TUTORIAL 06

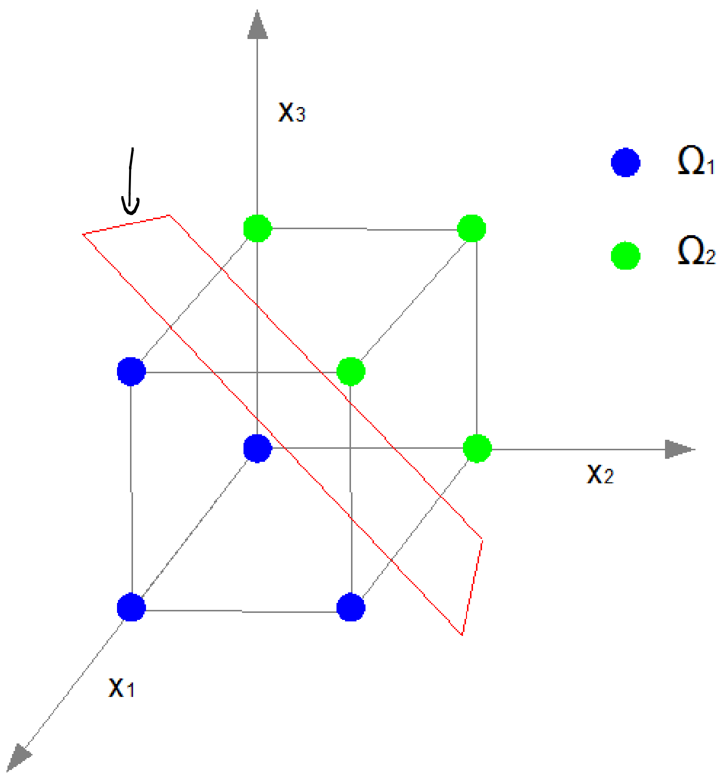
Notiztitel 06.06.2014

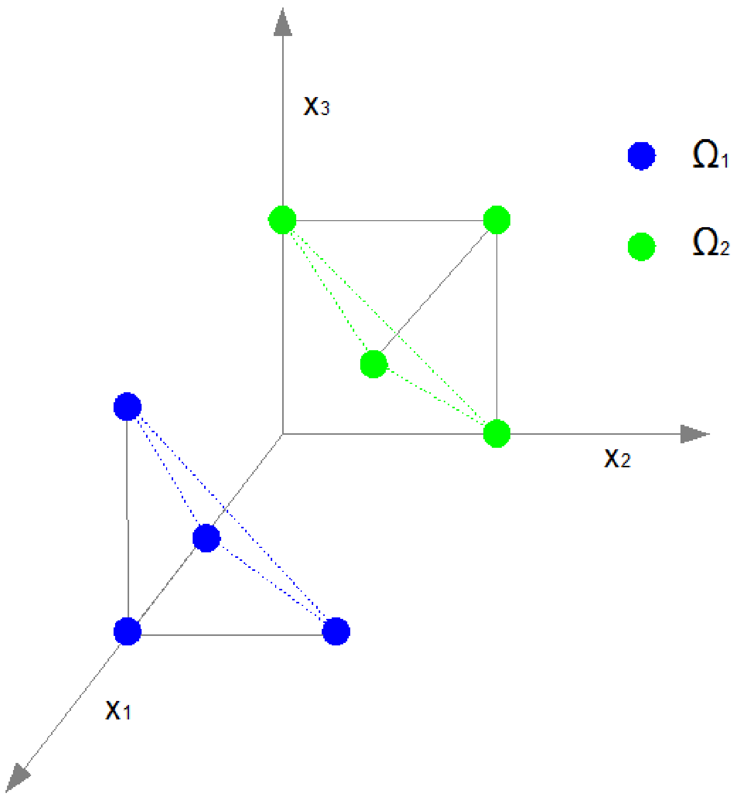
Outline:

- PCA
- Linear Discriminant Analysis (LDA)

Exercise 16

a)





b) Statistical features:

- $\underline{m}_x = E\{\underline{x}\} = \frac{1}{K} \sum_{k=1}^{K} \underline{x}(k)$ (Eq 2.64)

$$\underline{m}_\Omega = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

- $\underset{\sim}{C}_x = E\{(\underline{x} - \underline{m}_x) \cdot (\underline{x} - \underline{m}_x)^T\} =$

$= \frac{1}{K} \sum_{k=1}^{K} [\underline{x}(k) \cdot x(k)^T] - \underline{m}_x \underline{m}_x^T$ (Eq. 2.65)

$$C_{xx} = \frac{1}{8}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \right) - \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Eigen-values and Eigen-vectors of C_{xx}

$$\det\left(C_{xx} - \lambda I \right) \overset{!}{=} 0$$

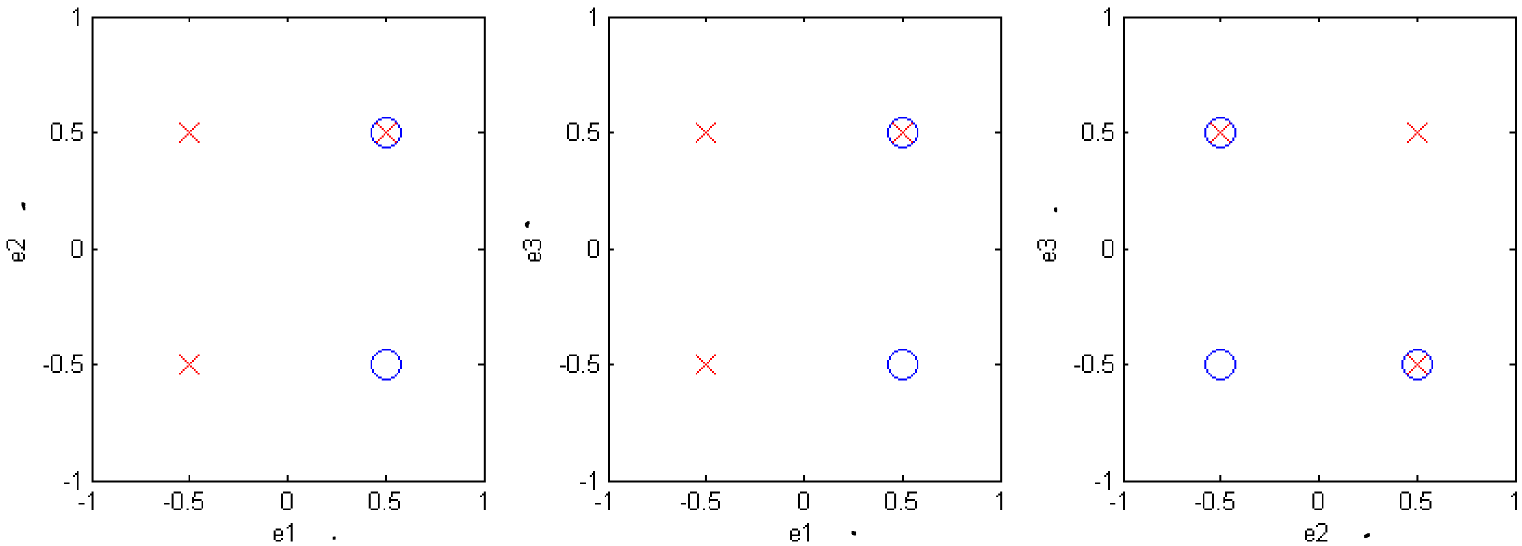
$$\det \begin{bmatrix} 0.25-\lambda & 0 & 0 \\ 0 & 0.25-\lambda & 0 \\ 0 & 0 & 0.25-\lambda \end{bmatrix}$$

$$\lambda_1 = 0.25$$
$$\lambda_2 = 0.25$$
$$\lambda_3 = 0.25$$

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Transform accorging to Eq. 2.71

$$y(k) = \underset{\sim}{U} \left(\underline{x}(k) - \underline{m}_x \right)$$



c) The patterns cannot be separated in the reduced feature space.

× Ω_1

○ Ω_2

d) LDA processing – statistical features.

- a priory – probability (2.78) $p_n = \frac{K_n}{K}$

$$p_{\Omega_1} = \frac{4}{8} = \frac{1}{2} \qquad p_{\Omega_2} = \frac{1}{2}$$

- class – centre (2.79) $\underline{m}_n = \frac{1}{\underline{K}_n} \sum_{k=1}^{\underline{K}_n} \underline{x}(k)$

$$\underline{m}_{\Omega_1} = \frac{1}{4} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \qquad \underline{m}_{\Omega_2} = \frac{1}{4} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

- Class – covariance matrix (2.80)

$$\underset{\sim}{C}_m = \frac{1}{\underline{K}_m} \sum_{k=1}^{K_m} \underline{x}(k) \cdot \underline{x}(k)^T - \underline{m}_m \underline{m}_m^T$$

$$\underset{\sim}{C}_{\Omega_1} = \frac{1}{16}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \qquad \underset{\sim}{C}_{\Omega_2} = \frac{1}{16}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

- Overall centre for all patterns: (2.81)

$$\underline{m} = \sum_{n=1}^{N} p_n \cdot \underline{m}_n \Rightarrow \underline{m}_{\Omega} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Instruction for LDA-procedure (pp. 37-38)

<u>Step 1</u>

- Intra class scattering matrix (2.82)

$$\underset{\sim}{C}_a = \sum_{n=1}^{N} p_n \cdot \underset{\sim}{C}_n \qquad \underset{\sim}{C}_a = \frac{1}{16}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

- Interclass scattering matrix (2.83)

$$\rightarrow \underset{\sim}{C}_b = \sum_{n=1}^{N} p_n \left(\underline{m}_n - \underline{m} \right) \left(\underline{m}_n - \underline{m} \right)^T$$

$$\underset{\sim}{C}_b = \frac{1}{16} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Step2

Eigen-values and Eigen-vectors of $\underset{\sim}{C}_a$

$$\underset{\sim}{\Lambda}_a = \frac{1}{16} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \underset{\sim}{U}_a = \begin{bmatrix} -0.58 & -0.15 & 0.80 \\ 0.58 & -0.77 & 0.22 \\ 0.58 & 0.62 & 0.54 \end{bmatrix}$$

$$\underset{\sim}{\Phi}_a = \underset{\sim}{\Lambda}_a^{-\frac{1}{2}} \underset{\sim}{U}_a \qquad (2.92)$$

$$\rightarrow \quad \Phi_a = \begin{bmatrix} -2.31 & -0.62 & 3.21 \\ 1.16 & -1.54 & 0.54 \\ 1.16 & 1.23 & 1.07 \end{bmatrix}$$

Step 3

Calculate C_b'

$$C_b' = \Phi_a \cdot C_b \cdot \Phi_a^T \qquad (\text{Eq. } 2.97)$$

$$C_b' = \begin{bmatrix} 1.5 & -0.66 & 0.35 \\ -0.66 & 0.29 & -0.16 \\ 0.35 & -0.16 & 0.08 \end{bmatrix}$$

Step 4

Eigenvalues and Eigenvectors of C_b'

Set transformation matrix Φ_b to Eigen-vectors U_b'

$$\Lambda_b' = \begin{bmatrix} 1.88 & 0 & 0 \\ 0 & \sim 0 & 0 \\ 0 & 0 & \sim 0 \end{bmatrix} \qquad U_b' = \begin{bmatrix} 0.90 & \sim 0 & 0.37 \\ -0.40 & 0.47 & 0.92 \\ 0.21 & 0.88 & 0.16 \end{bmatrix} = \Phi_b$$

Step 5

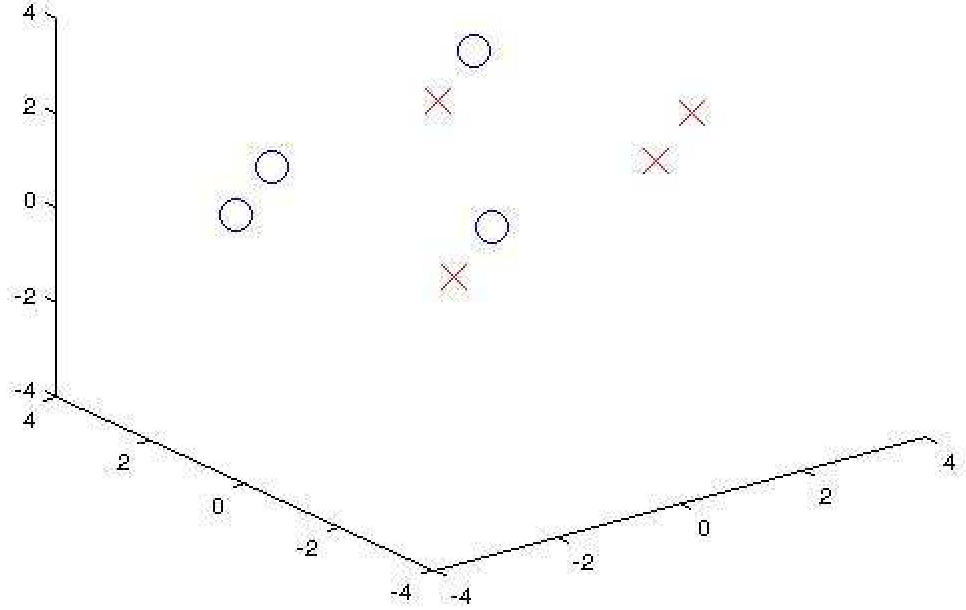
Calculate transformation matrix $\Phi = \Phi_b \cdot \Phi_a$ (2.90)

and transform patterns (Ω_1, Ω_2)

$$\Phi = \begin{bmatrix} -1.64 & 0.10 & 3.26 \\ 2.51 & 0.65 & -0.04 \\ 0.72 & -1.30 & 1.31 \end{bmatrix} \qquad x''(k) = \Phi \cdot x(k)$$

$$\Omega_1'' = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.64 \\ 2.51 \\ 0.72 \end{bmatrix}, \begin{bmatrix} -1.74 \\ 3.16 \\ -0.58 \end{bmatrix}, \begin{bmatrix} 1.62 \\ 2.45 \\ 2.03 \end{bmatrix} \right)$$

$$\Omega_2'' = \left(\begin{bmatrix} 1.53 \\ 3.13 \\ 0.73 \end{bmatrix}, \begin{bmatrix} 3.16 \\ 0.62 \\ 0.14 \end{bmatrix}, \begin{bmatrix} 3.26 \\ -0.04 \\ 1.31 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 0.65 \\ -1.30 \end{bmatrix} \right)$$


4
2
0
-2
-4
4
2
0
-2
-4
-4
-2
0
2
4

PCA - transform of $\{\Omega_1'', \Omega_2''\} = \Omega''$

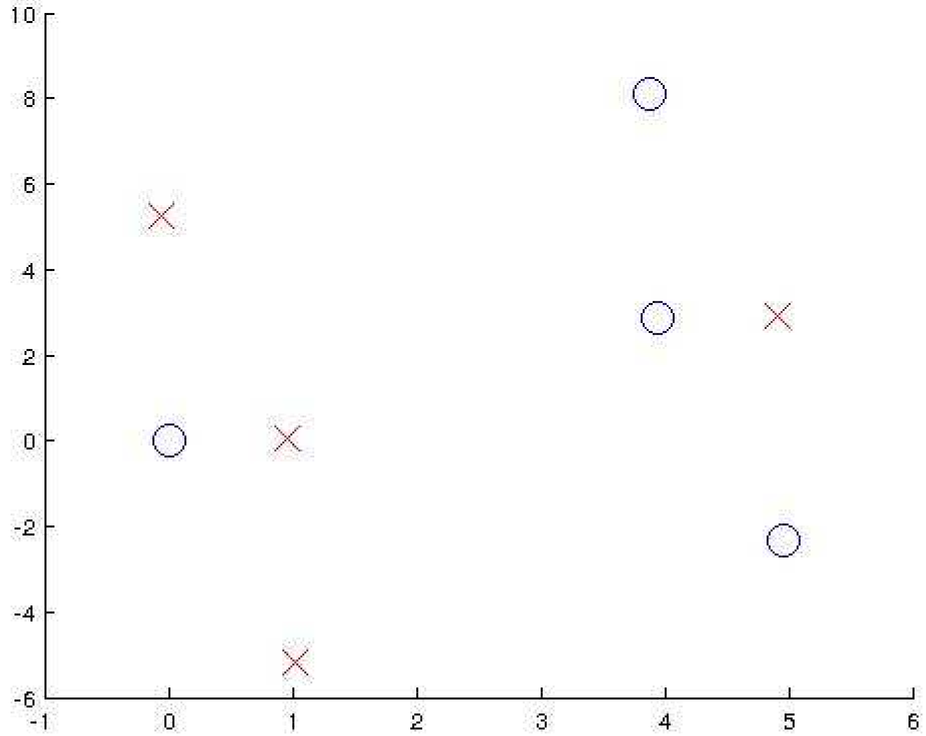
Eigen values of $\Omega'' \rightarrow$ $\Lambda_{\Omega''} = \begin{bmatrix} 0.43 & 0 & 0 \\ 0 & 1.57 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Eigen vectors of $\Omega'' \rightarrow$ $U_{\Omega''} = \begin{bmatrix} 0.39 & -0.21 & -0.90 \\ 0.48 & -0.78 & 0.40 \\ -0.78 & 0.59 & -0.21 \end{bmatrix}$

Transform according to Eq. 2.71 $\quad y(k) = U \cdot [x(k) - m_x]$

$${}^{2D}\Omega_1'' = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3.94 \\ 2.87 \end{bmatrix}, \begin{bmatrix} 4.96 \\ -2.32 \end{bmatrix}, \begin{bmatrix} 3.88 \\ 8.12 \end{bmatrix} \right)$$

"Ω_2" $= \left(\begin{bmatrix} 4.96 \\ 2.93 \end{bmatrix}, \begin{bmatrix} 0.97 \\ 0.06 \end{bmatrix}, \begin{bmatrix} -0.06 \\ 5.25 \end{bmatrix}, \begin{bmatrix} 1.02 \\ -5.19 \end{bmatrix} \right)$



The patterns Ω_1 and Ω_2 can be separated in the 2-D space applying a LDA-preprocessing

(see Matlab script "Exercise16.m")