TUTORIAL 02

Notiztitel 16.05.2014

Preprocessing:

Transformations in the frequency domain

- Discrete Fourier Transformation (DFT)
- Discrete Cosine Transformation (DCT)

Exercise 4

a) $$\sum_{k=0}^{N-1} e^{j \frac{2\pi m}{N} \cdot k} \cdot e^{-j \frac{2\pi m'}{N} \cdot k} =$$

$$= \sum_{k=0}^{N-1} e^{j \frac{2\pi k}{N} (n - m)}$$

• Case 1 $\quad n = m$

$$\sum_{k=0}^{N-1} e^{j \frac{2\pi k}{N} (n - m)} = \sum_{k=0}^{N-1} e^{0} = \sum_{k=0}^{N-1} 1 = N;$$

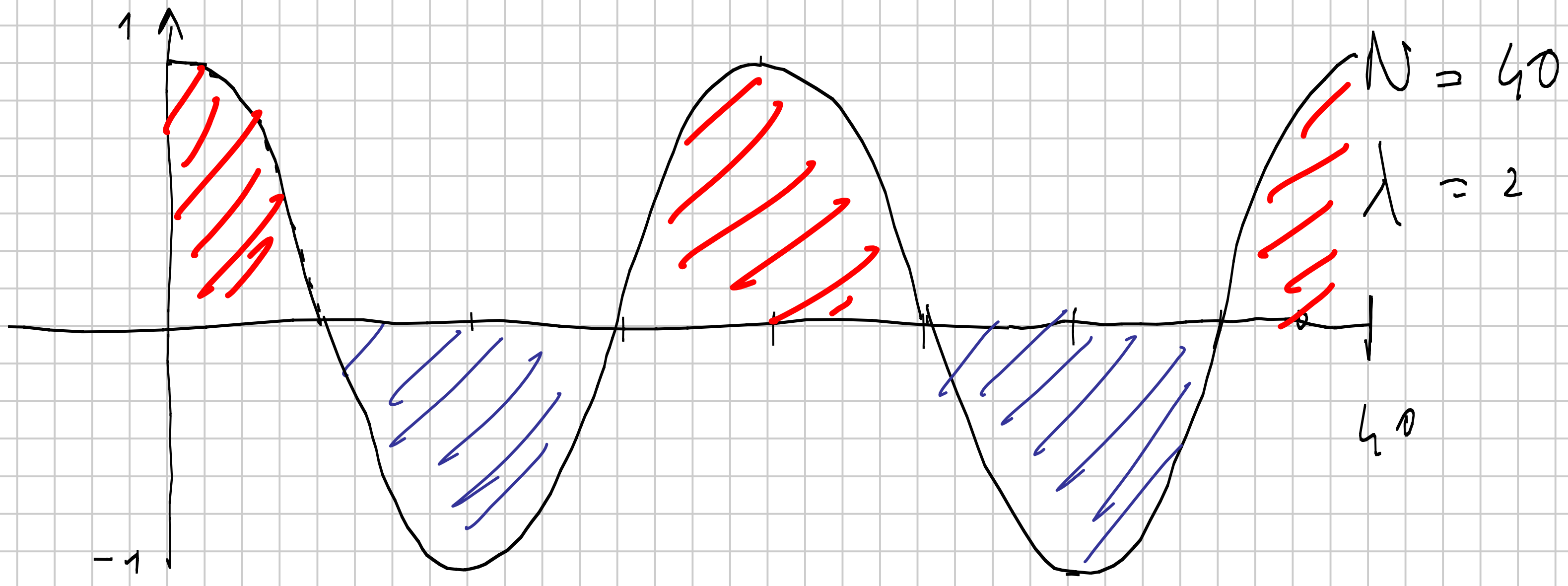
- Case 2 $\quad n \neq m$

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}(n-m)} \Rightarrow \quad \lambda = n - m$$

$$= \sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}\lambda} = \sum_{k=0}^{N-1} \cancel{\cos\left(\frac{2\pi \lambda k}{N}\right)} \to 0 + j \sum_{k=0}^{N-1} \cancel{\sin\left(\frac{2\pi \lambda k}{N}\right)} \to 0$$

$$\cos\left(\frac{2\pi \lambda k}{N}\right) \rightarrow \qquad \frac{N}{\lambda} \longrightarrow |\lambda| \text{ periods}$$

$$m \quad 0 \ldots\ldots N-1$$



If the periodic function is splitted into segments with the same width, for each positive value a negative value can be found as well.

$\Rightarrow$ The sum is 0 $\Rightarrow$ this is valid also for $\sin\left(\frac{2\pi\lambda K}{N}\right)$

$$= \sum_{k=0}^{N-1} e^{j\frac{2\pi k}{N}(n-m)} = 0 \qquad n \neq m$$

b) $$\sum_{k=0}^{N-1} \cos\left(\frac{2\pi n k}{N}\right) \cdot \cos\left(\frac{2\pi m}{N} k\right)$$

$$\cos\alpha = \frac{1}{2}\left(e^{j\alpha} + e^{-j\alpha}\right)$$

$$= \sum_{k=0}^{N-1} \frac{1}{2}\left[e^{j\frac{2\pi n k}{N}} + e^{-j\frac{2\pi n}{N}k}\right] \cdot \frac{1}{2}\left[e^{j\frac{2\pi m k}{N}} + e^{-j\frac{2\pi m k}{N}}\right]$$

$$= \frac{1}{4} \cdot \sum_{k=0}^{N-1} \left\{ e^{j\frac{2\pi k}{N}(n+m)} + e^{j\frac{2\pi k}{N}(n-m)} + e^{-j\frac{2\pi k}{N}(n+m)} + e^{-j\frac{2\pi k}{N}(n-m)} \right\}$$

- **Case 1** $\qquad m = m$

$$\sum_{k=0}^{N-1} \cos\left(\frac{2\pi m k}{N}\right) \cdot \cos\left(\frac{2\pi m k}{N}\right) =$$

$$= \frac{1}{4} \sum_{k=0}^{N-1} \left(e^{j\frac{2\pi k}{N} \cdot 2m} + e^{-j\frac{2\pi k}{N} \cdot 2m} + 1 + 1 \right) =$$

$$= \frac{1}{4} \sum_{k=0}^{N-1} \left(2 + \underbrace{\cancelto{0}{2 \cos\left(\frac{4\pi k m}{N}\right)}} \right) = \frac{1}{4} \sum_{k=0}^{N-1} 2 = \frac{N}{2}$$

- **Case 2** $\qquad m \neq m$

The sum $\to 0$

Exercise 5

$$\underline{Y(k)} = \frac{1}{N} \sum_{m=0}^{N-1} y(m) \cdot e^{j\frac{2\pi k}{N} \cdot m} \qquad (Eq\ 2.21)$$

Discrete - Fourier Transformation $\qquad (Eq\ 2.20)$

$$\underline{\underline{Y_{DFF}(m)}} = \sum_{k=0}^{N-1} y(k) \cdot e^{-j\frac{2\pi m}{N} \cdot k}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi m}{N} k} \cdot \sum_{m=0}^{N-1} y(m) \cdot e^{j\frac{2\pi k}{N} \cdot m}$$

Orthogonality from Ex. 4:

$$\sum_{k} e^{-\varphi_n} \, e^{\varphi_m} = \begin{cases} N & n = m \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{DFT}(m) = \frac{1}{N} \underbrace{\sum_{m=0}^{N-1} \sum_{k=0}^{N-1} e^{-j\frac{2\pi m}{N}\cdot k} \cdot e^{j\frac{2\pi m}{N} k}}_{N} \cdot y(m) =$$

$$= \frac{1}{N} \cdot N \cdot y(m) = y(m)$$

Exercise 6

a) $\underline{y(k)} = \begin{cases} a & \text{for } 0 \le k \le N, \quad N \text{ arbitrary} \\ 0 & \text{otherwise} \end{cases}$

DFT (Eq. 2.20) $\quad Y_{DFT}(m) = \sum_{k=0}^{N-1} y(k) \cdot e^{-j\frac{2\pi m}{N} \cdot k}$

$$Y_{DFT}(m) = \sum_{k=0}^{N-1} a \cdot e^{-j\frac{2\pi m}{N} k} = \quad 0 \text{ for } m \neq 0$$

$$= a \cdot \sum_{k=0}^{N-1} e^{-j\frac{2\pi m}{N} \cdot k} = a \sum_{k=0}^{N-1} \left(\cos\left(\cancel{-\frac{2\pi m k}{N}}\right)^{0} + j \sin\left(\cancel{-\frac{2\pi m k}{N}}\right)^{0} \right) =$$

$$= \begin{cases} a \cdot N & m = 0 \rightarrow \text{DC direct component} \\ 0 & \text{otherwise} \rightarrow \text{no oscillation component} \end{cases}$$

b)

$$y(k) = \cos \frac{2\pi}{N} \cdot k$$

$$Y_{DFT}(m) = \sum_{k=0}^{N-1} \cos\left(\frac{2\pi}{N} k\right) \cdot e^{-j\frac{2\pi}{N} m k}$$

with $\quad \cos\alpha = \frac{1}{2}\left(e^{j\alpha} + e^{-j\alpha}\right)$

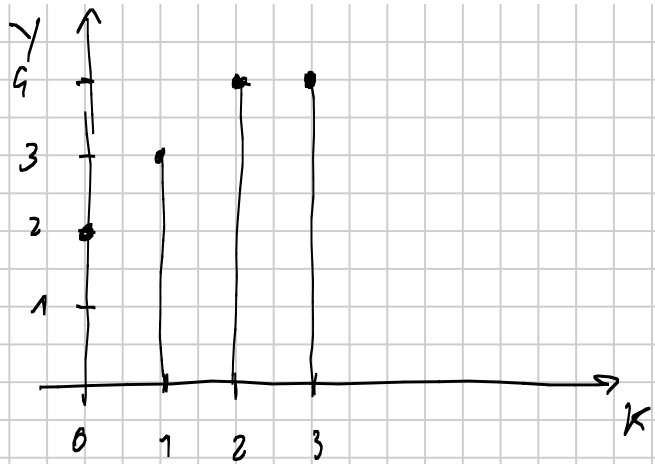
$$\Rightarrow Y_{DFT}(m) = \sum_{k=0}^{N-1} \frac{1}{2}\left(e^{j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}k}\right) \cdot e^{-j\frac{2\pi}{N} m k}$$

$$= \frac{1}{2} \sum_{k=0}^{N-1} \left\{ e^{j \frac{2\pi k}{N}(1-m)} + e^{-j\frac{2\pi k}{N}(1+m)} \right\}$$

$$\text{for} \begin{cases} m \neq 1 \quad \text{and} \quad m \neq -1 \implies Y_{DFT}(m) = 0 \\ m = 1 \qquad Y_{DFT}(1) = \frac{1}{2} \cdot \sum_{k=0}^{N-1} e^{0} = \frac{N}{2} \\ m = -1 \qquad Y_{DFT}(-1) = \frac{1}{2} \cdot \sum_{k=0}^{N-1} e^{0} = N/2 \end{cases}$$

$$\Rightarrow Y_{DFT}(m) = \begin{cases} N/2 & \text{for } m = -1 \\ N/2 & \text{for } m = 1 \\ 0 & \text{otherwise} \end{cases}$$

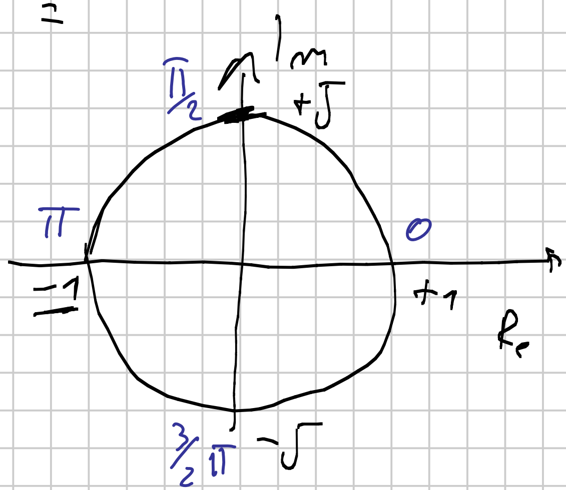
c) $$Y_{DFT}(m) = \sum_{k=0}^{3} y(k) \cdot e^{-j\frac{2\pi m}{4} \cdot k}$$



$$Y_{DFT}(0) = \sum_{k=0}^{3} y(k) \cdot e^{0} = 2+3+4+4 = 13$$

$$Y_{DFT}(1) = 2 \cdot e^{0} + 3 \cdot \underline{e^{(-)j\frac{\pi}{2}}} + 4 \cdot \underline{e^{-j\pi}} + 4 \cdot \underline{e^{-j\frac{3}{2}\pi}} =$$

$$= \underline{2} + 3 \cdot (-j) + 4 \cdot (-1) + 4 \cdot (+j) = -2 + j$$



$$Y_{DFT}(2) = 2 \cdot e^{0} + 3 \cdot \underline{e^{-j\pi}} + 4 \cdot \underline{e^{-j2\pi}} + 4 \cdot e^{-j3\pi} =$$

$$= 2 + 3(-1) + 4(+1) + 4 \cdot (-1) = -1$$

$$Y_{DFT}(3) = 2e^{0} + 3 \cdot e^{-j\frac{3}{2}\pi} + 4 \cdot e^{-j3\pi} + 4 \cdot e^{-j\frac{9}{2}\pi} =$$

$$= 2 + 3(+j) + 4(-1) + 4(-j) = -2 - j$$