TUTORIAL 05

Notiztitel 06.06.2014

Outline:

Special transformation in the time domain:

→ Principal Component Analysis (PCA)

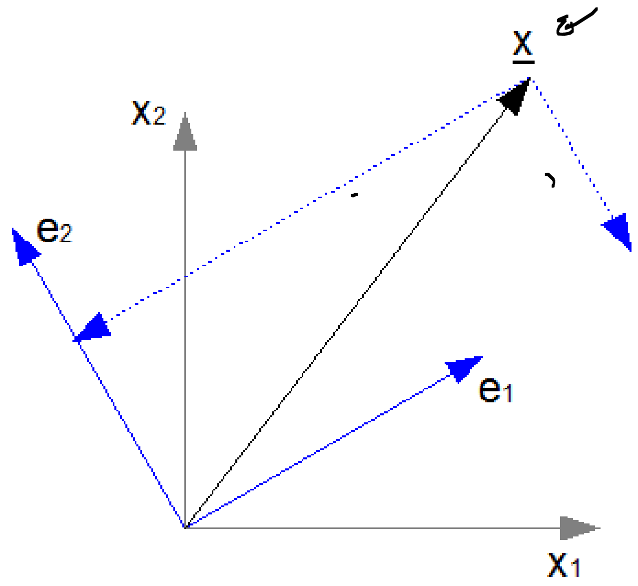
Exercise 14

EIGENVECTOR

$$\underline{e}_1^T \rightarrow \begin{bmatrix} e_{1,1} & \cdots & e_{1,N} \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\underline{y} = \underline{\Phi}_N \cdot \underline{x} = \begin{bmatrix} \underline{e}_1^T \\ \vdots \\ \underline{e}_N^T \end{bmatrix} \cdot \underline{x} = \begin{bmatrix} \underline{e}_1^T \underline{x} \\ \vdots \\ \underline{e}_N^T \underline{x} \end{bmatrix}$$

$\Rightarrow$ Projection of $\underline{x}$ to $\underline{e}_i$



$\Rightarrow$ Vector $\underline{y}$ contains in its components the projection of $\underline{x}$ to $\underline{e}_i$

$\Rightarrow$ $\underset{\sim}{\underline{\Phi}}^T$ is a rotation of the $\underline{x}$ - coordination - system to the $\underline{e}$ - coordination - system

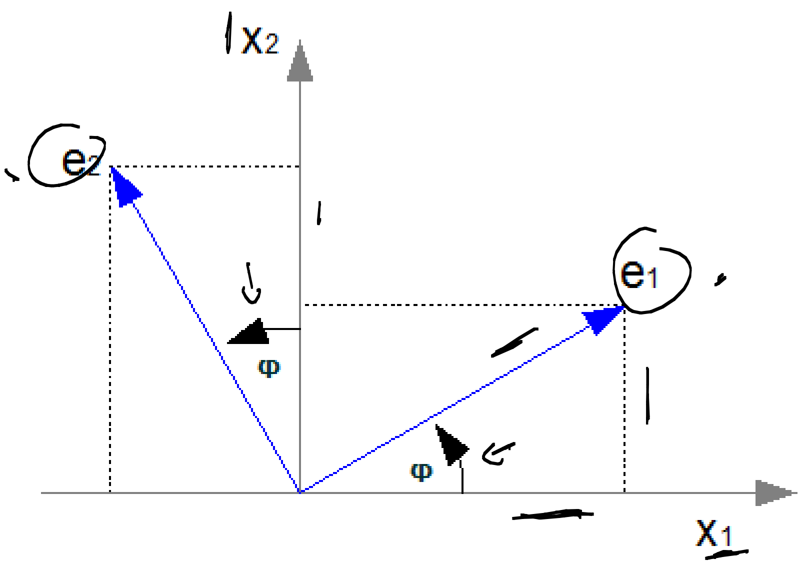
b) No, $|e_i|$ normalized! $|e_1| = 1$

if $|e_i| > 1 \rightarrow$ expansion

if $|e_1| < 1 \rightarrow$ compression

c) Projection:

$$\cos\varphi = \frac{|y|}{|x|}$$



$|e_1| = 1$

$|e_2| = 1$

for $\underline{e}_1$

$$\left.\begin{aligned} \cos\varphi &= \frac{x_1}{1} \Rightarrow x_1 = \cos\varphi \\ \sin\varphi &= \frac{x_2}{1} \Rightarrow x_2 = \sin\varphi \end{aligned}\right\} \underline{e}_1$$

for $\underline{e}_2$

$$\left.\begin{aligned} x_1 &= -\sin\varphi \\ x_2 &= \cos\varphi \end{aligned}\right\} \underline{e}_2$$

$$\underline{\Phi} = \left[\underline{e}_1 , \underline{e}_2 \right]^T = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$ PASSIVE

rotation of the coordination system - counterclockwise

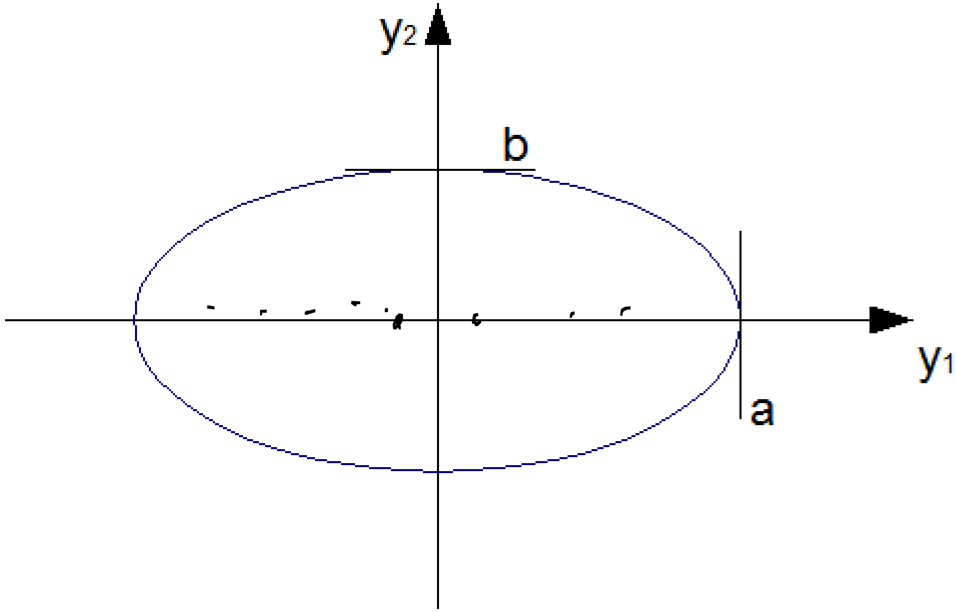
$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$ ACTIVE

rotation of a point P to P' by an angle θ - clockwise

EXERCISE 15

a) Principal Component Analysis (PCA) results in a rotation of the ellipse, in a way that the eigenvectors build the new coordinate system. At the same time the PCA generates a diagonal covariance matrix.

→ if it can be shown, that the covariance matrix of the rotated object has a diagonal form, a rotation by the eigen-vectors of the original covariance matrix has been performed. Then the eigenvectors are identical with the principal components of the ellipse:



Covariance of all points inside the ellipse: (Eq. 2.65)

$$C = E\left\{ (\underline{y} - \underline{m}_y)(\underline{y} - \underline{m}_y)^T \right\} = E\left\{ \underline{y} \cdot \underline{y}^T \right\} \quad \underline{m}_y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} E\{Y_1^2\} & E\{Y_1 \cdot Y_2\} \\ E\{Y_1 Y_2\} & E\{Y_2^2\} \end{bmatrix}$$

We have to show that

$$E\{Y_1 Y_2\} = 0$$

$$E\{Y_1 Y_2\} = \frac{1}{N} \sum_{i=1}^{N} Y_{1,i} \cdot Y_{2,i}$$ for N point inside the ellipse

for each pair of points: (Y_1, Y_2) an exact one symmetric pair can be found: $(-Y_1, Y_2)$

$$\Downarrow$$

$$C = \begin{bmatrix} E\{Y_1^2\} & 0 \\ 0 & E\{Y_2^2\} \end{bmatrix} \Leftarrow \sum_{i=1}^{N} Y_{1,i} Y_{2,i} = 0$$

b) Matematical formulation of $E\{Y_1 \; Y_2\}$

- $$E\{Y_1 Y_2\} = \frac{\iint_{ellipse} Y_1 \cdot Y_2 \, dY_1 \, dY_2}{\iint_{ellipse} dY_1 \, dY_2} \Rightarrow \pi a b$$

$$\iint_{ellipse} Y_1 Y_2 \, dY_1 \, dY_2 \leftarrow$$

Normal form of an ellipse:

$$\frac{Y_1^2}{a^2} + \frac{Y_2^2}{b^2} = 1$$

$$Y_2 = \pm \frac{b}{a} \sqrt{a^2 - Y_1^2}$$

$$\Rightarrow \int_{Y_1=-a}^{a} \int_{Y_2=-\frac{b}{a}\sqrt{a^2-Y_1^2}}^{\frac{b}{a}\sqrt{a^2-Y_1^2}} Y_1 Y_2 \, dY_1 \, dY_2$$

$$= \int_{Y_1=-a}^{a} Y_1 \left[\frac{Y_2^2}{2} \right]_{-b/a\sqrt{a^2-Y_1^2}}^{b/a\sqrt{a^2-Y_1^2}} dY_1 = 0$$

$$\cdot\ E\{Y_1^2\} = \frac{\iint_{ellipse} Y_1^2 \, dY_1 \, dY_2}{\iint_{ellipse} dY_1 \, dY_2} \rightarrow \pi \cdot a b$$

$$= \frac{1}{\pi a b} \int_{Y_1=-a}^{a} \int_{Y_2=-\frac{b}{a}\sqrt{a^2-Y_1^2}}^{\frac{b}{a}\sqrt{a^2-Y_1^2}} Y_1^2 \, dY_1 \, dY_2 =$$

$$= \frac{1}{\pi a b} \int_{y_1 = -a}^{a} y_1^2 \left[y_2 \right]_{-b/a\sqrt{a^2-y_1^2}}^{b/a\sqrt{a^2-y_1^2}} \, d y_1 =$$

$$= \frac{1}{\pi a b} \int_{y_1 = -a}^{a} y_1^2 \left(\frac{b}{a} \cdot \sqrt{a^2 - y_1^2} + \frac{b}{a} \sqrt{a^2 - y_1^2} \right) d y_1 =$$

$$= \frac{2}{\pi a^2} \int_{y_1 = -a}^{a} y_1^2 \sqrt{a^2 - y_1^2} \, d y_1$$

$$\int x^2 \cdot \sqrt{a^2 - x^2}\, dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \cdot \arcsin\frac{x}{a}\right)$$

$$\Rightarrow \frac{2}{\pi a^2}\left[\underbrace{-\frac{Y_1}{4}\cdot\sqrt{(a^2 - Y_1^2)^3}}_{0} + \frac{a^2}{8}\left(\underbrace{Y_1\sqrt{a^2 - Y_1^2}}_{0} + a^2 \arcsin\frac{Y_1}{a}\right)\right]_{-a}^{a} =$$

$$= \frac{2}{\pi a^2}\cdot\frac{a^2}{8}\left(a^2 \cdot \underbrace{\arcsin(1)}_{\pi/2} - a^2 \underbrace{\arcsin(-1)}_{-\pi/2}\right)$$

$$= \frac{a^2}{4\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{a^2}{4} \longrightarrow E\{Y_1^2\} \Rightarrow E\{Y_2^2\} = \frac{b^2}{4}$$

$$\underset{\sim}{C} = \begin{bmatrix} a^2/4 & 0 \\ 0 & b^2/4 \end{bmatrix}$$