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基础题目

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1.1 基础知识

符号：

$f^n(x)$ 表示 f 的 n 次复合，$\underbrace{f(f(\cdots f(x)))}_{n\text{层}}$。特别地，$f^0(x)=x$。

$\binom{m}{n}$ 表示 m 中取 n 的组合数，$\binom{m}{n}=C_m^n=\dfrac{m!}{n!(m-n)!}$。

常见函数的导数：

函数	导函数	定义域
ax^n	anx^{n-1}	$x\in\mathbb{R}, n\neq 0$
a^x	$a^x\ln a$	$x\in\mathbb{R}, a\neq 0$
e^x	e^x	$x\in\mathbb{R}$
$\ln x$	$\dfrac{1}{x}$	$x\in(0,+\infty)$
$\log_a x$	$\dfrac{1}{x\ln a}$	$x, a\in(0,+\infty)$
$\sin x$	$\cos x$	$x\in\mathbb{R}$
$\cos x$	$-\sin x$	$x\in\mathbb{R}$
$\tan x$	$\dfrac{1}{\cos^2 x}$	$x\neq\dfrac{(2n+1)\pi}{2}, n\in\mathbb{Z}$

复合函数的导数：

函数	$f(g(x))$	$f(x)+g(x)$	$f(x)g(x)$	$\dfrac{f(x)}{g(x)}$
导函数	$g'(x)f'(g(x))$	$f'(x)+g'(x)$	$f'(x)g(x)+f(x)g'(x)$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

函数积的多次导数（莱布尼兹公式）：

$$[f(x)g(x)]^{(n)}=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)=C_n^0f(x)g^{(n)}(x)+C_n^1f'(x)g^{(n-1)}(x)+\cdots+C_n^nf^{(n)}(x)g(x)$$

1.2 函数的概念

题目 001 已知函数 $f(x)$ 定义域为 $\left[-\dfrac{1}{2},\dfrac{3}{2}\right]$，$a>0$，求函数 $g(x)=f(ax)+f\left(\dfrac{x}{a}\right)$ 的定义域。

解答 过程略。

1. 当 $a\geqslant 1$ 时，所求定义域 $\left[-\dfrac{1}{2a},\dfrac{3}{2a}\right]$；
2. 当 $0<a<1$ 时，所求定义域 $\left[-\dfrac{a}{2},\dfrac{3a}{2}\right]$。

TAG 定义域 DIF 易

题目 002 求下面函数的值域：(1)$y=\dfrac{2\sin x-1}{2\sin x+1}$ (2)$y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

(3)$y=(x+1)(x+2)(x+3)(x+4), x\in[-3,3]$

解答

1. （方法一）[1]反解 $\sin x$ 得 $\sin x=\dfrac{1+y}{2(1-y)}$，由 $\sin x\in[-1,1]$ 知，严格地，
$$|\sin x|=\left|\frac{1+y}{2(1-y)}\right|\leqslant 1 \ \Leftrightarrow\ (1+y)^2\leqslant 4(1-y)^2 \ \Leftrightarrow\ y\in\left(-\infty,\frac{1}{3}\right]\cup[3,+\infty)$$
（方法二）记 $t=\sin x\in[-1,1]$，则
$$y=\frac{2t-1}{2t+1}=1-\frac{2}{2t+1}\in\left(-\infty,\frac{1}{3}\right]\cup[3,+\infty)$$
2. 两种方法同上，此处不赘述。$y\in(-1,1)$。
3. 对称展开得 $y=(x^2+5x+4)(x^2+5x+6)=(t+4)(t+6)=(t+5)^2-1$，其中 $t=x^2+5x\in\left[-\dfrac{25}{4},24\right]$，结合二次函数图像可知 $y=(t+5)^2-1\in[-1,840]$。

TAG 反函数 值域 DIF 易

题目 003 求下列函数的值域：(1)$f(x)=-\dfrac{x^2-x+2}{x^2+2}$ (2)$f(x)=x-\sqrt{1-x^2}$

解答

1. （方法一）$f(x)=-\dfrac{x^2-x+2}{x^2+2}=\dfrac{1}{\dfrac{2}{x}+x}-1$，分母是对勾函数[2]，其值域为 $\dfrac{2}{x}+x\in(-\infty,-2\sqrt{2})\cup(2\sqrt{2},+\infty)$，另外 $f(-1)=0$，因此得值域 $\left[-1-\dfrac{\sqrt{2}}{4},-1+\dfrac{\sqrt{2}}{4}\right]$。
（方法二）注意到分母不为 0。记 $f(x)=y$，则原式 $\Leftrightarrow\ (y+1)x^2-x+(2y+2)=0$，视之为关于 $x\in\mathbb{R}$ 的二次函数[3]，那末
$$\Delta=1-4(y+1)(2y+2)\geqslant 0\ \Leftrightarrow\ y\in\left[-1-\frac{\sqrt{2}}{4},-1+\frac{\sqrt{2}}{4}\right]$$
2. （方法一）设 $x=\sin t\in[-1,1]$[4]，则 $f(x)=\sin t-|\cos t|\in[-\sqrt{2},1]$。
（方法二）易知 $f(x)$ 在 $[0,1]$ 上有 $f(x)\geqslant f(-x)$ 且单调增，$x\in[-1,1]$，因此严格地 $f(x)\leqslant f(1)=1$。又
$$(y-x)^2=1-x^2\ \Leftrightarrow\ 2x^2-2xy+y^2-1=0\Rightarrow\ \Delta=8-4y^2\geqslant 0\ \Leftrightarrow\ f(x)=y\in[-\sqrt{2},\sqrt{2}]$$
注意到 $f\left(-\dfrac{\sqrt{2}}{2}\right)=-\sqrt{2}$，$f(x)$ 连续[5]，故值域 $f(x)\in[-\sqrt{2},1]$。

TAG 判别式 值域 DIF 易

题目 004

1. 已知函数 $f(x)=\log_a\left(x+\dfrac{a}{x}-4\right)$ 的定义域为 $\mathbb{R}$，其中 $a>0$，$a\neq 1$，求实数 a 的取值范围。
2. 已知函数 $f(x)=\log_3\dfrac{mx^2+8x+n}{x^2+1}$ 的定义域为 $\mathbb{R}$，值域为 $[0,2]$，求

[1] 但凡有单值反函数，那末反函数的定义域即为函数的值域，反函数的值域即为函数的定义域。

[2] 当我们上下同除 x 时，事实上假定了 $x\neq 0$，因此计算后须对 $x=0$ 时的取值进行补充。这样做是无妨的，原因有二，一是 $f(x)$ 是连续的，二是我们完全可以补充 $\dfrac{1}{\infty}=0$。

[3] 形如 $f(x)=\dfrac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$ 和 $f(x)=ax+b+c\sqrt{dx^2+ex+f}$ 的函数式均可尝试用判别式的方法求值域。

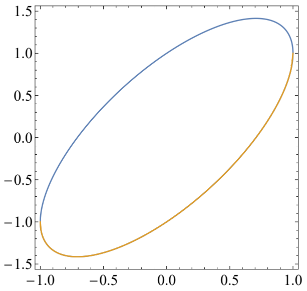

Figure 1.1: $2x^2-2xy+y^2=1$ 的图像。橙色部分即为 (2) 中 $f(x)$ 的图像。

[4] 为何设三角函数对于类似形式的函数可以起到等价化简的作用？因为这原本就是二次圆锥曲线的变形，本题 (2) 中的函数即是旋转的椭圆。

[5] 求最值和求值域是不同的，函数不一定能取到最大值和最小值之间的值，然而函数必然可以取到值域中的所有值。

m,n 的值。

3. 设函数 $f(x)=\log_3\dfrac{bx^2+ax+b}{x^2+x+1}$，已知 $a>b$，函数的值域为 $(-\infty,0]$，求 a 的取值范围。

解答

1. 值域为 $\mathbb{R}$ 意味着 $x+\dfrac{a}{x}-4$ 值域包含 $\mathbb{R}^+$，又 $a>0$，故这是一个对勾函数，其最小值为 $x+\dfrac{a}{x}-4\geqslant 2\sqrt{x\cdot\dfrac{a}{x}}-4=2\sqrt{a}-4$，那末只需使 $2\sqrt{a}-4\leqslant 0\Rightarrow$ $a\in(0,1)\cup(1,4]$ 。
2. 由于 $f(x)$ 定义域为 $\mathbb{R}$，故 $\forall x\in\mathbb{R}$，恒有
$$\frac{mx^2+8x+n}{\underbrace{x^2+1}_{>0}}>0\ \Leftrightarrow\ \text{分母}\Delta=64-4mn<0,m>0$$
又根据值域有[6]
$$\frac{mx^2+8x+n}{x^2+1}=k\in[1,9]\ \Leftrightarrow\ k^2-(m+n)k+mn-16\leqslant 0\ \forall k\in[1,9]$$
$$\Rightarrow\ m+n=10,mn-16=9\ \Rightarrow\ m=n=5$$
3. 由题条件[7]，
$$\frac{bx^2+ax+b}{\underbrace{x^2+x+1}_{>0}}=k\in(0,1]\ \Rightarrow\ (b-k)x^2+(a-k)x+(b-k)=0$$
当 $k\in(0,1]$ 时恒有解，$k>1$ 时无解。因此
$$\forall k\in(0,1],\ \Delta=(a-k)^2-4(b-k)^2\geqslant 0,\text{且}k=1\text{处取等，下同}$$
$$\Leftrightarrow\ \forall k\in(0,1],\ \left(k-\frac{a+2b}{3}\right)(k-(2b-a))\geqslant 0$$
由于 $a>b$，故 $\dfrac{a+2b}{3}>2b-a$，
$$\begin{cases}\dfrac{a+2b}{3}=1\\ 2b-a\leqslant 0\end{cases}\ \Rightarrow\ a\geqslant\frac{3}{2}$$

TAG 复合函数 DIF 中

题目 005 已知 $f(x)=\sqrt{4-x^2}$ 在闭区间 M 上的反函数是其本身，求 M 的取值集合。

解答 显然 $f(x)$ 函数图像为半径为 2，圆心在原点的圆的上半部分。结合函数与其反函数图像的几何关系[8]，可知 M 的取值集合为 $M\in\left\{[2\sin\theta,2\cos\theta]:\theta\in\left[0,\dfrac{\pi}{4}\right]\right\}$ 。

TAG 反函数 圆 DIF 易

题目 006 设定义域为 $\mathbb{R}$ 的函数 $f(x),g(x)$ 均有反函数，且函数 $f(x-1)$ 和 $g^{-1}(x-2)$ 的图像关于直线 $y=x$ 对称，若 $g(5)=2021$，求 $f(4)$ 的值。

解答 由题意 $(5,f(4))$ 与 $(f(4),g^{-1}(f(4)-2))$ 关于 $y=x$ 对称，故 $g^{-1}(f(4)-2)=5$，由于 $g(x)$ 有反函数，故 g 为双射[9]，因此 $f(4)-2=2021\Rightarrow$ $f(4)=2023$ 。

TAG 反函数 DIF 易

[6] 此过程中暗含使用了 $x\in\mathbb{R}$ 这一条件，因此我们可以毫无顾虑的使用判别式计算。同时闭区间端点对应的是边界条件，放在不等式上意味着「取等」。

[7] 本小题对定义域没有说明和限制，但并不意味着定义域为 $\mathbb{R}$。

[8] 函数与其反函数关于直线 $x=y$ 镜像对称。

[9] **双射：** 若关于映射 $f:A\to B$ 有以下两特点，则称映射 f 为双射，也即「一一映射」：

1. $\forall y\in B,\exists x\in A$, s.t. $f(x)=y$（满射）
2. 若 $x_1\in A,x_2\in A,f(x_1)=f(x_2)$，则必有 $x_1=x_2$（单设）

若一个函数有反函数，即一个映射有它的逆，那末它必是双射。

题目 007 设函数 $f:\mathbb{R}\to\mathbb{R}$ 满足 $f(0)=1$，且 $\forall x,y\in\mathbb{R}$ 都有 $f(xy+1)=f(x)f(y)-f(y)-x+2$，求 $f(x)$ 的表达式。

解答 令 $x=y=0$,可得 $f(1)=2$;令 $y=0$,可得 $f(1)=2=f(x)f(0)-f(0)-x+2=f(x)-x+1\Rightarrow$ $f(x)=x+1$ 。

TAG 抽象函数 DIF 易

题目 008 设定义在整数集上的函数 f 满足 $f(n)=\begin{cases}n+5 & n<2000\\ f(f(n-8)) & n\geqslant 2000\end{cases}$，求 $f(4096)$。

解答 我们使用符号 $f^k(x)$ 表记 $\overbrace{f(f(f(\cdots f(x))))}^{k次f复合}$，那末

$$\begin{aligned}&f(4096)=f^2(4088)=\cdots=f^{263}(2000)=f^{264}(1992)\\&=f^{262}(2002)=f^{263}(1994)=f^{261}(2004)=f^{262}(1996)=f^{261}(2001)\\&=f^{262}(1993)=f^{260}(2003)=f^{261}(1995)=f^{260}(2000)\end{aligned}$$

这里我们得到了第一个周期 $f^{263}(2000)=f^{260}(2000)$，因此可知 $\forall n\geqslant 2000,k\in\mathbb{Z}$，$f^{k+3}(n)=f^k(n)$，因此 $f(4096)=f^{261}(2004)=f^0(2004)=$ 2004 。事实上可以求出，$\forall n\geqslant 2000$，$f(n)=2002+(n+1)\%3$，其中「%」为取余符号。

TAG 抽象函数 DIF 中

题目 009 已知函数 $f(x)=\dfrac{2x^2+bx+c}{x^2+1}$ 的值域为 $[1,3]$，其中 $b<0$。

1. 求实数 b,c 的值。
2. 判断 $F(x)=\lg f(x)$ 在 $[-1,1]$ 上的单调性，并给出证明。
3. 若 $t\in\mathbb{R}$，求证：$\lg\dfrac{7}{5}\leqslant F\left(\left|t-\dfrac{1}{6}\right|-\left|t+\dfrac{1}{6}\right|\right)\leqslant\lg\dfrac{13}{5}$。

解答

1. 记 $y=f(x)$，定义域 $x\neq\pm1$。整理得 $(y-2)x^2-bx+(y-c)=0\Rightarrow\Delta=b^2-4(y-2)(y-c)\geqslant 0$，由于 $y=f(x)$ 的值域为 $[1,3]$，因此 $\Delta|_{y=1}=\Delta|_{y=3}=0\Rightarrow$ $b=-2,c=2$ 。
2. 由于 $\lg x$ 是增函数，因此其增减性与 $f(x)$ 一致。$f'(x)=\dfrac{2(x^2-1)}{(x^2+1)^2}<0$，因此 $F(x)=\lg x$ 在 $[-1,1]$ 上单调减。
3. 根据 $F(x)$ 的递减性，只需找到 $h(t)=\left|t-\dfrac{1}{6}\right|-\left|t+\dfrac{1}{6}\right|$ 的值域。首先 $h(t)$ 是连续的，

$$h(t)\leqslant\left|\left(t-\frac{1}{6}\right)-\left(t+\frac{1}{6}\right)\right|=\frac{1}{3}\qquad 当t=-\frac{1}{6}时取等$$

由于 $h(t)$ 是奇函数，因此其最小值为 $\min h(t)=-\max h(t)=-\dfrac{1}{3}$。综上，

$$\lg\frac{7}{5}=F\left(\frac{1}{3}\right)\leqslant F(h(t))\leqslant F\left(-\frac{1}{3}\right)=\lg\frac{13}{5}$$

TAG 绝对值不等式 值域 DIF 易

1.3 抽象函数

抽象函数相关题目在试题中出现的不多，然而由于它需要对函数概念有较深的理解，并有一定的函数分析能力，因此不乏不错的习题。抽象函数有一些常出现的模型，列举如下：

模型	方程式	备注
一次函数模型	$f(x+y)=f(x)+f(y)$	
幂函数模型	$f(xy)=f(x)f(y)$	
指数函数模型	$f(x+y)=f(x)f(y)$	
对数函数模型	$f(xy)=f(x)+f(y)$	
余切函数模型	$f(x+y)=\dfrac{f(x)-f(y)}{1-f(x)f(y)}$	$\cot x=\dfrac{1}{\tan x}$
正切函数模型	$f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$	
双曲正切函数模型	$f(x+y)=\dfrac{f(x)+f(y)}{1+f(x)f(y)}$	$\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

函数的性质：

奇偶性 $\forall x\in D$，若 $-x\in D$ 且 $f(x)=-f(-x)$，称函数为奇函数；$\forall x\in D$，若 $-x\in D$ 且 $f(x)=f(-x)$，称函数为偶函数。

有界性 若 $\exists M<\infty$，使得 $\forall x\in S\subset D$，$f(x)<M$，称 $f(x)$ 在 S 上有上界 M；若 $\exists N<\infty$，使得 $\forall x\in S\subset D$，$f(x)>N$，称 $f(x)$ 在 S 上有下界 N；若 $f(x)$ 在 S 上有上下界，称 $f(x)$ 在 S 上有界。

单调性 若 $\forall x_1,x_2\in S\subset D, x_1>x_2$，恒有 $f(x_1)-f(x_2)\geqslant 0$，称 $f(x)$ 在 S 上单调递增，若式中等号无法取得，称 $f(x)$ 在 S 上严格单调递增；若 $\forall x_1,x_2\in S\subset D, x_1>x_2$，恒有 $f(x_1)-f(x_2)\leqslant 0$，称 $f(x)$ 在 S 上单调递减，若式中等号无法取得，称 $f(x)$ 在 S 上严格单调递减。

连续性 固定 $x_0\in D$, 若 $\forall\varepsilon>0, \exists\delta(\varepsilon)>0$, s.t $\forall x\in(x_0,x_0+\delta(\varepsilon))$, $|f(x_0)-f(x)|<\varepsilon$, 称 $f(x)$ 在 $x=x_0$ 处右连续；固定 $x_0\in D$，若 $\forall\varepsilon>0$，$\exists\delta(\varepsilon)>0$，s.t $\forall x\in(x_0-\delta(\varepsilon),x_0)$，$|f(x_0)-f(x)|<\varepsilon$，称 $f(x)$ 在 $x=x_0$ 处左连续；若 $f(x)$ 在 $x=x_0$ 处左连续且右连续，称 $f(x)$ 在 $x=x_0$ 处连续。

可导性 固定 $x_0\in D$，若 $\exists a<\infty$，$\forall\varepsilon>0$，$\exists\delta(\varepsilon)>0$，s.t $\forall x\in(x_0,x_0+\delta(\varepsilon))$，$\left|\dfrac{f(x)-f(x_0)}{x-x_0}-a\right|<\varepsilon$，称 $f(x)$ 在 $x=x_0$ 处右导数存在且为 a；固定 $x_0\in D$，若 $\exists b<\infty$，$\forall\varepsilon>0$，$\exists\delta(\varepsilon)>0$，s.t $\forall x\in(x_0-\delta(\varepsilon),x_0)$，$\left|\dfrac{f(x_0)-f(x)}{x_0-x}-b\right|<\varepsilon$，称 $f(x)$ 在 $x=x_0$ 处左导数存在且为 b；若 $f(x)$ 在 $x=x_0$ 处左右导数都存在且 $a=b=A$，称 $f(x)$ 在 $x=x_0$ 处可导且导数为 A。x 到 $f(x)$ 在 x 处导数的映射称为 $f(x)$ 的导函数。可导必定连续。连续不一定可导。[10]

题目 010

1. 若函数 $f(x)$ 满足 $\forall x\neq 0$，$f(x)+2f\left(\dfrac{1}{x}\right)=3x$，求 $f(x)$ 的显性表达式。
2. 若函数 $f(x)$ 满足 $\forall x\neq 0$，$f(x)+f\left(\dfrac{x-1}{x}\right)=1+x$，求 $f(x)$ 的显性表达式。
3. 若函数 $f(x)$ 满足 $\forall x\neq 0$，

$$f\left(\frac{x-1}{x+1}\right)+f\left(-\frac{1}{x}\right)+f\left(\frac{1+x}{1-x}\right)=\cos x$$

[10] 连续性和可导性可在定义极限后改用极限语言叙述，可复习教材定积分一章。

求 $f(x)$ 的显性表达式。

解答

1. 令 $x=\frac{1}{x}$（此处为赋值），得 $2f(x)+f\left(\frac{1}{x}\right)=\frac{3}{x}$，与题中条件联立得 $f(x)=\frac{2}{x}-x$ 。

2. 记 $g(x)=\frac{x-1}{x}$，我们注意到这样一个事实[11]：$\forall n\in\mathbb{N}, g^{n+3}(x)=\underbrace{g(g(\cdots g(x)))}_{n+3\text{层复合}}=g^{n}(x)$，因此若记[12] $a=f(x)=f(g^{0}(x)), b=f(g(x)), c=f(g^{2}(x))$，赋值 $x=x, x=g(x), x=g^{2}(x)$，可得三个等式：

$$\begin{cases} a+b=f(x)+f(g(x))=1+x \\ b+c=f(g(x))+f(g^{2}(x))=1+g(x) \\ c+a=f(g^{2}(x))+f(g^{3}(x))=f(g^{2}(x))+f(x)=1+g^{2}(x) \end{cases}$$

解之得

$$a=f(x)=\frac{1}{2}\left[(1+x)-(1+g(x))+1+g^{2}(x)\right]=\frac{x^{3}-x^{2}-1}{2x(x-1)}$$

3. 记 $g(x)=\frac{1-x}{1+x}$，注意到这样一个事实：$g^{2}(x)=-\frac{1}{x}$，$g^{3}(x)=\frac{1+x}{1-x}$，$g^{4}(x)=x=g^{0}(x)$，因此原式可改写为：$f(g(x))+f(g^{2}(x))+f(g^{3}(x))=\cos x$，赋值 $x=x, x=g(x), x=g^{2}(x)$，可得四个等式：

$$\begin{cases} f(g(x))+f(g^{2}(x))+f(g^{3}(x))=\cos x \\ f(g^{2}(x))+f(g^{3}(x))+f(x)=\cos g(x) \\ f(g^{3}(x))+f(x)+f(g(x))=\cos g^{2}(x) \\ f(x)+f(g(x))+f(g^{2}(x))=\cos g^{3}(x) \end{cases}$$

解之得[13]

$$\begin{aligned} f(x)&=\frac{1}{3}\left[-2\cos x+\cos g(x)+\cos g^{2}(x)+\cos g^{3}(x)\right] \\ &=\frac{1}{3}\left[-2\cos x+\cos\frac{1-x}{1+x}+\cos\frac{1}{x}+\cos\frac{1+x}{1-x}\right] \end{aligned}$$

TAG 抽象函数 DIF 中

题目 011

1. 求所有的函数 $f:\mathbb{R}\to\mathbb{R}$，使得 $\forall x,y\in\mathbb{R}$，下式恒成立：
$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^{2}-y^{2})$$
2. 求所有的函数 $f:\mathbb{R}\to\mathbb{R}$，使得 $\forall x,y\in\mathbb{R}$，下式恒成立：
$$f(f(x)+y)=2x+f(f(f(y))-x)$$
3. 求所有的函数 $f:\mathbb{R}\to\mathbb{R}$，使得 $\forall x,y\in\mathbb{R}$，下式恒成立：
$$f(x+y)+f(x-y)=2f(x)\cos y$$

解答

1. [14]设 $u=x+y$，$v=x-y$，原式可化为

$$vf(u)-uf(v)=(u^{2}-v^{2})uv \Leftrightarrow \frac{f(u)}{u}-\frac{f(v)}{v}=u^{2}-v^{2}$$

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[11] 这一事实启示了我们，若有一个数列 $\{a_n\}$，其递推式为 $a_{n+1}=1-\frac{1}{a_n}$，那末 $\{a_n\}$ 为周期数列。这里结合数列知识可以展开很多，不过与章节内容不符，故不赘述。

[12] 以下两种记号应当作区分：$f^{n}(x)$ 表示 $\underbrace{f(f(\cdots f(x)))}_{n\text{层复合}}$；$[f(x)]^{n}$ 表示 $\underbrace{f(x)f(x)\cdots f(x)}_{n\text{个}f(x)\text{乘积}}$。只有在不造成混淆的情况下，可用第一种记号表示函数值的乘方。事实上，函数的复合被视为映射的乘积，这就是第一种记号的来源。

[13] 函数

$$f(x)=\frac{x\cos\frac{2\pi}{m}-\sin\frac{2\pi}{m}}{x\sin\frac{2\pi}{m}+\cos\frac{2\pi}{m}}$$

满足 $f^{m}(x)=x$，2,3 两小题出题背景即是此式。此式的背景为旋转变换，可自行查阅旋转变换相关的拓展知识。

[14] 本题背景为图像的旋转变换 $(x,y)\mapsto(x\cos\theta-y\sin\theta,\ x\sin\theta+y\cos\theta)$，以及线性组合。

显然 $f(x)=x^3$ 为一特解，我们需要找到所有解，即通解。假设有另一解 $f(x)=x^3+xg(x)$，$g(x)$ 为同定义域的未知函数，代入原式得

$$\forall u,v\in\mathbb{R},\ g(u)-g(v)=0 \ \Leftrightarrow\ g(x)\equiv C$$

其中 C 为一实常数。因此通解为 $f(x)=x^3+C,\ C\in\mathbb{R}$ 。

2. 我们先求 $f(0)$。令 $x=y=0$，得 $f^2(0)=f^3(0)\Rightarrow \forall n\geqslant 2, f^n(0)=f^2(0)$；令 $y=0,x=f^2(0)$，得 $f^4(0)=f^2(0)=2f^2(0)+f(0)\Rightarrow -f(0)=f^2(0)$；令 $x=0,y=-f(0)$，得

$$f(0)=f(f^2(-f(0)))=f^5(0)=-f(0)\ \Rightarrow\ f(0)=0$$

令 $y=0$，得 $f^2(x)=2x+f(-x)$；令 $y=x,x=0$，得 $f(x)=f^3(x)$；令 $y=-f(x)$，得

$$f(0)=0=2x+f(f^2(-f(x))-x)$$

令 $x=f(x),y=0$，得

$$f^3(x)=2f(x)+f(-f(x))\ \Rightarrow\ f(-f(x))=-f(x)$$

赋值 $x=f(x)$ 之后，似乎可以认为 $f(x)=x$ 了，不过因为 $f:\mathbb{R}\to\mathbb{R}$，我们须得[15]确认 $f(x)$ 可遍历 $\mathbb{R}$。回代之得 $\forall x\in\mathbb{R}$，

$$0=2x+f(f(f(-f(x)))-x)=2x+f(-f(x)-x)$$

故 $f(x)$ 可遍历 $\mathbb{R}$。因此可以确认只有 $f(x)=x$ 符合题意。

3. [16]分别赋值 $x=0,y=-x$；$x=\dfrac{\pi}{2},y=\dfrac{\pi}{2}+x$；$y=\dfrac{\pi}{2},x=\dfrac{\pi}{2}+y$，可得三个式子：

$$\begin{cases} f(x)+f(-x)=2f(0)\cos x\\ f(\pi+x)+f(x)=0\\ f(\pi+x)+f(-x)=-2f\left(\dfrac{\pi}{2}\right)\sin x\end{cases}$$

解之得 $f(x)=f(0)\cos x+f\left(\dfrac{\pi}{2}\right)\sin x$。由于 $f(0),f\left(\dfrac{\pi}{2}\right)$ 任意，因此

$$f(x)=A\cos x+B\sin x,\ A,B\in\mathbb{R}$$

对于第三小问，我们简要给出其出题背景的解释。原式可改写为

$$\frac{\dfrac{f(x+y)-f(x)}{y}-\dfrac{f(x)-f(x-y)}{y}}{y}=\frac{2f(x)}{y^2}[\cos y-1]$$

易证 f 有二阶导数，那末取 $y\to 0$，式子化为

$$f''(x)=\lim_{y\to 0}\text{左}=2f(x)\lim_{y\to 0}\frac{\cos y-1}{y^2}=-f(x)$$

这是标准的波动方程，其通解为

$$f(x)=A\sin x+B\cos x$$

TAG 抽象函数 DIF 难

题目 012 设 $f:\mathbb{Z}^{\geqslant 0}\to\mathbb{Z}^{\geqslant 0}$（$\mathbb{Z}^{\geqslant 0}$ 即非负整数集），$\forall n\in\mathbb{Z}^{\geqslant 0}$，$f(n)$ 满足：

(1) $[f(2n+1)]^2-[f(2n)]^2=6f(n)+1$

(2) $f(2n)\geqslant f(n)$

求 f 的值域中，满足小于 2021 的数的个数。

[15] 在做变量代换时，切记观察定义域是否有变动。

[16] 本题背景为波动方程。

题目 013 已知函数 $f(x)$ 的定义域为 $\mathbb{R}$，且 $\forall m,n\in\mathbb{R}$，$f(m+n)=f(m)+f(n)-1$，$f\left(-\frac{1}{2}\right)=0$，且 $\forall x>-\frac{1}{2}$，$f(x)>0$。求证 $f(x)$ 是严格单调递增函数。

解答 设 $x_1,x_2\in\mathbb{R}$，$x_1>x_2$，有[17]

$$\begin{aligned}f(x_1)-f(x_2)&=f(x_1-x_2+x_2)-f(x_2)=f(x_1-x_2)+f(x_2)-1-f(x_2)\\&=f(x_1-x_2)-1=f(x_1-x_2)+f\left(-\frac{1}{2}\right)-1\\&=f\left[-\frac{1}{2}+\underbrace{(x_1-x_2)}_{>0}\right]>0\quad\Rightarrow\quad f(x)\text{严格单调增}\end{aligned}$$

TAG 抽象函数 DIF 易

题目 014 设 $f_1(x),f_2(x)$ 定义在 $\mathbb{R}^+$ 上，且 $f_1(x)$ 单调增，$\forall x_1,x_2\in\mathbb{R}^+$，$|f_1(x_1)-f_1(x_2)|>|f_2(x_1)-f_2(x_2)|$。设 $f(x)=f_1(x)-f_2(x)$。

1. 求证：$f(x)$ 在 $\mathbb{R}^+$ 上严格单调递增。
2. 设 $F(x)=xf(x)$，$a>0$，$b>0$，求证：$F(a+b)>F(a)+F(b)$。

解答

1. 任取 $x_1>x_2>0$，那末

$$\begin{aligned}f(x_1)-f(x_2)&=\overbrace{[f_1(x_1)-f_1(x_2)]}^{>0}-[f_2(x_1)-f_2(x_2)]\\&\geqslant|f_1(x_1)-f_1(x_2)|-|f_2(x_1)-f_2(x_2)|>0\end{aligned}$$

因此 $f(x)$ 严格单调递增[18]。

2.

$$\begin{aligned}F(a+b)-F(a)-F(b)&=(a+b)f(a+b)-af(a)-bf(b)\\&=a\left[f(a+b)-f(a)\right]+b\left[f(a+b)-f(b)\right]>0\\&\Leftrightarrow\ F(a+b)>F(a)+F(b)\end{aligned}$$

TAG 抽象函数 DIF 易

题目 015 已知定义域 $\mathbb{R}$ 的函数 $f(x)$ 满足 $f(f(x)-x^2+x)=f(x)-x^2+x$。

1. 若 $f(2)=3$，求 $f(1)$；又若 $f(0)=a$，求 $f(a)$。
2. 若有且仅有一个实数 x_0，使得 $f(x_0)=x_0$，求函数 $f(x)$ 的表达式。

解答

1. 令 $x=2$ 得 $f(1)=1$；令 $x=0$ 得 $f(a)=a$。
2. 原表述意味着 $f(x)-x^2+x\equiv x_0$，故 $f(x_0)=x_0^2-x_0+x_0=x_0^2=x_0\Rightarrow x_0=0,1$。若 $x_0=0$，则 $f(x)=x^2-x$，$f(0)=0$，$f(2)=2$，这与 x_0 唯一性矛盾；若 $x_0=1$，则 $f(x)=x^2-x+1=x\Rightarrow x=1$，这是符合题意的。因此 $x_0=1$，$f(x)=x^2-x+1$。

TAG 抽象函数 不动点 DIF 易

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[17] 步骤暗示 (indicate) 了这样一个事实：将题干中 $-\frac{1}{2}$ 均更改为任意实数，$f(x)$ 增减性不变。这一性质由 $f(m+n)=f(m)+f(n)-1$（甚至这里的「1」也可更改）隐含保证。就此可稍作思考。

[18] 观察以下两个命题：

a) $\forall x_1,x_2\in D$，$|f_1(x_1)-f_1(x_2)|>|f_2(x_1)-f_2(x_2)|$
b) $\forall x_1,x_2\in D$，$|f'(x_1)|>|f'(x_2)|$

这两个式子都是同一个朴素性质的刻画：$f_1(x)$ 比 $f(x_2)$ 变化得快。我们对 (1) 同除 $|x_1-x_2|$，似乎可以推出 (2)，然而 (2) 是 (1) 的充分非必要条件，这是由于 (1) 不包含连续、可导的信息。这一点是重要的。

题目 016 设函数在 $\mathbb{R}$ 上满足 $f(2-x)=f(2+x)$，$f(7-x)=f(7+x)$，且在闭区间上，只有 $f(1)=f(3)=0$。

1. 判断 $f(x)$ 的奇偶性。
2. 求方程 $f(x)=0$ 在闭区间 $[-2021,2021]$ 上根的个数。

解答

1. 假设 $f(x)$ 为奇函数或偶函数，那末 $0=f(1)=f(-1)=f(5)$，这与 $f(x)$ 在 $[0,7]$ 上只有 $x=1,3$ 两个零点矛盾。因此 $f(x)$ 为非奇非偶函数。
2. 首先当且仅当 $m=5n+2, n\in\mathbb{Z}$ 时，$x=m$ 为 $f(x)$ 对称轴：

   ($\Rightarrow$) 记 $g(x)=f(x+2)$，则 $g(x)$ 满足 $g(x)$ 在 $[-2,5]$ 上有且仅有 $-1,1$ 两个零点，$g(x)=g(-x)$，$g(5-x)=g(5+x)$，$\forall k\in\mathbb{Z}, x\in\mathbb{R}$，

   $$\begin{aligned} g(5(2k)+x) &= g(10-5(2k)-x) = g(x+5(2k)-10)\\ &= \cdots = g(x) = g(-x) = \cdots = g(5(2k)-x)\\ g(5(2k+1)+x) &= g(10-5(2k+1)-x) = g(x+5(2k+1)-10)\\ &= \cdots = g(x+5) = g(5-x) = \cdots = g(5(2k+1)-x)\end{aligned}$$

   故 $\forall n\in\mathbb{Z}$，$x=5n$ 为 $g(x)$ 对称轴 $\Leftrightarrow$ $x=5n+2$ 为 $f(x)$ 对称轴。同时我们也说明了 $f(x)$ 是周期为 10 的周期函数。

   ($\Leftarrow$) 假设存在 $m=a\neq 5n, n\in\mathbb{Z}$ 为 $g(x)$ 对称轴，由周期性，不妨认为 $a\in[-5,5]$。

   a) 若 $a\in(0,2]$，则 $f(\underbrace{2a+1}_{\in(1,5]})=f(-1)=0$，矛盾。
   b) 若 $a\in[2,3]$，则 $f(1)=f(\underbrace{2a-1}_{\in[3,5]})=f(0)$，矛盾。
   c) 若 $a\in[3,5)$，则 $f(1)=f(2a-1)=f(\underbrace{11-2a}_{\in(1,5]})=0$，矛盾。

   结合 $g(x)$ 为偶函数可知不存在这样的额外对称轴。因此 $f(x)$ 不存在 $x=5n+2, n\in\mathbb{Z}$ 以外的对称轴。[19]因此 $f(x)$ 在每个周期内均有且仅有两个零点，在 $[-2021,2021]$ 上总的零点个数为 $\dfrac{2020}{10}\times 2+1=$ 405 。

TAG 抽象函数 周期性 轴对称 DIF 易

题目 017 已知定义在 $\mathbb{R}$ 上的函数 $f(x)$ 满足：

① 值域为 $(-1,1)$，且当 $x>0$ 时，$-1<f(x)<0$；

② 对于定义域内任意实数 x,y，均满足 $f(x+y)=\dfrac{f(x)+f(y)}{1+f(x)f(y)}$。

1. 试求 $f(0)$ 的值。
2. 判断和证明函数 $f(x)$ 的单调性。
3. 若函数存在反函数 $g(x)$，求证：
   $$g\left(\frac{1}{5}\right)+g\left(\frac{1}{11}\right)+\cdots+g\left(\frac{1}{n^2+3n+1}\right)>g\left(\frac{1}{2}\right)$$

解答

1. 令 $x=y=0$，得 $f(0)[(f(0))^2-1]=0$。又 $f(x)\in(-1,1)$，故 $f(0)=0$ 。
2. 首先，$f(x)$ 是奇函数：令 $y=-x$ 得
   $$f(x-y)=f(0)=0=\frac{f(x)+f(-x)}{1+f(x)f(-x)} \Rightarrow f(-x)=-f(x)$$

---

[19] 本题并不需要证明必要性，但这是一个重要的事实。事实上

$$f(a-x)=f(a+x)$$

意味着 $f(x)$ 关于 $x=a$ 轴对称；

$$f(a-x)=-f(a+x)$$

意味着 $f(x)$ 关于 $(a,0)$ 中心对称。

其次，$f(x)$ 在 $\mathbb{R}^+$ 上是减函数：令 $1 > x = x_1 > x_2 = y > 0$，得

$$\begin{aligned} 0 > f(x_1 - x_2) &= \frac{f(x_1) + f(-x_2)}{1 + f(x_1)f(-x_2)} \\ &= \frac{f(x_1) - f(x_2)}{1 + f(x_1)f(-x_2)} \\ &\Leftrightarrow\ f(x_1) - f(x_2) < 0 \end{aligned}$$

综上所述，可知 $f(x)$ 在 $\mathbb{R}$ 上是严格减函数。

3. 记 $f(x) = X, f(y) = Y$，则 $X = g(x), Y = g(Y)$，可以注意到[20]

$$g(X) + g(Y) = x + y = g(f(x + y)) = g\left(\frac{f(x) + f(y)}{1 + f(x)f(y)}\right) = g\left(\frac{X + Y}{1 + XY}\right)$$

由于 $f(x)$ 是严格单调减的奇函数，所以 $g(x)$ 也是严格单调减的奇函数，我们首先取 $Y = -Y$[21]，可得 $g(X) - g(Y) = g\left(\frac{Y - X}{XY - 1}\right)$，故

$$\begin{aligned} \text{原式左} &= g\left(\frac{3 - 2}{2 \times 3 - 1}\right) + g\left(\frac{4 - 3}{3 \times 4 - 1}\right) + \cdots + g\left(\frac{(n + 2) - (n + 1)}{(n + 1)(n + 2) - 1}\right) \\ &= [g(2) - g(3)] + [g(3) - g(4)] + \cdots + [g(n + 1) - g(n + 2)] \\ &= g(2) - g(n + 2) = g\left(\frac{(n + 2) - 2}{2(n + 2) - 1}\right) \\ &= g\underbrace{\left(\frac{1}{2 + 3/n}\right)}_{<1/2} > g\left(\frac{1}{2}\right) \end{aligned}$$

TAG 抽象函数 反函数 分析 DIF 难

至此为止我们始终根据抽象函数的性质进行计算和推导，然而应当有这样的问题：「满足所述性质的抽象函数是否存在？」「存在的话所有满足条件的函数构成的集合空间是怎样的？」「这样的空间有何性质？」「这样的函数是否可导？」等等。我们只添加 $f(x)$ 在 $x = 0$ 处连续且可导一个条件，从更高的角度看一下该函数究竟是什么函数。

令 $y = -x + \varepsilon$，有

$$\frac{f(x) - f(x - \varepsilon)}{\varepsilon} = [1 - f(x)f(x - \varepsilon)]\frac{f(\varepsilon)}{\varepsilon} \tag{※}$$

由 0 处可导性，$\lim\limits_{\varepsilon \to 0} f(\varepsilon) = f'(0) < \infty$。另外，结合 $f(0) = 0$ 和 0 处连续性，我们有：给定任意 $\delta > 0$，$\exists \varepsilon > 0$ 使得 $|f(\varepsilon)| < \delta/2$，进而

$$|f(x) - f(x - \varepsilon)| = |[1 - f(x)f(x - \varepsilon)]||f(\varepsilon)| < 2 \cdot \frac{\delta}{2} = \delta \ \Rightarrow\ f(x)\text{连续}$$

令※中 $\varepsilon \to 0$，可知 $f(x)$ 可导，且 $f'(x) = [1 - (f(x))^2]f'(0)$。求解该方程（称为微分方程）可得：

$$f(x) = \frac{e^{f'(0)x} - e^{-f'(0)x}}{e^{f'(0)x} + e^{-f'(0)x}} = \tanh(f'(0)x)$$

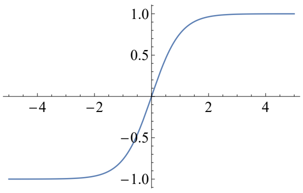

Figure 1.2: $f(x)$ 的一个图像

至此我们可以发现，函数的基本形态已经大体由 ② 确定了，① 中我们只使用了 $f(x)$ 有界这一条件，因此 ① 完全可以放宽为有界性。So what if $f'(0)$ remains undefined?

题目 018 函数 $f(x)$ 满足 ①$f(a + b) = f(a) \cdot f(b)$，②$f(4) = 16$，$m, n$ 为互质整数，$n \neq 0$，求 $f\left(\frac{m}{n}\right)$ 的值。

解答 令 $a = b = 0$ 得 $f(0) = [f(0)]^2 \Rightarrow f(0) = 0, 1$。若 $f(0) = 0$，再令 $b = 0$，可得 $f(a + 0) = f(a) = f(a)f(0) = 0 \Rightarrow f(x) \equiv 0$，这与 ② 矛盾。以下就 $f(0) = 1$ 时情况作讨论。令 $b = -a$ 可得 $f(-a) = \frac{1}{f(a)}$，再令 $a = b = \frac{x}{2}$，可得 $f(x) = \left[f\left(\frac{x}{2}\right)\right]^2 \geqslant 0$，再根据条件 ② 可知 $f(1) = 2$。

[20] 这里生动的说明了函数名为「Function」的原因。

[21] 这里的等号实际为「赋值」。

因此 $f\left(\frac{m}{n}\right)=\left[f\left(\frac{1}{n}\right)\right]^m=[f(1)]^{\frac{m}{n}}=2^{\frac{m}{n}}$。

TAG 抽象函数 分析 DIF 易

本题中抛却条件 ②，我们可以知道一个更纯粹的事实：满足 $f(a+b)=f(a)\cdot f(b)$ 的函数 $f(x)$ 至少在有理数域上为指数函数。同※一样地，我们添加 $f(x)$ 在 $x=0$ 处连续且可导以及全定义域上有界的条件，便可将这一事实拓展到整个实数域（忽略平凡解 $f(x)\equiv 0$）：注意到随着 $\varepsilon\to 0$，

$$|f(x+\varepsilon)-f(x)|=\underbrace{|f(x)|}_{<M}\underbrace{|[f(\varepsilon)-1]|}_{\to 0}\to 0$$

因此 $f(x)$ 在整个定义域内连续；注意到随着 $\varepsilon\to 0$，

$$\frac{f(x+\varepsilon)-f(x)}{\varepsilon}=f(x)\frac{f(\varepsilon)-f(0)}{\varepsilon}\to f(x)f'(0)<\infty$$

因此 $f(x)$ 在整个定义域内可导，且 $f'(x)=f(x)f'(0)$，解这一方程可得，$f(x)=e^{f'(0)x}$，这是指数函数。

题目 019 已知函数 $f(x)$ 满足：① 对任意 $x,y\in\mathbb{R}$，有 $f(xy)=f(x)f(y)$；②$f(-1)=1$，$f(27)=9$；③$\forall x\in[0,1),f(x)\in[0,1)$。

1. 判断 $f(x)$ 的奇偶性。
2. 判断 $f(x)$ 在 $[0,+\infty)$ 的单调性。
3. 若 $a\geqslant 0$，且 $f(a+1)\leqslant\sqrt[3]{9}$，求 a 的取值范围。

解答

1. 令 $y=-1$，得 $f(-x)=f(x)f(-1)=f(x)$，$f(x)$ 为偶函数。
2. 令 $x=y=0$，得 $f(0)=0$。$\forall x\in\mathbb{R}^+$，$f(x)=[f(\sqrt{x})]^2\geq 0$。若存在不为零的正实数 x_0 使得 $f(x_0)=0$，那末[22]$9=f(27)=f(x_0)f\left(\frac{27}{x_0}=0\right)$，矛盾，因此 $\forall x>0$，$f(x)>0$。

   令 $x_1>x_2>0$，$x=x_1$，$y=\frac{x_2}{x_1}$，得

   $$f\left(x_1\cdot\frac{x_2}{x_1}\right)=\underbrace{f(x_2)}_{>0}=\underbrace{f(x_1)}_{>0}\underbrace{f\left(\frac{x_2}{x_1}\right)}_{\in[0,1)}<f(x_1)$$

   因此 $f(x)$ 在 $[0,+\infty)$ 上（严格）单调增，在 $(-\infty,0]$ 上（严格）单调减。
3. $f((a+1)^2)=[f(a+1)]^3\leqslant 9\ \Leftrightarrow\ (a+1)^3\in[0,27]\ \Leftrightarrow\ a\in[1,2]$

TAG 抽象函数 分析 DIF 易

题目 020 设 A 是定义在 $[2,4]$ 上且满足如下条件的函数 $\varphi(x)$ 的集合：
① $\forall x\in[1,2]$，有 $\varphi(2x)\in(1,2)$。
② 存在常数 $L(0<L<1)$，使得对于任意 $x_1,x_2\in[1,2]$，都有 $|\varphi(2x_1)-\varphi(2x_2)|\leqslant L|x_1-x_2|$。

1. 设 $\varphi(2x)=\sqrt[3]{1+x}$，$x\in[2,4]$，证明：$\varphi\in A$。
2. 设 $\varphi\in A$，如果存在 $x_0\in(1,2)$，使得 $x_0=\varphi(2x_0)$，那么这样的 x_0 是唯一的。
3. 设 $\varphi\in A$，任取 $x_1\in(1,2)$，令 $x_{n+1}=\varphi(2x_n),n\in\mathbb{N}^+$，证明：给定正

[22] 应当注意到，根据题设，$f(x)<\infty$，即 $f(x)$ 定义域为 $\mathbb{R}$。

整数 k，那末对于任意的正整数 p，不等式

$$|x_{k+p}-x_p|\leqslant\frac{L^{k+1}}{1-L}|x_2-x_1|$$

成立。

解答

1. [23]显然 $x\in[1,2]$ 时,$\varphi(2x)=\sqrt[3]{1+x}\in[\sqrt[3]{2},\sqrt[3]{3}]\subset(1,2)$;对于任意 $x_1,x_2\in[1,2]$,有
$$\begin{aligned}\left|\sqrt[3]{1+x_1}-\sqrt[3]{1+x_2}\right| &= \frac{|x_1-x_2|}{|(1+x_1)^{2/3}+(1+x_2)^{2/3}+(1+x_1)^{1/3}(1+x_2)^{1/3}|}\\ &\leqslant\frac{|x_1-x_2|}{|3\times 2^{2/3}|}=L|x_1-x_2|\end{aligned}$$
其中 $L=\dfrac{1}{3\times 2^{2/3}}\in(0,1)$ 符合要求，因此 $\varphi\in A$。
2. 假设 $\exists x_1,x_2\in(1,2)$,$x_1\neq x_2$,$x_1=\varphi(2x_1)$,$x_2=\varphi(2x_2)$,那末 $|\varphi(2x_1)-\varphi(2x_2)|=|x_1-x_2|$，这与 $\varphi(x)\in A$ 前提下应满足的 ② 矛盾，因此唯一性得以说明。
3. 注意到这样一个事实：$\forall m\in\mathbb{N}^+$,
$$|x_{m+1}-x_m|=|\varphi(2x_m)-\varphi(2x_{m-1})|\leqslant L|x_m-x_{m-1}|\leqslant\cdots\leqslant L^{m-1}|x_2-x_1|$$
因此，
$$\begin{aligned}|x_{k+p}-x_k| &= |(x_{k+p}-x_{k+p-1})+(x_{k+p-1}-x_{k+p-2})+\cdots+(x_{k+1}-x_k)|\\ &\leqslant|x_{k+p}-x_{k+p-1}|+|x_{k+p-1}-x_{k+p-2}|+\cdots+|x_{k+1}-x_k|\\ &\leqslant(L_{k+p-2}+L_{k+p-3}+\cdots+L_{k-1})|x_2-x_1|\\ &=\frac{L_{k-1}(1-L^p)}{1-L}|x_2-x_1|\\ &\leqslant\frac{L_{k-1}}{1-L}|x_2-x_1|\end{aligned}$$

TAG 抽象函数 分析 DIF 中

[23] 本题出题背景涉及到了柯西列、李普希兹条件和柯西中值定理，但最终成题却非常简单。有兴趣可对这些名词稍作了解。应当注意到 $\dfrac{f(x_1)-f(x_2)}{x_1-x_2}$ 为割线斜率。若可导函数割线斜率有界，那末其导函数必有界。

1.4 求取参变量范围的综合题目

题目 021 已知函数 $f(x)=\dfrac{a}{2}x^2+(a+1)x+2\ln(x-1)$,

1. 若曲线 $y=f(x)$ 在点 $(2,f(2))$ 处的切线与直线 $2x-y+1=0$ 平行，求出这条切线的方程。
2. 讨论函数 $f(x)$ 的单调区间。
3. 若对于任意 $x\in(1,+\infty)$ 都有 $f(x)<-2$，求实数 a 的取值范围。

解答 易知 $f(x)$ 定义域 $x\in(1,+\infty)$,

1. $f'(x)=ax+a+1+\dfrac{2}{x-1}\Rightarrow f'(2)=3a+3=2$，故 $a=-\dfrac{1}{3}$，$f(2)=\dfrac{2}{3}$，故切线方程为
$$y-\frac{2}{3}=2(x-2)\quad\Leftrightarrow\quad 3y-6x+10=0$$
2. 由于 $a=-\dfrac{1}{3}$，故 $f(x)=-\dfrac{1}{6}x^2+\dfrac{2}{3}x+2\ln(x-1)$，$f'(x)=-\dfrac{1}{3}x+\dfrac{2}{3}+\dfrac{2}{x-1}$，$f''(x)=-\dfrac{1}{3}-\dfrac{2}{(x-1)^2}<0$，易知
$$f'(x)\leqslant 0\Leftrightarrow x\in[4,+\infty)\qquad f'(x)\geqslant 0\Leftrightarrow x\in(1,4]$$

故 $f(x)$ 在 $(1,4]$ 上单调增，在 $[4,+\infty)$ 上单调减。

3. （方法一）显然，$a<0$。分离变量得[24]

原式 $\Leftrightarrow\ a(x^2+2x)<-4-2x-4\ln(x-1)$

$$\Leftrightarrow\ a<-\frac{4+2x+4\ln(x-1)}{x^2+2x}=-g(x)\qquad x^2+2x=(x+1)^2-1>3>0$$

若我们能够求得 $\max g(x)$，则解即为 $a<-\max g(x)$。为方便书写，记 $\ln 0=-\infty$，

$$g'(x)=-\frac{\overbrace{2(x+1)}^{>0}(x^2-4+4(x-1)\ln(x-1))}{\underbrace{x^2(x+2)^2(x-1)}_{>0}}=-\frac{2(x+1)(x^2-4+4h(x))}{x^2(x+2)^2(x-1)}$$

注意到 $g'(2)=h(2)=h(1)=0$ [25]$g'(2)=h(2)=h(1)=0$。而 $h'(x)=\ln(x-1)+1$，故 $h(x)$ 在 $\left(1,1+\frac{1}{e}\right)$ 上单调减，在 $\left(1+\frac{1}{e},+\infty\right)$ 上单调增，进而

$$h(x)<0,x^2-4<0\ \Leftrightarrow\ x\in(1,2)\quad h(x)>0,x^2-4>0\ \Leftrightarrow\ x\in(2,+\infty)$$

可推知 $g(x)$ 在 $(1,2)$ 上单调增，在 $(2,+\infty)$ 上单调减，$a<-\max g(x)=-g(2)=$ -1 。

（方法二）显然，$a<0$。$f'(x)=0\ \Leftrightarrow\ x_1=-1$(舍) $x_2=\dfrac{a-1}{a}>1$，故 $f(x)$ 在 $(1,+\infty)$ 上必然先增后减，

$$\max f(x)=f\left(\frac{a-1}{a}\right)=-1-\frac{1}{2a}+\frac{3a}{2}+2\ln\left(-\frac{1}{a}\right)=g(a)<-2\ \Rightarrow\ a<0$$

$$g'(a)=\frac{3}{2}+\frac{1}{2a^2}-\frac{2}{a}>0\ \Rightarrow\ g(a)\text{单调增}$$

注意到 $g(-1)=-2$，故所求 a 的取值范围为 $a<-1$ 。

TAG 参数范围 DIF 易

[24] 一般都是有两种方法求解此类题。一是将 $f(x)$ 的最大值以 a 表达出来为 $g(a)$，再解关于 a 的不等式方程；二是分离变量为 $f_1(a)<f_2(x)$ 后解不等式 $f_1(a)<\min f_2(x)$。

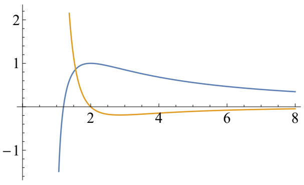

Figure 1.3: $g'(x)$ 的图像

[25] 当 x 接近 0 时，$x\ln x$ 接近 0。这一点可用不等式

$$x-1\geqslant\ln x\geqslant-\frac{2}{\sqrt{x}}$$

证明。因此我们始终「不妨可设」$0\ln 0=0$ 进行补充定义。为使叙述简便和理解方便，方法一过程中补充定义了边界值，实际考试中应当小心仔细说明。

题目 022 设函数 $f(x)=2\ln(x-1)-(x-1)^2$，

1. 求函数的单调递增区间。
2. 若关于函数 x 的方程 $f(x)+x^2-3x-a=0$ 在区间 $[2,4]$ 上有两个相异的实根，求实数 a 的取值范围。

解答

1. $f'(x)=\dfrac{2}{x-1}-2(x-1)$，故结合定义域 $x>1$ 知 $f'(x)\geqslant 0\Rightarrow x\in(1,2]$，$f'(x)\leqslant 0\Rightarrow x\in[2,+\infty)$，$f(x)$ 在 $(1,2]$ 上单调增，在 $[2,\infty)$ 上单调减。
2. 改写原式为 $a=f(x)+x^2-3x=2\ln(x-1)-x-1=g(x)$，由 $g'(x)=\dfrac{2}{x-1}-1$ 知

$$g(x)\text{单调增}\ \Leftrightarrow\ g'(x)\geqslant 0\ \Leftrightarrow x\in(1,3]\quad g(x)\text{单调减}\ \Leftrightarrow\ g'(x)\leqslant 0\ \Leftrightarrow x\in[3,+\infty)$$

因此 $\max g(x)=g(3)=2\ln 2-4$。注意到当 x 趋近于 1 或 $-\infty$ 时，$g(x)$ 均趋近于 $-\infty$，我们可大致知道 $g(x)$ 的图像形状，因此易知 a 的取值范围为 $a<2\ln 2-4$ 。

TAG 参数范围 最值 DIF 易

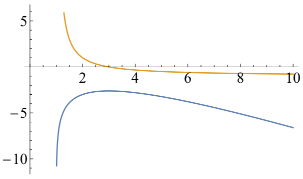

Figure 1.4: $g(x)=2\ln(x-1)-x-1$（蓝）及其导函数（橙）的图像

题目 023 已知函数 $f(x)=x^3-ax^2-a^2x+1$，$g(x)=1-4x-ax^2$，其中实数 $a\neq 0$，

1. 求函数 $f(x)$ 的单调区间。

2. 当函数 $y=f(x)$ 与 $y=g(x)$ 的图像只有一个公共点且 $g(x)$ 存在最小值时，记 $g(x)$ 的最小值为 $h(a)$，求 $f(a)$ 的值域。
3. 若 $f(x)$ 与 $g(x)$ 在区间 $(-a,-a+2)$ 内均为增函数，求 a 的取值范围。

解答

1. $f'(x)=3x^2-2ax-a^2=(3x+a)(x-a)=0\Rightarrow x_1=-\dfrac{a}{3}$，$x_2=a$，显然 $a\neq 0$ 时 $x_1\neq x_2$。

   a) 当 $a>0$ 时，$x_2>x_1$，$f(x)$ 在 $\left(-\infty,-\dfrac{a}{3}\right]$ **和** $[a,+\infty)$ 上单调增，在 $\left[-\dfrac{a}{3},a\right]$ 上单调减。
   b) 当 $a<0$ 时，$x_2<x_1$，$f(x)$ 在 $(-\infty,a]$ **和** $\left[-\dfrac{a}{3},+\infty\right)$ 上单调增，在 $\left[a,-\dfrac{a}{3}\right]$ 上单调减。

2. $g(x)$ 存在最小值意味着 $a>0$，由于 $f(x)$ 与 $g(x)$ 的次数不一，因此从两者图像上寻找「只有一个公共点」的情形是困难的，我们须得设差值函数 $m(x)=f(x)-g(x)=x^3+(4-a^2)x$，公共点条件即转化为 $m(x)$ 有且仅有一个零点[26]。由于 0 为 $m(x)$ 的一个零点，$m(x)$ 为中心对称的奇函数，故若使 $m(x)$ 零点唯一，$m(x)$ 必然单调增，即

$$m'(0)=[3x^2+(4-a^2)]_{x=0}=4-a^2\geqslant 0\ \Rightarrow\ a\in(0,2]$$

[26] 涉及到函数图像相交的问题，均可转化为「交点处为差值函数零点 ⇔ 交点处函数值相同」的问题，这一过程将减少函数个数，从而简化问题。对于特殊图像，也可从图像特征入手。

3. 记 $I=(-a,-a+2)$。$g'(x)=-4-2ax$，使 $g(x)$ 在 I 上单调增有

$$\begin{cases}g'(-a)=-4+2a^2\geqslant 0\\ g'(-a+2)=2a^2-4a-4\geqslant 0\end{cases}\quad\Leftrightarrow\quad a\in(-\infty,-\sqrt{2}]\cup[1+\sqrt{3},+\infty)$$

在这一范围基础上，注意到 I 的长度为 $|I|=2$，而 $f'(x)$ 两根 x_1,x_2 满足 $x_1+x_2=\dfrac{2a}{3}$，$x_1x_2=-\dfrac{a^2}{3}$，故

$$|x_1-x_2|=\sqrt{(x_1+x_2)^2-4x_1x_2}=\frac{4a^2}{3}\geqslant\frac{4\times(\sqrt{2})^2}{3}=\frac{16}{3}>2=|I|$$

因此 $f(x)$ 在 I 上不可能有三段增减区间，故只需使

$$\begin{cases}f'(-a)=4a^2\geqslant 0\\ f'(-a+2)=4(a-3)(a-1)\geqslant 0\end{cases}\quad\Leftrightarrow\quad a\in(-\infty,1]\cup[3,+\infty)$$

综上，可知 a 的取值范围为 $a\in(-\infty,-\sqrt{2}]\cup[3,+\infty)$。

TAG 取值范围 交点 DIF 中

题目 024 设 a 为实数，记函数 $f(x)=a\sqrt{1-x^2}+\sqrt{1+x}+\sqrt{1-x}$ 的最大值为 $g(a)$。

1. 设 $t=\sqrt{1+x}+\sqrt{1-x}$，求 t 的取值范围，并把 $f(x)$ 改写为 t 的函数 $m(t)$。
2. 求出 $g(a)$。
3. 求出满足 $g(a)=g\left(\dfrac{1}{a}\right)$ 的所有实数 a。

解答

1. 显然 $x\in[-1,1]$，$t\geqslant 0$，$t^2=2+2\sqrt{1-x^2}\in[2,4]\Rightarrow t\in[\sqrt{2},2]$。同时我们得

到 $\sqrt{1-x^2} = \frac{1}{2}t^2 - 1$，进而

$$f(x) = a\left(\frac{1}{2}t^2 - 1\right) + t = \frac{a}{2}t^2 + t - a = m(t)$$

2. $m(t)$ 对称轴 $t_0 = -\frac{1}{a}(a \neq 0)$，$t \in [\sqrt{2}, 2]$。
   - a) 若 $a = 0$，则 $m(t)$ 退化为单调增的一次函数[27]，最大值 $g(0) = m(2) = 2$。
   - b) 若 $a > 0$，则 $m(t)$ 为开口向上，对称轴在 y 轴左侧的二次函数，最大值 $g(a) = m(2) = a + 2$。
   - c) 若 $a \in \left[-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right]$，则 $m(t)$ 为开口向下，对称轴 $t_0 \in [\sqrt{2}, 2]$ 的二次函数，最大值 $g(a) = m(t_0) = -a - \frac{1}{2a}$。
   - d) 若 $a \in \left(-\frac{\sqrt{2}}{2}, 0\right)$，则 $m(t)$ 为开口向下，对称轴 $t_0 < \sqrt{2}$ 的二次函数，最大值 $g(a) = m(\sqrt{2}) = \sqrt{2}$。
   - e) 若 $a \in \left(-\infty, -\frac{1}{2}\right)$，则 $m(t)$ 为开口向下，对称轴 $t_0 > 2$ 的二次函数，最大值 $g(a) = m(2) = 2 + a$。

因此：$g(a) = \begin{cases} a + 2 & a \in \left(-\frac{1}{2}, +\infty\right) \\ -a - \frac{1}{2a} & a \in \left[-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right] \\ \sqrt{2} & a \in \left(-\infty, -\frac{\sqrt{2}}{2}\right) \end{cases}$。

3. 由 $(a+2)|_{a=-1/2} = \left(-a - \frac{1}{2a}\right)|_{a=-1/2} = \frac{3}{2}$，$\sqrt{2} = \left(-a - \frac{1}{2a}\right)|_{a=-1/\sqrt{2}} = \sqrt{2}$ 知 $g(a)$ 是连续的。分段求导可知，$g(a)$ 当 $a \geqslant -\frac{\sqrt{2}}{2}$ 时严格单调增，那末

$$g(a) = g\left(\frac{1}{a}\right) \Leftrightarrow a = \frac{1}{a} \text{ 或 } a, \frac{1}{a} \leqslant -\frac{\sqrt{2}}{2} \Leftrightarrow a \in \left[-\sqrt{2}, -\frac{\sqrt{2}}{2}\right] \cup \{1\}$$

TAG 取值范围 DIF 易

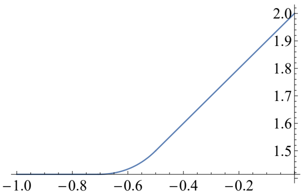

Figure 1.5: $g(x)$ 在 $[-1, 0]$ 上的图像。可以看到是连续的。

题目 025 已知函数 $f(x) = \frac{1+x}{1-x}e^{-ax}$，

1. 设 $a > 0$，讨论 $y = f(x)$ 的单调性。
2. 若对任意 $x \in (0, 1)$ 恒有 $f(x) > 1$，求 a 的取值范围。

解答 $f'(x) = \dfrac{a\left(x^2 + \frac{2-a}{a}\right)}{(x-1)^2}e^{-ax}$

1.
   - a) 若 $a \in (0, 2]$，则 $\frac{2-a}{a} \geqslant 0$，$f'(x) \geqslant 0$ 恒成立，$f(x)$ 在 $(-\infty, 1)$ **和** $(1, +\infty)$ 上单调增。
   - b) 若 $a > 2$，则 $f'(x) = 0 \Leftrightarrow x_1 = -\sqrt{\frac{a-2}{a}}$，$x_2 = \sqrt{\frac{a-2}{a}} < 1$，因此 $f(x)$ 在 $\left(-\infty, -\sqrt{\frac{a-2}{a}}\right]$、$\left[\sqrt{\frac{a-2}{a}}, 1\right)$ **和** $(1, +\infty)$ 上单调增，在 $\left[-\sqrt{\frac{a-2}{a}}, \sqrt{\frac{a-2}{a}}\right]$ 单调减。

2. （方法一）$(0, 1)$ 上，$f'(x) = \dfrac{2 + a\overbrace{(x^2-1)}^{\in(-1,0)}}{(x-1)^2}e^{-ax}$

[27] 一次函数可以视为退化的二次函数，直线和双直线可以视为退化的二次曲线。基于这一点我们应当有这样的先验直觉：$g(a)$ 必然连续。事实上也的确连续。

a) 若 $a \leqslant 2$，则 $2 + a(x^2 - 1) \geqslant 0 \Rightarrow f'(x)$ 单调增 $\Rightarrow f(x) > f(0) = 1$，$x \in (0,1)$，这是符合题意的。

b) 若 $a > 2$，则 $f'(0) < 0$，$f(x)$ 先严格单调减后单调增，而 $f(0) = 1$，因此必存在 $1 > x_0 > 0$ s.t. $f(x_0) < 1$，这是不符合题意的，因此 $a \leqslant 2$ 。

（方法二 分离变量法）易知 $x \in (0,1)$ 则 $\dfrac{1+x}{1-x} > 0$。$f(x)$ 取对数知 $\forall x \in (0,1)$，$\ln f(x) = \ln(1+x) - \ln(1-x) - ax > \ln 1 = 0$，

$$a < \min_{x\in(0,1)} \frac{\ln(1+x)}{x} - \frac{\ln(1-x)}{x} = \min_{x\in(0,1)} g(x)$$

$g'(x) = \dfrac{1}{x^2}\left[\dfrac{2x}{1-x^2} + \ln\left(\dfrac{1-x}{1+x}\right)\right] = \dfrac{1}{x^2}h(x)$，注意到 $h'(x) = \dfrac{4x^2}{(x^2-1)^2} > 0$，$h(0) = 0$，故在 $(0,1)$ 上 $g'(x) > 0$，$g(x)$ 单调增。现在我们须找出 x 趋近于 0 时，$g(x)$ 趋近的值。事实上，可利用以下不等式[28]：

$$\begin{aligned}
& x - \frac{x^2}{2} \leqslant \ln(1+x) \leqslant x \\
\Rightarrow\ & \left(x - \frac{x^2}{2}\right) - (-x) \leqslant \ln(1+x) - \ln(1-x) \leqslant x - \left(-x - \frac{x^2}{2}\right) \\
\Leftrightarrow\ & 2x - \frac{x^2}{2} \leqslant \ln\left(\frac{1+x}{1-x}\right) \leqslant 2x + \frac{x^2}{2} \qquad \text{这是一个非常常用的不等式} \\
\Leftrightarrow\ & 2 - \frac{x}{2} \leqslant g(x) \leqslant 2 + \frac{x}{2}
\end{aligned}$$

因此[29]当 x 趋于 0 时，$g(x)$ 趋于 2，而这一最小值是取不到的。因此 $a \leqslant 2$ 。

CEE 高考题目 TAG 取值范围 极限 DIF 易

1.5 引出不等式证明的综合题目

关于此类题目一些常用不等式的统一证明：

1. [*IEQ 1*] $\mathbf{e^x \geqslant 1 + x}$

   该式截取自 e^x 的麦克劳林展开式

   $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

   前两项，截取更多项时证明相仿，只需多求几次导数，不赘述。

   令 $g(x) = e^x - (1+x)$，则 $g'(x) = e^x - 1$，$g''(x) = e^x > 0$，因此 $g'(x)$ 单调增，又 $g'(0) = 0$，因此 $g(x)$ 在 $(-\infty, 0)$ 上单调递减，在 $(0, +\infty)$ 上单调递增，$\forall x \in \mathbb{R}$，

   $$g(x) \geqslant g(0) = 0 \ \Leftrightarrow\ e^x \geqslant 1 + x$$

   当仅当 $x = 0$ 时取等。

2. [*IEQ 2*] $\ln(\mathbf{1} + \mathbf{x}) \leqslant \mathbf{x}\ (x > -1), \ln(\mathbf{1} + \mathbf{x}) \geqslant \mathbf{x} - \dfrac{\mathbf{x^2}}{\mathbf{2}}\ (x \geqslant 0)$

   该式截取自 $\ln(1+x)$ 的麦克劳林展开式

   $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

   前两项，截取更多项时证明相仿，只需多求几次导数，不赘述，这里我们证第二式。

   令 $g(x) = \ln(1+x) - x + \dfrac{x^2}{2!}$，则 $g'(x) = \dfrac{1}{1+x} - 1 + x$，$g''(x) = -\dfrac{1}{(1+x)^2} + 1 \geqslant 0$，$g'(0) = 0$，因此 $g(x)$ 在 $[0, +\infty)$ 上单调增，$\forall x \geqslant 0$，

   $$g(x) \geqslant 0 \ \Rightarrow\ \ln(1+x) \geqslant x - \frac{x^2}{2}$$

   当仅当 $x = 0$ 时取等。

---

[28] 这里的不等式均源于多项式级数展开的截取，证明大同小异，非常简单，故不再证明。下诸题同。

[29] 「趋于」这一词引入于数学教材定积分一章，是否在考试中可以使用视要求而定。不能使用的情况下可以如下叙述：

a) $g(x)$ 是连续的，且单调增至无穷。对于任意大于 2 的值，譬如 $2+\epsilon$，$\epsilon > 0$，均可取适当的 $x > 0$，使得 $g(x) \leqslant 2 + \dfrac{x^2}{2} < 2 + \epsilon$，这样由 $g(x)$ 单调增知 $g(x)$ 必可取到 $2+\epsilon$。

b) 假设 $g(x)$ 可取到任意小于 2 的值，譬如 $g(x_0) = 2 - \epsilon, \epsilon > 0$，均可取适当的 $x_0 > x_1 > 0$，使得 $g(x_1) \geqslant 2 - \dfrac{x^2}{2} > 2 - \epsilon$，这与 $g(x)$ 单调增矛盾。

c) 因此 $g(x)$ 必然严格大于 2。

3. [*IEQ 3*] $\ln\mathbf{x} \leqslant \mathbf{1}-\dfrac{\mathbf{1}}{\mathbf{x}}$

令 $g(x)=\ln x-1+\dfrac{1}{x}$，可证 $g(x)\geqslant g(1)=0$，因此 $\forall x\in\mathbb{R}^{+}$，$\ln x\geqslant 1-\dfrac{1}{x}$。

题目 026 已知函数 $f(x)=\dfrac{a(1-x)}{x}\ln(1-x), a\in\mathbb{R}$，$e$ 为自然常数。

1. 求 $f(x)$ 在区间 $[1-e^2,1-e]$ 上的最值。
2. 比较 $\left(1+\dfrac{1}{2!}\right)\left(1+\dfrac{1}{3!}\right)\cdots\left(1+\dfrac{1}{n!}\right)$ 与 e 的大小。

解答

1. 导数 $f'(x)=-\dfrac{a(x+\ln(1-x))}{x^2}$。由 [IEQ 2]，$\ln(1-x)\leqslant -x$，因此
   a) 若 $a=0$，则 $f(x)\equiv 0$，所求最值 $\max f(x)=0$。
   b) 若 $a>0$，则 $x+\ln(1-x)\leqslant 0$，$f'(x)\geqslant 0$，$f(x)$ 单调增，所求最值
   $$\max_{[1-e^2,1-e]} f(x)=f(1-e)=\frac{ae}{1-e}\qquad \min_{[1-e^2,1-e]} f(x)=f(1-e^2)=\frac{2ae^2}{1-e^2}$$
   c) 若 $a<0$，则 $f'(x)\leqslant 0$，$f(x)$ 单调减，所求最值
   $$\min_{[1-e^2,1-e]} f(x)=f(1-e)=\frac{ae}{1-e}\qquad \max_{[1-e^2,1-e]} f(x)=f(1-e^2)=\frac{2ae^2}{1-e^2}$$
2. 由 [IEQ 2]，$\ln(1+x)\leqslant x$，因此
$$\begin{aligned}
&\ln\left[\left(1+\frac{1}{2!}\right)\left(1+\frac{1}{3!}\right)\cdots\left(1+\frac{1}{n!}\right)\right]\\
=&\ln\left(1+\frac{1}{2!}\right)+\ln\left(1+\frac{1}{3}\right)+\cdots+\ln\left(1+\frac{1}{n!}\right)\\
\leqslant&\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\\
\leqslant&\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n-1)n}\\
=&1-\frac{1}{n}<1=\ln e\\
\Rightarrow&\left(1+\frac{1}{2!}\right)\left(1+\frac{1}{3!}\right)\cdots\left(1+\frac{1}{n!}\right)<e
\end{aligned}$$

TAG 不等式 DIF 易

题目 027 已知函数 $f(x)=\ln ax-\dfrac{x-1}{x}$。

1. 求此函数的单调区间与最值。
2. 求证：对于任意正整数 n，均有：
$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}>\ln\frac{e^n}{n!}$$
3. 当 $a=1$ 时，过点 $(1,-1)$ 是否存在 $y=f(x)$ 图像的切线？若存在，有多少条？若不存在，阐明理由。

解答

1. $f'(x)=\dfrac{1}{x}\left(1-\dfrac{1}{x}\right)=0\Rightarrow x=1$。若 $a>0$，则 $x>0$，$f(x)$ 在 $(0,1)$ 上单调递减，在 $[1,+\infty)$ 上单调递增。最小值 $\min f(x)=f(1)=\ln a$，无最大值；若 $a<0$，则 $x<0$，$f(x)$ 在 $\mathbb{R}^{-}$ 上单调递减，无最值。
2. 由上小题结论（也即 [IEQ 3]），若取 $a>0$，则 $\ln ax-1+\dfrac{1}{x}\geqslant\ln a\Rightarrow\dfrac{1}{x}\geqslant 1-\ln x$，

当仅当 $x=1$ 时取等，因此

$$1+\frac{1}{2}+\cdots+\frac{1}{n} > n-\ln\frac{1}{n!}=\ln\frac{e^n}{n!}$$

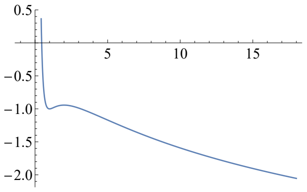

Figure 1.6: $g(x)$ 的图像

3. $a=1$ 条件下，$x>0$，$f(x)$ 在 $(x_0,f(x_0))$ 处切线方程可写为：$y-f(x_0)=\frac{x_0-1}{x_0^2}(x-x_0)$。假使该切线过点 $(1,-1)$，则

$$-1-f(x_0)=\frac{x_0-1}{x_0^2}(1-x_0) \quad\Rightarrow\quad 1-\frac{3}{x_0}+\frac{1}{x_0^2}-\ln x_0=g(x_0)=0$$

由于 $g'(x)=-\frac{2}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-\frac{1}{2}\right)=0\Rightarrow x_1=1,x_2=2$，故 $g(x)$ 在 $(0,1)$ 上单调减，在 $(1,2)$ 上单调增，在 $(2,+\infty)$ 上单调减，$g(1)=-1<0$，$g(0)=+\infty$，$g(2)=-\ln 2-\frac{1}{4}<0$，$g(+\infty)=-\infty$，因此 $g(x)$ 有且仅有一个零点，这意味着所求切线存在且仅存在 1 条。

TAG 不等式 切线方程 DIF 易

题目 028 设函数 $f(x)=1-e^{-x}$，函数 $g(x)=\frac{x}{ax+1}$，其中 $a\in\mathbb{R}$，e 为自然常数。

1. 当 $a=0$ 时，求函数 $h(x)=f'(x)g(x)$ 的极值。
2. 若 $f(x)\leqslant g(x)$ 在 $[0,+\infty)$ 上恒成立，求实数 a 的取值范围。
3. 设 $n\in\mathbb{N}^+$，求证：（$\exp x=e^x$）

$$\exp\left(2n-\sum_{k=1}^{n}\frac{4}{k+1}\right)\leqslant n!\leqslant\exp\left(\frac{n(n-1)}{2}\right)$$

解答

1. $f'(x)=e^{-x}$。$a=0$ 时，$h(x)=\frac{x}{x+1}e^{-x}$，

$$h'(x)=\frac{1-x-x^2}{(1+x)^2}e^{-x}=0\Rightarrow x_1=\frac{-1-\sqrt{5}}{2},\ x_2=\frac{-1+\sqrt{5}}{2},\ f''(x_1)>0,\ f''(x_2)<0$$

因此极小值 $f(x_1)=\frac{3+\sqrt{5}}{2}e^{\frac{1+\sqrt{5}}{2}}$，极大值

$$f(x_2)=\frac{3-\sqrt{5}}{2}e^{\frac{1-\sqrt{5}}{2}}$$

2. 显然首先 $a\geqslant 0$。

什么**什么**什么什么什么什么什么什么 TAG 参数范围 DIF 易 Many modern printed textbooks have adopted a layout with prominent margins where small figures, tables, remarks and just about everything else can be displayed. Arguably, this layout helps to organise the discussion by separating the main text from the ancillary material, which at the same time is very close to the point in the text where it is referenced.

This document does not aim to be an apology of wide margins, for there are many better suited authors for this task; the purpose of all these words is just to fill the space so that the reader can see how a book written with the kaobook class looks like. Meanwhile, I shall also try to illustrate the features of the class.

The main ideas behind kaobook come from this blog post, and actually the name of the class is dedicated to the author of the post, Ken Arroyo Ohori, which has kindly allowed me to create a class based on his thesis. Therefore, if you want to know more reasons to prefer a 1.5-column layout for your books, be sure to read his blog post.

Another source of inspiration, as you may have noticed, is the Tufte-Latex Class. The fact that the design is similar is due to the fact that it is very difficult to improve something which is already so good. However, I like to think that this class is more flexible than Tufte-Latex. For instance, I have tried to use only standard packages and to implement as little as possible from scratch;[30] therefore, it should be pretty easy to customise anything, provided that you read the documentation of the package that provides that feature.

In this book I shall illustrate the main features of the class and provide information about how to use and change things. Let us get started.

1.6 What This Class Does

The `kaobook` class focuses more about the document structure than about the style. Indeed, it is a well-known LaTeX principle that structure and style should be separated as much as possible (see also Section Section 1.7 on the following page). This means that this class will only provide commands, environments and in general, the opportunity to do things, which the user may or may not use. Actually, some stylistic matters are embedded in the class, but the user is able to customise them with ease.

The main features are the following:

Page Layout The text width is reduced to improve readability and make space for the margins, where any sort of elements can be displayed.

Chapter Headings As opposed to Tufte-Latex, we provide a variety of chapter headings among which to choose; examples will be seen in later chapters.

Page Headers They span the whole page, margins included, and, in twoside mode, display alternatively the chapter and the section name.[31]

Matters The commands `\frontmatter`, `\mainmatter` and `\backmatter` have been redefined in order to have automatically wide margins in the main matter, and narrow margins in the front and back matters. However, the page style can be changed at any moment, even in the middle of the document.

Margin text We provide commands `\sidenote` and `\marginnote` to put text in the margins.[32]

Margin figs/tabs A couple of useful environments is `marginfigure` and `margintable`, which, not surprisingly, allow you to put figures and tables in the margins (*cfr.* Figure 1.7).

Margin toc Finally, since we have wide margins, why don't add a little table of contents in them? See `\margintoc` for that.

Hyperref `hyperref` is loaded and by default we try to add bookmarks in a sensible way; in particular, the bookmarks levels are automatically reset at `\appendix` and `\backmatter`. Moreover, we also provide a small package to ease the hyperreferencing of other parts of the text.

Bibliography We want the reader to be able to know what has been cited without having to go to the end of the document every time, so citations go in the margins as well as at the end, as in Tufte-Latex. Unlike that class, however, you are free to customise the citations as you wish.

[30] This also means that understanding and contributing to the class development is made easier. Indeed, many things still need to be improved, so if you are interested, check out the repository on github!

[32] Sidenote
bered while r

The order of the title pages, table of contents and preface can be easily changed, as in any LaTeX document. In addition, the class is based on KOMA-Script's `scrbook`, therefore it inherits all the goodies of that.

1.7 What This Class Does Not Do

As anticipated, further customisation of the book is left to the user. Indeed, every book may have sidenotes, margin figures and so on, but each book will have its own fonts, toc style, special environments and so on. For this reason, in addition to the class, we provide only sensible defaults, but if these features are not needed, they can be left out. These special packages are located in the `style` directory, which is organised as follows:

kao.sty This package contains the most important definitions of macros and specifications of page layout. It is the heart of the `kaobook`.

kaobiblio.sty Contains commands to add citations and customise the bibliography.

packages.sty Loads additional packages to decorate the writing with special contents (for instance, the `listing` package is loaded here as it is not required in every book). There are also defined some useful commands to print the same words always in the same way, *e.g.* latin words in italics or `packages` in verbatim.

kaorefs.sty Some useful commands to manage labeling and referencing, again to ensure that the same elements are referenced always in a consistent way.

environments.sty Provides special environments, like boxes. Both simple and complex environments are available; by complex we mean that they are endowed with a counter, floating and can be put in a special table of contents.[33]

theorems.sty The style of mathematical environments. Actually, there are two such packages: one is for plain theorems, *i.e.* the theorems are printed in plain text; the other uses `mdframed` to draw a box around theorems. You can plug the most appropriate style into its document.

In the rest of the book, I shall assume that the reader is not a novice in the use of LaTeX, and refer to the documentation of the packages used in this class for things that are already explained there. Moreover, I assume that the reader is willing to make minor edits to the provided packages for styles, environments and commands, if he or she does not like the default settings.

1.8 How to Use This Class

Either if you are using the template from latextemplates, or if you cloned the GitHub repository, there are infinite ways to use the `kaobook` class in practice, but we will discuss only two of them. The first is to find the `main.tex` file which I used to write this book, and edit it; this will probably involve a lot of text-deleting, copying-and-pasting, and rewriting. The second way is to start almost from scratch and use the `skeleton.tex` file, which is a cleaned-up version of the `main.tex`; even if you choose the second way, you may find it useful to draw inspiration from the `main.tex` file.

To compile the document, assuming that its name is `main.tex`, you will have to run the

[33] See Chapter 7 on page 42 for some examples.

following sequence of commands:

```
pdflatex main # Compile template
makeindex main.nlo -s nomencl.ist -o main.nls # Compile nomenclature
makeindex main # Compile index
biber main # Compile bibliography
makeglossaries main # Compile glossary
pdflatex main # Compile template again
pdflatex main # Compile template again
```

You may need to compile the template some more times in order for some errors to disappear. For any support requests, please ask a question on `tex.stackexchange.org` with the tag “kaobook”, open an issue on GitHub, or contact the author via e-mail.

Class Options

2

In this chapter I will describe the most common options used, both the ones inherited from `scrbook` and the `kao`-specific ones. Options passed to the class modifies its default behaviour; beware though that some options may lead to unexpected results...

2.1 KOMA Options

The `kaobook` class is based on `scrbook`, therefore it understands all of the options you would normally pass to that class. If you have a lot of patience, you can read the KOMA-Script guide.[1] Actually, the reading of such guide is suggested as it is very instructive.

Every KOMA-Script option you pass to the class when you load it is automatically activated. In addition, in `kaobook` some options have modified default values. For instance, the font size is 9.5pt and the paragraphs are separated by space,[2] not marked by indentation.

2.2 `kao` Options

In the future I plan to add more options to set the paragraph formatting (justified or ragged) and the position of the margins (inner or outer in twoside mode, left or right in oneside mode).[3]

I take this opportunity to renew the call for help: everyone is encouraged to add features or reimplement existing ones, and to send me the results. You can find the GitHub repository at `https://github.com/fmarotta/kaobook`.

To Do

Implement the `justified` and `margin` options. To be consistent with the KOMA-Script style, they should accept a simple switch as a parameter, where the simple switch should be `true` or `false`, or one of the other standard values for simple switches supported by KOMA-Script. See the KOMA-Script documentation for further information.

The above box is an example of a `kaobox`, which will be discussed more thoroughly in Chapter 7 (Mathematics and Boxes) on page 42. Throughout the book I shall use these boxes to remarks what still needs to be done.

2.3 Other Things Worth Knowing

A bunch of packages are already loaded in the class because they are needed for the implementation. These include:

- etoolbox

[1] The guide can be downloaded from `https://ctan.org/pkg/koma-script?lang=en`.

[2] To be precise, they are separated by half a line worth of space: the `parskip` value is "half".

[3] As of now, paragraphs are justified, formatted with `\singlespacing` (from the `setspace` package) and `\frenchspacing`.

- calc
- xifthen
- xkeyval
- xparse
- xstring

Many more packages are loaded, but they will be discussed in due time. Here, we will mention only one more set of packages, needed to change the paragraph formatting (recall that in the future there will be options to change this). In particular, the packages we load are:

- ragged2e
- setspace
- hyphenat
- microtype
- needspace
- xspace
- xcolor (with options `usenames,dvipsnames`)

Some of the above packages do not concern paragraph formatting, but we nevertheless grouped them with the others. By default, the main text is justified and formatted with singlespacing and frenchspacing; the margin text is the same, except that the font is a bit smaller.

As a last warning, please be aware that the `cleveref` package is not compatible with `kaobook`. You should use the commands discussed in Section 5.3 instead.

2.4 Document Structure

We provide optional arguments to the `\title` and `\author` commands so that you can insert short, plain text versions of this fields, which can be used, typically in the half-title or somewhere else in the front matter, through the commands `\@plaintitle` and `\@plainauthor`, respectively. The PDF properties `pdftitle` and `pdfauthor` are automatically set by hyperref to the plain values if present, otherwise to the normal values.[4]

There are defined two page layouts, `margin` and `wide`, and two page styles, `plain` and `fancy`. The layout basically concern the width of the margins, while the style refers to headers and footer; these issues will be discussed in Chapter 6 (Page Design) on page 37.[5]

The commands `\frontmatter`, `\mainmatter`, and `\backmatter` have been redefined in order to automatically change page layout and style for these sections of the book. The front matter uses the `margin` layout and the `plain` page style. In the mainmatter the margins are wide and the headings are fancy. In the appendix the style and the layout do not change; however we use `\bookmarksetup{startatroot}` so that the bookmarks of the chapters are on the root level (without this, they would be under the preceding part). In the backmatter the margins shrink again and we also reset the bookmarks root.

[4] We think that this is an important point so we remark it here. If you compile the document with pdflatex, the PDF metadata will be altered so that they match the plain title and author you have specified; if you did not specify them, the metadata will be set to the normal title and author.

[5] For now, suffice it to say that pages with the `margin` layout have wide margins, while with the `wide` layout the margins are absent. In `plain` pages the headers and footer are suppressed, while in `fancy` pages there is a header.

Margin Stuff

3

Sidenotes are a distinctive feature of all 1.5-column-layout books. Indeed, having wide margins means that some material can be displayed there. We use margins for all kind of stuff: sidenotes, marginnotes, small tables of contents, citations, and, why not?, special boxes and environments.

3.1 Sidenotes

Sidenotes are like footnotes, except that they go in the margin, where they are more readable. To insert a sidenote, just use the command `\sidenote{Text of the note}`. You can specify a mark[O] with

`\sidenote[mark]{Text}`, but you can also specify an offset, which moves the sidenote upwards or downwards, so that the full syntax is:

```
\sidenote[mark][offset]{Text}
```

[O] This sidenote has a special mark, a big O!

If you use an offset, you always have to add the brackets for the mark, but they can be empty.[1]

In `kaobook` we copied a feature from the `snotez` package: the possibility to specify a multiple of `\baselineskip` as an offset. For example, if you want to enter a sidenote with the normal mark and move it upwards one line, type:

```
\sidenote[][*-1]{Text of the sidenote.}
```

As we said, sidenotes are handled through the `sidenotes` package, which in turn relies on the `marginnote` package.

3.2 Marginnotes

This command is very similar to the previous one. You can create a marginnote with `\marginnote[offset]{Text}`, where the offset argument can be left out, or it can be a multiple of `\baselineskip`, *e.g.*

```
\marginnote[-12pt]{Text} or \marginnote[*-3]{Text}
```

While the command for margin notes comes from the marginnote package, it has been redefined in order to change the position of the optional the text of the note, whereas in the

To Do

A small thing that needs to be done is to renew the `\sidenote` command so that it takes only one optional argument, the offset. The special mark argument can go somewhere else. In other words, we want the syntax of `\sidenote` to resemble that of `\marginnote`.

We load the packages `marginnote`, `marginfix` and `placeins`. Since `sidenotes` uses `marginnote`, what we said for marginnotes is also valid for sidenotes. Side- and marginnotes are shifted slightly upwards (`\renewcommand{\marginnotevadjust}{3pt}`) in order to align them to the bottom of the line of text where the note is issued. Impor-

---

[1] If you want to know more about the usage of the `\sidenote` command, read the documentation of the `sidenotes` package.

tantly, both sidenotes and marginnotes are defined as floating if the optional argument (*i. e.* the vertical offset) is left blank, but if the offset is specified they are not floating. Recall that floats cannot be nested, so in some rare cases you may encounter errors about lost floats; in those cases, remember that sidenotes and marginnotes are floats. To solve the problem, it may be possible to transform them into non-floating elements by specifying an offset of 0pt.

3.3 Footnotes

Even though they are not displayed in the margin, we will discuss about footnotes here, since sidenotes are mainly intended to be a replacement of them. Footnotes force the reader to constantly move from one area of the page to the other. Arguably, marginnotes solve this issue, so you should not use footnotes. Nevertheless, for completeness, we have left the standard command `\footnote`, just in case you want to put a footnote once in a while.[*]

3.4 Margintoc

Since we are talking about margins, we introduce here the `\margintoc` command, which allows one to put small table of contents in the margin. Like other commands we have discussed, `\margintoc` accepts a parameter for the vertical offset, like so: `\margintoc[offset]`.

The command can be used in any point of the document, but we think it makes sense to use it just at the beginning of chapters or parts. In this document I make use of a KOMA-Script feature and put it in the chapter preamble, with the following code:

```
\setchapterpreamble[u]{\margintoc}
\chapter{Chapter title}
```

The font used in the margintoc is the same as the one for the chapter entries in the main table of contents at the beginning of the document.

3.5 Marginlisting

On some occasions it may happen that you have a very short piece of code that doesn’t look good in the body of the text because it breaks the flow of narration: for that occasions, you can use a `marginlisting`. The support for this feature is still limited, especially for the captions, but you can try the following code:

```
print(”Hello World!”)
```

```
\begin{marginlisting}[-0.5cm]
\caption{My caption}
\vspace{0.2cm}
\begin{lstlisting}[language=Python,style=kaolstplain]
... code ...
\end{lstlisting}
\end{marginlisting}
```

---

[*] And this is how they look like. Notice that in the PDF file there is a back reference to the text; pretty cool, uh?

Unfortunately, the space between the caption and the listing must be adjusted manually; if you find a better way, please let me know.

Not only textual stuff can be displayed in the margin, but also figures. Those will be the focus of the next chapter.

4 Figures and Tables

4.1 Normal Figures and Tables

Figures and tables can be inserted just like in any standard LaTeX document. The `graphicx` package is already loaded and configured in such a way that the figure width is equal to the textwidth and the height is adjusted in order to maintain the original aspect ratio. As you may have imagined, the captions will be positioned...well, in the margins. This is achieved with the help of the `floatrow` package.

Here is a picture of Mona Lisa (Figure 4.1), as an example. The captions are formatted as the margin- and the side-notes; If you want to change something about captions you can use the command `\captsetup` from the `caption` package. Remember that if you want to reference a figure, the label must come *after* the caption!

Figure 4.1: It's Mona Lisa again. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

The credits for the image above the chapter title go to: Bushra Feroz, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=68724647

While the format of the caption is managed by `caption`, its position is handled by the `floatrow` package. Achieving this result has been quite hard, but now I am pretty satisfied. In two-side mode, the captions are printed in the correct margin.

Tables can be inserted just as easily as figures, as exemplified by the following code:

Listing 4.1: Caption of a listing.

```
\begin{table}
\begin{tabular}{ c c c c }
    \toprule
    col1 & col2 & col3 & col 4 \\
    \midrule
    \multirow{3}{4em}{Multiple row} & cell2 & cell3 & cell4\\ &
    cell5 & cell6 & cell7 \\ &
    cell8 & cell9 & cell10 \\
    \multirow{3}{4em}{Multiple row} & cell2 & cell3 & cell4 \\ &
    cell5 & cell6 & cell7 \\ &
    cell8 & cell9 & cell10 \\
    \bottomrule
\end{tabular}
\end{table}
```

which results in the useless Table Table 4.1.

Table 4.1: A useless table.

col1	col2	col3	col 4
Multiple row	cell2	cell3	cell4
	cell5	cell6	cell7
	cell8	cell9	cell10
Multiple row	cell2	cell3	cell4
	cell5	cell6	cell7
	cell8	cell9	cell10

I don't have much else to say, so I will just insert some blind text. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

4.2 Margin Figures and Tables

Marginfigures can be inserted with the environment `marginfigure`. In this case, the whole picture is confined to the margin and the caption is below it. Figure 1.7 is obtained with something like this:

Listing 4.2: Another caption.

```
\begin{marginfigure}
    \includegraphics{monalisa}
    \caption[The Mona Lisa]{The Mona Lisa.}
    \labfig{marginmonalisa}
\end{marginfigure}
```

There is also the `margintable` environment, of which Table 4.2 is an example. Notice how you can place the caption above the table by just placing the `\caption` command before beginning the `tabular` environment. Usually, figure captions are below, while table captions are above. This rule is also respected for normal figures and tables: the captions are always on the side, but for figure they are aligned to the bottom, while for tables to the top.

Table 4.2: Another useless table.

col1	col2	col3
Multiple row	cell2	cell3
	cell5	cell6
	cell8	cell9

Marginfigures and tables can be positioned with an optional offset command, like so:

```
\begin{marginfigure}[offset]
    \includegraphics{seaside}
\end{marginfigure}
```

Offset ca be either a measure or a multiple of `\baselineskip`, much like with `\sidenote`, `\marginnote` and `\margintoc`. If you are wondering how I inserted this orange bubble, have a look at the `todo` package.

Improve this part.

4.3 Wide Figures and Tables

With the environments `figure*` and `table*` you can insert figures which span the whole page width. For example, here are a wide figure and a wide table.

Figure 4.2: A wide seaside, and a wide caption. Credits: By Bushra Feroz, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=68724647

Table 4.3: A wide table with invented data about three people living in the UK. Note that wide figures and tables are centered and their caption also extends into the margin.

Name	Surname	Job	Salary	Age	Height	Country
Alice	Red	Writer	4.000 £	34	167 cm	England
Bob	White	Bartender	2.000 £	24	180 cm	Scotland
Drake	Green	Scientist	4.000 £	26	175 cm	Wales

It is the user's responsibility to adjust the width of the table, if necessary, until it is aesthetically pleasing. The previous table was obtained with the following code:

Listing 4.3: How to typeset a wid
ble

```
\begin{table*}[h!]
    \caption{A wide table with invented data about three people living in the
      UK. Note that wide figures and tables are centered and their caption
```

```
     also extends into the margin.}
    \begin{tabular}{p{2.0cm} p{2.0cm} p{2.0cm} p{2.0cm} p{2.0cm} p{2.0cm} p
     {1.5cm}}
        \toprule
        Name    & Surname   & Job       & Salary          & Age   & Height
       & Country \\
        \midrule
        Alice   & Red       & Writer    & 4.000 \pounds    & 34    & 167 cm
        & England \\
        Bob     & White     & Bartender & 2.000 \pounds    & 24    & 180 cm
        & Scotland \\
        Drake   & Green     & Scientist & 4.000 \pounds    & 26    & 175 cm
        & Wales \\
        \bottomrule
    \end{tabular}
\end{table*}
```

You may have noticed the full width image at the very beginning of this chapter: that, however, is set up in an entirely different way, which you'll read about in Chapter Chapter 6 on page 37.

`kaobook` also supports paginated tables (have a look at the `longtable` package). The `longtable`[1] environment behaves a bit differently from `table`, in that `longtable` encompasses both `table` and `tabular`, so that you can write, *e.g.*,

Listing 4.4: Example of a longtable

```
\begin{longtable}{|l c c|}
    \hline
    One & Two & Three \\
    Left & Center & Center \\
    \hline
    \caption{Caption of the longtable.}
\end{longtable}
```

to obtain the following table:

One	Two	Three
Left	Center	Center

Table 4.4: Caption of the longtable.

The caption of a `longtable` is always positioned below the table, and it has the same width as the text (it doesn't extend into the margin). However, sometimes you may need a `longtable` that is so wide that it trespass into the margins; in those cases, you may want to also increase the width of the caption. To do so, you'll have to write two additional commands, one before and one after the `longtable`:

Listing 4.5: Increasing the width of the caption of a `long`

```
\floatsetup[longtable]{margins=centering,LTcapwidth=table} % Add this line
     before the longtable to increase the caption width
\begin{longtable}{lp{8cm}p{5cm}p{2cm}}
...
\end{longtable}
\floatsetup[longtable]{margins=raggedright,LTcapwidth=\textwidth} % Add this
     line after the longtable to revert the previous change
```

Having seen figures and tables, it is now time to tackle hyperreferences.

[1] Interestingly, `longtables` may require up to four rounds of compilation before they are typeset correctly.

References

5

5.1 Citations

To cite someone [**Visscher2008**, **James2013**] is very simple: just use the `\sidecite` command. It does not have an offset argument yet, but it probably will in the future. This command supports multiple entries, as you can see, and by default it prints the reference on the margin as well as adding it to the bibliography at the end of the document. Note that the citations have nothing to do with the text,[**James2013**] but they are completely random as they only serve the purpose to illustrate the feature.

Visscher2008, **James2013**

James2013

For this setup I wrote a separate package, `kaobiblio`, which you can find in the `styles` directory and include in your main tex file. This package accepts all the options that you can pass to `biblatex`, and actually it passes them to `biblatex` under the hood. Moreover, it also defines some commands, like `\sidecite`, and environments that can be used within a `kao` book.[1]

If you want to use `bibtex` instead of `biblatex`, pass the option `backend=bibtex` to `kaobiblio`.

As you have seen, the `\sidecite` command will print a citation in the margin. However, this command would be useless without a way to customise the format of the citation, so the `kaobook` provides also the `\formatmargincitation` command. By "renewing" that command, you can choose which items will be printed in the margins. The best way to understand how it works is to see the actual definition of this command.

```
\newcommand{\formatmargincitation}[1]{%
    \parencite{#1}: \citeauthor*{#1} (\citeyear{#1}), \citetitle{#1}%
}
```

Thus, the `\formatmargincitation` accepts one parameter, which is the citation key, and prints the parencite followed by a colon, then the author, then the year (in brackets), and finally the title.[**Battle2014**] Now, suppose that you wish the margin citation to display the year and the author, followed by the title, and finally a fixed arbitrary string; you would add to your document:

Battle2014

```
\renewcommand{\formatmargincitation}[1]{%
    \citeyear{#1}, \citeauthor*{#1}: \citetitle{#1}; very interesting!%
}
```

The above code results in citations that look like the following.[**Zou2005**] Of course, changing the format is most useful when you also change the default bibliography style. For instance, if you want to use the "philosophy-modern" style for your bibliography, you might have something like this in the preamble:

Zou2005

```
\usepackage[style=philosophy-modern]{styles/kaobiblio}
\renewcommand{\formatmargincitation}[1]{%
    \sdcite{#1}%
}
\addbibresource{main.bib}
```

[1] For this reason you should always use `kaobiblio` instead of `biblatex`, but the syntax and the options are exactly the same.

The commands like `\citeyear`, `\parencite` and `\sdcite` are just examples. A full reference of the available commands can be found in this cheatsheet, under the "Citations" section.

Finally, to compile a document containing citations, you need to use an external tool, which for this class is biber. You need to run the following (assuming that your tex file is called main.tex):

```
$ pdflatex main
$ biber main
$ pdflatex main
```

5.2 Glossaries and Indices

The `kaobook` class loads the packages `glossaries` and `imakeidx`, with which you can add glossaries and indices to your book. For instance, I previously defined some glossary entries and now I am going to use them, like this: computer. `glossaries` also allows you to use acronyms, like the following: this is the full version, Frame per Second (FPS), and this is the short one FPS. These entries will appear in the glossary in the backmatter.

Unless you use Overleaf or some other fancy IDE for LaTeX, you need to run an external command from your terminal in order to compile a document with a glossary. In particular, the commands required are:[2]

```
$ pdflatex main
$ makeglossaries main
$ pdflatex main
```

Note that you need not run `makeglossaries` every time you compile your document, but only when you change the glossary entries.

To create an index, you need to insert the command `\index{subject}` whenever you are talking about "subject" in the text. For instance, at the start of this paragraph I would write `index{index}`, and an entry would be added to the Index in the backmatter. Check it out!

A nomenclature is just a special kind of index; you can find one at the end of this book. To insert a nomenclature, we use the package `nomencl` and add the terms with the command `\nomenclature`. We put then a `\printnomenclature` where we want it to appear.

In theory, you would need to run an external command for the index as well, but luckily the package we suggested, `imakeidx`, can compile the index automatically.

Also with this package we need to run an external command to compile the document, otherwise the nomenclature will not appear:

```
$ pdflatex main
$ makeindex main.nlo -s nomencl.ist -o main.nls
$ pdflatex main
```

These packages are all loaded in packages.sty, one of the files that come with this class. However, the configuration of the elements is best done in the main.tex file, since each book will have different entries and styles.

Note that the `nomencl` package caused problems when the document was compiled, so,

[2] These are the commands you would run in a UNIX system, but see also Section 5.4 (A Final Note on Compilation); I have no idea about how it works in Windows.

to make a long story short, I had to prevent `scrhack` to load the hack-file for `nomencl`. When compiling the document on Overleaf, however, this problem seem to vanish.

This brief section was by no me
a complete reference on the sub
therefore you should consult the
umentation of the above packag
gain a full understanding of how t
work.

5.3 Hyperreferences

Together with this class we provide a handy package to help you referencing the same elements always in the same way, for consistency across the book. First, you can label each element with a specific command. For instance, should you want to label a chapter, you would put `\labch{chapter-title}` right after the `\chapter` directive. This is just a convenience, because `\labch` is actually just an alias to `\label{ch:chapter-title}`, so it spares you the writing of "ch:". We defined similar commands for many typically labeled elements, including:

- Page: `\labpage`
- Part: `\labpart`
- Chapter: `\labch`
- Section: `\labsec`
- Figure: `\labfig`
- Table: `\labtab`
- Definition: `\labdef`
- Assumption: `\labassum`
- Theorem: `\labthm`
- Proposition: `\labprop`
- Lemma: `\lablemma`
- Remark: `\labremark`
- Example: `\labexample`
- Exercise: `\labexercise`

Of course, we have similar commands for referencing those elements. However, since the style of the reference should depend on the context, we provide different commands to reference the same thing. For instance, in some occasions you may want to reference the chapter by name, but other times you want to reference it only by number. In general, there are four reference style, which we call plain, vario, name, and full.

The plain style references only by number. It is accessed, for chapters, with `\refch{chapter-title}` (for other elements, the syntax is analogous). Such a reference results in: Chapter 5.

The vario and name styles rest upon the `varioref` package. Their syntax is `\vrefch{chapter-title}` and `\nrefch{chapter-title}`, and they result in: Chapter Chapter 5 on page 31, for the vario style, and: Chapter 5 (References), for the name style. As you can see, the page is referenced in `varioref` style.

The full style references everything. You can use it with `\frefch{chapter-title}` and it looks like this: Chapter 5 (References) on page 31.

Of course, all the other elements have similar commands (*e.g.* for parts you would use `\vrefpart{part-title}` or something like that). However, not all elements implement all the four styles. The commands provided should be enough, but if you want to see what is available or to add the missing ones, have a look at the attached package.

In order to have access to all these features, the `kaorefs` should be loaded in the preamble of your document. It should be loaded last, or at least after `babel` (or `polyglossia`) and `plaintheorems` (or `mdftheorems`). Options can be passed to it like to any other package; in particular, it is possible to specify the language of the captions. For instance, if you specify “italian” as an option, instead of “Chapter” it will be printed “Capitolo”, the Italian analog. If you know other languages, you are welcome to contribute the translations of these captions! Feel free to contact the author of the class for further details.

The `kaorefs` package also include `cleveref`, so it is possible to use `\cref` in addition to all the previously described referencing commands.

5.4 A Final Note on Compilation

Probably the easiest way to compile a latex document is with the `latexmk` script, as it can take care of everything, if properly configured, from the bibliography to the glossary. The command to issue, in general, is:

```
latexmk [latexmk_options] [filename ...]
```

`latexmk` can be extensively configured (see `https://mg.readthedocs.io/latexmk.html`). For convenience, I print here an example configuration that would cover all the steps described above.

```
# By default compile only the file called ’main.tex’
@default_files = (’main.tex’);

# Compile the glossary and acronyms list (package ’glossaries’)
add_cus_dep( ’acn’, ’acr’, 0, ’makeglossaries’ );
add_cus_dep( ’glo’, ’gls’, 0, ’makeglossaries’ );
$clean_ext .= ” acr acn alg glo gls glg”;
sub makeglossaries {
   my ($base_name, $path) = fileparse( $_[0] );
   pushd $path;
   my $return = system ”makeglossaries”, $base_name;
   popd;
   return $return;
}

# Compile the nomenclature (package ’nomencl’)
add_cus_dep( ’nlo’, ’nls’, 0, ’makenlo2nls’ );
sub makenlo2nls {
    system( ”makeindex -s nomencl.ist -o \”$_[0].nls\” \”$_[0].nlo\”” );
}
```

However, if you’d rather not use an external package and want to do everything manually, here are some tips.[3]

Compiling the examples in the kaobook repository

To compile the examples, and in particular the documentation, that are in the `examples` directory of the kaobook repository on GitHub, do as follows. `cd` into the root directory of the repository, and run `pdflatex -output-directory examples/documentation main.tex`. With this trick, you can compile the documentation using the class files pertaining to the repository (and not, say, those in your texmf tree). The “-output-directory” option works with the other LaTeX-related commands such as biber and makeglossaries.

A note of warning: sometimes LaTeX needs more than one run to get the correct position

[3] As the author only uses Linux and compiles everything from the command line, he doesn’t know how the compilation works in Windows or Mac. The tips, therefore, refer to the usage with Linux from the command line.

of each element; this is true in particular for the positioning of floating elements like figures, tables, and margin notes. Occasionally, LaTeX can need up to four re-runs, so If the alignment of margin elements looks odd, or if they bleed into ther main text, try runnign pdflatex one more time.

Design and Additional Features

6 Page Design

6.1 Headings

So far, in this document I used two different styles for the chapter headings: one has the chapter name, a rule and, in the margin, the chapter number; the other has an image at the top of the page, and the chapter title is printed in a box (like for this chapter). There is one additional style, which I used only in the appendix (on page ??page:appendix); there, the chapter title is enclosed in two horizontal rules, and the chapter number (or letter, in the case of the appendix) is above it.[1]

Every book is unique, so it makes sense to have different styles from which to choose. Actually, it would be awesome if whenever a `kao`-user designs a new heading style, he or she added it to the three styles already present, so that it will be available for new users and new books.

The choice of the style is made simple by the `\setchapterstyle` command. It accepts one option, the name of the style, which can be: “plain”, “kao”, or “lines”.[2] If instead you want the image style, you have to use the command `\setchapterimage`, which accepts the path to the image as argument; you can also provide an optional parameter in square brackets to specify the height of the image.

Let us make some examples. In this book, I begin a normal chapter with the lines:

```
1 \setchapterstyle{kao}
2 \setchapterpreamble[u]{\margintoc}
3 \chapter{Title of the Chapter}
4 \labch{title}
```

In Line 1 I choose the style for the title to be “kao”. Then, I specify that I want the margin toc. The rest is ordinary administration in LaTeX, except that I use my own `\labch` to label the chapter. Actually, the `\setchapterpreamble` is a standard KOMA-Script one, so I invite you to read about it in the KOMA documentation. Once the chapter style is set, it holds until you change it.[3] Whenever I want to start a chapter with an image, I simply write:

```
\setchapterimage[7cm]{path/to/image.png} % Optionally specify the height
\setchapterpreamble[u]{\margintoc}
```

[1] To be honest, I do not think that mixing heading styles like this is a wise choice, but in this document I did it only to show you how they look.

[2] Plain is the default LaTeX title style; the other ones are self explanatory.

[3] The `\margintoc` has to be specified at every chapter. Perhaps in the future this may change; it all depends on how this feature will be welcomed by the users, so keep in touch with me if you have preferences!

```
\chapter{Catchy Title} % No need to set a chapter style
\labch{catchy}
```

If you prefer, you can also specify the style at the beginning of the main document, and that style will hold until you change it again.

6.2 Headers & Footers

Headers and footers in KOMA-Script are handled by the `scrlayer-scrpage` package. There are two basic style: “scrheadings” and “plain.scrheadings”. The former is used for normal pages, whereas the latter is used in title pages (those where a new chapter starts, for instance) and, at least in this book, in the front matter. At any rate, the style can be changed with the `\pagestyle` command, *e.g.* `\pagestyle{plain.scrheadings}`.

In both styles, the footer is completely empty. In plain.scrheadings, also the header is absent (otherwise it wouldn’t be so plain...), but in the normal style the design is reminiscent of the “kao” style for chapter titles.

To Do

The `twoside` class option is still unstable and may lead to unexpected behaviours. As always, any help will be greatly appreciated.

6.3 Table of Contents

Another important part of a book is the table of contents. By default, in `kaobook` there is an entry for everything: list of figures, list of tables, bibliographies, and even the table of contents itself. Not everybody might like this, so we will provide a description of the changes you need to do in order to enable or disable each of these entries. In the following Table 6.1, each item corresponds to a possible entry in the TOC, and its description is the command you need to provide to have such entry. These commands are specified in the attached style package,[4] so if you don’t want the entries, just comment the corresponding lines.

In a later section, we will see how you can define your own floating environment, and endow it with an entry in the TOC.

Of course, some packages, like those for glossaries and indices, will try to add their own entries. In such cases, you have to follow the instructions specific to that package. Here, since we have talked about glossaries and notations in Chapter 5, we will briefly see how to configure them.

For the `glossaries` package, use the “toc” option when you load it: `\usepackage[toc]{glossaries}`. For `nomencl`, pass the “intoc” option at the moment of loading the package. Both `glossaries` and `nomencl` are loaded in the attached “packages” package.

Additional configuration of the table of contents can be performed through the packages

Table 6.1: Commands to add a particular entry to the table of contents.

Entry	Command to Activate
Table of Contents	`\setuptoc{toc}{totoc}`
List of Figs and Tabs	`\PassOptionsToClass{toc=listof}{\@baseclass}`
Bibliography	`\PassOptionsToClass{toc=bibliography}{\@baseclass}`

[4] In the same file, you can also choose the titles of these entries.

`etoc`, which is loaded because it is needed for the margintocs, or the more traditional `tocbase`. Read the respective documentations if you want to be able to change the default TOC style.[5]

6.4 Paper Size

Recent versions of Kaobook support paper sizes different from the default A4. It is possible to pass the name of the paper as an option to the class, as we are accustomed for any other LaTeX class. For example, the class option `b5paper` would set the paper size to the B5 format.

We also support the paper sizes specified in this web page and some additional sizes requested by the users, with the option names specified in Table 6.2.

Table 6.2: Some non-standard pa sizes supported by kaobook.

Dimension	Optic
12.0cm x 19.0cm	small
13.5cm x 21.5cm	pock
14.8cm x 21.0cm	a5pa
15.5cm x 22.0cm	juve
17.0cm x 17.0cm	small
21.0cm x 15.0cm	appe
17.0cm x 22.0cm	cook
19.0cm x 27.0cm	illust
17.0cm x 17.0cm	photo
16.0cm x 24.0cm	f24pa

For instance, to use the "smallpocketpaper" add the correct description at the beginning of the documentclass instruction:

```
\documentclass[
        smallpocketpaper,
        fontsize=10pt,
        twoside=false,
        %open=any,
        secnumdepth=1,
]{kaobook}
```

6.5 Page Layout

Besides the page style, you can also change the width of the content of a page. This is particularly useful for pages dedicated to part titles, where having the 1.5-column layout might be a little awkward, or for pages where you only put figures, where it is important to exploit all the available space.

In practice, there are two layouts: "wide" and "margin". The former suppresses the margins and allocates the full page for contents, while the latter is the layout used in most of the pages of this book, including this one. The wide layout is also used automatically in the front and back matters.

Sometimes it is desirable to increase the width for just one or a few paragraphs; the `widepar` environment does that: wrap your paragraphs in this environment, and they will occupy the full width of the page.

To change page layout, use the `\pagelayout` command. For example, when I start a new part, I write:

```
\pagelayout{wide}
\addpart{Title of the New Part}
\pagelayout{margin}
```

Beyond these two basic layouts, it is also possible to finely tune the page layout by redefining the `\marginlayout` command. This command is called internally by the higher-level `\pagelayout`, and it is responsible for setting the width of the margins and of the text. The default definition is:

```
\newcommand{\marginlayout}{%
    \newgeometry{
        top=27.4mm,             % height of the top margin
        bottom=27.4mm,          % height of the bottom margin
```

---

[5] 5: (And please, send me a copy of what you have done, I'm so curious!)

```
        inner=24.8mm,           % width of the inner margin
        textwidth=107mm,        % width of the text
        marginparsep=8.2mm,     % width between text and margin
        marginparwidth=49.4mm,  % width of the margin
    }%
}
```

so if you want to, say, decrease the width of the margin while increasing the width of the text, you could write in the preamble of your document something like:

```
\renewcommand{\marginlayout}{%
    \newgeometry{
        top=27.4mm,             % height of the top margin
        bottom=27.4mm,          % height of the bottom margin
        inner=24.8mm,           % width of the inner margin
        textwidth=117mm,        % width of the text
        marginparsep=8.2mm,     % width between text and margin
        marginparwidth=39.4mm,  % width of the margin
    }%
}
```

where the text width has been increased by 10mm and the margin width has been decreased by 10mm.

6.6 Numbers & Counters

In this short section we shall see how dispositions, sidenotes and figures are numbered in the `kaobook` class.

By default, dispositions are numbered up to the section in `kaobook` and up to the subsection in `kaohandt`. This can be changed by passing the option `secnumdepth` to`kaobook` or `kaohandt` (e.g. 1 corresponds to section and 2 corresponds to subsections).

The sidenotes counter is the same across all the document, but if you want it to reset at each chapter, just uncomment the line

`\counterwithin*{sidenote}{chapter}`

in the `styles/style.sty` package provided by this class.

Figure and Table numbering is also per-chapter; to change that, use something like:

`\renewcommand{\thefigure}{\arabic{section}.\arabic{figure}}`

6.7 White Space

Attention! This section may be incomplete.

One of the things that I find most hard in LaTeX is to finely tune the white space around objects. There are not fixed rules, each object needs its own adjustment. Here we shall see how some spaces are defined at the moment in this class.

Space around sidenotes and citations marks

There should be no space before or after sidenotes and citation marks, like so:

sidenote[6]sidenote
citation[**James2013**]citation

[6] This paragraph can be used to diagnose any problems: if you see whitespace around sidenotes or citation marks, probably a % sign is missing somewhere in the definitions of the class macros.

Space around figures and tables

```
\renewcommand\FBaskip{.4\topskip}
\renewcommand\FBbskip{\FBaskip}
```

Space around captions

```
\captionsetup{
    aboveskip=6pt,
    belowskip=6pt
}
```

Space around displays (*e.g.* equations)

```
\setlength\abovedisplayskip{6pt plus 2pt minus 4pt}
\setlength\belowdisplayskip{6pt plus 2pt minus 4pt}
\abovedisplayskip 10\p@ \@plus2\p@ \@minus5\p@
\abovedisplayshortskip \z@ \@plus3\p@
\belowdisplayskip \abovedisplayskip
\belowdisplayshortskip 6\p@ \@plus3\p@ \@minus3\p@
```

Mathematics and Boxes | 7

7.1 Theorems

Despite most people complain at the sight of a book full of equations, mathematics is an important part of many books. Here, we shall illustrate some of the possibilities. We believe that theorems, definitions, remarks and examples should be emphasised with a shaded background; however, the colour should not be to heavy on the eyes, so we have chosen a sort of light yellow.[1]

[1] The boxes are all of the same colour here, because we did not want our document to look like Harlequin.

Definition 7.1.1 *Let (X, d) be a metric space. A subset $U \subset X$ is an open set if, for any $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$. We call the topology associated to d the set τ_d of all the open subsets of (X, d).*

Definition 7.1.1 is very important. I am not joking, but I have inserted this phrase only to show how to reference definitions. The following statement is repeated over and over in different environments.

Theorem 7.1.1 *A finite intersection of open sets of (X, d) is an open set of (X, d), i.e τ_d is closed under finite intersections. Any union of open sets of (X, d) is an open set of (X, d).*

Proposition 7.1.2 *A finite intersection of open sets of (X, d) is an open set of (X, d), i.e τ_d is closed under finite intersections. Any union of open sets of (X, d) is an open set of (X, d).*

You can even insert footnotes inside the theorem environments; they will be displayed at the bottom of the box.

Lemma 7.1.3 *A finite intersection[a] of open sets of (X, d) is an open set of (X, d), i.e τ_d is closed under finite intersections. Any union of open sets of (X, d) is an open set of (X, d).*

---

[a] I'm a footnote

You can safely ignore the content of the theorems…I assume that if you are interested in having theorems in your book, you already know something about the classical way to add them. These example should just showcase all the things you can do within this class.

Corollary 7.1.4 (Finite Intersection, Countable Union) *A finite intersection of open sets of (X, d) is an open set of (X, d), i.e τ_d is closed under finite intersections. Any union of open sets of (X, d) is an open set of (X, d).*

Proof. The proof is left to the reader as a trivial exercise. Hint: Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language. ☐

Definition 7.1.2 *Let (X, d) be a metric space. A subset $U \subset X$ is an open set if, for any $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$. We call the topology associated to d the set τ_d of all the open subsets of (X, d).*

Here is a random equation, just because we can:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}}$$

Example 7.1.1 Let (X, d) be a metric space. A subset $U \subset X$ is an open set if, for any $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$. We call the topology associated to d the set τ_d of all the open subsets of (X, d).

Remark 7.1.1 Let (X, d) be a metric space. A subset $U \subset X$ is an open set if, for any $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$. We call the topology associated to d the set τ_d of all the open subsets of (X, d).

As you may have noticed, definitions, example and remarks have independent counters; theorems, propositions, lemmas and corollaries share the same counter.

Remark 7.1.2 Here is how an integral looks like inline: $\int_a^b x^2 dx$, and here is the same integral displayed in its own paragraph:

$$\int_a^b x^2 dx$$

There is also an environment for exercises.

Exercise 7.1.1 Prove (or disprove) the Riemann hypothesis.

We provide two files for the theorem styles: plaintheorems.sty, which you should include if you do not want coloured boxes around theorems; and mdftheorems.sty, which is the one used for this document.[2] You may want to edit these files according to your taste and the general style of the book. However, there is an option to customise the background colour of the boxes in : when you load this package, you can pass it the `background=mycolour` option (replace “mycolour” with the actual colour, for instance, “red!35!white”). This will change the colour of all the boxes, but it is also possible to override the default colour only for some elements. For instance, the `propositionbackground=mycolour` option will change the colour for propositions only. There are similar options for theorem, definition, lemma, corollary, remark, and example.

[2] The plain one is not showed, but actually it is exactly the same as this one, only without the yellow boxes.

7.2 Boxes & Custom Environments [3]

Say you want to insert a special section, an optional content or just something you want to emphasise. We think that nothing works better than a box in these cases. We used `mdframed` to construct the ones shown below. You can create and modify such environments by editing the provided file environments.sty.

Title of the box

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

If you set up a counter, you can even create your own numbered environment.

Comment 7.2.1

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

7.3 Experiments

to wrap marginnotes inside boxes, too. Audacious readers are encouraged own experiments and let me know the outcomes.

title of margin note

Margin note inside a kaobox. (Actually, kaobox inside a marginnote!)

I believe that many other special things are possible with the kaobook class. During its development, I struggled to keep it as flexible as possible, so that new features could be added without too great an effort. Therefore, I hope that you can find the optimal way to express yourselves in writing a book, report or thesis with this class, and I am eager to see the outcomes of any experiment that you may try.

APPENDIX

A

Heading on Level 0 (chapter)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.1 Heading on Level 1 (section)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Heading on Level 2 (subsection)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Heading on Level 3 (subsubsection)

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

Heading on Level 4 (paragraph) Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

A.2 Lists

Example for list (itemize)

- First item in a list
- Second item in a list
- Third item in a list
- Fourth item in a list
- Fifth item in a list

Example for list (4*itemize)

- First item in a list
  - First item in a list
    - First item in a list
      - First item in a list
      - Second item in a list
    - Second item in a list
  - Second item in a list
- Second item in a list

Example for list (enumerate)

1. First item in a list
2. Second item in a list
3. Third item in a list
4. Fourth item in a list
5. Fifth item in a list

Example for list (4*enumerate)

1. First item in a list
   a) First item in a list

        i. First item in a list
            A. First item in a list
            B. Second item in a list
        ii. Second item in a list
    b) Second item in a list
2. Second item in a list

Example for list (description)

First item in a list
Second item in a list
Third item in a list
Fourth item in a list
Fifth item in a list

Example for list (4*description)

First item in a list
    First item in a list
        First item in a list
            First item in a list
            Second item in a list
        Second item in a list
    Second item in a list
Second item in a list

B

Fonts Testing

B.1 Font Sizes

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

The quick brown fox jumps over the lazy dog.

B.2 Font Families

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

The quick brown fox jumps over the lazy dog. Medium.

The quick brown fox jumps over the lazy dog. Bold.

The quick brown fox jumps over the lazy dog. Upright.

The quick brown fox jumps over the lazy dog. Italics.

The quick brown fox jumps over the lazy dog. Slanted.

The quick brown fox jumps over the lazy dog. Small Caps.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

The quick brown fox jumps over the lazy dog. Medium.

The quick brown fox jumps over the lazy dog. Bold.

The quick brown fox jumps over the lazy dog. Upright.

The quick brown fox jumps over the lazy dog. Italics.

The quick brown fox jumps over the lazy dog. Slanted.

The quick brown fox jumps over the lazy dog. Small Caps.

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

The quick brown fox jumps over the lazy dog. Medium.

The quick brown fox jumps over the lazy dog. Bold.

The quick brown fox jumps over the lazy dog. Upright.

The quick brown fox jumps over the lazy dog. Italics.

The quick brown fox jumps over the lazy dog. Slanted.

The quick brown fox jumps over the lazy dog. Small Caps.

Greek Letters with Pronunciations

Character	Name	Character	Name
α	alpha *AL-fuh*	ν	nu *NEW*
β	beta *BAY-tuh*	ξ, Ξ	xi *KSIGH*
γ, Γ	gamma *GAM-muh*	o	omicron *OM-uh-CRON*
δ, Δ	delta *DEL-tuh*	π, Π	pi *PIE*
ϵ	epsilon *EP-suh-lon*	ρ	rho *ROW*
ζ	zeta *ZAY-tuh*	σ, Σ	sigma *SIG-muh*
η	eta *AY-tuh*	τ	tau *TOW (as in cow)*
θ, Θ	theta *THAY-tuh*	υ, Υ	upsilon *OOP-suh-LON*
ι	iota *eye-OH-tuh*	ϕ, Φ	phi *FEE, or FI (as in hi)*
κ	kappa *KAP-uh*	χ	chi *KI (as in hi)*
λ, Λ	lambda *LAM-duh*	ψ, Ψ	psi *SIGH, or PSIGH*
μ	mu *MEW*	ω, Ω	omega *oh-MAY-guh*

Capitals shown are the ones that differ from Roman capitals.

Alphabetical Index