(I) 1. $\underset{\sim}{A}$ is symmetric,

$$\langle \underset{\sim}{X}, \underset{\sim}{Y} \rangle_{\underset{\sim}{A}} = tr(\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{Y}^T) = tr((\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{Y}^T)^T)$$

$$= tr(\underset{\sim}{Y}\underset{\sim}{A}\underset{\sim}{X}^T) = \overline{\langle \underset{\sim}{Y}, \underset{\sim}{X} \rangle_{\underset{\sim}{A}}} \quad \cdots\cdots \; ①$$

$$\langle \underset{\sim}{X}, \underset{\sim}{X} \rangle_{\underset{\sim}{A}} = tr(\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{X}^T) \qquad \underset{\sim}{X} = \begin{bmatrix} — x_1^T — \\ — x_2^T — \\ \vdots \\ — x_n^T — \end{bmatrix}$$

$$= \sum_{i=1}^{n} \underline{x}_i \underset{\sim}{A} \underline{x}_i^T > 0$$

when $\underset{\sim}{A}$ pdf and $\underline{x}_i \underset{\sim}{A} \underline{x}_i^T = 0$

if and only if $\underline{x}_i = \underline{0}$ $\cdots\cdots$ ②

$$\langle a\underset{\sim}{X}, \underset{\sim}{Y} \rangle_{\underset{\sim}{A}} = tr(a\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{Y}^T) = a\,tr(\underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{Y}^T) = a\langle \underset{\sim}{X}, \underset{\sim}{Y} \rangle_{\underset{\sim}{A}}$$

$\cdots\cdots$ ③

$$\langle \underset{\sim}{X_1} + \underset{\sim}{X_2}, \underset{\sim}{Y} \rangle_{\underset{\sim}{A}} = tr((\underset{\sim}{X_1} + \underset{\sim}{X_2})\underset{\sim}{A}\underset{\sim}{Y}^T) = tr(\underset{\sim}{X_1}\underset{\sim}{A}\underset{\sim}{Y}^T)$$

$$+ tr(\underset{\sim}{X_2}\underset{\sim}{A}\underset{\sim}{Y}^T)$$

$$= \langle \underset{\sim}{X_1}, \underset{\sim}{Y} \rangle + \langle \underset{\sim}{X_2}, \underset{\sim}{Y} \rangle \cdots ④$$

① conjugate symmetry
② positive - definitness
③, ④ Linearity

$\Rightarrow$ $\langle \underset{\sim}{X}, \underset{\sim}{Y} \rangle_{\underset{\sim}{A}}$ ∃ inner product

(I) 2. $\langle \cdot , \cdot \rangle_{H_1}$ and $\langle \cdot , \cdot \rangle_{H_2}$ are two inner product in Hilbert space H_1, H_2

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} = \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2}$$

$$= \overline{\langle y_1, x_1 \rangle_{H_1}} + \overline{\langle y_2, x_2 \rangle_{H_2}}$$

$$= \overline{\langle (y_1, y_2), (x_1, x_2) \rangle_{H_1 \oplus H_2}} \cdots ①$$

$$\langle a(x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} = \langle (ax_1, ax_2), (y_1, y_2) \rangle_{H_1 \oplus H_2}$$

$$= \langle ax_1, y_1 \rangle_{H_1} + \langle ax_2, y_2 \rangle_{H_2} = a \langle x_1, y_1 \rangle_{H_1} + a \langle x_2, y_2 \rangle_{H_2}$$

$$= a \left[\langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2} \right]$$

$$= a \langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} \cdots ②$$

$$\langle (x_1, x_2) + (x_3, x_4), (y_1, y_2) \rangle_{H_1 \oplus H_2} = \langle (x_1 + x_3, x_2 + x_4), (y_1, y_2) \rangle_{H_1 \oplus H_2}$$

$$= \langle x_1 + x_3, y_1 \rangle_{H_1} + \langle x_2 + x_4, y_2 \rangle_{H_2} = \langle x_1, y_1 \rangle_{H_1} + \langle x_3, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2} + \langle x_4, y_2 \rangle_{H_2}$$

$$③ \cdots = \langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} + \langle (x_3, x_4), (y_1, y_2) \rangle_{H_1 \oplus H_2}$$

$$\langle (x_1, x_2), (x_1, x_2) \rangle_{H_1 \oplus H_2} = \langle x_1, x_1 \rangle_{H_1} + \langle x_2, x_2 \rangle_{H_2}$$

for $\forall\ x_1 \neq 0, x_1 \in H_1$, $\langle x_1, x_1 \rangle_{H_1} > 0$

for $\forall\ x_2 \neq 0, x_2 \in H_2$, $\langle x_2, x_2 \rangle_{H_2} > 0$

④ $\Rightarrow \langle (x_1, x_2), (x_1, x_2) \rangle_{H_1 \oplus H_2} > 0$ for $\forall\ (x_1, x_2) \neq (0, 0)$

With ①②③④, designed product is inner product, so $H_1 \oplus H_2$ is Hilbert space.

②

(1). 2 (following part)

Let $\{(y_1, y_2), y_1 \in H_1 \wedge y_2 \in H_2\}$ subspace be the orthogonal complement of subspace $\{(x_1, 0) : x_1 \in H_1\}$

So, $\langle (y_1, y_2), (x_1, 0) \rangle_{H_1 \oplus H_2} \stackrel{!}{=} 0$

$$\langle (y_1, y_2), (x_1, 0) \rangle_{H_1 \oplus H_2} := 0$$

$$= \langle y_1, x_1 \rangle_{H_1} + \langle y_2, 0 \rangle_{H_2}$$

$$\langle y_1, x_1 \rangle_{H_1} := 0 \implies y_1 = 0.$$

$\implies$ orthognal complement is

$$\{(0, y_2) : y_2 \in H_2\}.$$

(I) 3. a)

P_2 is the polynomial transformation (Map)

$\langle P, g\rangle = P(1)g(1) + P(0)g(0) + P(-1)g(-1)$

$= g(1)P(1) + g(0)P(0) + g(-1)P(-1)$

$= \langle g, P\rangle = \overline{\langle g, P\rangle}$ ----- ①

$\langle P, P\rangle = P^2(1) + P^2(0) + P^2(-1)$

$\langle P, P\rangle = 0$ if and only if P_2's polynomial C_0, C_1, C_2 all 0 ------ ②

$\langle aP, g\rangle = aP(-1)g(-1) + aP(0)g(0) + aP(1)g(1)$

$= a\langle P, g\rangle$ ---- ③

$\langle x+y, g\rangle = (x+y)(-1)g(-1) + (x+y)(0)g(0) + (x+y)(1)g(1)$

$= x(-1)g(-1) + x(0)g(0) + x(1)g(1) + y(-1)g(-1) + y(0)g(-1) + y(1)g(1)$

$= \langle x, g\rangle + \langle y, g\rangle$ ----- ④

With ① ② ③ ④, $\langle P, g\rangle$ can be proved to be a inner product.

(II) 1. $\forall V \in \mathbb{R}^2$, $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

① $V_2 > 0$, $D_v f(0,0) = \lim_{t \to 0} \frac{t\sqrt{V_1^2+V_2^2} - 0}{t} = \sqrt{V_1^2+V_2^2}$

② $V_2 = 0$ $D_v f(0,0) = \frac{tV_1 - 0}{t} = V_1$

③ $V_2 < 0$ $D_v f(0,0) = \frac{-t\sqrt{V_1^2+V_2^2} - 0}{t} = -\sqrt{V_1^2+V_2^2}$

Every directional derivative exsists on $(0,0)$

In order to prove undifferentiability is to prove $D_{\underline{V_1}+\underline{V_2}} f(0,0) \neq D_{\underline{V_1}} f(0,0) + D_{\underline{V_2}} f(0,0)$

Take $\underline{V_1} = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix}$ $\underline{V_2} = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix}$, $V_{12} = -V_{22} \neq 0$

$D_{\underline{V_1}+\underline{V_2}} f(0,0) = V_{11} + V_{12}$

$D_{\underline{V_1}} f(0,0) + D_{\underline{V_2}} f(0,0) = \sqrt{V_1^2 + V_2^2} - \sqrt{V_1^2 + V_2^2}$
$= 0$

$\Rightarrow D_{\underline{V_1}+\underline{V_2}} f(0,0) \neq D_{\underline{V_1}} f(0,0) + D_{\underline{V_2}} f(0,0)$

$\Rightarrow f$ is not differentiable in $(0,0)$

(1) 3, b)

$P_3 := f: \mathbb{R} \to \mathbb{R};\ f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3,\ c_i \in \mathbb{R}, i = 0 \dots 4$

Assume that $\langle p, q \rangle = p(-1)\, q(-1) + p(0)\, q(0)$
$+ p(1)\, q(1)$

is still a inner product

and $q, p \in P_3$

$\Rightarrow$ If and only if $c_0, c_1, c_2, c_3 = 0$

$\langle p, p \rangle = 0 \quad = (c_0 + c_1 + c_2 + c_3)^2 + c_0^2 + (c_0 - c_1 + c_2 - c_3)^2$

but $0 \neq c_0 := -c_2$ and $c_1, c_3 := 0$

$\langle p, p \rangle = 0 \Rightarrow$ Assumption does not hold

$\Rightarrow$ When $p, q \in P_3$,

$p(-1)\, q(-1) + q(0)\, p(0) + p(-1)\, q(-1)$

is not inner product.

(II) 2. $f(0,0)=0$

$$f(0^+,0^+)=f(0^-,0^-)=\lim_{\substack{x\to 0^+,0^- \\ y\to 0^+,0^-}} \frac{1}{\sqrt{\frac{1}{x^2}+\frac{1}{y^2}}}=0$$

$$f(0^+,0^-)=f(0^-,0^+)=\lim_{\substack{x\to 0^+,0^- \\ y\to 0^-,0^+}} \frac{-1}{\sqrt{\frac{1}{x^2}+\frac{1}{y^2}}}=0$$

$\Rightarrow f$ is continuous

when $(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial x}=\frac{\partial \frac{xy}{\sqrt{x^2+y^2}}}{\partial x}, \quad \frac{\partial f}{\partial y}=\frac{\partial \frac{xy}{\sqrt{x^2+y^2}}}{\partial y}$$

obviously, it's differentiable.

when $x=0$:

$$\left.\frac{\partial f}{\partial y}\right|_{x=0}=\lim_{t\to 0^+}\frac{f(0,t)-f(0,0)}{t}=0$$

$$=\lim_{t\to 0^-}\frac{f(0,t)-f(0,0)}{t}=0$$

So, f is partial differentiable for y

and same prove can be done for x

so, f is partial differentiable.

(II) 2. (following part)

$$D_V f(0,0) = \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^2 v_1 v_2}{t\sqrt{v_1^2+v_2^2}}}{t}$$

$$= \frac{v_1 + v_2}{\sqrt{v_1^2 + v_2^2}}$$

$$D_f(0,0) \cdot \underline{V} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot * \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

If f was differentiable, $Df(0,0) \cdot V = D_V f(0,0)$
$= f(v)$

(II) 3, Use the standard inner product

$A, B \in \mathbb{R}^{m\times n}$, $<A, B> = tr(A^T B)$

and $\|A\|_F^2 = tr(A^T A) = <A, A>$

$$D_{f \text{ in } V} H = \lim_{t\to 0} \frac{f(U, V+tH) - f(U,V)}{t},$$

is the directional derivative of f in (U, V) w.r.t V in direction H

$$D_{f \text{ in } V} H = \frac{d}{dt}\Big|_{t=0} f(U, V+tH)$$

$$= \frac{d}{dt}\Big|_{t=0} \Big[<(UV^T - Y) + tUH^T, (UV^T - Y) + tUH^T> + \frac{\lambda}{2} \cdot 2 \cdot <V, tH> \Big]$$

$$= 2 <UV^T - Y, UH^T> + \lambda <V, H>$$

(II), 3. (following part).

$$= 2 \langle H^T, U^T(UV^T - Y) \rangle + \lambda \langle H, V \rangle$$

$$= \langle H, 2(VU^T - Y^T)U + \lambda V \rangle$$

(II) 4, Because of the differentiability of f, $V \in R^3$

Method 1: $\Rightarrow D_V f(x_0, y_0, z_0) = Df(x_0, y_0, z_0) \cdot V$

and differential map should be the partial derivative.

Method 2: $\Rightarrow$ The differential map of f ~~in~~ (x_0, y_0, z_0) is the partial derivate of f at (x_0, y_0, z_0)

$$Df(x_0, y_0, z_0) = \left[\frac{\partial f}{\partial x}\Big|_{x=x_0} \quad \frac{\partial f}{\partial y}\Big|_{y=y_0} \quad \frac{\partial y}{\partial z}\Big|_{z=z_0} \right]$$

$$= \begin{bmatrix} e^{x_0+z_0} + 2x_0y_0 & x_0^2 & e^{x_0+z_0} \\ 1 & 2y_0 & 2z_0 \end{bmatrix}$$

which is also the Jacobi of f

(II) 5. Directional Derivative of f is

$$D_V f(\underset{\sim}{X}) = \frac{d}{dt}\Big|_{t=0} \left[(\underset{\sim}{X} + t\underset{\sim}{V}) \underset{\sim}{A} (\underset{\sim}{X} + t\underset{\sim}{V})^T - (\underset{\sim}{X} + t\underset{\sim}{V}) \underset{\sim}{B} \right]$$

$$= \underset{\sim}{V}\underset{\sim}{A}\underset{\sim}{X}^T + \underset{\sim}{X}\underset{\sim}{A}\underset{\sim}{V}^T - \underset{\sim}{V}\underset{\sim}{B}$$

$Df: X \mapsto Df(\underset{\sim}{X})$, map a element from original space (here $R^{m\times n}$) to a linear map $\mathcal{L}$, which maps $\underset{\sim}{V}$ (direction) to the directional derivative.

$$Df(\underset{\sim}{X_0}) = \underset{\sim}{[\cdot]} \underset{\sim}{A} \underset{\sim}{X_0}^T + \underset{\sim}{X_0} \underset{\sim}{A} \underset{\sim}{[\cdot]}^T - \underset{\sim}{[\cdot]} \underset{\sim}{B}.$$

Because f is differentiable, f is continuous, the uniqueness is also satisfied.