Lehrstuhl für STEUERUNGS- UND REGELUNGSTECHNIK Technische Universität München Prof. Dongheui Lee	MACHINE LEARNING IN ROBOTICS Exercises: Markov processes & HMM

Exercise 1

Let us call $\boldsymbol{P}$ the transition matrix of an irreducible Markov chain with finitely many states.

a) Show that $\boldsymbol{Q} = \frac{1}{2}(\boldsymbol{I} + \boldsymbol{P})$, where $\boldsymbol{I}$ stands for the identity matrix, is the transition matrix of an irreducible and aperiodic Markov chain.

b) Show that $\boldsymbol{P}$ and $\boldsymbol{Q} = \frac{1}{2}(\boldsymbol{I} + \boldsymbol{P})$ have the same stationary distributions.

c) Discuss, physically, how the two chains are related.

Solution Exercise 1

a) Let p_{ij} be the entries of $\boldsymbol{P}$. Then the entries q_{ij} of $\boldsymbol{Q}$ are

$$\begin{cases} q_{ij} = \dfrac{1}{2}p_{ij}, & \text{if } i \neq j \\ q_{ii} = \dfrac{1}{2}(1 + p_{ii}) & \text{otherwise} \end{cases}$$

The graph of the new chain has more arrows than the original one. Hence it is also irreducible. But the new chain also has self-loops for each i because $q_{ii} > 0$ for all i. Hence it is aperiodic.

b) Let π be a stationary distribution for $\boldsymbol{P}$. Then $\pi\boldsymbol{P} = \pi$. We must show that $\pi\boldsymbol{Q} = \pi$.

$$\pi\boldsymbol{Q} = \frac{1}{2}(\pi\boldsymbol{I} + \pi\boldsymbol{P}) = \frac{1}{2}(\pi + \pi) = \pi$$

c) The physical meaning of the new chain is that it represents a slowing down of the original one. Indeed, all outgoing probabilities have been halved, while the probability of staying at the same state has been increased. The chain performs the same transitions as the original one but stays longer at each state.

Exercise 2

A protocol for data transmission shall be analysed using a Markov chain with 3 states. The probability for the transition from $state1$ (*check interface for incoming data*) to $state2$ (*check address*) is 0.1. The address is correct with probability 0.4. In this case, there is a transition to $state3$ (*message received*). Otherwise, the system returns to $state1$. If a message was received and there is no further message (probability 0.7), the system leaves $state3$ and enters in the $state1$. If there is a further message, it enters in the $state2$.

a) Specify the matrix of transition probabilities.

b) Draw the corresponding Markov chain.

c) What is the probability for the system to be in $state1$?

Solution Exercise 2

a)

$$\boldsymbol{P} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{pmatrix}$$

remember: $\sum_j \boldsymbol{P}_{ij} = 1 \forall i$

b) The resulting Markov chain is shown in Fig. 1

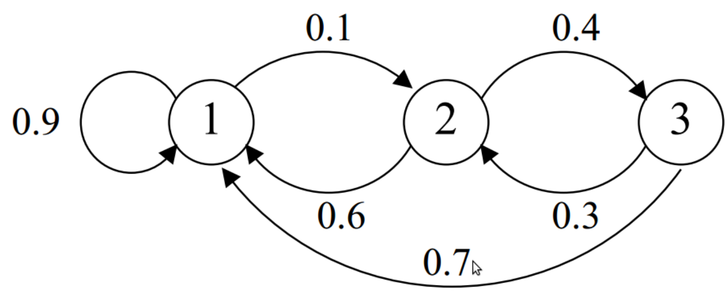

Figure 1: Markov chain

c) Given is a set of equations $\boldsymbol{p} = \boldsymbol{P}^T \boldsymbol{p}$:

$$p1 = 0.9p1 + 0.6p2 + 0.7p3 \tag{1}$$

$$p2 = 0.1p1 + 0.3p3 \tag{2}$$

$$p3 = 0.4p2 \tag{3}$$

and the normalization equation

$$\sum_i pi = p1 + p2 + p3 = 1 \tag{4}$$

Solving 4 for $p2$ and inserting it in 2 and 3, respectively, gives:

$$p3 = \frac{2}{7} - \frac{2}{7}p1 \tag{5}$$

$$p1 = \frac{10}{11} - \frac{13}{11}p3 \tag{6}$$

Inserting 5 in 6 gives the stationary probability to be in $state1$:

$$p1 = \frac{44}{51} = 0.8627.$$

Exercise 3

The EM algorithm finds parameters $\boldsymbol{\theta}$ which maximize

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{i-1}) = \mathbb{E}_{\boldsymbol{Z}}[\ln p(\boldsymbol{X}, \boldsymbol{Z}|\boldsymbol{\theta})|\boldsymbol{X}, \boldsymbol{\theta}^{i-1}]$$

Similarly, the Baum-Welch algorithm finds the model parameters λ which maximize

$$Q(\lambda|\lambda^{i-1}) = \mathbb{E}_{\mathcal{Q}}[\log p(\mathcal{O}, \mathcal{Q}|\lambda)|\mathcal{O}, \lambda^{i-1}] = \sum_{\forall \mathcal{Q}} \log\left[p(\mathcal{O}, \mathcal{Q}|\lambda)\right] p(\mathcal{Q}|\mathcal{O}, \lambda^{i-1})$$

where $\mathcal{O} = [o_1, o_2, \ldots, o_T]$.

Show that the maximization of $Q(\lambda|\lambda^{i-1})$, with the constraint $\sum_{i=1}^{N} \hat{\pi}_i = 1$, leads to the update equation

$$\hat{\pi}_i = \gamma_1(i) = \sum_{j=1}^{N} \xi_1(i,j) = \sum_{j=1}^{N} p(q_1 = s_i, q_2 = s_j|\mathcal{O}, \lambda^{i-1}) = p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})$$

Solution Exercise 3

We want to find the parameters λ that maximize:

$$\underset{\lambda}{\operatorname{argmax}}\, Q(\lambda|\lambda^{i-1}) = \underset{\lambda}{\operatorname{argmax}} \sum_{\forall \mathcal{Q}} \log\left[p(\mathcal{O}, \mathcal{Q}|\lambda)\right] p(\mathcal{Q}|\mathcal{O}, \lambda^{i-1})$$

The term $p(\mathcal{O}, \mathcal{Q}|\lambda)$ can be written as

$$p(\mathcal{O}, \mathcal{Q}|\lambda) = \pi_{q_1} b_{q_1}(o_1) \prod_{t=2}^{T} a_{q_{t-1},q_t} b_{q_t}(o_t)$$

Taking the $\log$ gives us

$$\log p(\mathcal{O}, \mathcal{Q}|\lambda) = \log \pi_{q_1} + \sum_{t=1}^{T} \log b_{q_t}(o_t) + \sum_{t=2}^{T} \log a_{q_{t-1},q_t}$$

and substituting into $Q(\lambda|\lambda^{i-1})$, we get

$$Q(\lambda|\lambda^{i-1}) = \sum_{\forall \mathcal{Q}} \log \pi_{q_1} p(\mathcal{Q}|\mathcal{O}, \lambda^{i-1}) + \sum_{\forall \mathcal{Q}} \sum_{t=1}^{T} \log b_{q_t}(o_t) p(\mathcal{Q}|\mathcal{O}, \lambda^{i-1}) + \sum_{\forall \mathcal{Q}} \sum_{t=2}^{T} \log a_{q_{t-1},q_t} p(\mathcal{Q}|\mathcal{O}, \lambda^{i-1})$$

We can insert the constraint $\sum_{i=1}^{N} \hat{\pi}_i = 1$ in the previous expression using Lagrange multipliers, obtaining the Lagrangian

$$L(\lambda|\lambda^{i-1}) = Q(\lambda|\lambda^{i-1}) - \lambda_\pi \left(\sum_{i=1}^{N} \hat{\pi}_i - 1 \right)$$

that can be maximized by zeroing the partial derivatives with respect to $\hat{\pi}_i$ and λ_π

$$\begin{cases} \dfrac{\partial L(\lambda|\lambda^{i-1})}{\partial \hat{\pi}_i} = \dfrac{\partial Q(\lambda|\lambda^{i-1})}{\partial \hat{\pi}_i} - \lambda_\pi = 0 \\ \dfrac{\partial L(\lambda|\lambda^{i-1})}{\partial \lambda_\pi} = -\left(\displaystyle\sum_{i=1}^{N} \hat{\pi}_i - 1 \right) = 0 \end{cases} \tag{7}$$

Solving the first equation in (7) yields

$$\frac{\partial L(\lambda|\lambda^{i-1})}{\partial \hat{\pi}_i} = \frac{\partial}{\partial \hat{\pi}_i}\left(\sum_{\forall \mathcal{Q}} \log \hat{\pi}_{q_1} p(\mathcal{Q}|\mathcal{O}, \lambda^{i-1})\right) - \lambda_\pi = \frac{\partial}{\partial \hat{\pi}_i}\left(\sum_{j=1}^{N} \log \hat{\pi}_j p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})\right) - \lambda_\pi$$
$$= \frac{p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})}{\hat{\pi}_i} - \lambda_\pi = 0 \longrightarrow \hat{\pi}_i = \frac{p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})}{\lambda_\pi}$$

and substituting $\hat{\pi}_i$ in the second equation in (7)

$$\sum_{i=1}^{N} \hat{\pi}_i - 1 = \sum_{i=1}^{N} \frac{p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})}{\lambda_\pi} - 1 = \frac{\sum_{i=1}^{N} p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})}{\lambda_\pi} - 1 = \frac{1}{\lambda_\pi} - 1 = 0 \longrightarrow \lambda_\pi = 1$$

Hence, the value of $\hat{\pi}$ that maximize $Q(\lambda|\lambda^{i-1})$ is

$$\hat{\pi}_i = p(q_1 = s_i|\mathcal{O}, \lambda^{i-1})$$

Exercise 4

A Hidden Markov Model with 2 states $\{s_1 = H, s_2 = C\}$ and 3 possible observations based on the number of observed sizes $\{small, medium, large\}$ is given with transition probability matrix:

$$A = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

and observation matrix:

$$B = \begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{pmatrix}$$

The prior probabilities of states are $\pi_1 = 0.6, \pi_2 = 0.4$.

a) Compute the probability of the state sequence $\{q_1 = H\ q_2 = H\ q_3 = C\ q_4 = C\}$.

b) Compute the probability $P(o_1 = S\ o_2 = M\ o_3 = S\ o_4 = L \mid q_1 = H\ q_2 = H\ q_3 = C\ q_4 = C)$.

Solution Exercise 4

a) $P(q_1)P(q_2|q_1)P(q_3|q_2)P(q_4|q_3) = 0.6 * 0.7 * 0.3 * 0.6 = 0.0756$.

b) Compute the probability $P(o_1|q_1)P(o_2|q_2)P(o_3|q_3)P(o_4|q_4) = 0.1 * 0.4 * 0.7 * 0.1 = 0.0028$.

Exercise 5

Two HMMs with different structure are shown in Fig. 1.

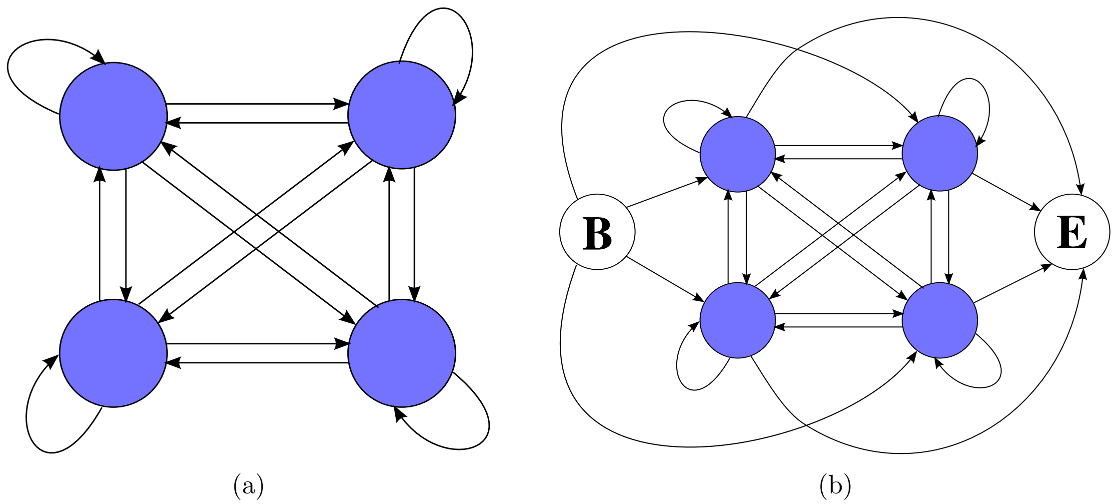

Figure 2: HMMs with different structures.

a) Consider the HMM in Fig. 2(a). Show that the sum of the probability of all possible state sequences $\mathcal{Q} = [q_1, \ldots, q_L]$ of length L is equal to 1.

b) Assume that the HMM has a begin (B) and an end (E) state, as in Fig. 2(b). The end state has probability ε. Show that the sum of the probability over all state sequences $\mathcal{Q} = [q_1, \ldots, q_L]$ of length L (and properly terminating by making a transition to the end state) is $p = \varepsilon(1-\varepsilon)^{L-1}$. Use this result to show that the sum of the probability over all possible state sequences of any length is 1.

Hint: Use the result $\sum_{i=0}^{\infty} x^i = 1/(1-x)$ for $0 < x < 1$.

Solution Exercise 5

a) The probability for a sequence $\mathcal{Q} = [q_1, \ldots, q_L]$ with length L is

$$p(\mathcal{Q}) = p(q_L|q_{L-1}, \ldots, q_2|q_1, q_1) = p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_2|q_1)p(q_1)$$

The sum of the probability of all possible state sequences of length L is

$$\sum_{\forall \mathcal{Q}, L} p(\mathcal{Q}) = \sum_{q_1}\sum_{q_2}\cdots\sum_{q_L} p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_2|q_1)p(q_1)$$

where $\forall \mathcal{Q}, L$ indicates we are considering all the sequences of length L.

Note that, for a particular q_i, the sum of transition probability for all possible q_{i+1} is 1, i.e.

$$\sum_{q_L} p(q_L|q_{L-1}) = 1, \sum_{q_{L-1}} p(q_{L-1}|q_{L-2}) = 1, \cdots, \sum_{q_2} p(q_2|q_1) = 1$$

Then

$$\begin{aligned}\sum_{\forall \mathcal{Q}, L} p(\mathcal{Q}) &= \sum_{q_1}\sum_{q_2}\cdots\sum_{q_L} p(q_L|q_{L-1})p(q_{L-1}|q_{L-2})\ldots p(q_2|q_1)p(q_1) \\ &= \sum_{q_1}\sum_{q_2} p(q_2|q_1)p(q_1) = \sum_{q_1} p(q_1) = 1\end{aligned}$$

b) The sum of the probability of all possible state sequences of length L and terminating in E is

$$\sum_{\forall \mathcal{Q}, L} p(\mathcal{Q}) = \sum_{q_1} \sum_{q_2} \cdots \sum_{q_L} p(q_E|q_L) p(q_L|q_{L-1}) p(q_{L-1}|q_{L-2}) \ldots p(q_1)$$

where $\forall \mathcal{Q}, L$ indicates we are considering all the sequences of length L.

Note that

$$\sum_{q_1} p(q_1) = 1 \qquad \text{and} \qquad p(q_E|q_L) = \varepsilon$$

and that, for a particular q_i, the sum of transition probability for all possible q_{i+1} (except for $i+1 = E$) is $1 - \varepsilon$, i.e.

$$\sum_{q_L} p(q_L|q_{L-1}) = 1 - \varepsilon, \sum_{q_{L-1}} p(q_{L-1}|q_{L-2}) = 1 - \varepsilon, \cdots, \sum_{q_2} p(q_2|q_1) = 1 - \varepsilon$$

Then

$$\begin{aligned} \sum_{\forall \mathcal{Q}, L} p(\mathcal{Q}) &= \sum_{q_1} \sum_{q_2} \cdots \sum_{q_L} p(q_E|q_L) p(q_L|q_{L-1}) p(q_{L-1}|q_{L-2}) \ldots p(q_1) \\ &= \varepsilon \sum_{q_1} \sum_{q_2} \cdots \sum_{q_L} p(q_L|q_{L-1}) p(q_{L-1}|q_{L-2}) \ldots p(q_1) = \varepsilon (1 - \varepsilon)^{L-1} \end{aligned}$$

The sum of the probability over all possible state sequences of any length is

$$\sum_{\forall \mathcal{Q}} p(\mathcal{Q}) = \sum_{L=1}^{\infty} \sum_{\forall \mathcal{Q}, L} p(\mathcal{Q}) = \sum_{L=1}^{\infty} \varepsilon (1 - \varepsilon)^{L-1} = \varepsilon \sum_{L-1=0}^{\infty} (1 - \varepsilon)^{L-1} = \varepsilon \frac{1}{\varepsilon} = 1$$