CS 303, Paul Cao
Plan: Maximum flow problem (26.1-26.2)

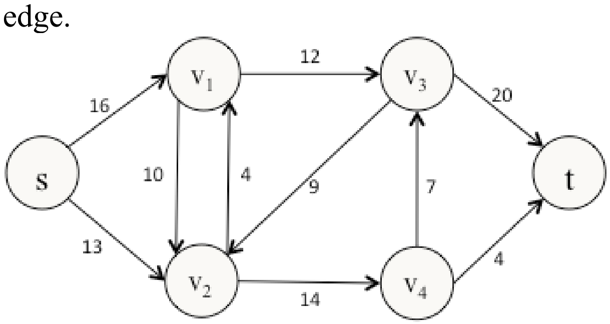
1. What is maximum flow problem?
A:/ How can we maximize the flow in a network from a source or set of sources to a destination of set of destinations?

Background
- The problem reportedly rose to prominence in relation to the rail networks of the Soviet Union, during the 1950's. The US wanted to know how quickly the Soviet Union could get supplies through its rail network to its satellite states in Eastern Europe.
- In addition, the US wanted to know which rails it could destroy most easily to cut off the satellite states from the rest of the Soviet Union. It turned out that these two problems were closely related, and that solving the **max flow problem** also solves the **min cut problem** of figuring out the cheapest way to cut off the Soviet Union from its satellites.

2. Problem set up
- A Network is a directed graph G
- Edges represent pipes that carry flow
- Each edge <u,v> has a maximum capacity $c_{<u,v>}$
- A source node s in which flow arrives
- A sink node t out which flow leaves

The solution uses a graph to model material that flows through conduits.
- Each edge represents one conduit, and has a capacity, which is an upper bound on the flow rate = units/time.
- Can think of edges as pipes of different sizes.
- Want to compute max rate that we can ship material from a designated source to a designated sink.
- Each edge (u,v) has a nonnegative capacity c(u,v).
- If (u,v) is not in E, assume c(u,v)=0.
- We have a source s, and a sink t.
- Assume that every vertex v in V is on some path from s to t.

e.g. find the maximum flow from s to t. The number on edges represents the capacity of that edge.



$c(s,v_1)=16$; $c(v_1,s)=0$; $c(v_2,v_3)=0$

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Q:/ What is a flow in the network?
A:/ For each edge (u,v), the flow f(u,v) is a real-valued function that must satisfy 3 conditions:

$\forall u, v$

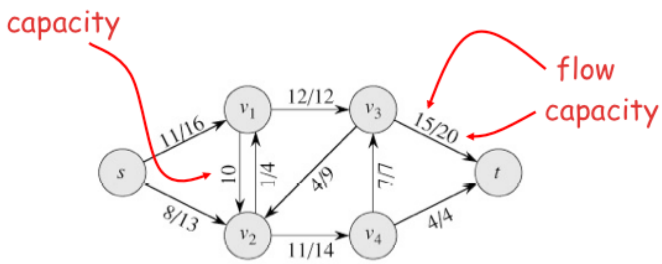
Capacity conservation: $\forall u,v \in V, f(u,v) \le c(u,v)$

Skew symmetry: $\forall u,v \in V, f(u,v) = -f(v,u)$

Flow conservation: $\forall u \in V - \{s,t\}, \sum_{v \in V} f(u,v) = 0$

Q:/ What is f(u,u)?
A:/ It is 0 from skew symmetry

Example



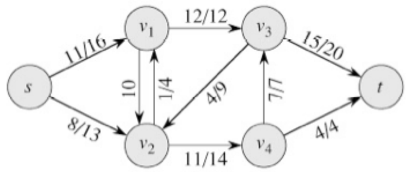
- $f(v_2, v_1) = 1$, $c(v_2, v_1) = 4$.
- $f(v_1, v_2) = -1$, $c(v_1, v_2) = 10$.
- $f(v_3, s) + f(v_3, v_1) + f(v_3, v_2) + f(v_3, v_4) + f(v_3, t) =$
  $0 \quad + \quad (-12) \quad + \quad 4 \quad + \quad (-7) \quad + \quad 15 \quad = \; 0$

The overall flow value of a network is

$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,u)$

This is the total flow leaving s = the total flow arriving in t.

Example



$|f| = f(s, v_1) + f(s, v_2) + f(s, v_3) + f(s, v_4) + f(s, t) =$

$\quad 11 + 8 + 0 + 0 + 0 = 19$

$|f| = f(s, t) + f(v_1, t) + f(v_2, t) + f(v_3, t) + f(v_4, t) =$

$\quad 0 + 0 + 0 + 15 + 4 = 19$

Assumption

- We assume that there is only flow in one direction at a time. e.g.Sending 7 trucks from Ashland to Columbus and 3 trucks from Columbus to Ashland has the same net effect as sending 4 trucks from Ashland to Columbus.

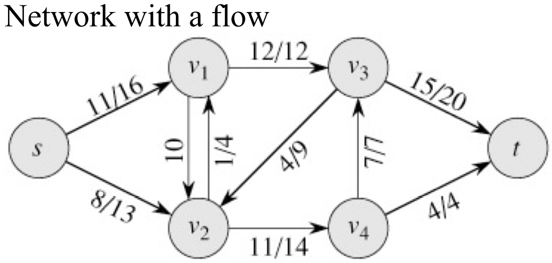
Residual network

$c_f(u,v) = c(u,v) - f(u,v)$

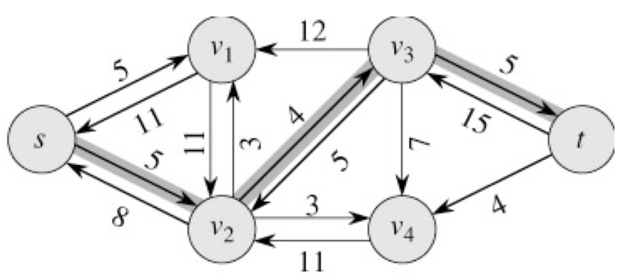
note: The residual network of a graph *G* induced by a flow *f* is the graph including only the edges with positive residual capacity.

e.g.

Network with a flow



Residual network



Q:/ Why do we need to study the residual network?

A:/ To find if there are possibilities of improving the flow augmented path

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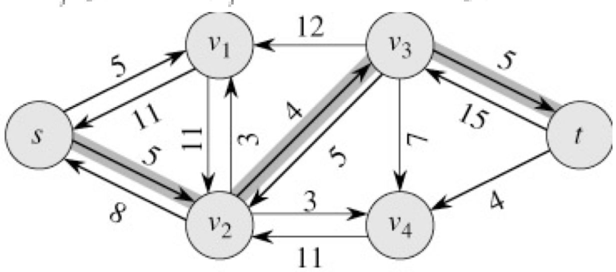
- An augmenting path p is a simple path from s to t on the residual network.
- We can put more flow from s to t through p.
- We call the maximum capacity by which we can increase the flow on p the residual capacity of p.

$c_f(p) = \min\{c_f(u,v) : (u,v) \text{ is on } p\}$



e.g. in the residual network, the capacity of the augmenting path is 4.

3. Ford-Fulkerson method

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FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p
      do augment flow f along p
return f
```

e.g

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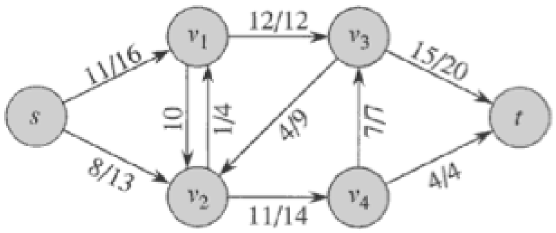
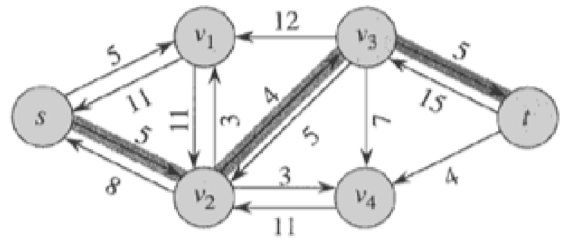
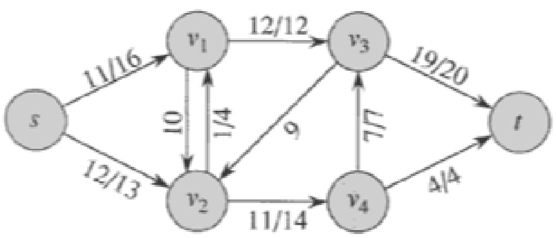


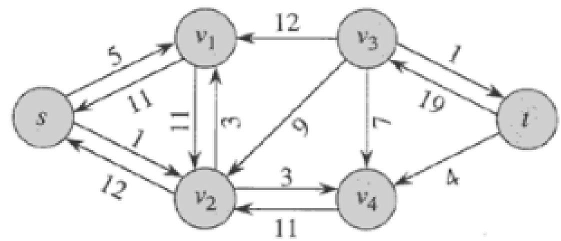
Flow(1)



Residual(1)



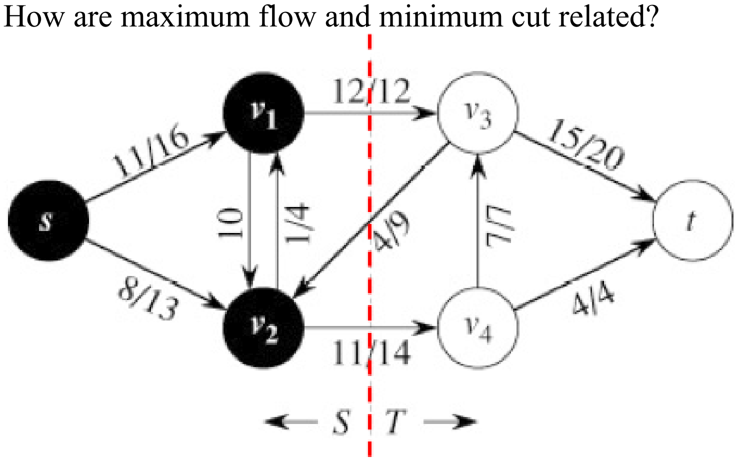
Flow(2)



Residual(2)

No more augmenting paths $\Rightarrow$ max flow attained.

How are maximum flow and minimum cut related?



A cut (S,T) of a flow network is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.

In this example, f(S,T) = 12 – 4 + 11 = 19, c(S,T)= 12+ 0 + 14 = 26

minimum cut is the flow through the cut from s to t in the maximum flow network

Theorem:

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If f is a flow in a flow network G=(V,E), with source s and sink t, then the following conditions are equivalent:

1. *f* is a maximum flow in *G*.
2. The residual network G_f contains no augmented paths.
3. $|f| = c(S,T)$ for some cut (S,T) (a min-cut).

Ford –Fulkerson algorithm

FORD-FULKERSON(G, s, t)

for each edge $(u, v) \in E[G]$
    do $f[u, v] \leftarrow 0$
        $f[v, u] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f
    do $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
        for each edge (u, v) in p
            do $f[u, v] \leftarrow f[u, v] + c_f(p)$
                $f[v, u] \leftarrow -f[u, v]$