CS 303, Paul Cao,
Plan: Greedy Algorithm (MST Construction)
Review:

Graphs review (App. B)

<u>Digraph:</u> (directed graph) $G=(V,E)$

- set V of vertices
- set E of edges $\in V \times V$ (ordered pair of vertices)

<u>undirected graph:</u> E contains undirected pair of vertices

- $|E| \in O(V^2)$
  $\hookrightarrow$ because ${}_2C_n$ at most
- if G is connected, then $|E| > |V|-1$
- $\lg |E| \in \Theta(\lg |V|)$ for connected undirected graphs

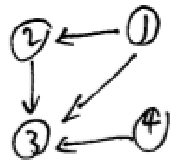
<u>Graph representation</u>

① Adjacency Matrix of $G=(V,E)$
where $V=\{1,2,\cdots,n\}$ is the $n \times n$ matrix of A
given by

$$A[i,j] = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

- for weighted graph. 1 will be replaced w/ weight

e.g.



A

	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

storage for adjacency matrix : $\Theta(V^2)$

good for dense representation

↳ such as a complete graph

② adjacency list of $v \in V$ is the list adj[v] of vertices adjacent to v

e.g. Adj[1] = {2, 3}

2 = {3}

3 = { }

4 = {3}

good for sparse graph

↳ such as trees, internet nodes,

Q:/ what's the length of the list.

$$|Adj[v]| = \begin{cases} \text{degree}(v) & \text{for undirected} \\ \text{out-degree}(v) & \text{for directed} \end{cases}$$

handshaking Lemma (undirected graph)

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

Thus for undirected graph, adjacency list cost $\Theta(|V|+|E|)$ storage. same for digraphs

Adjacency list usually out-performs adjacency matrix but the matrix is used some of the very important algorithms due to the easiness of matrix operations.

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Minimum spanning tree (MST)

Why?
Given a set of points (cities), how do you use the minimum amount of wire to connect these points?

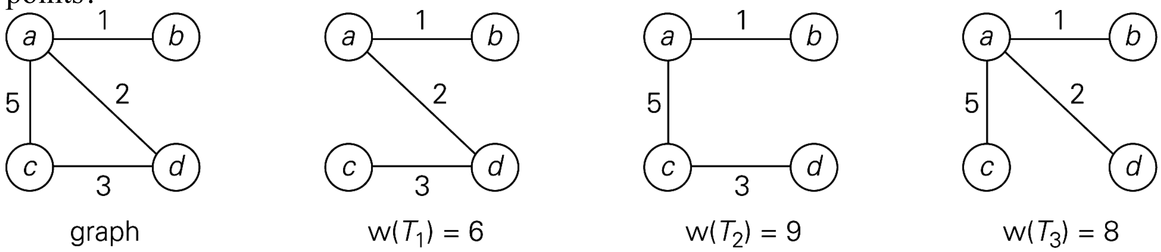


FIGURE 9.1 Graph and its spanning trees; T_1 is the minimum spanning tree

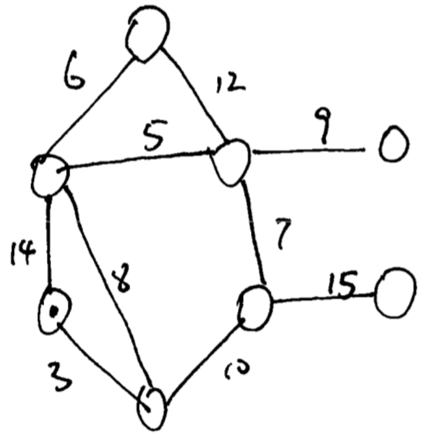
Definitions
<u>Spanning tree</u> of a connected graph *G*: a connected acyclic subgraph of *G* that includes all of *G*'s vertices

<u>Minimum spanning tree</u> of a weighted, connected graph *G*: a spanning tree of *G* of minimum total weight

Problem:
Input: Connected undirected graph G=(V,E) with weight function N:E → R
For simplicity, all weights are distinct (i.e. the function is one to one)
Output: A spanning tree T(connects all vertices) of minimum weight

$$W(T) = \sum_{(u,v)\in T} w(u,v)$$

There are many applications of MST on clustering, and point connecting problem
e.g.



edges have to be in

- include 9 & 15
- 3 has to be there because the dot node has two choices to be connected.
- 6 is there for a similar reason
- 5, 7, 8 have to be there because we have 4 connected components to be connected.

Prim's algorithm

Idea: build the MST by adding in the **nearest** node into the tree in each iteration. (the greedy idea comes from the nearest)

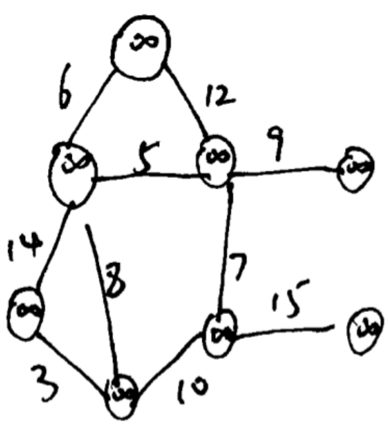
Procedure

- Start with tree T_1 consisting of one (any) vertex and "grow" tree one vertex at a time to produce MST through a series of expanding subtrees $T_1, T_2, \ldots, T_n$
- On each iteration, construct T_{i+1} from T_i by adding vertex not in T_i that is closest to those already in T_i (this is a "greedy" step!)
- Stop when all vertices are included

Pseudo-code

```
PRIM(G, w, r)
  Q = Ø
  for each u ∈ G.V
      u.key = ∞
      u.π = NIL
      INSERT(Q, u)
  DECREASE-KEY(Q, r, 0)        // r.key = 0
  while Q ≠ Ø
      u = EXTRACT-MIN(Q)
      for each v ∈ G.Adj[u]
          if v ∈ Q and w(u, v) < v.key
              v.π = u
              DECREASE-KEY(Q, v, w(u, v))
```

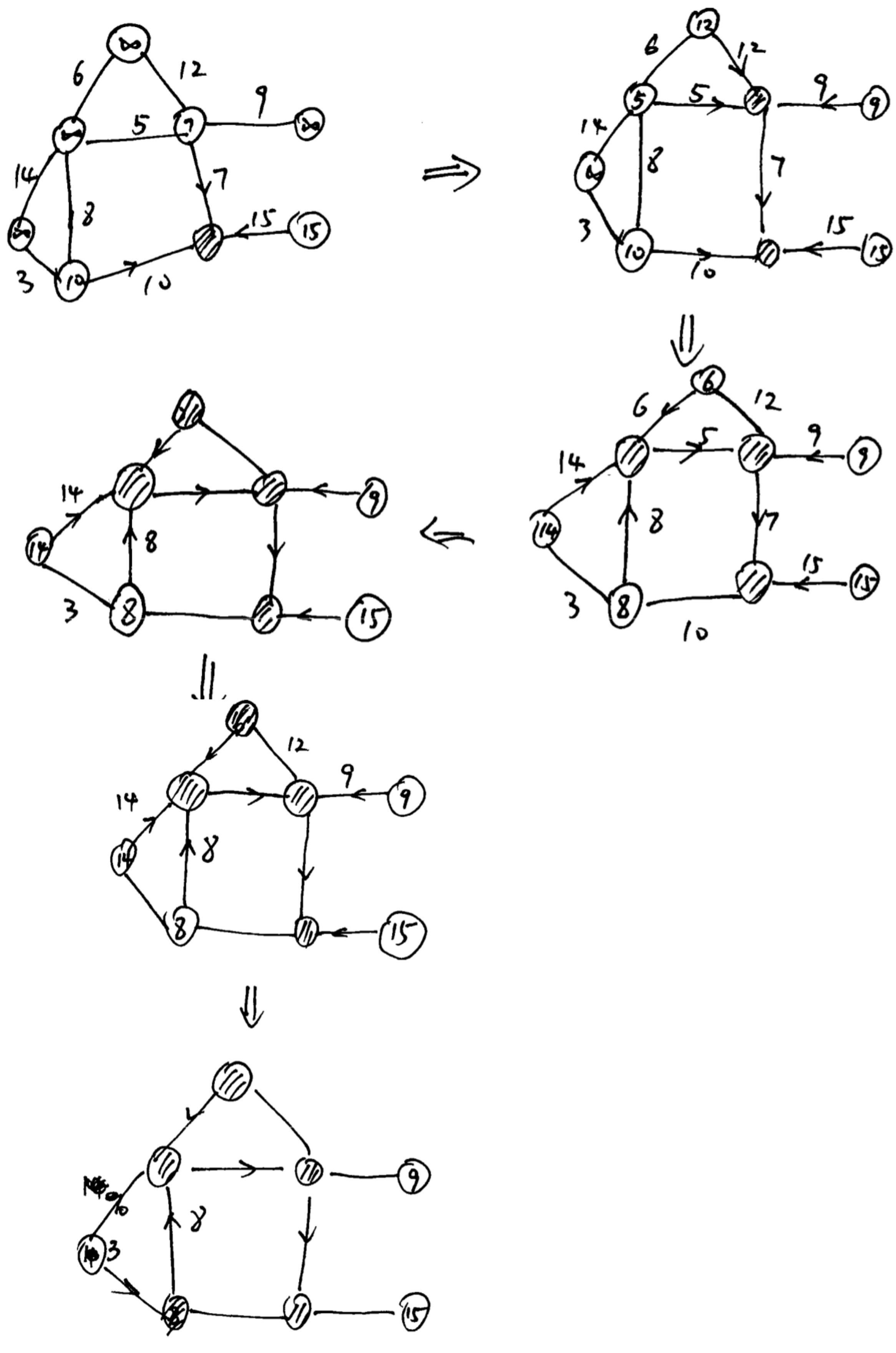
E.g. Find the MST on the example graph above



: already part of MST

: in Q

→ pointer



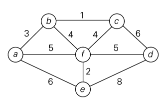
Then just add 9 and 15 to the tree

Notes

- Proof by induction that this construction actually yields MST
- Needs priority queue for locating closest fringe vertex

- Efficiency
    - $O(n^2)$ for weight matrix representation of graph and array implementation of priority queue
    - $O(m \log n)$ for adjacency list representation of graph with n vertices and m edges and min-heap implementation of priority queue

Exercise



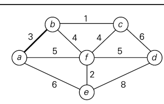
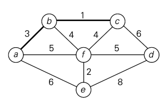
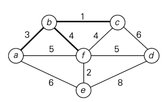
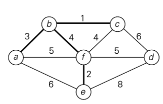
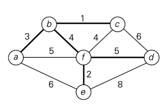
Tree vertices	Remaining vertices	Illustration
a(−, −)	**b(a, 3)** c(−, ∞) d(−, ∞) e(a, 6) f(a, 5)	
b(a, 3)	**c(b, 1)** d(−, ∞) e(a, 6) f(b, 4)	
c(b, 1)	d(c, 6) e(a, 6) **f(b, 4)**	
f(b, 4)	d(f, 5) **e(f, 2)**	
e(f, 2)	**d(f, 5)**	
d(f, 5)		

FIGURE 9.2 Application of Prim's algorithm. The parenthesized labels of a vertex in the middle column indicate the nearest tree vertex and edge weight; selected vertices and edges are shown in bold.

Kruskal's algorithm

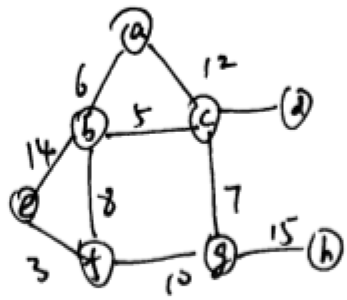
Idea: Since MST is a tree and we can build this tree, why not order the edges and build up the tree with edges that don't make circles.

Steps

- Sort the edges in nondecreasing order of lengths
- "Grow" tree one edge at a time to produce MST through a series of expanding forests F_1, F_2, …, F_{n-1}

- On each iteration, add the next edge on the sorted list unless this would create a cycle. (If it would, skip the edge.)

e.g. same graph

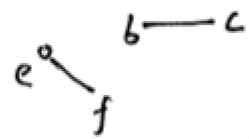


Sort by edge: ef bc ab cg bf cd be gh

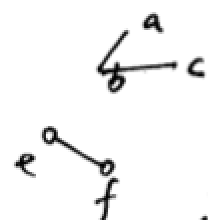
grow by edge:

ef ✓

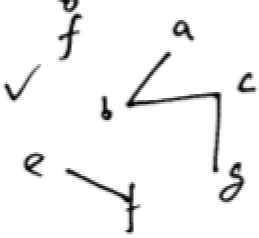
ef bc ✓



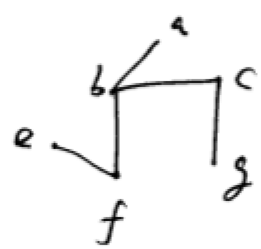
ef bc ab ✓



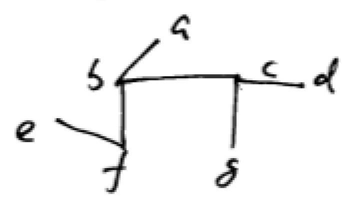
ef bc ab cg ✓



ef bc ab cg bf ✓

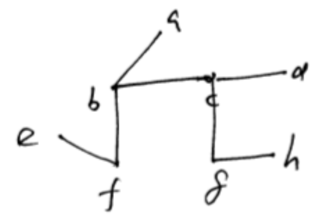


ef bc ab cg bf cd ✓



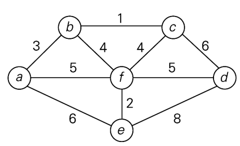
ef bc ab cg bf cd ~~be~~ X
↳ will introduce a cycle

ef bc ab cg bf cd gh ✓



done

Exercise



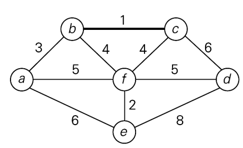
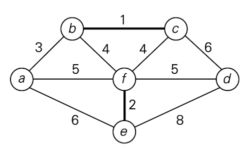
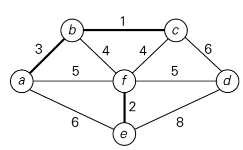
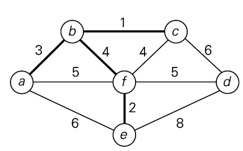
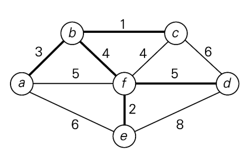
Tree edges	Sorted list of edges	Illustration
	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
bc 1	bc **ef** ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
ef 2	bc ef **ab** bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
ab 3	bc ef ab **bf** cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
bf 4	bc ef ab bf cf af **df** ae cd de 1 2 3 4 4 5 5 6 6 8	
df 5		

FIGURE 9.4 Application of Kruskal's algorithm. Selected edges are shown in bold.