CS 303, Paul Cao,
Plan: Simplex method for set of linear equations (ch 29)

Example
In the voting season, a politician has to determine how to spend his money. In Ohio, supposely there are three types of voting districts – urban (100k voters), suburban(200k voters), and rural (50k voters). You need to win at least 50% of each category to get elected (more fair than what we have now?)

A few topics that you have to cover (numbers in the table represents the number of votes (in thousand) you will win or lose if you spend 1000 dollars on the topic)

policy	urban	suburban	rural
build roads	−2	5	3
gun control	8	2	−5
farm subsidies	0	0	10
gasoline tax	10	0	−2

Now how can you achieve the goal with min amount of money spent?
A:/ try and error. Try 20000 for buidling roads, no ad for gun control, 4000 for farm subsidy and 9000 on gasoline tax. This will put you over.

Or formulate the problem in this way
let x1 be the dollar amount spent on building roads, x2 for gun control, x3 for farm subsidies and x4 for gasoline tax (the unit is 1000)
then we want to make sure
-2x1+8x2+0x3+10x4>=50
5x1+2x2+0x3+0x4>=100
3x1-5x2+10x3-2x4>=25
x1, x2, x3, x4 >=0

while minimize (x1+x2+x3+x4)

Linear programming
Linear programming (LP) problem is to optimize a linear function of several variables subject to linear constraints:

maximize (or minimize) $c_1 x_1 + \ldots + c_n x_n$
subject to $a_{i1}x_1 + \ldots + a_{in} x_n \le (\text{ or } \ge \text{ or } =)\ b_i,\ i = 1,\ldots,m$ $\quad x_1 \ge 0, \ldots , x_n \ge 0$

The function $z = c_1 x_1 + \ldots + c_n x_n$ is called the *objective function*; constraints $x_1 \ge 0, \ldots , x_n \ge 0$ are called *nonnegativity constraints*

Applications of linear programming

- Diet Problem (Find the cheapest combination of foods that will satisfy all your nutritional requirements) http://www-neos.mcs.anl.gov/CaseStudies/dietpy/WebForms/index.html
- Crew Scheduling for Airline companies (minimize the cost while covering each flight; avoid overstressing the pilots)
- Traveling Salesman Problem (Find the shortest route that visits each city exactly once and return to the original city (cook paper, 24000 cities)

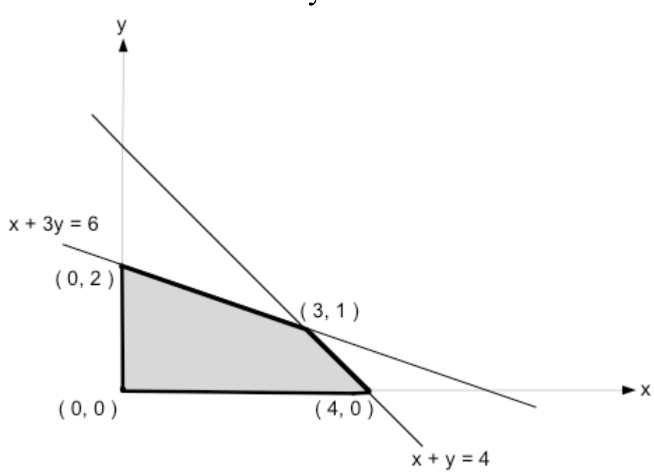
e.g.

maximize $3x + 5y$

subject to

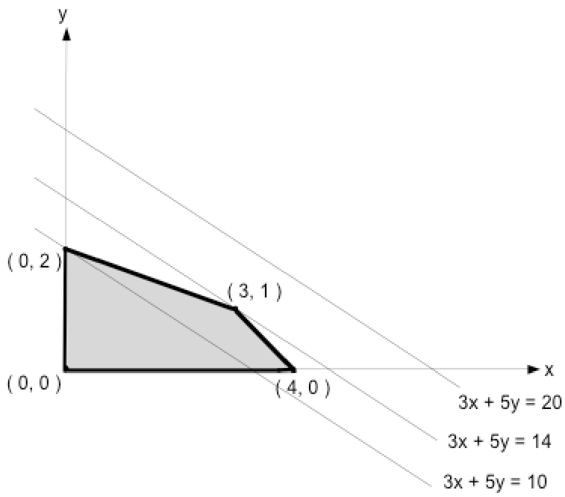
$$x + y \le 4$$
$$x + 3y \le 6$$
$$x \ge 0,\ y \ge 0$$

The constraints basically tell us where our solution(s) might reside



The shaded region is called the *feasible region*. *Feasible region* is the set of points defined by the constraints.

By trying different values of the objective function, we can see from the following figure that the optimum solution is (3,1) with the optimal value as 14.



Fact:
Extreme Point Theorem: Any LP problem with a nonempty bounded feasible region has an optimal solution; moreover, an optimal solution can always be found at an *extreme point* of the problem's feasible region.

There might be three possible outcomes for any LP problem

1. has a finite optimal solution, which may no be unique. (e.g. the example we saw)
2. unbounded: the objective function of maximization (minimization) LP problem is unbounded from above (below) on its feasible region (e.g. maximize 3x+y given the constraints as x>=0, y>=0)
3. infeasible: there are no points satisfying all the constraints, i.e. the constraints are contradictory region (e.g. maximize 3x+y given the constraints as x>=0, y>=0, y<=x-2, y>=-x+1)

3. Simplex method
Simplex method is one of the methods that can solve LP problems. It is considered to be one of the most important algorithms ever invented (invented by George Dantzig in 1947)

Based on the iterative improvement idea:
Generates a sequence of adjacent points of the problem’s feasible region with improving values of the objective function until no further improvement is possible

It requires the LP to follow the following requirements.

- must be a maximization problem
- all constraints (except the nonnegativity constraints) must be in the form Ax<=b
- all the variables must be required to be nonnegative

Thus, the general linear programming problem in standard form with *m* constraints and *n* unknowns ($n \geq m$) is

maximize $c_1 x_1 + \ldots + c_n x_n$
subject to $a_{i1}x_1 + \ldots + a_{in} x_n \le b_i$, $i = 1,\ldots,m$, $x_1 \ge 0, \ldots, x_n \ge 0$

Note: Every LP problem can be represented in such form

e.g.

Transform to

maximize $3x + 5y$	maximize $3x + 5y + 0u + 0v$
subject to $x + y \le 4$	subject to $x + y + u = 4$
$x + 3y \le 6$	$x + 3y + v = 6$
$x \ge 0$, $y \ge 0$	$x \ge 0$, $y \ge 0$, $u \ge 0$, $v \ge 0$

Variables u and v, transforming inequality constraints into equality constrains, are called *slack variables*

Simplex approach starts with a **basic feasible solution**
A *basic solution* to a system of m linear equations in n unknowns ($n \ge m$) is obtained by setting $n - m$ variables to 0 and solving the resulting system to get the values of the other m variables.

The variables set to 0 are called *nonbasic*; the variables obtained by solving the system are called *basic*. A basic solution is called *feasible* if all its (basic) variables are nonnegative.

In the previous example, (0, 0, 4, 6) is basic feasible solution (x, y are nonbasic; u, v are basic)

Fact: There is a 1-1 correspondence between extreme points of LP's feasible region and its basic feasible solutions.

This feasible solution will be used to form the initial tableau.

basic variables

	x	y	u	v	
u	1	1	1	0	4
v	1	3	0	1	6
objective row	−3	−5	0	0	0

basic feasible solution (0, 0, 4, 6)

value of z at (0, 0, 4, 6)

Given this initial tableau, the simplex method tries to find the extreme points that are connected to the current solution in the feasible region. This process repeats until the optimum solution is found.

Steps

- Step 0 [Initialization] Present a given LP problem in standard form and set up initial tableau.

- Step 1 [Optimality test] If all entries in the objective row are nonnegative — stop: the tableau represents an optimal solution.

- Step 2 [Find entering variable] Select (the most) negative entry in the objective row. Mark its column to indicate the entering variable and the pivot column.

- Step 3 [Find departing variable] For each positive entry in the pivot column, calculate the θ-ratio by dividing that row's entry in the rightmost column by its entry in the pivot column. (If there are no positive entries in the pivot column — stop: the problem is unbounded.) Find the row with the smallest θ-ratio, mark this row to indicate the departing variable and the pivot row.

- Step 4 [Form the next tableau] Divide all the entries in the pivot row by its entry in the pivot column. Subtract from each of the other rows, including the objective row, the new pivot row multiplied by the entry in the pivot column of the row in question. Replace the label of the pivot row by the variable's name of the pivot column and go back to Step 1.

Example

Simplex solution

The extreme points that are covered are (0,0), then to (0,2) then to (3,1)

Notes
- Finding an initial basic feasible solution may pose a problem
- Theoretical possibility of cycling
- Typical number of iterations is between m and 3m, where m is the number of equality constraints in the standard form
- Worse-case efficiency is exponential
- Other interior-point algorithms such as Karmarkar's algorithm (1984) have polynomial worst-case efficiency and have performed competitively with the simplex method in empirical tests

Exercise
maximize 3x+y
subject to –x+y<=1, 2x+y<=4, x>=0, y>=0

First, rearrange the constraints and objective function

maximize :3x+y+0u+0v

subject to

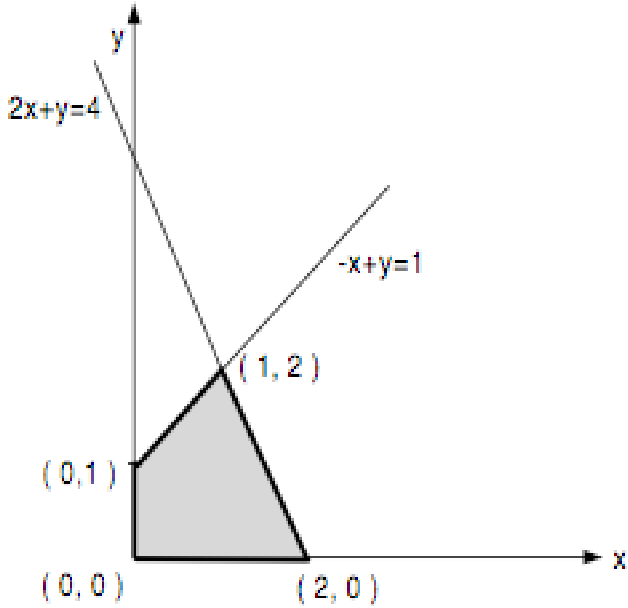
CS 303, Paul Cao,
-x+y+u=1
2x+y+v=4
x>=0, y>=0, u>=0, v>=0

Basic feasible solution (0,0,1, 4)

	x	y	u	v		
u	−1	1	1	0	1	
v	2	1	0	1	4	$Q_v = \frac{4}{2}$
	−3	−1	0	0	0	

	x	y	u	v	
u	0	$\frac{3}{2}$	1	$\frac{1}{2}$	3
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	2
	0	$\frac{1}{2}$	0	$\frac{3}{2}$	6

thus the final result is x=2 and y=0, the maximal value of the objective is 6. It makes sense if we look at the problem graphically.


y
2x+y=4
-x+y=1
(1, 2)
(0,1)
(0, 0)
(2, 0)
x