *congestion*, which is given by

$$\rho(f) \coloneqq \max_{(Z,Z') \in \mathcal{E}(P)} \frac{1}{p(Z)P(Z, Z')} \sum_{\gamma \in \Gamma, (Z,Z') \in \gamma} f(\gamma)L(\gamma), \tag{48}$$

A fundamental result in the analysis of mixing times of Markov chains is that there is a flow such that the mixing time is captured by the congestion of that flow.

Theorem 12 ([Sin92; Gur16]). *For a lazy, ergodic, reversible Markov chain P, we have*

$$\Omega\left(\inf_f \rho(f)\right) \leq \tau(P) \leq \ln\left(\frac{2e}{\min_{S \in \Omega} p(S)}\right) \rho(f). \tag{49}$$

The mixing time of a Markov chains is therefore equivalent to the congestion of some flow.

To show rapid mixing, we will construct a "good" flow for the RC model ϕ.

Theorem 13. *Suppose that $h \in \{0, 1\}^{c \times n}$ is Δ-graphic or Δ-cographic. Then there is a flow F for the RC model ϕ with congestion*

$$\rho(F) \leq O(c 2^{2\Delta+4} n^4). \tag{50}$$

Combining this with Theorem 12, we arrive at:

Corollary 14. *Suppose that $h \in \{0, 1\}^{c \times n}$ is Δ-graphic or Δ-cographic. Then the code SW chain for h mixes in* $\mathsf{poly}(n)$ *time.*

We prove this theorem by coupling the RC model on h to a generalization of the even subgraph model, which we dub the *even cover model*. The even cover model is defined by the distribution

$$\xi_{h,p}(A) \propto \left(\frac{p}{1-p}\right)^{|A|} \mathbb{1}(1_A \in \ker(h^T)), \tag{51}$$

where 1_A is the indicator of $A \subset E$ with $\mathrm{supp}(1_A) = A$. In words, this is a weighted distribution over subsets of *hyperedges* ($[c]$) such that every vertex ($[n]$) is incident to an even number of hyperedges.

The connection between even cover model and the RC model is made clear in the following lemma.

Lemma 15 (Coupling of even covers and RC). *There are two couplings of the even-cover model to the random cluster model. Let $h^\perp$ be a column-generator matrix of* $\mathrm{col}(h)^\perp$.

- *(primal lift) Let $A \sim \xi_{h,p/2}$, and add each $e \in E \setminus A$ iid. with probability* $(p/2)/(1 - p/2)$ *to obtain B. Then $B \sim \phi_p$.*
- *(dual lift) Let $A \sim \xi_{h^\perp,(1-p)/(2-p)}$, and add each $e \in E \setminus A$ iid. with probability* $1 - p$ *to obtain B. Then $E \setminus B \sim \phi_p$.*

We call $\xi_{h,p/2}$ the primal coupling *to ϕ and $\xi_{h^\perp,(1-p)/(2-p)}$ the* dual coupling *to ϕ.*

We prove Lemma 15 in Appendix B.

Our high-level strategy for constructing flows for the RC model follows similarly to Guo and Jerrum [GJ17]. They begin with a parity check matrix g that is an edge-vertex incidence matrix and consider the primal coupling $\xi_{h,p/2}$. In this case, the primal coupling is a weighted distribution over even *subgraphs* of the original graph, i.e., subgraphs that have even degree on every vertex. They call this set of even subgraphs Ω_0. Then, they enlarge the state space to also include subgraphs with exactly *two* odd degree vertices, which they call Ω_2. This enlargement allows any two states in $\Omega_0 \cup \Omega_2$ to be connected by sequences of single edge-flips, and in addition, this joined state space has good flows [JS93]. Finally, they show how to "lift" these flows to the RC model by *mimicking* edge additions and deletions from the even subgraph model to additions and deletions in the RC model to construct a good flow for the RC model.

Our proof strategy for constructing good flows for Δ-graphic and Δ-cographic h proceeds by adapting the above strategy starting from the primal or dual coupling g, except with modifications tailored to our specific setting. First, when Δ is nonzero, the even subgraph space of g, Ω_0, *contains* the set of even covers of h or $h^\perp$, i.e., $\ker(h^T) \subset \Omega_0$ if using the primal coupling or $col(h) \subset \Omega_0$ if using the dual coupling, rather than a strict equality. Below, we show that this difference can be upper bounded by 2^Δ. Second, in the case that g is the dual coupling to h, edges (non-edges) in g correspond to non-edges (edges) in h. We fix this by *reversing* the mimicking rules done in the primal case.

We now proceed to the argument.

Let p' be the weight parameter for our primal (dual) coupling to ϕ and g be the primal (dual) coupling to h. For the primal case, $p' = p/2$ and for the dual case, $p' = (1-p)/(2-p)$. We want to construct a set of flows between pairs of states in Ω_0, the even subgraphs of g. To achieve this, we need to relax the state space so that all even subgraphs are connected by single-edge flips. This is achieved by the *worm distribution*: the idea is to extend Ω_0 to those subgraphs for which all but two vertices are required to have even degree, but the remaining two vertices (called "holes") are allowed to have odd degree. Because even subgraphs are collections of loops, we can thereby move between any two even subgraphs $S, T \subset E$ via single-edge flips by unwinding the loops in $S \oplus T$.

Formally, we let $w(S) = (p'/(1-p'))^{|S|}$ and define the even-subgraphs $\Omega_0 = \ker(g^T)$, and the worm-space

$$\Omega_2 := \bigcup_{u,v\in[n']} \Omega(u,v), \quad \text{with } \Omega(u,v) = \ker((g^T)_{[n']\setminus\{u,v\}}) \setminus \Omega_0 \tag{52}$$

the configurations in which the two vertices u, v have odd degree. The worm distribution

$$\omega_{g,p'}(S) \propto w(S)\mathbb{1}(S \in \Omega_0) + \frac{1}{\binom{n'}{2}} w(S)\mathbb{1}(S \in \Omega_2) \tag{53}$$

penalizes those near-even-subgraph configurations such that the probability weight on Ω_0 and on Ω_2 is roughly the same.

We will lift a good flow for the above worm distribution to a good flow for the RC model. However, we can only couple (Lemma 15) the RC model to the parts of the worm distribution that arise from $\ker(h^T) \subset \Omega_0$ if primal and $col(h) \subset \Omega_0$ if dual, where the difference in the dimension between the two subspaces is governed by Δ. Below, we show that the effect of this dimension mismatch on the resulting "lifted" distributions is bounded by 2^Δ.

In detail, consider the following modified primal lift: Sample $S \sim \omega_{g,p/2}$. Add each edge $e \in E \setminus A$ iid. with probability $(p/2)/(1-p/2)$ to obtain B. Let the resulting marginal distribution on B be $\hat{\phi}_{\uparrow}$. Similarly, for the dual lift, sample $S \sim \omega_{g,(1-p)/(2-p)}$. Add each edge $e \in E \setminus A$ iid. with probability $1-p$, to obtain B'. Let the resulting marginal distribution on $E \setminus B'$ be $\hat{\phi}_{\downarrow}$. Then $\hat{\phi}_{\uparrow}$ and $\hat{\phi}_{\downarrow}$ are both "close to" the RC model ϕ in the following way.

Lemma 16. *If h is Δ-graphic or Δ-cographic, then*

$$\frac{\hat{\phi}_{\uparrow}(B)}{\phi(B)} \leq 2^{\Delta+1} \quad \text{and} \quad \frac{\hat{\phi}_{\downarrow}(B)}{\phi(B)} \leq 2^{\Delta+1}, \tag{54}$$

respectively, for any B.

We prove Lemma 16 in Appendix B.

[DH: Try and get rid of "if primal", "if dual"] Next, we relate good flows for the worm model to good flows for the RC model. To this end, let us define the type of flow we need.

Definition 17. *Let ξ be an even cover model, i.e. $\xi = \xi_{h,p/2}$ if primal or $\xi = \xi_{h^{\perp},(1-p)/(2-p)}$ if dual. A flow for the even cover model ξ through the worm space Ω_w is a function f on Ω_w^* that is nonzero only on paths with start and end point $I, F \in \ker(h^T)$ if primal and $I, F \in col(h)$ if dual, i.e., $f(\gamma) > 0 \Rightarrow \gamma \in \Gamma(I,F)$ and satisfies*

$$\sum_{\gamma \in \Gamma(I,F)} f(\gamma) = \xi(I)\xi(F). \tag{55}$$

The following lemma, which uses the flows through the worm model due to [JS93; GJ17]], gives us a good flow for the even cover model.

Lemma 18 (SC dynamics for the worm model mixes rapidly)**.** *Suppose that h is Δ-graphic (Δ-cographic) and primal (dual) coupled to $g \in \{0,1\}^{c \times n'}$. Then there is a good flow f for $\xi_{h,p/2}$ (resp. $\xi_{h^{\perp},(1-p)/(2-p)}$) through Ω_w that satisfies*

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq 2^{\Delta+1}(n')^4 \omega_{g,p/2}(W) \text{ if primal} \tag{56}$$

and

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq 2^{\Delta+1}(n')^4 \omega_{g,(1-p)/(2-p)}(W) \text{ if dual} \tag{57}$$

for any $W' = W \oplus e$. In the special case that $W' = W \cup e$, then

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq 2^{\Delta+1}(n')^4 \omega_{g,p/2}(W)(\frac{p/2}{1-p/2}) \text{ if primal} \tag{58}$$

and

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq 2^{\Delta+1}(n')^4 \omega_{g,(1-p)/(2-p)}(W)(1-p) \text{ if dual.} \tag{59}$$

We prove Lemma 18 in Appendix C.

We are now ready to prove the existence of a good flow for the RC model for Δ-graphic and Δ-cographic h.

Proof of Theorem 13. Suppose that h is Δ-graphic (Δ-cographic) with primal (dual) coupling g. We show how to construct a good flow for ϕ_p.

Let $\gamma = (W_0, \ldots, W_\ell) \in \Gamma$ with $\ell = L(\gamma)$ be a path through Ω_w sampled from the flow guaranteed by Lemma 18. We will construct a distribution over paths $(Z_0, \ldots Z_{\ell'})$ for the RC model based on γ.

Given $W_0 \in \Omega_w$, we construct $Z_0 \subset E$ using the coupling of Lemma 15. That is, we add every $e \in E \setminus W_0$ with probability $(p/2)/(1-p/2)$ (resp. $(1-p)$) to W_0 (resp. and take the complement) to obtain Z_0 if doing a primal (dual) lift.

In other words, for $w \subset z$, let

$$\delta_\uparrow(w, z) = \left(\frac{p/2}{1-p/2}\right)^{|z \setminus w|} \left(1 - \frac{p/2}{1-p/2}\right)^{|E \setminus z|}$$

and for $z \subset \bar{w}$,

$$\delta_\downarrow(w, z) = (1-p)^{|\bar{z} \setminus w|} \, (p)^{|E \setminus \bar{z}|} \, .$$

Then $\Pr[Z_0 = Z] = \delta_\uparrow(W_0, Z)$ for any $Z \supset W_0$ if doing a primal lift and $\Pr[Z_0 = Z] = \delta_\downarrow(W_0, Z)$ if doing a dual lift.

We now "follow" the underlying path in γ as follows:

- If $W_{k+1} = W_k$, let $Z_{k+1} = Z_k$.
- If $W_{k+1} = W_k \cup e$ for $e \notin W_k$,
  - If primal: let $Z_{k+1} = Z_k \cup e$.
  - If dual: let $Z_{k+1} = Z_k \setminus e$
- If $W_{k+1} = W_k \setminus e$ for $e \in W_k$,
  - If primal: resample the edge, i.e., let $Z_{k+1} = Z_k$ with probability $\frac{p/2}{1-p/2}$ and $Z_{k+1} = Z_k \setminus e$ with probability $1 - \frac{p/2}{1-p/2}$.
  - If dual: resample the edge, i.e., let $Z_{k+1} = Z_k$ with probability p and $Z_{k+1} = Z_k \setminus e$ with probability $1 - p$.

One may check that this ensures that $\Pr(Z_k = Z | \gamma) = \delta_\uparrow(W_k, Z)$ for a primal lifting and $\Pr(Z_k = Z | \gamma) = \delta_\downarrow(W_k, Z)$ for a dual lifting.

To finish the construction of the flow, observe that after this procedure, at the end of γ, $Z_{L(\gamma)}$ remains correlated with Z_0. To remove the correlation, we re-randomize the edges $\{e_1, \ldots, e_k\} = E \setminus W_\ell$ not in W_ℓ with $k = |E \setminus W_\ell|$. Therefore let

- If primal: $Z_{\ell+i+1} = Z_{\ell+i} \setminus e_i$ with probability $1 - \frac{p/2}{1-p/2}$ and $Z_{\ell+i+1} = Z_{\ell+i} \cup e_i$ with probability $\frac{p/2}{1-p/2}$.
- If dual: $Z_{\ell+i+1} = Z_{\ell+i} \setminus e_i$ with probability p and $Z_{\ell+i+1} = Z_{\ell+i} \cup e_i$ with probability p.

We therefore obtain a path $\lambda = (Z_0, Z_1, \ldots, Z_{\ell+k})$ to which we assign the weight

$$F(\lambda) = \sum_{\gamma \in \Gamma} f(\gamma) \Pr(Z = \lambda | \gamma) \tag{60}$$

and observe that $L(\lambda) \leq L(\gamma) + c$.

We check that F is a valid flow for ϕ: Let $\Lambda(I,F)$ be the set of paths from I to $F \subset E$, and let ξ be $\xi_{h,p/2}$ if doing a primal lift and $\xi_{h^\perp,(1-p)/(2-p)}$ if doing a dual lift.

$$\sum_{\lambda\in\Lambda(I,F)} F(\lambda) = \sum_{\substack{U\subset I, V\subset F\\ U,V\in\Omega_0}} \sum_{\gamma\in\Gamma(U,V)} f(\gamma)\Pr(Z_0 = I, Z_{-1} = F|\gamma) \tag{61}$$

$$= \sum_{\substack{U\subset I, V\subset F\\ U,V\in\Omega_0}} \sum_{\gamma\in\Gamma(U,V)} \xi(U)\xi(V)\delta_{\uparrow/\downarrow}(U,I)\delta_{\uparrow/\downarrow}(V,F) \tag{62}$$

$$= \phi(I)\phi(F), \tag{63}$$

where we have used that $\Pr(Z_0 = I, Z_{-1} = F|\gamma) = \delta_{\uparrow/\downarrow}(U,I)\delta_{\uparrow/\downarrow}(V,F)$, that f is a flow for ξ, and that

$$\phi(S) = \sum_{W\subset S, S\in\Omega_0} \xi(W)\delta_{\uparrow/\downarrow}(W,S). \tag{64}$$

We now bound the flow through any edge (Z,Z') in the primal lift. First, let us define $i(\gamma,w)$ to be the index of state w in path γ, and let $k(w,e)$ be the index of edge e in $E\setminus W_\ell$. Also define $p' = \frac{p/2}{1-p/2}$

If $Z' = Z\cup e$,

$$\sum_{\lambda\ni(Z,Z')} F(\lambda) = \sum_{W\subset Z}\Bigg(\sum_{\gamma\ni(W,W\cup e)} f(\gamma)\Pr\Big(Z_{i(\gamma,W)} = Z, Z_{i(\gamma,W)+1} = Z'|\gamma\Big) \tag{65}$$

$$+ \sum_{\gamma:\gamma_{-1}=W} f(\gamma)\Pr\Big(Z_{\ell+k(W,e)} = Z, Z_{\ell+k(W,e)+1} = Z'|\gamma\Big)\Bigg) \tag{66}$$

$$= \sum_{W\subset Z}\left(\sum_{\gamma\ni(W,W\cup e)} f(\gamma)\delta_\uparrow(W,Z) + \sum_{\gamma:\gamma_{-1}=W} f(\gamma)\delta_\uparrow(W,Z)p'\right) \tag{67}$$

$$= \sum_{W\subset Z}\delta_\uparrow(W,Z)\left(\sum_{\gamma\ni(W,W\cup e)} f(\gamma) + \sum_{\gamma:\gamma_{-1}=W} f(\gamma)p'\right) \tag{68}$$

$$\le \sum_{W\subset Z}\delta_\uparrow(W,Z)\left(2^{\Delta+1}n^4\omega_{g,p/2}(W)p' + \xi(W)p'\right) \tag{69}$$

$$\le 2^{2\Delta+2}n^4p'\phi_p(Z) + p'\phi_p(Z) \tag{70}$$

$$\le 2^{2\Delta+3}n^4\phi_p(Z)p' \tag{71}$$

If $Z' = Z\setminus e$,

$$
\begin{aligned}
\sum_{\lambda \ni (Z,Z')} F(\lambda) = \sum_{W \subset Z} \Bigg(& \sum_{\gamma \ni (W, W \cup e)} f(\gamma) \Pr\Big(Z_{i(\gamma,W)} = Z, Z_{i(\gamma,W)+1} = Z' | \gamma\Big) && (72)\\
& + \sum_{\gamma : \gamma_{-1} = W} f(\gamma) \Pr\Big(Z_{\ell+k(W,e)} = Z, Z_{\ell+k(W,e)+1} = Z' | \gamma\Big)\Bigg) && (73)\\
= \sum_{W \subset Z} & \left(\sum_{\gamma \ni (W, W \setminus e)} f(\gamma) \delta_{\uparrow}(W,Z)(1-p') + \sum_{\gamma : \gamma_{-1} = W} f(\gamma) \delta_{\uparrow}(W,Z)(1-p') \right) && (74)\\
= \sum_{W \subset Z} & \delta_{\uparrow}(W,Z) \left(\sum_{\gamma \ni (W, W \setminus e)} f(\gamma)(1-p') + \sum_{\gamma : \gamma_{-1} = W} f(\gamma)(1-p') \right) && (75)\\
\leq \sum_{W \subset Z} & \delta_{\uparrow}(W,Z) \left(2^{\Delta+1} n^4 \omega_{g,p/2}(W)(1-p') + \xi(W)(1-p') \right) && (76)\\
\leq 2^{2\Delta+2} & n^4 (1-p') \phi_p(Z) + (1-p')\phi_p(Z) && (77)\\
\leq 2^{2\Delta+3} & n^4 \phi_p(Z)(1-p') && (78)
\end{aligned}
$$

If $Z' = Z$,

$$
\begin{aligned}
\sum_{\lambda \ni (Z,Z')} F(\lambda) = \sum_{W \subset Z} \Bigg(& \sum_{\gamma \ni W} f(\gamma) \Pr\Big(Z_{i(\gamma,W)} = Z, Z_{i(\gamma,W)+1} = Z | \gamma\Big) && (79)\\
& + \sum_{\gamma : \gamma_{-1} = W} \sum_{i=1}^{|E \setminus W|} f(\gamma) \Pr(Z_{\ell+i} = Z, Z_{\ell+i+1} = Z | \gamma)\Bigg) && (80)\\
\leq \sum_{W \subset Z} & \left(\sum_{\gamma \ni W} f(\gamma) \delta_{\uparrow}(W,Z) + c \sum_{\gamma : \gamma_{-1} = W} f(\gamma) \delta_{\uparrow}(W,Z) \right) && (81)\\
= \sum_{W \subset Z} & \delta_{\uparrow}(W,Z) \left(\sum_{\gamma \ni W} f(\gamma) + c \sum_{\gamma : \gamma_{-1} = W} f(\gamma) \right) && (82)\\
\leq \sum_{W \subset Z} & \delta_{\uparrow}(W,Z) \left(2^{\Delta+1} n^4 \omega_{g,p/2}(W) + \xi(W) c \right) && (83)\\
\leq 2^{2\Delta+2} & n^4 \phi_p(Z) + c\phi_p(Z) && (84)\\
\leq 2^{2\Delta+3} & n^4 \phi_p(Z) c && (85)
\end{aligned}
$$

We now bound the congestion.
If $Z' = Z \cup e$,

$$
\begin{aligned}
\frac{1}{\phi(Z) P(Z,Z')} \sum_{\gamma, (Z,Z') \in \gamma} F(\gamma) L(\gamma) & \leq \frac{c}{\phi(Z) P(Z,Z')} 2^{2\Delta+3} n^4 \phi_p(Z) p' && (86)\\
& \leq \frac{c}{\phi(Z) \min(1, \frac{p}{1-p})} 2^{2\Delta+3} n^4 \phi_p(Z) p' && (87)\\
& \leq c 2^{2\Delta+3} n^4 && (88)
\end{aligned}
$$

If $Z' = Z \setminus e$,

$$\frac{1}{\phi(Z)P(Z,Z')}\sum_{\gamma,(Z,Z')\in\gamma}F(\gamma)L(\gamma)\leq\frac{c}{\phi(Z)P(Z,Z')}2^{2\Delta+3}n^4\phi_p(Z)(1-p') \tag{89}$$

$$\leq\frac{c}{\phi(Z)\min(1,\frac{1-p}{p})}2^{2\Delta+3}n^4\phi_p(Z)p' \tag{90}$$

$$\leq c2^{2\Delta+4}n^4 \tag{91}$$

If $Z'=Z$,

$$\frac{1}{\phi(Z)P(Z,Z')}\sum_{\gamma,(Z,Z')\in\gamma}F(\gamma)L(\gamma)\leq\frac{c}{\phi(Z)P(Z,Z')}2^{2\Delta+3}n^4\phi_p(Z)c \tag{92}$$

$$\leq c^2 2^{2\Delta+3}n^4 \tag{93}$$

Therefore, the congestion over any edge (Z,Z') is upper bounded by $c^2 2^{2\Delta+4}n^4$. The proof for dual lifting congestion follows analogously, except with $p'=1-p$.

□

4.4 Beyond provable mixing times

[DH: Do something with this]

The conditions we find in Theorem 2 are sufficient conditions for rapid mixing of the code SW algorithm, but we do not believe them to be necessary. Indeed, Goldberg and Jerrum [GJ13] show that the associated approximation problem for the Tutte polynomial can be solved for so-called regular matroids, leading us to conjecture that the code SW algorithm also mixes quickly in this case. [DH: Move this citation up to the intro.] Drawing on the analogy to the classical (ferromagnetic) Ising and Potts models, the only known obstacle to rapid mixing in those cases are first-order phase transitions [GJ99; BCT12; GŠV19; GLP18]. We therefore conjecture that the code SW algorithm also mixes quickly for all codes and all temperatures other than around first-order phase transition points. Nonetheless, we believe that our proof strategy using couplings to distributions over even covers/subgraphs cannot be used to prove this, since the required worm dynamics for even covers will mix slowly for matroids that are far from being graphic.

We proved rapid mixing for ferromagnetic matroid models, i.e., where every check has to be satisfied. For arbitrary models in which some checks may be required to be unsatisfied the Gibbs sampling problem is #BIS-hard, where #BIS is the problem of counting independent sets in a bipartite graph [GJ12]. Our reduction of quantum to classical Gibbs sampling for commuting Pauli Hamiltonians implies that preparing their Gibbs state at any temperature is also #BIS-hard.

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A Code Swendsen-Wang dynamics are faster than single-check dynamics

In this section, we prove Lemma 7 by directly following the proof of Ullrich [Ull14].

For convenience of the proof, we define the (lazy) *single-check (SC) dynamics* of the random cluster model via the following update rule.

1. With probability $1/2$ let $B = A$.
2. Otherwise, choose a uniformly random check $e \in E$.
   - If $k(A) = k(A \cup e)$, let $B = A \cup e$ with probability p, and $B = A \setminus e$ with probqability $1 - p$.
   - If $k(A) \neq k(A \cup e)$, let $B = A \cup e$ with probability $p/2$, and $B = A$ with probability $1 - p/2$.
3. Output B.

We use the standard equivalences between two Markov chains, and apply it to the SC update compared to the Metropolis update. To do this, we define the *gap* of a Markov chain with transition matrix P on state space Ω as

$$\Delta(P) = 1 - \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P, \lambda \neq 1\}. \tag{94}$$

The spectral gap is an equivalent characterization of the mixing time as [LPW09, Theorem 12.3 & 12.4], stated here, following [Ull14]

$$\Delta(P)^{-1} - 1 \leq \tau(P) \leq \log\left(\frac{2e}{\min_{S \in \Omega} p(S)}\right) \Delta(P)^{-1} \tag{95}$$

Theorem 19 ([DS93])**.** *Let P and Q be the transition matrices of two Markov chains over a state space X with the same unique fixed point π, satisfying*

$$P(x, y) \leq Q(x, y) \leq cP(x, y) \tag{96}$$

for all $x \neq y \in X$ and some $c > 0$. Then

$$\Delta(P) \leq \Delta(Q) \leq c\Delta(P). \tag{97}$$

We now observe that

$$P_{\mathrm{SC}}(A, B) \leq P_{\mathrm{Metropolis}}(A, B) \leq 2P_{\mathrm{SC}}(A, B). \tag{98}$$

and subsequently follow the proof of Ullrich [Ull14] for the SC dynamcis. To this end, let us introduce some notation. Define the function space $L_2(\pi) := (\mathbb{R}^\Omega, \pi)$ in which the inner product is given by

$$\langle f, g\rangle_\pi = \sum_{x \in \Omega} f(x) f(g) \pi(x), \tag{99}$$

and $\|f\|_\pi := \langle f, f\rangle_\pi$. Define the operator P as

$$P : L_2(\pi) \to L_2(\pi) \tag{100}$$
$$Pf(x) := \sum_{y \in \Omega} P(x, y) f(y). \tag{101}$$

Furthermore we have the operator norm of P, $\|P\|_\pi := \|P\|_{L_2(\pi)\to L_2(\pi)} = \max_{\|f\|_\pi \le 1} \|Pf\|_\pi$.

To compare SW and SC dynamics, we define two mappings. The first one, $M : \Omega \to X \times \Omega$ lifts a RC configuration to a FK configuration, the second one, T_e updates the FK model.

$$M(B,(x,A)) := 2^{-k(A)}\mathbb{1}(A = B)\mathbb{1}(x \in X(A)), \tag{102}$$

$$T_e((x,A),(y,B)) := \mathbb{1}(x = y)\begin{cases} p, & B = A \cup e \text{ and } e \in E(x) \\ 1-p, & B = A \setminus e \text{ and } e \in E(x) \\ 1, & B = A \text{ and } e \notin E(x) \\ 0, & \text{otherwise.} \end{cases} \tag{103}$$

As above, these define operators $M : L_2(\mu) \to L_2(\phi)$, and $T_e : L_2(\mu) \to L_2(\mu)$. The adjoint of M is given by $M^*((x,A),B) = \mathbb{1}(A = B)$, since

$$\begin{aligned}
\langle f, Mg\rangle_\phi &= \sum_{B\in\Omega} f(B)Mg(B)\phi(B) && (104)\\
&\propto \sum_{B\in\Omega} f(B)\left(\sum_{(x,A)\in X\times\Omega} 2^{-k(A)}\mathbb{1}(A = B)\mathbb{1}(x \in X(A))\right)\left(\frac{p}{1-p}\right)^{|B|} 2^{k(B)} && (105)\\
&= \sum_{(x,A)\in X\times\Omega}\left(\sum_{B\in\Omega}\mathbb{1}(A = B)f(B)\right) g(x,A)\left(\frac{p}{1-p}\right)^{|A|}\mathbb{1}(A \subset E(x)) && (106)\\
&= \sum_{(x,A)\in X\times\Omega}\left(\sum_{B\in\Omega}\mathbb{1}(A = B)f(B)\right) g(x,A)\mu(x,A) = \langle M^* f, g\rangle_\mu && (107)
\end{aligned}$$

.

Lemma 20. *Let M, M^* and T_e be the operators defined above. We have*

i. M^*M and T_e are self-adjoint in $L_2(\mu)$.

ii. $MM^*(A,B) = \mathbb{1}(A = B)$ and thus $M^*MM^*M = M^*M$.

iii. $T_eT_e = T_e$ and $T_eT_{e'} = T_{e'}T_e$ for all $e, e' \in E$.

iv. $\|T_e\|_\mu = 1$ and $\|M^*M\|_\mu = 1$.

Proof. (i) Self-adjointness of M: $\langle f, M^*Mg\rangle_\mu = \langle M^*Mf, g\rangle_\mu$.

To show self-adjointness of T_e, we can use that Hence, we can use that

$$\mu(x, B \setminus e)\mathbb{1}(e \in B)\mathbb{1}(e \in E(x)) = \left(\frac{1-p}{p}\right)\mu(x,B)\mathbb{1}(e \in E(x)) \tag{108}$$

$$\mu(x, B \cup e)\mathbb{1}(e \notin B)\mathbb{1}(e \in E(x)) = \left(\frac{p}{1-p}\right)\mu(x,B)\mathbb{1}(e \in E(x)) \tag{109}$$

to find

$$\begin{aligned}
&\sum_A f(x,A)pg(x,A\cup e)\mathbb{1}(e \notin A)\mathbb{1}(e \in E(x))\mu(x,A) && (110)\\
&= \sum_B f(x,B\setminus e)pg(x,B)\mathbb{1}(e \in B)\mathbb{1}(e \in E(x))\mu(x,B\setminus e) && (111)\\
&= \sum_B f(x,B\setminus e)(1-p)g(x,B)\mu(x,B), && (112)
\end{aligned}$$

and likewise

$$\sum_A f(x,A)(1-p)g(x,A\setminus e)\mathbb{1}(e\in A)\mathbb{1}(e\in E(x))\mu(x,A) \tag{113}$$
$$=\sum_B f(x,B\cup e)(1-p)g(x,B)\mathbb{1}(e\notin B)\mathbb{1}(e\in E(x))\mu(x,B\cup e) \tag{114}$$
$$=\sum_B f(x,B\cup e)pg(x,B)\mu(x,B)\mathbb{1}(e\in E(x)). \tag{115}$$

(iii) This follows from the fact that the transition probabilities depend only on the coordinates x of a FK configuration, which are not changed by the update.

(iv) Follows from (i-iii) since $\|T_e\|_\mu = \|T_e^2\|_\mu$ by (iii) and $\|T_e^2\|_\mu = \|T_e\|_\mu^2$ by self-adjointness of T_e.

□

Next, we express the Swendsen-Wang and single-check updates in terms of M and T_e.

Lemma 21 ([Ull14]). *Let T_e, M, M^* be the operators defined above. Then*

(i) $P_{SW} = M\left(\prod_{e\in E} T_e\right) M^*$

(ii) $P_{SC} = \frac{I}{2} + \frac{1}{2|E|} M\left(\sum_{e\in E} T_e\right) M^*$

Proof. To see (i), we observe that M and T_e generate the update steps of the SW algorithm via left-multiplication. Given B applying M from the right chooses a uniformly random bit string x compatible with B. Likewise, T_e updates the cluster configuration by generating a new cluster configuration in which a check $e\in E(x)$ is kept/added to the cluster with probability p or removed with probability $1-p$.

For $B\subset E(x)$, we have that

$$\left(\prod_{e\in E} T_e\right)((x,B),(x,A)) = \sum_C \underbrace{\left(\prod_{e\notin E(x)} T_e\right)((x,B),(x,C))}_{\mathbb{1}(B=C)}\left(\prod_{e\notin E(x)} T_e\right)((x,C),(x,A)) \tag{116}$$
$$= p^{|A|}(1-p)^{|E(x)|-|A|}\mathbb{1}(A\subset E(x)) \tag{117}$$

which gives, using that M^* acts trivially,

$$\sum_x M(B,(x,B))\left(\prod_{e\in E} T_e\right)((x,B),(x,A))$$
$$= \sum_x \mathbb{1}(x\in X(A)\cap X(B))p^{|A|}(1-p)^{|E(x)|-|A|}2^{-k(B)} \equiv P_{SW}(B,A) \tag{118}$$

To see (ii), we need to show that the transition depends on whether or not the endpoints of e are connected through A.

For a pair (x,B) with $x\in X(B)$ we have

$$T_e((x,B),(x,A)) = p\mathbb{1}(e\in E(x))(\mathbb{1}(A=B\cup e)) - \mathbb{1}(A=B\setminus e)) \tag{119}$$
$$+\mathbb{1}(e\notin E(x))\mathbb{1}(A=B)+\mathbb{1}(e\in E(x))\mathbb{1}(A=B\setminus e) \tag{120}$$
$$= p\mathbb{1}(e\in E(x))(\mathbb{1}(A=B\cup e)) - \mathbb{1}(A=B\setminus e)) + \mathbb{1}(A=B\setminus e) \tag{121}$$

Now, let us compute

$$MT_eM^*(A,B) = \sum_{x\in X(A)} 2^{-k(A)} T_e((x,B),(x,A)) \tag{122}$$

$$= \mathbb{1}(B = A\setminus e) + p(\mathbb{1}(A = B\cup e)) - \mathbb{1}(A = B\setminus e)) \sum_{x\in X(A)} 2^{-k(A)} \mathbb{1}(e\in E(x)), \tag{123}$$

where we find with $\mathbb{1}_e(A) = \mathbb{1}(\ker(h_A) = \ker(h_{A\cup e}))$

$$\sum_{x\in X(A)} 2^{-k(A)} \mathbb{1}(e\in E(x)) = \mathbb{1}_e(A) + \frac{1}{2}(1-\mathbb{1}_e(A)) = \frac{1}{2}(1+\mathbb{1}_e(A)). \tag{124}$$

This gives exactly the transition matrix P_e from above. □

Proof of Lemma 7. The remainder of the proof follows from Sec. 6.1 of Ullrich [Ull14]. □

B Coupling to syndromes and even covers

We will now show that the RC measure is coupled both to the syndrome/satisfied check Gibbs distribution and the even-cover model. We will need the following lemma.

Lemma 22. *Let h, g be matrices.*

$$\dim(\ker(A)\cap \mathrm{col}(B)) = \dim(\ker(AB)) - \dim(\ker(B)) \tag{125}$$

Proof of Lemma 15. (dual lift) Note that the distribution of syndromes S is given by

$$\zeta(S) \propto e^{-2\beta|S|}\mathbb{1}(S\in \mathrm{col}(h)) = e^{-2\beta|S|}\mathbb{1}(S\in\ker(g^T)) \propto \xi_{g,(1-p)/(2-p)}, \tag{126}$$

where g is such that $\mathrm{col}(g) = \mathrm{col}(h)^\perp$.

This yields

$$\Pr(B) = \sum_S \zeta(S)(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(S\subset B) \tag{127}$$

$$= \sum_S (1-p)^{|S|}(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(1_S\in \mathrm{col}(h)\wedge S\subset B) \tag{128}$$

$$= \left(\frac{1-p}{p}\right)^{|B|}\sum_S \mathbb{1}(1_S\in(\mathrm{col}(h)\cap\ker(\mathbb{1}_{E\setminus B})) \tag{129}$$

$$= \left(\frac{p}{1-p}\right)^{|E\setminus B|} 2^{\dim(\ker(h_{E\setminus B}))-\dim(\ker(h))} \propto \phi(E\setminus B), \tag{130}$$

where we have used that $S\subset B \Leftrightarrow 1_S\in\ker(\mathbb{1}_{E\setminus B})$, where $\mathbb{1}_{E\setminus B}$ is the projector onto the rows indexed by $E\setminus B$ in Eq. (129), and Lemma 22 in Eq. (130).

(primal lift) We have

$$\Pr(B) = \sum_S \xi(S)\left(\frac{p}{1-p}\right)^{|B\setminus S|}\left(1-\frac{p}{1-p}\right)^{|E\setminus B|}\mathbb{1}(S\subset B\wedge S \text{ is even}) \tag{131}$$

$$= \sum_S \left(\frac{p}{1-p}\right)^{|S|}\left(\frac{p}{1-p}\right)^{|B\setminus S|}\left(1-\frac{p}{1-p}\right)^{|E\setminus B|}\mathbb{1}(1_S\in\ker(\mathbb{1}_{E\setminus B})\cap\ker(h^T)) \tag{132}$$

$$= \left(\frac{p}{1-2p}\right)^{|B|}\sum_S \mathbb{1}(1_S\in\underbrace{\ker(\mathbb{1}_{E\setminus B})}_{=\mathrm{col}(\mathbb{1}_B)}\cap\ker(h^T)) \tag{133}$$

$$= \left(\frac{p}{1-2p}\right)^{|B|}2^{\dim(\ker(h_B^T))-\dim(\ker\mathbb{1}_B)} \tag{134}$$

$$= \left(\frac{p}{1-2p}\right)^{|B|}2^{\dim(\ker(h_B))+e-v-(e-|B|)}\propto\phi_{2p}(B), \tag{135}$$

where in Eq. (134) we used Lemma 22 and in the last step we used the dimension formula

$$\dim(\mathrm{col}(h)) = v-\dim(\ker(h)) = e-\dim(\ker(h^T)) = \dim(\mathrm{col}(h^T)). \tag{136}$$

□

Proof of Lemma 16. We follow the same argument as in the proof of Lemma 15.

(dual lift) Let $H=(h|c_1,\ldots,c_\Delta)\in\{0,1\}^{c\times(n+\Delta)}$ be a column generator matrix of $\ker(g^T)$ where we have added linearly-independent columns $c_1,\ldots,c_\ell\notin\mathrm{col}(h)$ to h, and let $p'=(1-p)/(2-p)$. We follow a calculation analogous to [GJ17, Lemma 3.1]

$$\phi_\uparrow(E\setminus B) = \sum_S \omega_{g,p'}(S)(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(S\subset B) \tag{137}$$

$$\propto\sum_S(1-p)^{|S|}(1-p)^{|B\setminus S|}p^{|E\setminus B|}\mathbb{1}(S\subset B)(\mathbb{1}(S\in\Omega_0)+\frac{1}{n^2}\mathbb{1}(S\in\Omega_2)) \tag{138}$$

$$\propto\left(\frac{p}{1-p}\right)^{|E\setminus B|}\sum_S\mathbb{1}(S\subset B)(\mathbb{1}(S\in\Omega_0)+\frac{1}{n^2}\mathbb{1}(S\in\Omega_2)) \tag{139}$$

$$\begin{aligned}&= \left(\frac{p}{1-p}\right)^{|E\setminus B|}\sum_S\mathbb{1}(1_S\in(\underbrace{\ker(g^T)}_{=\mathrm{col}(H)}\cap\ker(\mathbb{1}_{E\setminus B}))\\&\qquad+\frac{1}{n^2}\sum_{i\neq j\in[n+\Delta]}\mathbb{1}(1_S\in(\underbrace{\ker((g^T)_{V(i,j)}))}_{=\mathrm{col}(H(V(i,j)))})\cap\ker(\mathbb{1}_{E\setminus B}))\end{aligned} \tag{140}$$

$$\begin{aligned}&= \left(\frac{p}{1-p}\right)^{|E\setminus B|}\Big(2^{\dim(\ker(H_{E\setminus B}))-\dim(\ker(H))}\\&\qquad+\frac{1}{n^2}\sum_{i\neq j\in[n+\Delta]}2^{\dim(\ker(H(V_{i,j})_{E\setminus B}))-\dim(\ker(H(V_{i,j})))}\Big)\end{aligned} \tag{141}$$

$$\leq\left(\frac{p}{1-p}\right)^{|E\setminus B|}2^{-\dim(\ker(h))}2^{\Delta}2^{\dim(\ker(h_{E\setminus B}))}\left(1+\binom{n}{2}/n^2\right) \tag{142}$$

$$\propto\frac{3}{2}2^{\Delta}\phi(E\setminus B) \tag{143}$$

where we have defined $V(i,j)=[n+\Delta]\setminus\{i,j\}$. In line (142), we used that $\dim(\mathrm{col}(H_A))\geq\dim(\mathrm{col}(h_A))$ for $A\subset E$ and therefore, by the dimension formula, $\dim(\ker(H_A))\leq\dim(\ker(h_A))+$

Δ. Moreover, $\dim(\ker(H(V(i,j))_A)) \leq \ker(H_A)$. Similarly, since $\dim(\mathrm{col}(H)) = \dim(\mathrm{col}(h)) + \Delta$, we have $\dim(\ker(H)) = \dim(\ker(h))$. This yields the claim for the dual lift.

For the primal lift, we follow the same reasoning. $\square$

C Flows for the even subgraph model

In this section, we construct the flows for Lemma 18. These flows are the *unwinding* flows by Jerrum and Sinclair and the proof follows a standard canonical paths argument [JS93].

Proof of Lemma 18. We prove this for the primal case. The dual follows analogously.

For shorthand, let $p' = p/2$, $\xi = \xi_{h,p'}$, and $\omega = \omega_{g,p'}(W)$.

The flows we construct are as follows. For any pair of states $A, B \in \mathrm{col}(h)$, which we can interpret as two even subgraphs in the graph described by g due to the Δ-graphic property, we will construct a path from A to B through the state space Ω_w.

Consider the symmetric difference $A \oplus B$, which is also an even subgraph in g. Place a canonical ordering on cycles in g and a canonical ordering of edges within each cycle, so that $A \oplus B$ is a disjoint union of cycles, ordered by this canonical ordering. Use the ordering of edges and cycles to decompose $A \oplus B$ into a sequence of edges $(e_1, \ldots, e_\ell)$ for $\ell \leq c$. Then traverse from A to B by taking the path determined by this sequence of edges $\gamma_{A,B} = (A, A \oplus e_1, A \oplus e_1 \oplus e_2, \ldots, A \oplus e_1 \oplus \ldots e_{\ell-1}, B)$. All states along this path are in Ω_w because at most two vertices have odd degree at all times. Assign this path a weight of $f(\gamma_{AB}) = \xi(A)\xi(B)$.

Now we bound the flow through any transition from W to $W' = W \oplus e$. For configurations $A, B \in \mathrm{col}(h)$, let $\Phi(A, B) = W \oplus A \oplus B$. This is an injective map because given (W, W') and $U = \Phi(A, B)$, we can recover A and B in the following way: Since $U \oplus W = A \oplus B$, then there is a canonical ordering on the edges in $U \oplus W$, including e. For any edge before e, its status is that in B and for any edge after e, its status is that in A. Finally, because $U \oplus W$ gives you the symmetric difference between A and B, then one can infer the remaining parts of A and B.

Recall that $w(A)$ is the weight function $(p')^{|A|}(1-p')^{|E\setminus A|}$, and define $Z_\xi = \sum_{A \in \mathrm{col}(h)} w(A)$, $Z_0 = \sum_{A \in \Omega_0} w(A)$, and $Z_2 = \sum_{A \in \Omega_2} w(A)$.

We first bound the ratios between these quantities. First define the unnormalized vector $|T\rangle = (|0\rangle + \frac{p'}{1-p'}|1\rangle)^{\otimes n'}$ and $S = \sum_{A \in col(h)} |A\rangle$. Note that $p' \leq 1/2$, so $H|T\rangle$ (the n'-qubit Hadamard transform) only has positive coefficients, and similarly $H|S\rangle$. Then $Z_\xi = \langle T|S\rangle = (\langle T| H)(H|S\rangle)$ is a summation of only positive elements. Letting X_a be the X-Pauli tensor product described by a, then $\langle T| X_a |S\rangle = \langle T| HHX_a |S\rangle = \langle T| HZ_aH |S\rangle \leq Z_\xi$ because this is now the same sum but with potential minus signs. Then letting D be all 2^Δ affine shifts for $col(h) \subset \ker(g)$ to cover $\ker(g)$, then $Z_0 = \sum_{a \in D} \langle T| X_a |S\rangle \leq \sum_{a \in D} Z_\xi \leq 2^\Delta Z_\xi$. Similarly, Guo and Jerrum showed that $Z_2 \leq \binom{n'}{2} Z_0$ in Lemma 8 of [GJ16].

This allows us to bound the flow through this transition:

$$\sum_{\gamma \ni (W,W')} f(\gamma) = \sum_{\substack{A,B \in \mathrm{col}(h): \\ \gamma_{AB} \ni (W,W')}} \xi(A)\xi(B) \tag{144}$$

$$= \sum_{\substack{A,B \in \mathrm{col}(h): \\ \gamma_{AB} \ni (W,W')}} \frac{w(A)w(B)}{Z_\xi^2} \tag{145}$$

$$= \sum_{\substack{A,B \in \mathrm{col}(h): \\ \gamma_{AB} \ni (W,W')}} \frac{w(W)w(\Phi(A,B))}{Z_\xi^2} \tag{146}$$

$$\leq w(W) \sum_{U \in \Omega_w} \frac{w(U)}{Z_\xi^2} \tag{147}$$

$$= w(W) \frac{Z_0 + Z_2}{Z_\xi^2}, \tag{148}$$

where the third equality uses that in every index in which A and B agree, W and $\Phi(A, B)$ agree with A and B, and the inequality used the fact that Φ is an injection.

Continuing,

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq \frac{w(W)}{Z_\xi} \frac{Z_0 + Z_2}{Z_\xi} \tag{149}$$

$$\leq \left(\binom{n'}{2} \omega(W) \right) \left(2^\Delta + \binom{n'}{2} 2^\Delta \right) \tag{150}$$

$$\leq 2^{\Delta+1} (n')^4 \omega(W). \tag{151}$$

In the special case that $W' = W \cup e$, then let $\Phi(A, B) = W' \oplus A \oplus B$ instead. Following the same proof, we see

$$\sum_{\gamma \ni (W,W')} f(\gamma) \leq w(W') \frac{Z_0 + Z_2}{Z_\xi^2} \tag{152}$$

$$\leq w(W) \frac{Z_0 + Z_2}{Z_\xi^2} \frac{p'}{1 - p'} \tag{153}$$

$$\leq 2^{\Delta+1} (n')^4 \omega(W) \frac{p'}{1 - p'} \tag{154}$$

□