TUTORIAL 09

Notiztitel 27.06.2014

Exam: 11th July 11:30 - 12:45

N1189 and N1095

- openbook, pocket calculator
- Programmable calculator are **NOT** allowed

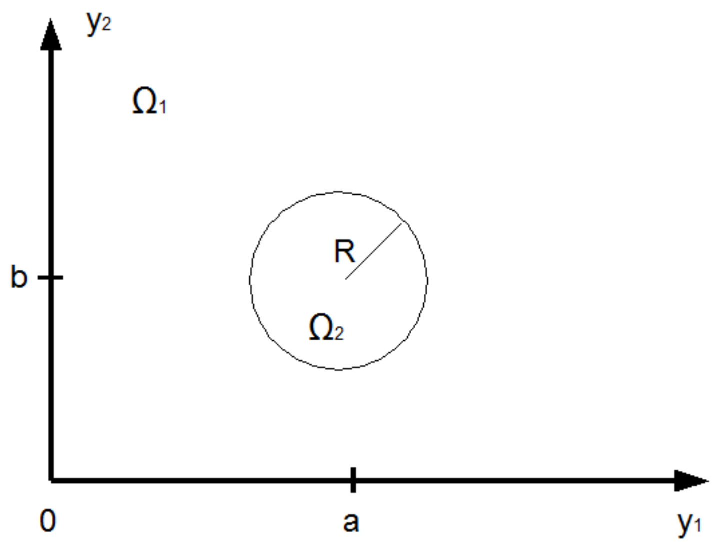
Outline:

Pattern classification

- Polynomial classifiers

Exercise 20

$$\underline{d} = W y$$



a) For the decision function (Eq. 3.20)

$d_1 > d_2$ if the point is outside

$d_1 \leq d_2$ if the point is inside

For two classes this can be rewritten as:

$d_1' = d_1 - d_2$

$d_2' = d_2 - d_1 = -d_1'$

$d_1' > 0$ points outside

$d_1' \leq 0$ points inside

Use a circle equation:

$$d_1' = (y_1 - a)^2 + (y_2 - b)^2 - r^2$$

$$d_2' = -d_1' = -(y_1 - a)^2 - (y_2 - b)^2 + r^2$$

$\Rightarrow$ class boundary: $d_1' = d_2'$

$$2(y_1 - a)^2 + 2(y_2 - b)^2 = 2r^2$$

$$(y_1 - a)^2 + (y_2 - b)^2 = r^2$$

$$d_1' = y_1^2 + y_2^2 - 2a y_1 - 2b y_2 + a^2 + b^2 - r^2$$

$$d_2' = -d_1'$$

$\Rightarrow$ classification with $\underline{d} = \underset{\sim}{W}\, \underline{y}$ (Eq. 3.27) with (Eq. 3.31)

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & -2a & -2b & (a^2+b^2-r^2) \\ -1 & -1 & +2a & +2b & (r^2-a^2-b^2) \end{bmatrix}}_{\underset{\sim}{W}} \begin{bmatrix} y_1^2 \\ y_2^2 \\ y_1 \\ y_2 \\ 1 \end{bmatrix}$$

b)

$$Y_1^T = [a, b]$$

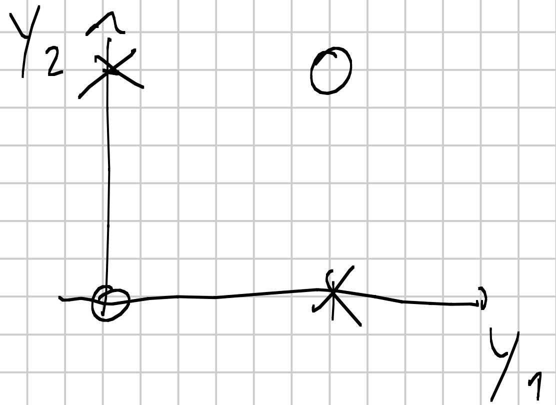
$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2a & -2b & (a^2+b^2-r^2) \\ -1 & -1 & 2a & 2b & (r^2-a^2-b^2) \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ a \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} -r^2 \\ r^2 \end{bmatrix}$$

$\underline{\underline{\text{Maximum}}}$ $\sim$ $r^2 \Rightarrow Y_1^T \Rightarrow \Omega_2$

$$Y_2^T = [a, 2b]$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} b^2 - r^2 \\ r^2 - b^2 \end{bmatrix} \qquad b^2 > r^2 \Rightarrow Y_2^T \rightarrow \Omega_1$$

EXERCISE 21



o Ω_1

x Ω_2

a) Classification with $\underline{d} = \underset{\sim}{W}\, \underline{y}$

2 - class problem : $\underline{d}$ two components

$\underset{\sim}{W}$ two rows

$\underset{\sim}{W}$ can be calculated with $\underset{\sim}{W} = \underset{\sim}{X}\, \underset{\sim}{C}^{-1}$

Target

Where $\underset{\sim}{X} = E\left\{ \tilde{d}\ \underset{\sim}{Y}^T \right\}$

input

$\underset{\sim}{C} = E\left\{ \underset{\sim}{Y}\, \underset{\sim}{Y}^T \right\}$ - covariance matrix

- linear classifier $\quad y^T = [y_1, y_2]$

$Y_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad Y_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad Y_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad Y_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

desired output:

$$\hat{d}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{d}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{d}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \hat{d}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Calculation of $\underset{\sim}{X}$

$$\underset{\sim}{X} = \frac{1}{\underline{K}} \sum_{k=1}^{\underline{K}} \hat{d}_k \, y_k^T = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{C} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underset{\sim}{C}^{-1} = \frac{1}{a_{11} a_{22} - a_{12} a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\underset{\sim}{C} = \frac{1}{\underline{K}} \sum_{k=1}^{\underline{K}} y_k \, y_k^T = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{W} = \underset{\sim}{X} \cdot \underset{\sim}{C}^{-1} = \frac{1}{4} \cdot \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

classification: $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rightsquigarrow$

Output always $d_1 = d_2 \Rightarrow$ no decision

$\Rightarrow$ classification does not work!

b) $\underline{y}^T = [y_1, y_2, y_1 \cdot y_2]$

$$\Rightarrow \quad y_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad y_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad y_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

desired output

$$\hat{d}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{d}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{d}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \hat{d}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{X} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\underset{\sim}{C} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \underset{\sim}{C}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\underset{\sim}{W} = \underset{\sim}{X} \cdot \underset{\sim}{C}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_1 Y_2 \end{bmatrix}$$

$$d_1 = Y_1 \cdot Y_2 \leftarrow$$

$$d_2 = Y_1 + Y_2 - 2 Y_1 \cdot Y_2$$

Y_1 : $\underline{d}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → wrong

Y_2 : $\underline{d}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → correct

Y_3 = $d_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → correct

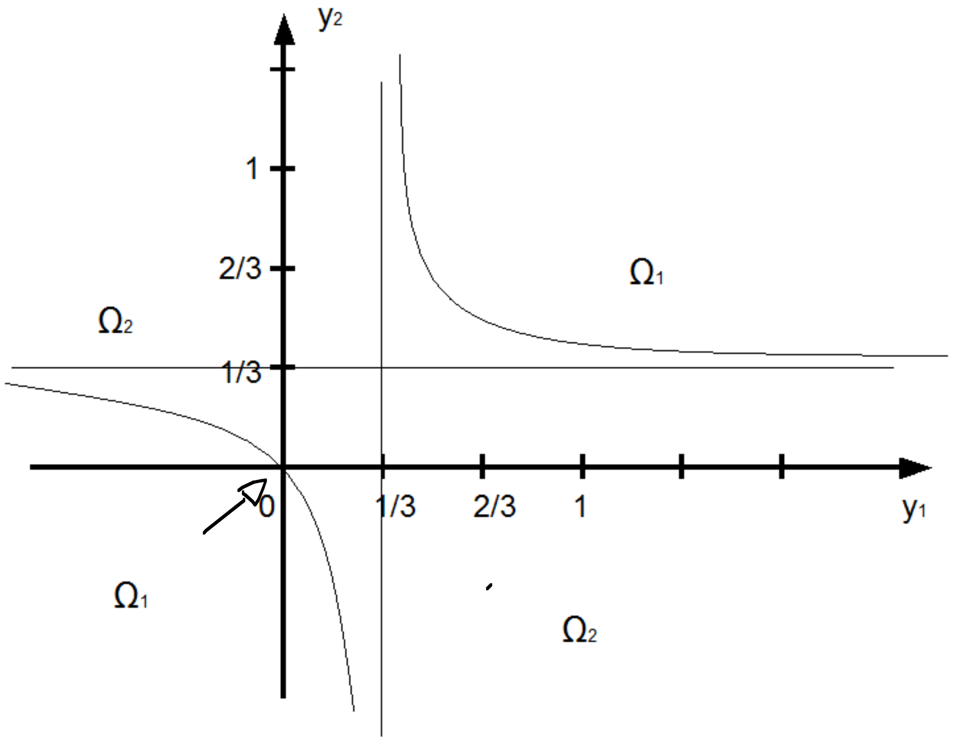
→ Wrong classification only for Y_1

Y_4 : $d_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ⇒ correct

Class boundary for $y^T = [y_1, y_2, y_1 \cdot y_2]$

$d_1 = d_2$

$y_1 y_2 = y_1 + y_2 - 2 y_1 y_2 \Rightarrow y_2 = \frac{y_1}{3y_1 - 1}$



c) class boundaries for $\underline{y}^T = [y_1^2, y_2^2, y_1 \cdot y_2]$

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \text{from b)}$$

$$y_1 \cdot y_2 = y_1^2 + y_2^2 - 2 y_1 y_2$$

$$(y_1 - 1.5 y_2)^2 - 1.25 y_2^2 = 0 \Rightarrow \text{through origin!}$$

$$\Rightarrow y_1 \Rightarrow \text{wrong!}$$

d) bias $+0.5$ applied to d_1:

$$d_1 = y_1 y_2 + 0.5 \qquad d_2 = y_1 + y_2 - 2 y_1 y_2$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0.5 \\ 1 & 1 & -2 & 0 \end{bmatrix}}_{\underset{\sim}{W}} \begin{bmatrix} y_1 \\ y_2 \\ y_1 \cdot y_2 \\ 1 \end{bmatrix}$$

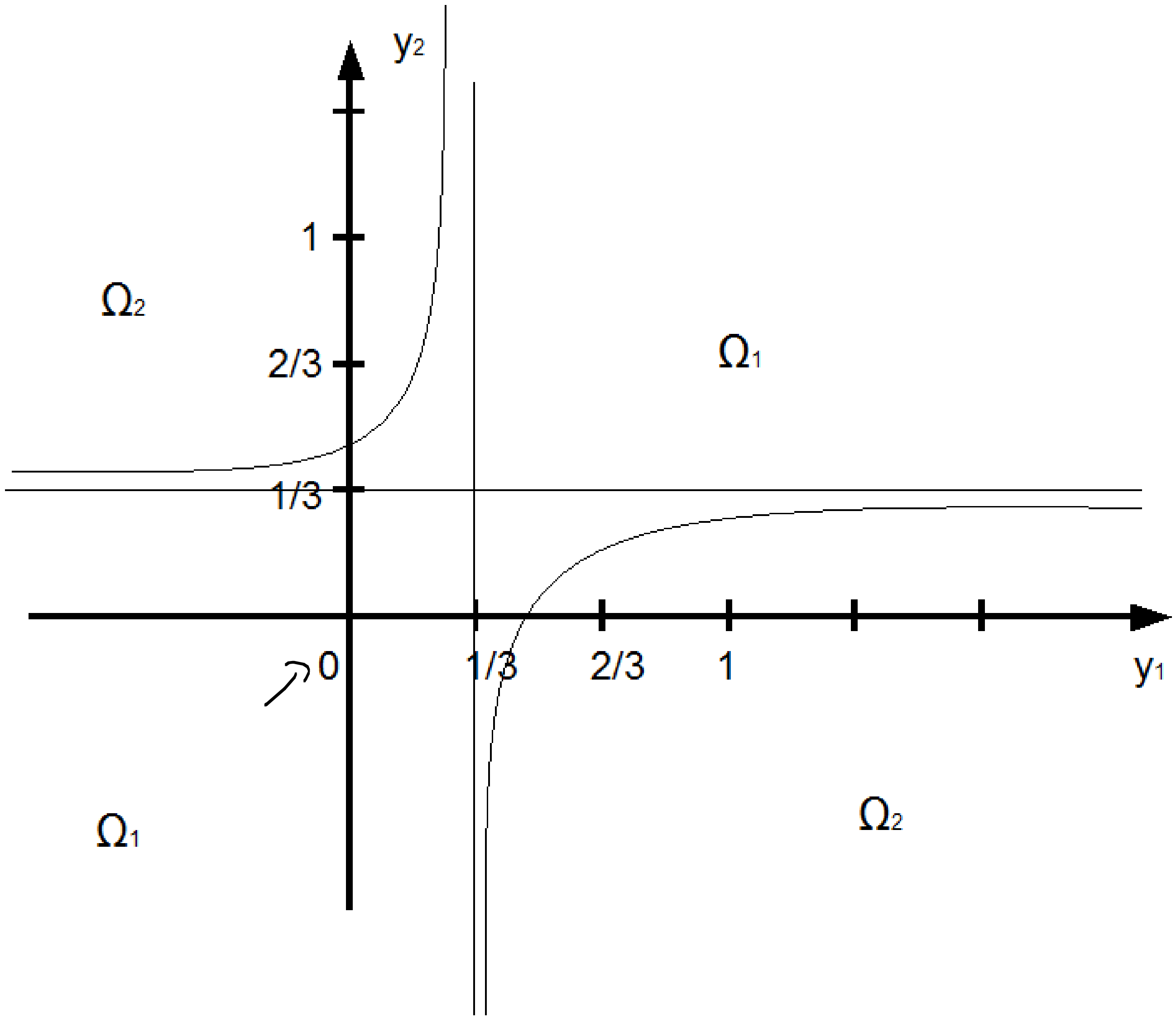
$$d_1 = d_2$$

$$y_1 \cdot y_2 + 0.5 = y_1 + y_2 - 2 y_1 y_2 \qquad \Big| -0.5$$

$$y_1 y_2 = y_1 + y_2 - 2 y_1 y_2 - 0.5 \qquad \Big| : y_2$$

$$y_1 = \frac{y_1}{y_2} + 1 - 2 y_1 - \frac{0.5}{y_2} \qquad \Big| -1 + 2 y_1$$

$$3 y_1 - 1 = \frac{y_1 - 0.5}{y_2} \quad \Rightarrow \quad y_2 = \frac{y_1 - 0.5}{3 y_1 - 1}$$



y2
1
2/3
1/3
0
1/3
2/3
1
y1
Ω2
Ω1
Ω1
Ω2