Lecture 29 All-pairs shortest paths

review: single source shortest path for $G = (V, E)$

- unweighted - BFS - $O(V+E)$
- non-negative weight - Dijkstra's - $O(V^2)$
- general weights - Bellman-Ford - $O(VE)$
- 24.2 talks about DAG - after class reading
  topological sorting then run Bellman-Ford just one-round.

All-pairs shortest paths

find the shortest path between any two vertices in G

Ideas:

- unweighted:
  - try $|V| \times$ BFS → efficiency $O(V(V+E)) \xrightarrow[\text{graph}]{\text{connected}} O(VE)$

    Good enough

- non-negative edge weights
  - try $|V| \times$ Dijkstra → efficiency $O(V \cdot V^2) = O(V^3)$

    Good enough

- general case
  - try $|V| \times$ Bellman-Ford → efficiency $O(V^2E) \simeq O(V^4)$ for dense G

    we'll improve upon this case w/ DP because of possible reuse

Problem:

<u>Input</u>: digraph $G=(V,E)$ w/ $V=\{1,2,\cdots,n\}$

edge weight function $W: E \to \mathbb{R}$

<u>output</u>: $n \times n$ matrix of shortest-path weights

$\delta(i,j)$ for all $i,j \in V$

Dynamic Programming

- let A be weighted adjacency matrix $A=(a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} W(i,j) & \text{if } (i,j)\in E \\ \infty & \text{otherwise} \end{cases}$$

- define $d_{ij}^{(m)}$ = weight of shortest path from i to j using $\le m$ edges.

Q:/ what's the goal?

A:/ $d_{ij}^{(n)} = \delta(i,j)$

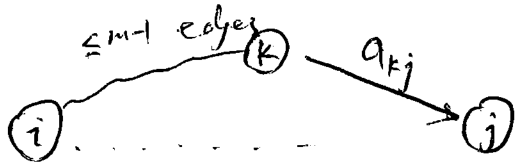
- recursion

$$d_{ij}^{(0)} = \begin{cases} 0 & \text{if } i=j \\ \infty & \text{if otherwise} \end{cases}$$

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$

for $m=1,2,\cdots,n$

Proof:



$d_{ij}^{(m)}$ is the smallest of all #

Algorithm

for $m \leftarrow 1$ to n
do for $i \leftarrow 1$ to n
do for $j \leftarrow 1$ to n
do for $k \leftarrow 1$ to n
if $d_{ij} \geq d_{ik} + a_{kj}$
then $d_{ij} = d_{ik} + a_{kj}$ relaxation.

efficiency : $O(n^4)$ same as Bellman-Ford.

Q:/ what does the above look like?

A:/ similar to matrix multiplication.

matrix multiplication.

A, B, C $n \times n$ matrices

$$C_{ij} = \sum_{k} a_{ik} \cdot b_{kj}$$

change $+$ to $\min$
$\cdot$ to $+$

then we have a very important similarity.

Improve this leads to Floyd-Marshall

trick: redefine the subproblem

- define $\quad c_{ij}^{(k)}$ = weight of shortest-path from i to j w/ intermediate vertices in set $\{1, 2, \ldots, k\}$

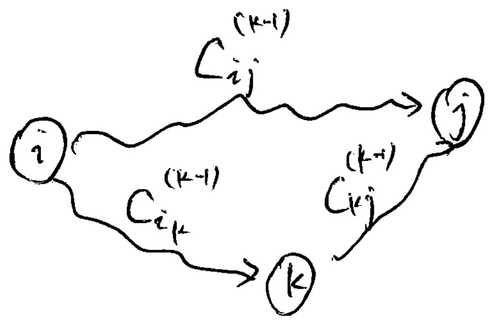
$\Rightarrow \delta(i,j) = c_{ij}^{(n)}$

- recursion

$$c_{ij}^{(0)} = A$$

$$c_{ij}^{(k)} = \min\left(c_{ij}^{(k-1)},\ c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right)$$

Proof:



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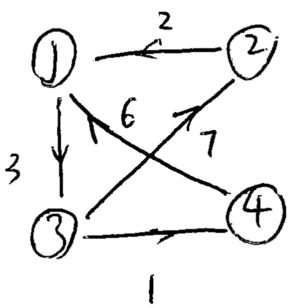
Floyd-Marshall (A)

$C \leftarrow A$
for $k \leftarrow 1$ to n
do for $i \leftarrow 1$ to n
  do for $j \leftarrow 1$ to n
    do if $c_{ij} > c_{ik} + c_{kj}$
      then $c_{ij} = c_{ik} + c_{kj}$

$O(n^3)$ better than Bellman-Ford.

e.g.



$$A = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} = C^{(0)}$$

step 1:
$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array}$$

w/ 1 as intermediate

Q/ what won't change?

A/ C_{1i} or C_{i1} for $i = 1, 2, \cdots n$. Basically the 1st row & col

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \boxed{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \boxed{9} & 0 \end{bmatrix}$$

step 2:
$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \boxed{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ 9 & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \boxed{10} & 3 & \boxed{4} \\ 2 & 0 & 5 & \boxed{6} \\ 9 & 7 & 0 & 1 \\ 6 & \boxed{16} & 9 & 0 \end{bmatrix}$$

Step 4:

$$\begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 9 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ \boxed{7} & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix}$$ done.