23.04.2014

Frequency discretization:

$$\omega_n = \frac{2\pi}{T} \cdot \frac{n}{N} \qquad n = 0, 1, \dots N-1$$

$$\omega_n (n = N) = \frac{2\pi}{T} \cdot \frac{N}{N} = \frac{2\pi}{T} \quad (\equiv \text{Sampling frequency})$$

Sampling theoreme:

$$\underbrace{\frac{2\pi}{T}}_{\omega_s} \geq 2\,\omega_{max} \Rightarrow \omega_{max} = \pi / T$$

Consider case $n = N/2$

$$\omega_n(n = N/2) = \frac{2\pi}{T} \cdot \frac{\cancel{N}}{2\cancel{N}} = \frac{\pi}{T}$$

$$= \frac{1}{2}\omega_s = \omega_{max}$$

$\Rightarrow$ Basically only one half of the n discrete frequency sample points contains all the information in the spectrum.

$\Rightarrow$ Large window size N in the time domain, will lead to many discrete frequencies in the frequency domain, and therefore to a fine frequency resolution.

$\Rightarrow$ Small window size leads to a bad (coarse) frequency resolution.

2.2.2 Time-Frequency-Transformations

typical frequency transformations:

$$y(t) \xrightarrow{\mathcal{F}} Y(\omega)$$

$$y(k) \xrightarrow{\tilde{\mathcal{F}}} Y(n)$$

time-frequency transformation:

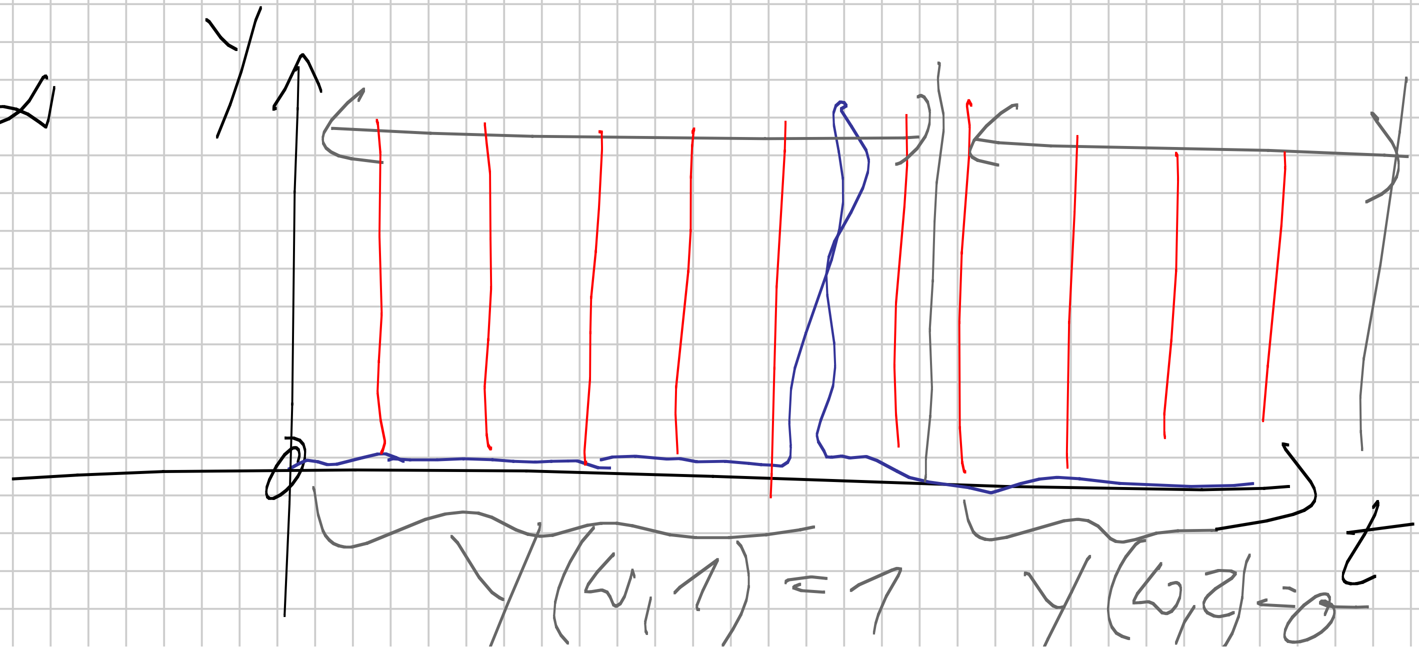
$$y(t) \xrightarrow{\mathcal{F}} Y(\omega, t)$$

$$y(k) \xrightarrow{\bar{\mathcal{F}}} Y(n, k)$$

Most simple case of a time-frequency-Transformation: Short-Time-Fourier-Transformation (STFT)

$$Y(\omega,\tau) = \int_{-\infty}^{\infty} y(t)\cdot s(t-\tau)\, e^{-j\omega t}\, dt$$

Window size:



⇒ large time window results into bad time resolution, but good frequency resolution

⇒ Small window size ⟶ good (fine) time resolution, but a bad frequency resolution

⇒ Uncertainty Principle in Signal Processing

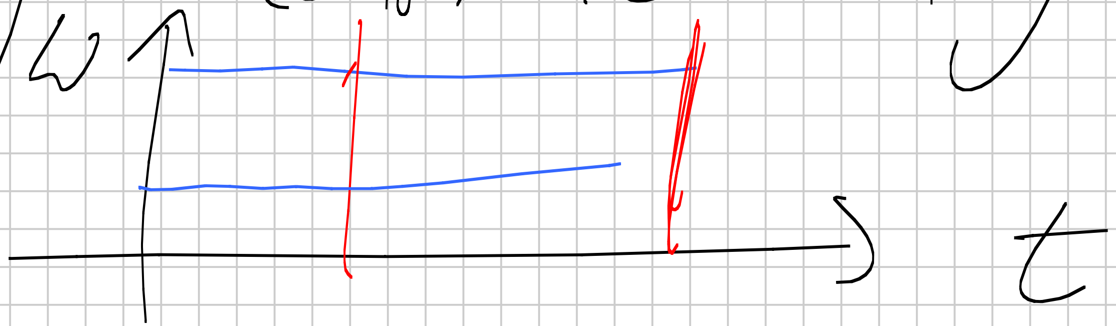
$$\underbrace{\Delta t}_{\text{time resolution}} \cdot \underbrace{\Delta \omega}_{\text{frequency resol.}} \geq \frac{1}{2}$$

(another uncertainty principle in Physics:
$\Delta x \cdot \Delta p \geq \hbar$) $\rightarrow$ Heisenberg's law.

2.2.2.1 Wavelet-Transformation (WT)

Basic idea: Frequency-dependent time-frequency resolution

Normally: constant analysis window



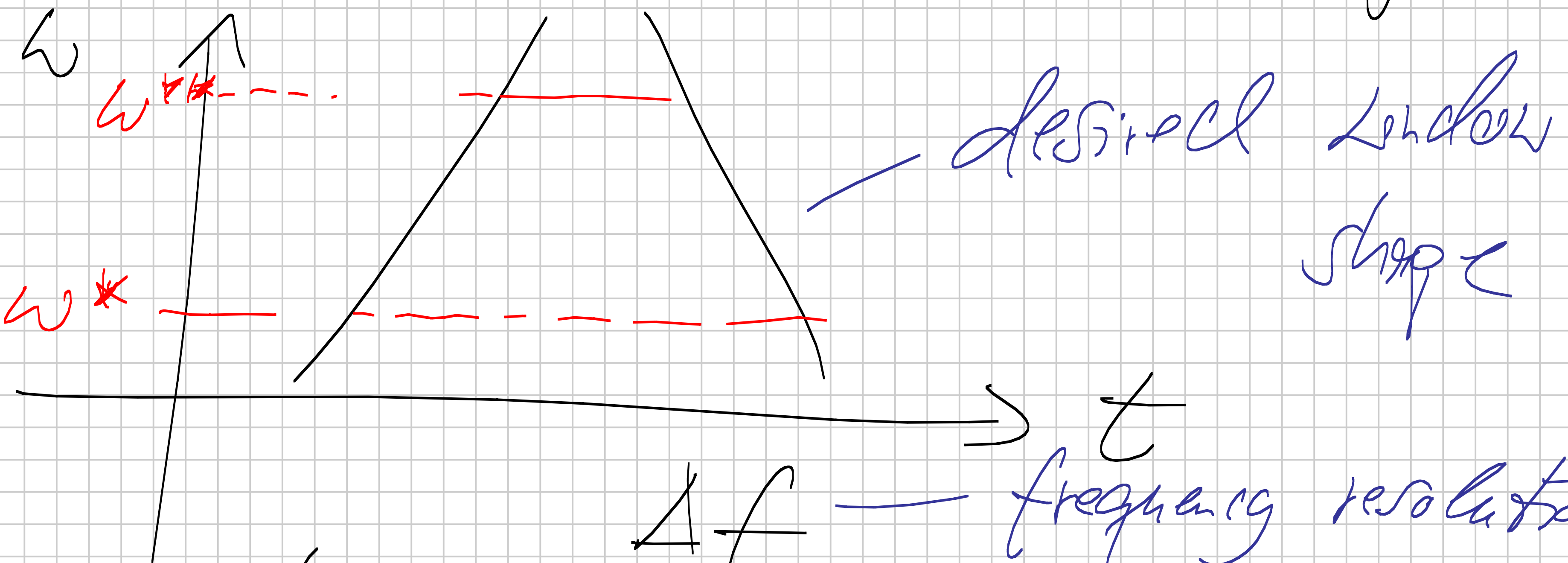
⇒ Automatic adjustment of analysis window based the current analysis frequency

Favorite time-frequency resolution:

- in case of low frequencies, a good frequency resolution is desired
  $\Delta\omega$ is small → Δt is large
- in case of high frequencies, a

lower frequency resolution is acceptable, for the advantage of high time resolution

$\Rightarrow \Delta t$ small $\rightarrow \Delta \omega$ large



$$\frac{\Delta f}{f} \approx \text{const.}$$

Δf — frequency resolution

f — analysis frequency

means also:

How can that be achieved: With a special of the basis functions

Basis functions are now called Wavelets

basis wavelet:

$$\psi(c,t) = \underbrace{\frac{1}{\sqrt{c}}}\,\psi\left(\frac{t-\tau}{c}\right)$$

scaling factor

energy scaling factor

time-shift of analysis window

Frequency of wavelet:

realized via scaling factor c

examples: $y(t) = \underbrace{t^2}_{c=1}$ $\quad y(t) = \underbrace{t^2/2}_{c=2}$

$$y(t) = \sin t \qquad y(t) = \sin t/2$$

$\Longrightarrow$ large $c \longrightarrow$ smaller frequency

small $c \longrightarrow$ larger frequency

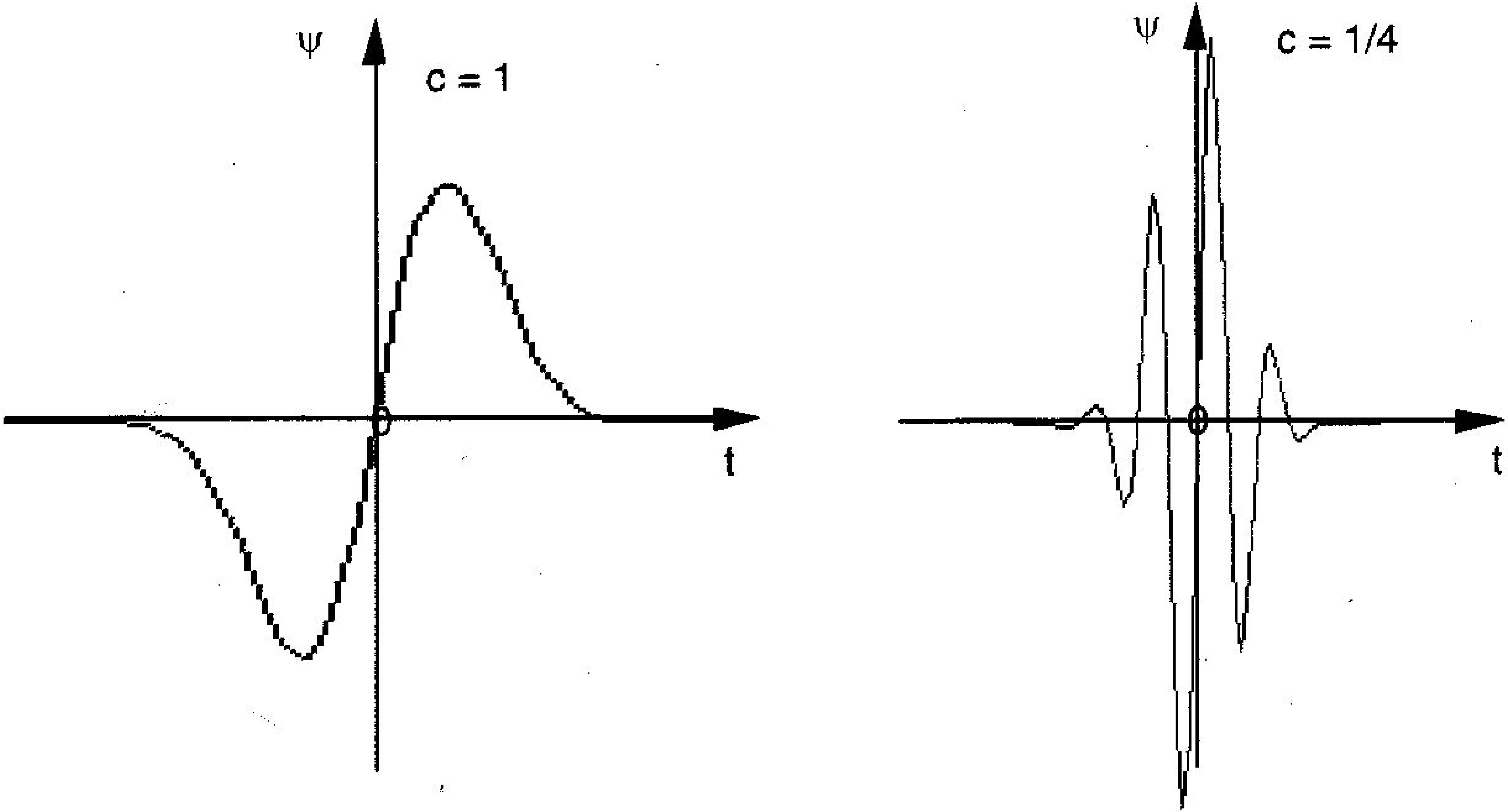
$$f = f_0 / c \Rightarrow c = \frac{f_0}{f}$$

Window size of a wavelet:

example of wavelet with time decay

$$\psi(c,t) = e^{-t^2/c} \cdot \sin\left(\frac{\omega t}{c}\right)$$

now, consider this wavelet for the scaling factors $c = 1$, and $c = 1/4$



Now we can formulate our WT:

STFT: $Y(\omega,\tau) = \int_{-\infty} y(t) \cdot s(t-\tau) e^{-j\omega t} dt$

$\psi(t)$

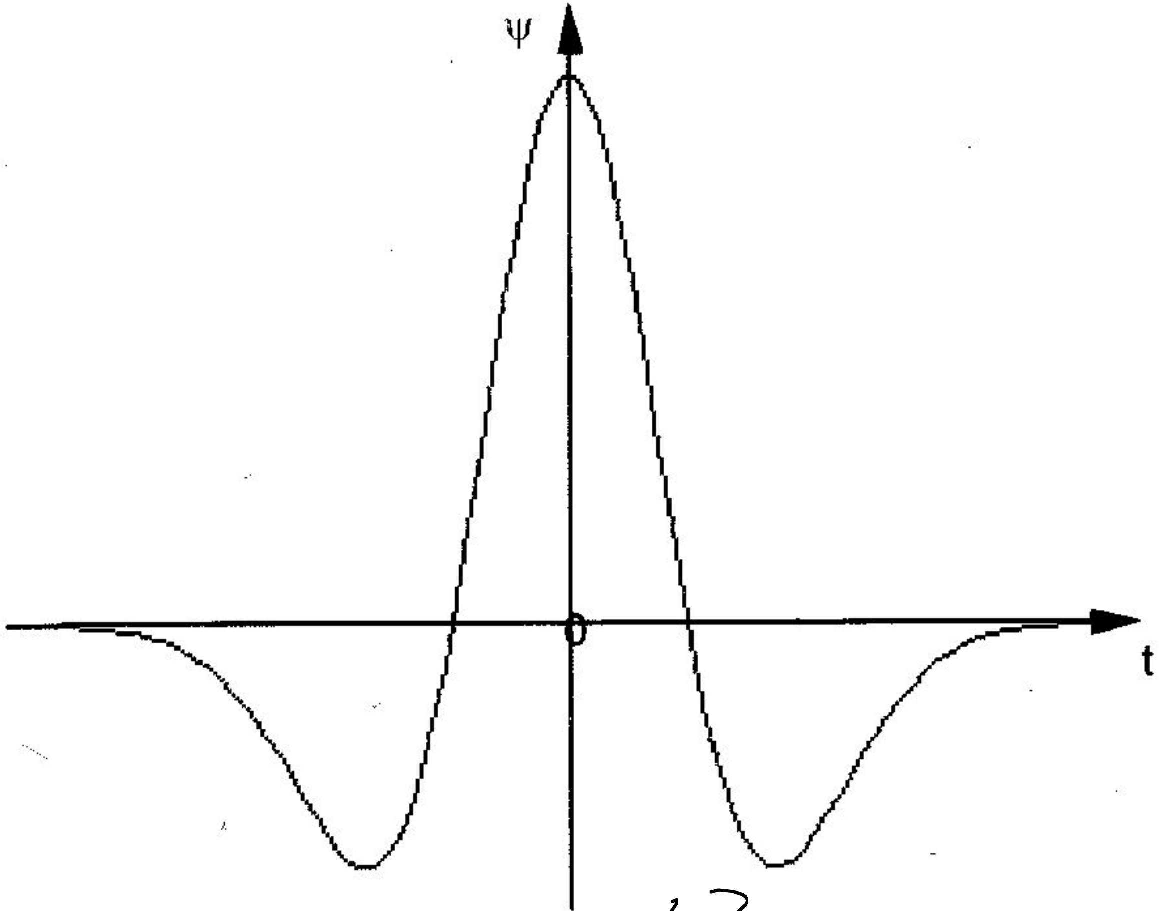
→ obviously our formula for the WT is:

$$Y_w(c,\tau) = \frac{1}{\sqrt{c}} \int_{-\infty}^{\infty} y(t)\,\psi\left(\frac{t-\tau}{c}\right) dt$$

Popular Wavelets:

1) "Mexican-Hat-Wavelet"

$$\psi(t) = \left(1 - \frac{t^2}{\sigma^2}\right) \cdot e^{-\frac{t^2}{2\sigma^2}}$$



2) $$\psi(t) = e^{-\frac{\pi t^2}{\sigma^2}} \left(e^{j\omega t} - e^{-\frac{\omega^2}{2}} \right)$$

($\Rightarrow$ complex wavelet function)

Reverse transformation:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi_F(\omega)|^2}{\omega} d\omega$$

ψ_F = Fourier-transform of basis wavelet $\psi(t)$

$$y(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_w(c,\tau)\,\psi(t,c,\tau)\frac{1}{c^2}\,dc\,d\tau$$

Energy Conservation:

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y_w(c,\tau)|^2 \cdot \frac{1}{c^2}\,dc\,d\tau$$