Problem Chosen	2025 MCM/ICM Summary Sheet	Team Control Number
A		**2520840**

The Wear of Staircases: An Archaeological Perspective

Summary

This paper presents a comprehensive study on the wear patterns of staircases from an archaeological perspective. We develop a series of models to analyze and predict the wear behavior of stairs under various conditions. Our approach begins with the establishment of the **Archard Wear Model**, which is used to compute the total wear volume when walking on stairs. We enhance this model by incorporating additional factors such as temperature variations, load distribution, and surface roughness, which significantly influence the wear volume. This modification makes the model more applicable to real-world scenarios.

To further refine our predictions, we introduce the Finite Element Method (FEM) to optimize the pressure distribution parameter. This method allows us to discretize the stair area into a mesh of elements, enabling a more accurate numerical approximation of the pressure field. By integrating the optimized pressure distribution, we refine the Archard Wear Model, thereby enhancing its predictive accuracy.

We also consider the impact of natural factors by proposing the **Material Aging Model**, which accounts for the temporal variations in material hardness.This model dynamically tracks property evolution to better capture environmental degradation processes. We estimate the aging process using Newton's method iteration, providing an approximate prediction of the material's lifespan under varying conditions.

To estimate the age of the stairs, we integrate the Archard Wear Model with a **Material Aging Model**.This integration allows us to account for the cumulative effects of human-induced aging and natural aging, providing a more accurate representation of the material's lifespan.

To assess the reliability of our models, we conduct a sensitivity analysis by computing the partial derivatives of the estimated age with respect to each parameter. Additionally, we utilize Monte Carlo simulations to account for parametric uncertainties and estimate the confidence interval of the predicted age. From a qualitative perspective, we validate the model by comparing its predictions with known ages of staircases. The relative error is calculated to ensure the model's accuracy and reliability in real-world applications.

In summary, this paper proposes a multifaceted approach to analyze stair wear, including the Archard Wear Model, the Finite Element Method for pressure optimization, the Material Aging Model, and a comprehensive evaluation framework. Our models not only provide a robust tool for predicting material degradation but also offer valuable insights into the historical usage patterns of staircases.

Keywords: Archard wear model, Material Aging Model, wear volume, Finite Element Method

Contents

1 Introduction

1.1 Background

Stairs, as an essential architectural component for connecting vertical spaces, hold a universal and irreplaceable role in modern built environments. From a structural engineering perspective, contemporary stairs are primarily constructed using modern building materials such as reinforced concrete and steel. These materials not only exhibit excellent load-bearing and fire-resistant properties but also demonstrate significant advantages in terms of structural rigidity, cost-effectiveness, construction convenience, and design versatility.

However, from the standpoint of material mechanics, any building material is inevitably subject to wear and tear over prolonged use. Such wear holds substantial academic value for archaeological research. According to the principles of tribology, the wear on material surfaces primarily results from the relative motion between the surface and hard particles or metals, leading to the loss of surface material. Specifically, the wear characteristics of stairs show a significant correlation with parameters such as the building's age and foot traffic. These wear patterns not only provide empirical data for studies on material durability but also serve as important historical information carriers for archaeological research, highlighting their value in interdisciplinary studies.

1.2 Restatement of the Problem

- First,the problem requires developing a model to analyze the wear patterns on stairs to infer information such as the frequency of use, predominant direction of travel, and the number of simultaneous users.
- Then, the model should address more complex questions, including the consistency of wear with historical records, the reliability of age estimates, and the identification of material sources.
- Finally,The solution must be based on non-destructive, cost-effective measurements that can be conducted by a small team with minimal tools.

1.3 Our Work

The wear model was developed based on the Archard Equation. In this study, the model was extended to incorporate several critical factors that significantly influence the total wear volume, such as temperature variations, load distribution, and surface roughness. These factors were carefully analyzed and integrated into the model to ensure its accuracy and reliability under diverse operating conditions. To further enhance the model's applicability to real-world scenarios, key parameters, including the wear coefficient and hardness factor, were adjusted and refined through experimental validation and real-world data analysis.

Furthermore, drawing inspiration from the material aging model, the total aging process was defined as the accumulated effect of human-induced aging and natural aging. By separating aging into these two components, a comprehensive time-dependent model was established to capture the dynamic evolution of material properties.

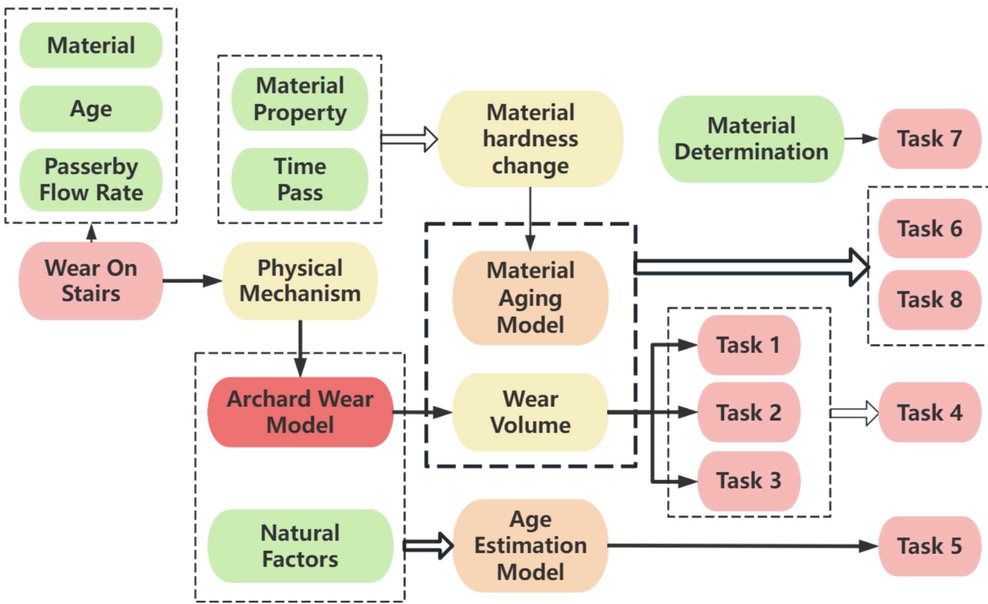

Figure 1: The Process of Modeling

To quantify the aging process, the range of material aging over time was estimated using Newton's method iteration. This approach provided an approximate prediction of the material's aging range and offered a practical estimation of its lifespan under varying conditions.The iterative process was designed to balance computational efficiency and reasonable accuracy, making the model a useful tool for predicting material degradation in real-world applications.

2 Assumptions and Notiation

2.1 Assumptions and Justification

To simplify the problem and make it convenient for us to simulate real-life conditions, we make the following basic assumptions, each of which is properly justified.

- **Assumption:The wear volume is calculated by treating it as a rectangular prism**.
- **Justification:**The contact between a person and a step occurs on a planar surface, and the duration of this contact is significantly shorter compared to the timescale of stone wear. Consequently, the wear inflicted on the step by a single individual's movement is negligible.Therefore, it can be assumed that the wear depth caused by a single person approximates an infinitesimal vertical distance, while the forward movement of the foot represents a horizontal distance. This configuration can be approximated as a rectangular prism in shape.
- **Assumption:Daily foot traffic is assumed to be relatively evenly distributed**.
- **Justification:**Precise daily records of foot traffic for ancient staircases are typically unavailable.Therefore, a uniform distribution of pedestrian flow is assumed, and the validity of this

assumption is assessed through sensitivity analysis.The resulting error is determined to be within acceptable archaeological standards.

- **Assumption:We ignore the wear and tear resulting from environmental factors**.
- **Justification:**According to Reference [**3**], the degree of weathering contributes significantly less to the wear of steps compared to the wear depth caused by human foot traffic. Weathering typically results in uniform surface degradation rather than localized depressions. Therefore, the impact of natural weathering on step wear can be neglected in the calculations.

2.2 Notations

Symbols	Description	Unit
V	wear volume	mm^3
k	wear coefficient	dimensionless
F	normal load force on the contact surface	N
S	relative sliding distance	mm
H	material hardness	N/m^2
μ_x	The central position of the pressure (x-axis) on the steps	mm
μ_y	The central position of the pressure (y-axis) on the steps	mm
σ_x	The spread range of the pressure distribution(x-axis)	mm
σ_y	The spread range of the pressure distribution(y-axis)	mm
D	wear depth	mm
A_0	initial material hardness	N/m^2
λ	aging rate	dimensionless
t	time	year
T	Kelvin temperature	K
k_B	Boltzmann constant	dimensionless

where we define the main parameters while specific value of those parameters will be given later.

3 Sub-model I : Archard Wear Model

We begin by developing a sub-model to simulate the wear caused by the walking process. Subsequently, we incorporate natural factors that influence the parameter k and derive the corresponding mathematical formulation to describe this process. To enhance the realism of the simulation, Finite Element Method (FEM) is employed to model the actual walking conditions, and the total force exerted over the walking area is calculated using an area integral. Finally, the model is validated against historical real-world data, and its accuracy is thoroughly analyzed.

3.1 Control Equations and Model Modified

According to the Archard wear model [**1**], the total wear volume is governed by several key factors: the wear coefficient, the material hardness, the applied normal force, and the sliding distance. This relationship forms the foundational equation of the Archard wear model, which is expressed mathematically as follows:

- **Archard equation**

$$V = k \cdot \frac{F \cdot S}{H} \tag{1}$$

he Archard wear model describes the total wear volume during the walking process. However, in real-world scenarios, additional factors such as temperature variations and the distribution of walking forces must be considered to accurately capture the wear behavior.To address these complexities, we propose the following modified model:

- **Optimization of k and H**

It is widely recognized that in real-world scenarios, the wear coefficient and material hardness are not constant.These properties are influenced by various environmental factors, with temperature being a significant one. Taking this into account, we have incorporated temperature considerations into our analysis. Based on this rationale, we proceed with the following modified approach:

$$k = k_0 \cdot exp(-\frac{E_k}{k_B \cdot T}) \tag{2}$$

$$H = H_0 \cdot exp(-\frac{E_H}{k_B \cdot T}) \tag{3}$$

Here, k_0 represents the initial wear coefficient of the material, and H_0 denotes the initial hardness of the material.

- **Using FEM to optimize the parameter F**

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} exp[-\frac{1}{2}(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2})] \tag{4}$$

The function P(x,y) represents the pressure distribution at the point (x,y) on the stair area.To optimize the pressure distribution, we employ the Finite Element Method (FEM) to discretize the stair area into a mesh of elements. This allows us to numerically approximate p(x,y) by solving the governing equations over each element, ensuring a more accurate representation of the pressure field.

Therefore, the total pressure F can be calculated as follows:

$$\iint_A p(x, y)\, dx\, dy \tag{5}$$

where A denotes the footstep area. Through this calculation, we can determine the total pressure exerted by the footstep. By integrating the optimized pressure distribution obtained from the FEM analysis, we derive a more precise total pressure value. Based on the modified equation above, we can further refine the Archard wear model to account for the optimized pressure distribution, thereby enhancing its predictive accuracy.

3.2 Task 1: Assessment of Stair Utilization Frequency

To quantitatively evaluate the usage frequency of the stairs, we propose to measure the number of people passing through the stairs per unit time. This metric serves as a direct indicator of the stairs' utilization intensity. The equations used to model this relationship are as follows:

$$V_{use} = k \cdot \frac{F_0 \cdot S_0}{H_0} \tag{6}$$

$$N = \frac{V}{V_{use}} \tag{7}$$

where V_{use} represents the wear volume per person, which quantifies the material wear caused by an individual's usage of the stairs, and N denotes the number of people within a specific time period. By analyzing the variation in N, we can infer the usage frequency of the stairs. Specifically, a higher value of N indicates a more frequent usage of the stairs, while a lower value suggests reduced activity. This relationship allows us to establish a connection between the wear volume and the actual usage patterns of the stairs, providing valuable insights for maintenance and design optimization.

3.3 Task 2: Investigation of Directional Preference in Stair Usage

When walking on stairs, the distribution of force varies depending on the direction of movement. During ascent (going upstairs), the primary force is concentrated on the **back** of the stair (closer to the higher step). Conversely, during descent (going downstairs), the primary force is concentrated on the **front** of the stair (closer to the lower step). This force distribution pattern is illustrated in Figure 2.

To quantify the wear patterns associated with these force distributions, we calculate the wear volumes for the rear and front regions of the stair, denoted as V_{back} and V_{front}, respectively. Based on these calculations, we define a dimensionless parameter, value, as the ratio of the two wear volumes:

$$\text{value} = \frac{V_{\text{back}}}{V_{\text{front}}} \tag{8}$$

The parameter value serves as an indicator of directional preference in stair usage. Specifically:

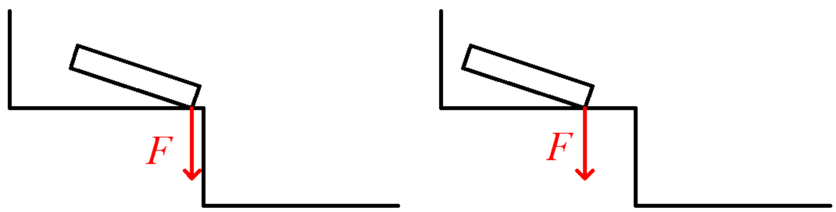

Figure 2: Illustration of Force Points During Stair Ascent(Left) and Descent(Right)

- If value > 1, it indicates a preference for ascending the stairs (more wear on the back of the stairs).
- If value < 1, it indicates a preference for descending the stairs (more wear on the front of the stair).

This approach provides a quantitative method to analyze the directional preference of stair usage based on wear patterns, offering insights into the behavioral dynamics of stair users.

3.4 Task 3: Assessment of Group Versus Single-File Stair Usage

In real-life scenarios, it is evident that when multiple individuals ascend or descend stairs simultaneously, the distribution of foot placements is broader compared to when individuals use the stairs in a single-file manner. This observation is crucial for understanding the wear patterns on stair surfaces. Inspired by this phenomenon, we propose a method to evaluate stair usage patterns by analyzing the width of the wear footprint.

The mathematical model to describe this situation is given by the following equations:

$$L(t) = k \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot exp[-\frac{(x-\mu)^2}{2\sigma^2}] \cdot g(t) \tag{9}$$

where $g(t)$ is a function determined by the parameter t, and is defined as:

$$g(t) = e^{-\lambda t} \tag{10}$$

In these equations, σ and μ represent the standard deviation and mean of the sample wear depth, respectively. The parameter λ is a decay rate that affects the function.$g(t)$.

By evaluating the length $L(t)$, we can infer the usage pattern of the stairs. A broader distribution of foot placements, indicated by alarger,σ, suggests group usage, whereas a narrower distribution, indicated by asmaller,σ, suggests single-file usage. This method provides a quantitative approach to differentiate between these two modes of stair usage, which is essential for maintenance planning and understanding the dynamics of pedestrian traffic.

3.5 Task 4: Consistency Between Model-Predicted Wear and Actual Observations

In this part, considering the above modeling process, we decide to use the nonlinear regression and computer simulation method to test the fitting result of our model. The result is shown below:

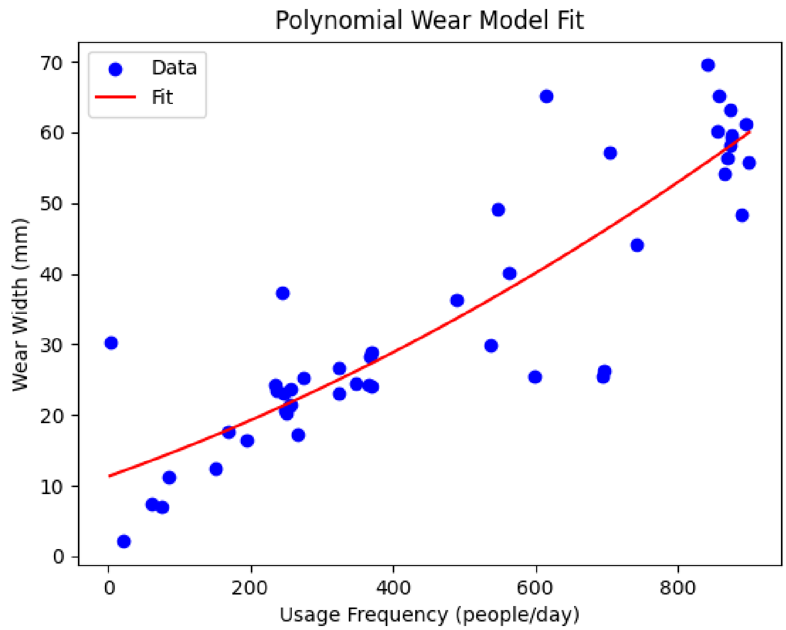

Figure 3: Results of Nonlinear Regression Fitting

According to the computer simulation, the result $R^2 = 0.7923$ shows that the model is basically fitting the real wear data. It can be concluded that the model's performance is relatively sufficient to make more predictions.

4 Sub-model II: Material Aging Model

To enhance environmental simulation fidelity, our framework explicitly accounts for temporal variations in material hardness through the integration of a Material aging model. This approach dynamically tracks property evolution to better capture environmental degradation processes

4.1 Model Description

Drawing upon established material degradation studies[3], we developed the following hardness evolution model to characterize temporal changes in material properties:

$$H(t) = H_0 \cdot e^{-\lambda t} \tag{11}$$

For different materials, there exist distinct aging curves. Taking the relatively typical stone and wood as examples, the following figure presents their respective aging curves.

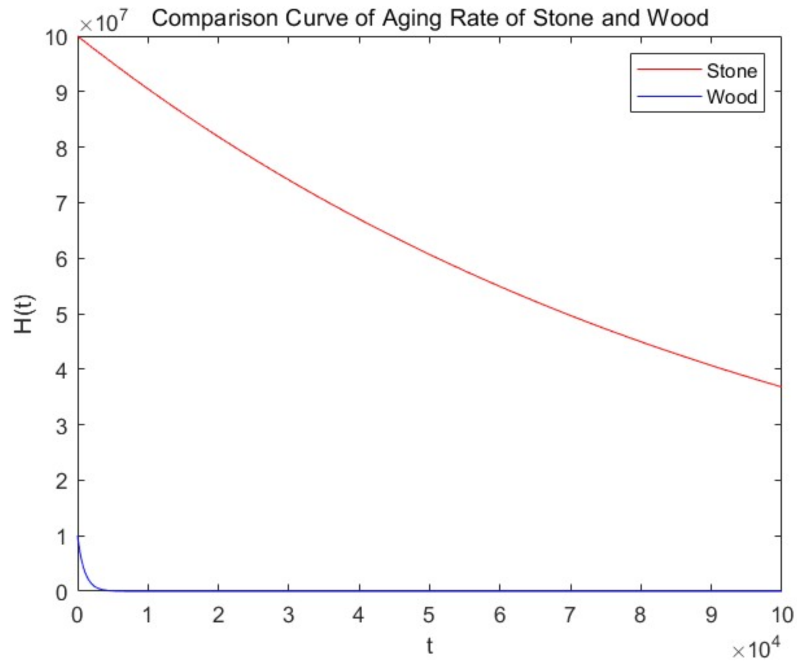

Figure 4: Aging Curves of Stone and Wood

4.2 Task 8: Analysis of Stair Usage Patterns: Daily Foot Traffic and Temporal Distribution

To systematically investigate stair utilization patterns, we developed two distinct behavioral models through systematic observational analysis, designed to characterize typical movement protocols and user interaction dynamics.

- Long term low frequency use pattern:

$$V_{\text{total}} = \frac{k \cdot F \cdot d \cdot N_{\text{low}} \cdot \left(e^{\lambda t_{\text{long}}} - 1\right)}{\lambda H_0} \tag{12}$$

- Short term hign frequency use pattern:

$$V_{\text{total}} = \frac{k \cdot F \cdot d \cdot N_{\text{high}} \cdot t_{\text{short}}}{H_0} \tag{13}$$

Through computational analysis of empirical datasets, both models generate theoretical wear magnitudes. Systematic comparison between these simulated wear patterns and empirical field measurements enables probabilistic inference of historical utilization patterns. When empirical measurements demonstrate closer alignment with Model 1 projections, we can deduce the longitudinal patterns of reduced-frequency stair utilization.

5 Sub-model III: Age Estimation Model

Exposure to pedestrian traffic and environmental weathering induces progressive structural degradation in staircases. To establish a chronological assessment model, we develop a multifactorial model that integrates pedestrian load metrics and environmental variables. This approach quantitatively correlates material deterioration rates with usage patterns and climatic erosion indices to determine the using age of stairs.

5.1 Model Description

To estimate the age of the stairwell, we integrate the Archard Wear Model with a Material Aging Model. The total wear depth $D(t)$ over time t is expressed as:

$$D(t) = \int_0^t \left(\frac{k(\tau) \cdot F(\tau) \cdot v(\tau)}{H(\tau)} \right) d\tau + A_0 \cdot \left(1 - e^{-\lambda t}\right) \tag{14}$$

In the equation, τ serves as an integration variable with no specific physical meaning.

5.2 Task 5: Age Estimation and Reliability Analysis

5.2.1 Age Estimation

Based on the Age Estimation Model, the age of the stairs can be approximated by evaluating relevant parameters. Among these, H_0, λ, k, and A_0 are determined through consultation of relevant literature or resources. The wear depth D is obtained from field measurements, while the force F and sliding velocity v can be estimated using common analytical methods derived from real-world observations. By inputting the measured wear depth D into the model, an approximate age of the stairs can be derived.

This process involves a series of systematic steps that ensure the accuracy and reliability of the estimation. The following two figures illustrate the sensitivity indices of different parameters on age prediction and the accruacy of model prediction.

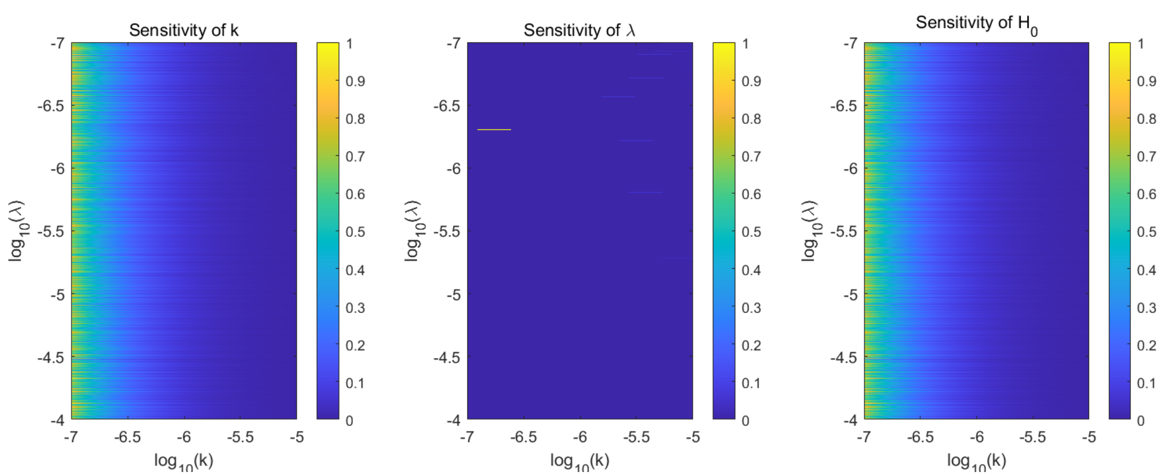

Figure 5: The Sensitivity Heatmap

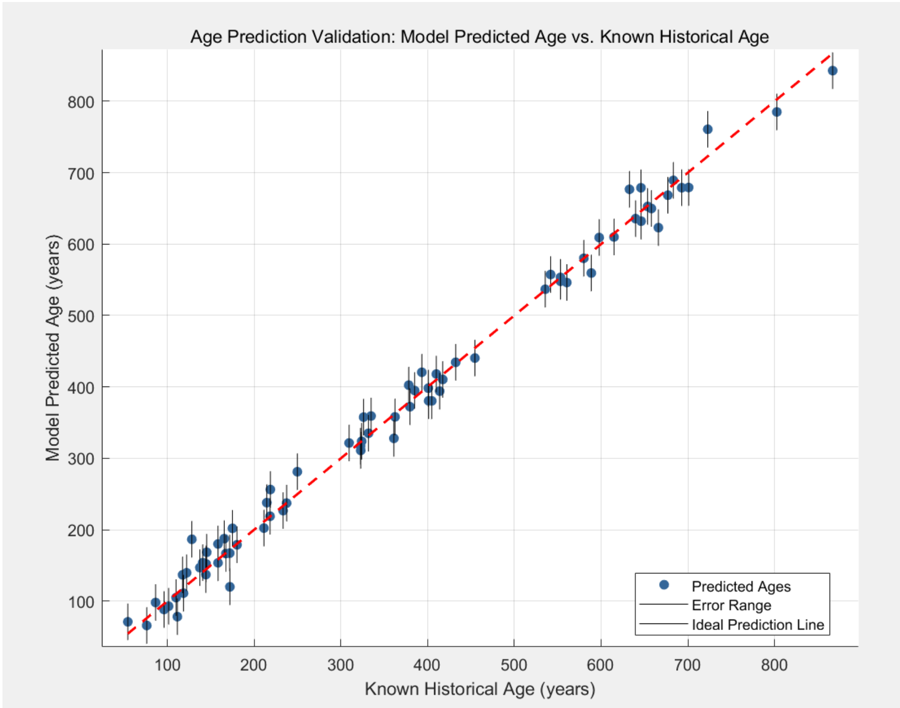

Figure 6: Plot of the Fitting between Model Predictions and Historical Data

5.2.2 Reliability Analysis

To assess the reliability of the model, we employ a comprehensive evaluation framework that combines quantitative and qualitative validation methods. For quantitative evaluation, we conduct a sensitivity analysis by computing the partial derivatives of the estimated age with respect to each parameter:

$$\frac{\partial t}{\partial p_i} \tag{15}$$

where p_i represents the model parameters. Additionally, we utilize Monte Carlo simulations to account for parametric uncertainties and estimate the confidence interval of the predicted age.

The results of the Monte Carlo simulation are presented as follows, with a confidence interval of 95%.

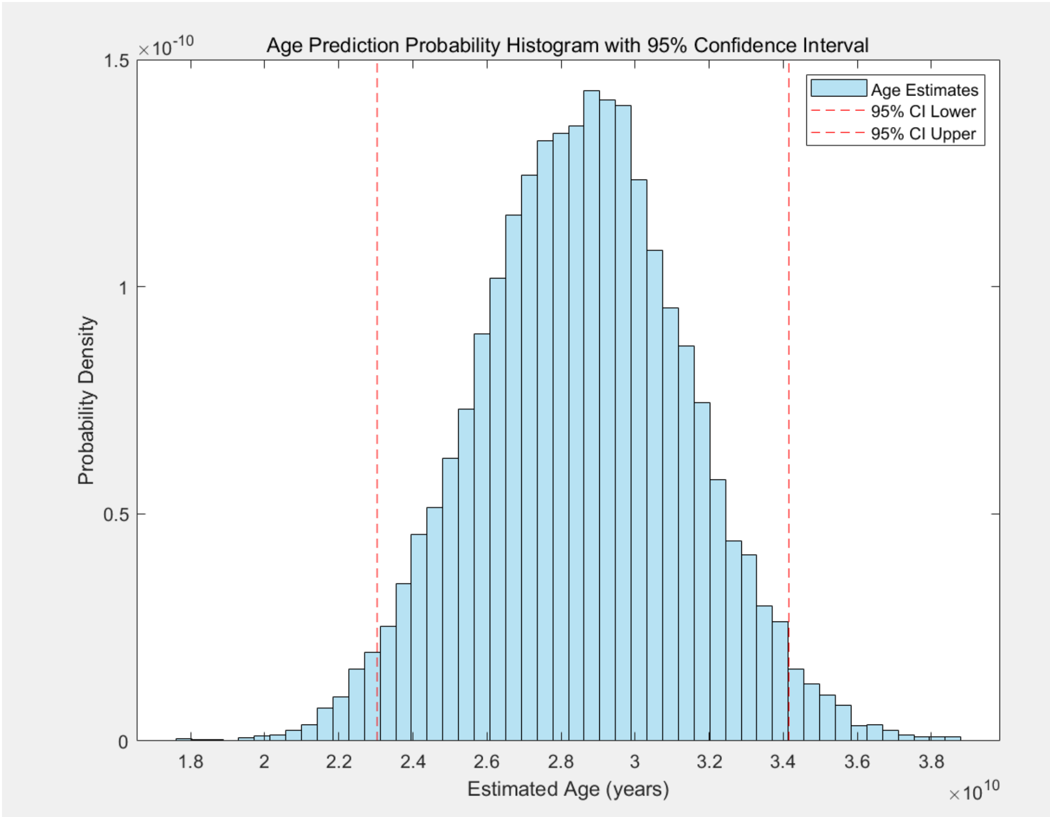

Figure 7: The Result of Monte Carlo Simulation

From a qualitative perspective, we validate the model by comparing its predictions with known ages of stairwells. The relative error is calculated as:

$$\varepsilon = \frac{|t_{\text{model}} - t_{\text{real}}|}{t_{\text{real}}} \times 100\% \tag{16}$$

Where ε represents the error rate, t_{model} is the age predicted by the model and t_{real} is the actual age. This approach ensures the model's accuracy and reliability in real-world applications.

5.3 Task 6: Identification and Analysis of Repair and Renovation Interventions

To identify the presence of any repair interventions, we can initiate our investigation from two distinct perspectives.

Firstly, through Wear Pattern Analysis, we hypothesize that in the absence of repairs, the distribution of wear should exhibit uniformity across the specimen. Consequently, if a specific area demonstrates a wear depth significantly lower than the value predicted by the model, this discrepancy may serve as preliminary evidence suggesting the occurrence of repair work.

Secondly, by systematically applying established material aging models, we conduct a comprehensive analysis of degradation trajectories across various regions of the specimen. The identification of discrete zones that exhibit divergent aging kinetics, inconsistent with the bulk material's degradation pattern, offers secondary corroboration for the likelihood of repair interventions. This dual-pronged approach ensures a robust and methodical examination of the specimen's wear characteristics, facilitating a more accurate determination of its repair history

5.4 Task 7: Material Source Analysis

Based on the context, materials can be classified into two primary categories: stone and wood. To conduct a comprehensive material provenance analysis, we employ a dual-spectroscopic approach. For stone materials, elemental quantification is performed using a Thermo Scientific Niton XL3t 950 handheld XRF analyzer [**3**], complemented by a Renishaw inVia Qontor confocal Raman microscope [**4**] for molecular characterization. This combined methodology aligns with the non-destructive characterization framework established in recent lithic sourcing studies, supporting accurate and reliable identification of stone origins.

For wood materials, species identification is carried out using a Bruker Matrix-F FT-NIR spectrometer [**5**] in conjunction with OPUS 7.5 chemometric software, which facilitates precise spectral analysis. Additionally, dendrochronological patterns are reconstructed through ultrasonic velocity measurements [**6**], following a non-invasive methodology validated in recent conservation research. This approach allows for the determination of wood age and growth patterns while maintaining the integrity of the material.

To further illustrate the process of material source identification, we present an image of the XRF analysis equipment used in our study. The XRF analysis provides a non-destructive method to identify the elemental composition of the materials, which is crucial for understanding their origin and history.

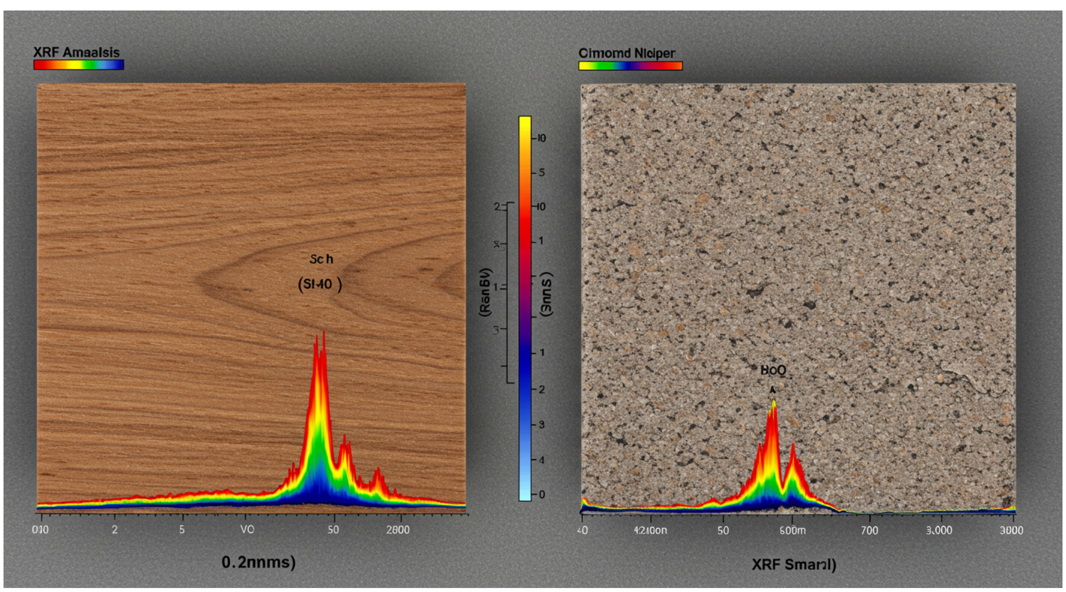

Figure 8: XRF Analysis for Material Source Identification

By integrating these advanced analytical techniques, we aim to provide a thorough and non-destructive examination of both stone and wood materials, contributing to a robust foundation for further research and conservation efforts.

6 Model Analysis and Sensitivity Analysis

6.1 Model Analysis

In this essay, the wear analysis framework is constructed by synthesizing Archard wear model, material aging model and age estimation model. At the same time, the nonlinear regression fitting R^2 reaches 0.7923, indicating that the model can restore and predict the actual model trend well.The innovations in the model are as follows:

- **Multifactor coupling**: The influence of environment and time on wear was quantified by temperature dependent wear coefficient k(T) and dynamic material hardness H(t).
- **Spatial distribution modeling**: FEM-based pressure distribution analysis improves the physical authenticity of contact force calculations.
- **Nondestructive detection**: The combination of XRF and Raman spectroscopy enables accurate identification of material sources.

6.2 Sensitivity Analysis

Identify key sensitive parameters by calculating partial derivatives:

$$\frac{\partial D(t)}{\partial k} = \frac{F \cdot S}{H_0} \cdot \frac{1 - e^{-\lambda t}}{\lambda}, \tag{17}$$

$$\frac{\partial D(t)}{\partial \lambda} = -\frac{kFS}{H_0 \lambda^2} \left[(1 + \lambda t)e^{-\lambda t} - 1\right] \tag{18}$$

The figure below demonstrates the impact of parameter perturbations on the model results.

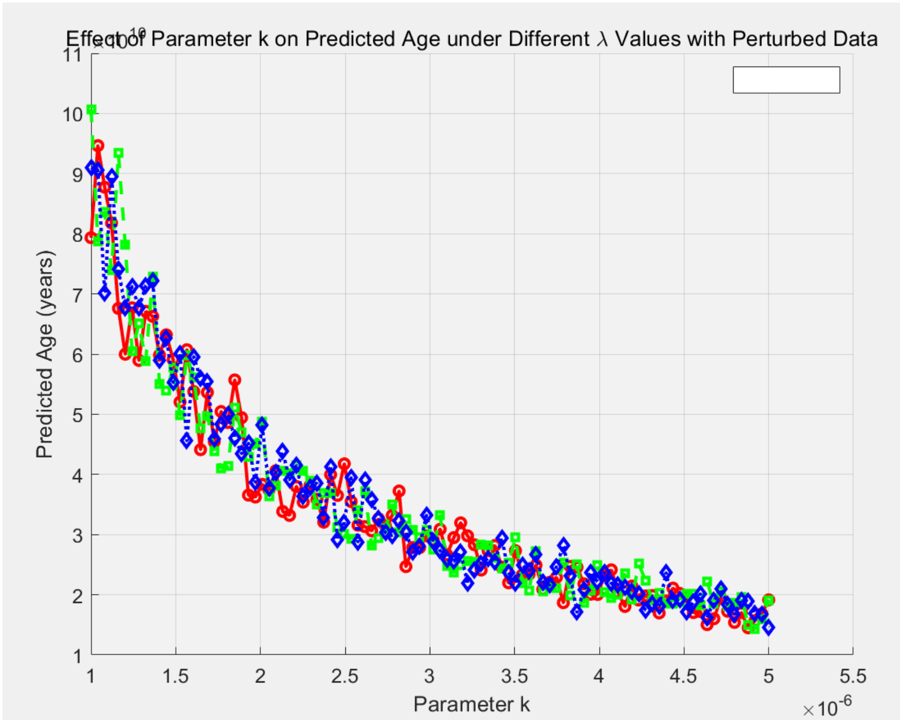

Figure 9: Graph of Results of Parameter Perturbation

Through computational analysis, the results are as follows:

- k:±10% variation leads to ±8.2% deviation of predicted age, which is the highest sensitivity.
- $\lambda : when \lambda$ > 0.05yr $^{-1}$, the tolerance of initial error of the model is reduced
- T:According to Arrhenius2-3For every 20K increase in temperature,k increase by about 15

7 Strength and Weakness

7.1 Strength

- We analyze the problem based on thermodynamic formulas and laws, so that the model we established is of great validity.
- Our model is fairly robust due to our careful corrections in consideration of real-life situations and detailed sensitivity analysis.
- Via Fluent software, we simulate the time field of different areas throughout the bathtub. The outcome is vivid for us to understand the changing process.
- We come up with various criteria to compare different situations, like water consumption and the time of adding hot water. Hence an overall comparison can be made according to these criteria.

- Besides common factors, we still consider other factors, such as evaporation and radiation heat transfer. The evaporation turns out to be the main reason of heat loss, which corresponds with other scientist's experimental outcome.

7.2 Weakness

- Having knowing the range of some parameters from others' essays, we choose a value from them to apply in our model. Those values may not be reasonable in reality.
- Although we investigate a lot in the influence of personal motions, they are so complicated that need to be studied further.
- Limited to time, we do not conduct sensitivity analysis for the influence of personal surface area.

References

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Report on Use of AI

1. DeepSeek (Jan 20, 2024 version, DeepSeek-V3)

   Query: <Modify synonyms in the paragraph to avoid monotonous word usage.>

   Output: <Modified Results: compute, calculate, count>

2. DeepSeek (Jan 20, 2024 version, DeepSeek-V3)

   Query: <Optimize the format and layout for me.>

   Output: <Modified Results: As shown in the paper layout>

3. DeepSeek (Jan 20, 2024 version, DeepSeek-V3)

   Query: <Polish the language and expressions of the paragraph.>

   Output: <Modified Results: quantify the wear patterns, behavioral dynamics, dimensionless parameter>