General

Topology

William Day

Jorge Guzman

Luke Jeevanayagam

GEORGIA GWINNETT COLLEGE

Contents

Chapter 1

Motivation & Course Goal

Higher mathematics concerns itself with similarity and sameness: what it takes for mathematical objects or structures to be considered alike or identical. From our coursework in Foundations, we know that two sets are identical if and only if they have precisely the same elements. But sets can also be compared in terms of size, or *cardinality*. Given two sets A and B, a surjection from A to B tells us that $|A| \geq |B|$, an injection from A to B tells us that $|A| \leq |B|$, and a bijection from A to B tells us that $|A| = |B|$.[1] Thus, in the study of set theory, the *objects* of study are sets and the *maps* (functions) relating them tell us whether sets are alike or identical with respect to size.

Now consider linear algebra. The main *objects* in linear algebra are vector spaces, and the *maps* relating them are linear transformations. Notice that vector spaces are more than just ordinary sets; they are sets with *structure*, as specified by the vector space axioms. The axioms tell us how the vectors behave with respect to the operations of addition $(+)$ and scalar multiplication $(\cdot)$. The maps are *structure-preserving*, in the following sense: Given a linear transformation $L : V \to V'$, we have

$$\begin{aligned} L(\vec{v} + \vec{w}) &= L(\vec{v}) + L(\vec{w}) \text{ and} \\ L(\alpha\vec{v}) &= \alpha L(\vec{v}) \end{aligned}$$

[1] Draw bubble diagrams of functions with arrows to convince yourself of this.

for all $\vec{v}, \vec{w} \in V$ and $\alpha \in \mathbb{R}$. Such a map guarantees that the vector spaces V and V' are structurally similar. If the map is also bijective, then the spaces are structurally identical. Put simply, a vector space is more than just a random set of vectors. It is inseparable from the axioms that govern operations on its vectors. In fact, we wrap up the notion of a vector space by writing it as a triple $(V, +, \cdot)$.

If we have taken a semester of abstract algebra, then we know that the main *objects* in group theory are groups, and the *maps* relating them are homomorphisms and isomorphisms. Thus groups too are more than just ordinary sets; they are sets with *structure*, as specified by the group axioms. The axioms tell us how the group elements behave with respect to the group operation $(*)$. The maps are *structure-preserving*, in the following sense: Given a group homomorphism $\varphi : G \to G'$, we have

$$\varphi(x * y) = \varphi(x) * \varphi(y)$$

for all $x, y \in G$. A homomorphism guarantees that the groups G and G' are structurally similar. A bijective homomorphism, or isomorphism, guarantees that the spaces are structurally identical. Hence a group is inseparable from the axioms that govern the operation on its elements. We wrap up the notion of a group by writing it as a pair $(G, *)$.

Topology, too, is concerned with similarity and sameness, but in a different respect. Roughly, topology asks which properties of a geometric object are preserved under *continuous deformations* such as stretching, twisting, bending, and the like. A *topological space* is a set endowed with a set of privileged subsets called *open sets*. Given a set X and a set $\mathcal{T}$ of open sets, we denote the topological space as $(X, \mathcal{T})$. If Y is another set and $\mathcal{S}$ is its set of open sets, then $(Y, \mathcal{S})$ is a topological space. Like abstract algebra, topology has its distinctive maps. From our calculus courses, we are already very familiar with the continuous map, which shows that two topological spaces are structurally similar. The map called a *homeomorphism* shows that they are structurally identical. Though the elements of two topological spaces may bear little resemblance to one another, a homeomorphism between them shows that they are topologically indistinguishable.

The entire purpose of this course is to prove that two topological spaces are structurally identical. Specifically, we will prove that the middle-thirds Cantor topological space, endowed with the subspace topology, is homeomorphic to the countable product of the set $\{0, 1\}$, endowed with the discrete topology. Over the course of the semester, it will become clear what is meant by all of this terminology. The homeomorphism we prove will provide a surprising and beautiful example of how two seemingly unlike spaces are *topologically* identical.

Chapter 2

Set & Function

Definition 2.1: Let A and B be subsets of a set X. We define the following:

1. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
2. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
3. $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

Theorem 2.2: Let A and B be subsets of a set X. Then

1. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$
2. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof. 1. Let A and B be subsets of a set X. We need to show that $X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$ and $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$. First, let's show that $X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$. Let $y \in X \setminus (A \cup B)$. This means $y \in X$ and $y \notin (A \cup B)$. the fact that $y \notin (A \cup B)$ implies $y \notin A$ and $y \notin B$. Then, let $y \in (X \setminus A) \cap (X \setminus B)$. If $y \in (X \setminus A)$, then $y \in X$ and $y \notin A$. If

$y \in (X \setminus B)$, then $y \in X$ and $y \notin B$. Since $y \in (X \setminus A) \cap (X \setminus B)$, we have $y \in X$ and $y \notin A$ and $y \notin B$.

Now, we will show that $(X \setminus A) \cap (X \setminus B) \subseteq X \setminus (A \cup B)$. Let $k \in (X \setminus A) \cap (X \setminus B)$. If $k \in (X \setminus A)$, $k \in X$ and $k \notin A$. If $k \in (X \setminus B)$, then $k \in X$ and $k \notin B$. Since $X \setminus (A \cup B) \subseteq (X \setminus A) \cap (X \setminus B)$ and $X \setminus A) \cap (X \setminus B)) \subseteq (X \setminus (A \cup B)$, it follows that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

2. Let A and B be subsets of a set X. We need to show that $X \setminus (A \cap B) \subseteq (X \setminus A) \cup (X \setminus B)$ and $(X \setminus A) \cup (X \setminus B) \subseteq X \setminus (A \cup B)$. We will first show that $X \setminus (A \cap B) \subseteq (X \setminus A) \cup (X \setminus B)$. Let $y \in (X \setminus A) \cup (X \setminus B)$. Then $y \in X$ and $y \notin A$ OR $y \in X$ and $y \notin B$. Then we will show that $X \setminus A) \cup (X \setminus B) \subseteq (X \setminus (A \cap B)$. Let $k \in ((X \setminus A)) \cup (X \setminus B)$. If $k \in (X \setminus A)$, then $k \in X$ and $k \notin A$. If $k \in (X \setminus B)$, then $k \in X$ and $k \notin B$. Since $k \in (X \setminus A) \cup (X \setminus B)$, it follows that $k \in X$ for each circumstance and either $k \notin A$ and $k \in B$ OR $k \in A$ and $k \notin B$. Since $X \setminus (A \cap B) \subseteq (X \setminus A) \cup (X \setminus B)$ and $(X \setminus A) \cup (X \setminus B) \subseteq (X \setminus (A \cup B))$, we have

   $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

$\square$

Definition 2.3: Let Λ be an indexing set. For each $\alpha \in \Lambda$, let A_α be a set. We define the following:

1. $\bigcup_{\alpha \in \Lambda} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Lambda\}$
2. $\bigcap_{\alpha \in \Lambda} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in \Lambda\}$

Theorem 2.4: Let $\{A_\alpha \mid \alpha \in \Lambda\}$ be a collection of subsets of a universal set U. Then

1. $\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right)^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c$
2. $\left(\bigcap_{\alpha \in \Lambda} A_\alpha\right)^c = \bigcup_{\alpha \in \Lambda} A_\alpha^c$

Proof. 1. To prove this set equality, we can choose an element $x \in \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c$ and show that it must also be in the intersection of the complement and vice versa.

Let x be an arbitrary element, where $x \in \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c$. Because x is defined as being in the complement, it is also true that $x \notin \bigcup_{\alpha\in\Lambda} A_\alpha$. This means that $x \notin A_\alpha$ for all $\alpha \in \Lambda$. Therefore, we can say that $x \in A_\alpha^c$ for all $\alpha \in \Lambda$, or $x \in \bigcap_{\alpha\in\Lambda} A_\alpha^c$. Thus, we have that $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c \subseteq \bigcap_{\alpha\in\Lambda} A_\alpha^c$.

Next, if we take a different arbitrary element $y \in \bigcap_{\alpha\in\Lambda} A_\alpha^c$, we have that $y \in A_\alpha^c$ for all $\alpha \in \Lambda$. This means that $y \notin A_\alpha$ for all $\alpha \in \Lambda$. Because our element y is not found in all sets, therefore we have $y \notin \bigcup_{\alpha\in\Lambda} A_\alpha$. This means that $y \in \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c$, and therefore $\bigcap_{\alpha\in\Lambda} A_\alpha^c \subseteq \left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c$.

Thus, we have proven that $\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c = \bigcap_{\alpha\in\Lambda} A_\alpha^c$.

2. To show this set equality, let x be an arbitrary element, where $x \in \left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c$. Because x is defined as being in the complement, it is also true that $x \notin \bigcap_{\alpha\in\Lambda} A_\alpha$. This means that $x \notin A_\alpha$ for some $\alpha \in \Lambda$. Therefore we can say that $x \in A_\alpha^c$ for some $\alpha \in \Lambda$, or $x \in \bigcup_{\alpha\in\Lambda} A_\alpha^c$. Thus we have that $\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha^c$. Next, if we take a different arbitrary element $z \in \bigcup_{\alpha\in\Lambda} A_\alpha^c$, we have that $z \in A_\alpha^c$ for some $\alpha \in \Lambda$. This means that $z \notin A_\alpha$ for some $\alpha \in \Lambda$ because z is in the complement. Because our element z is not found in one of the sets, it cannot be in all of them, therefore we have $z \notin \bigcap_{\alpha\in\Lambda} A_\alpha$. This means that $z \in \left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c$, and therefore $\bigcup_{\alpha\in\Lambda} A_\alpha^c \subseteq \left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c$. Thus we have proven that $\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c = \bigcup_{\alpha\in\Lambda} A_\alpha^c$.

$\square$

Definition 2.5: Let X and Y be sets and let $f \subseteq X \times Y$.

1. Then f is a *function* from X to Y if, for each $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in f$. In this case we write $f : X \to Y$, and $y = f(x)$.

2. With f defined as above, let $A \subseteq X$ and $B \subseteq Y$. Then the *image* of A under f is the set

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

The *inverse image* of B under f is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Theorem 2.6: Let $f : X \to Y$ be a function, and let B and C be subsets of Y. Then

1. $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$
2. $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$
3. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$
4. $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$

Proof. 1. Let $f : X \to Y$ be a function, and let B and C be subsets of Y. Then let $j \in f^{-1}(B \cup C)$. Since $f^{-1}(B \cup C)$, we know $f(j) \in (B \cup C)$. Thus $f(j) \in B$ or $f(j) \in C$. So $j \in f^{-1}(B)$ or $j \in f^{-1}(C)$. So $j \in f^{-1}(B) \cup f^{-1}(C)$. Therefore, $f^{-1}(B \cup C) \subseteq f^{-1}(B) \cup f^{-1}(C)$.

Conversely, let $j \in f^{-1}(B) \cup f^{-1}(C)$. So $j \in f^{-1}(B)$ or $j \in f^{-1}(C)$ which implies $f(j) \in B$ or $f(j) \in C$. So $f(j) \in (B \cup C)$. Thus $j \in f^{-1}(B \cup C)$. Therefore, $f^{-1}(B) \cup f^{-1}(C) \subseteq f^{-1}(B \cup C)$. Since $f^{-1}(B \cup C) \subseteq f^{-1}(B) \cup f^{-1}(C)$ and $f^{-1}(B) \cup f^{-1}(C) \subseteq f^{-1}(B \cup C)$, we have $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$.

2. We have

$$\begin{aligned} x \in f^{-1}(B \cap C) &\Longleftrightarrow f(x) \in B \cap C \\ &\Longleftrightarrow f(x) \in B \text{ and } f(x) \in C \\ &\Longleftrightarrow x \in f^{-1}(B) \text{ and } x \in f^{-1}(C) \\ &\Longleftrightarrow x \in f^{-1}(B) \cap f^{-1}(C). \end{aligned}$$

   The equality follows.

3. We have

$$\begin{aligned} x \in f^{-1}(Y \setminus C) &\Longleftrightarrow f(x) \in Y \setminus C \\ &\Longleftrightarrow f(x) \in Y \text{ and } f(x) \notin C \\ &\Longleftrightarrow x \in f^{-1}(Y) \text{ and } x \notin f^{-1}(C) \\ &\Longleftrightarrow x \in X \text{ and } x \notin f^{-1}(C) \\ &\Longleftrightarrow x \in X \setminus f^{-1}(C). \end{aligned}$$

   The equality follows.

4. We have

$$\begin{aligned} x \in f^{-1}(B \setminus C) &\Longleftrightarrow f(x) \in B \setminus C \\ &\Longleftrightarrow f(x) \in B \text{ and } f(x) \notin C \\ &\Longleftrightarrow x \in f^{-1}(B) \text{ and } x \notin f^{-1}(C) \\ &\Longleftrightarrow x \in f^{-1}(B) \setminus f^{-1}(C). \end{aligned}$$

   The equality follows.

$\square$

Theorem 2.7: Let $f : X \to Y$ be a function, and let A_1 and A_2 be subsets of X. Then

1. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

3. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$, but the reverse inclusion fails
4. $f(A_1) \setminus f(A_2) \subseteq f(A_1 \setminus A_2)$, but the reverse inclusion fails

Proof. Assume the hypotheses.

1. Assume that $A_1 \subseteq A_2$. Let $f(x) \in f(A_1)$ for some $x \in A_1$. Since $A_1 \subseteq A_2$, we have $x \in A_2$, and thus $f(x) \in f(A_2)$. We conclude that $f(A_1) \subseteq f(A_2)$.

2. Let $f(x) \in f(A_1 \cup A_2)$ for some $x \in A_1 \cup A_2$. Then $x \in A_1$ or $x \in A_2$. Thus we have $f(x) \in f(A_1)$ or $f(x) \in f(A_2)$. This shows that $f(x) \in f(A_1) \cup f(A_2)$.

   Conversely, let $f(x) \in f(A_1) \cup f(A_2)$, and suppose, without loss of generality, that $f(x) \in f(A_1)$. Then $f(x) = f(a)$ for some $a \in A_1$. Since $A_1 \subseteq A_1 \cup A_2$, it follows that $f(x) = f(a) \in f(A_1 \cup A_2)$.

3. Let $f(x) \in f(A_1 \cap A_2)$ for some $x \in A_1 \cap A_2$. Then $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$, so we have $f(x) \in f(A_1) \cap f(A_2)$. This proves the inclusion.

   To see that equality does not hold, take $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $A_1 = [-1, 1)$ and let $A_2 = (-1, 1]$. Then $A_1 \cap A_2 = (-1, 1)$, and we have $f(A_1 \cap A_2) = [0, 1)$. But $f(A_1) = [0, 1] = f(A_2)$, and hence $f(A_1) \cap f(A_2) = [0, 1]$. Therefore, $f(A_1) \cap f(A_2) \nsubseteq f(A_1 \cap A_2)$.

4. Let $f(x) \in f(A_1) \setminus f(A_2)$ for some $x \in X$. Then $f(x) \in f(A_1)$ and $f(x) \notin f(A_2)$. Thus $f(x) = f(a)$ for some $a \in X$ such that $a \in A_1$ but $a \notin A_2$. Therefore, $f(x) \in f(A_1 \setminus A_2)$.

$\square$

Theorem 2.8: Let $f : X \to Y$ be a function. Let A be a subset of X, and let B be a subset of Y. Prove that each of the following inclusions hold, but that the reverse inclusions fail. Can you conjecture conditions on f under which any of the inclusions become equalities? If so, state and prove your conjectures.

1. $A \subseteq f^{-1}(f(A))$
2. $f(f^{-1}(B)) \subseteq B$
3. $f(X) \setminus f(A) \subseteq f(X \setminus A)$

Proof. Let $f : X \to Y$ be a function. Let A be a subset of X, and let B be a subset of Y.

1. Let $x \in A$. Then $f(x) \in f(A)$. Therefore
$$x \in f^{-1}(f(x)) \subseteq f^{-1}(f(A)).$$
2. Let $y \in f(f^{-1}(B))$. Then $y = f(x)$ for some $x \in f^{-1}(B)$. Thus $y = f(x) \in B$.
3. Let $f(x) \in f(X) \setminus f(A)$. Then $f(x) \in f(X)$ but $f(x) \notin f(A)$. Hence $x \notin A$, so $f(x) \in f(X \setminus A)$.

$\square$

Definition 2.9: Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then g *composed with* f, denoted $g \circ f$, is the function $g \circ f : X \to Z$ given by $(f \circ g)(x) = f(g(x))$.

Theorem 2.10: Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let B be a subset of Z. Then
$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)).$$

Proof. Let $x \in X$. We have

$$\begin{aligned} x \in (g \circ f)^{-1}(B) &\iff (g \circ f)(x) \in B \\ &\iff g(f(x)) \in B \\ &\iff f(x) \in g^{-1}(B) \\ &\iff x \in f^{-1}(g^{-1}(B)). \end{aligned}$$

This proves the equality. □

Definition 2.11: Let $f : A \to B$ be a function.

1. The function f is an injection iff $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$.
2. The function f is a surjection iff for each $y \in B$ there exists $x \in A$ such that $f(x) = y$.
3. The function f is a bijection iff it is both an injection and a surjection.

Theorem 2.12: Let $f : A \to B$ and $g : B \to C$ be functions.

1. If $g \circ f : A \to C$ is surjective, then g is surjective.
2. If $g \circ f : A \to C$ is injective, then f is injective.
3. If f and g are bijective, then $g \circ f : A \to C$ is bijective.

Proof.

1. Assume that $g \circ f : A \to C$ is surjective. Let $z \in C$. Since $g \circ f$ is surjective, it follows that there exists $x \in A$ such that $g(f(x)) = (g \circ f)(x) = z$. This implies that there exists $y = f(x) \in B$ such that $g(y) = z$. Thus g is surjective.
2. Assume that $g \circ f : A \to C$ is injective. Let $x, y \in A$ and assume that $x \neq y$. Then

$$g(f(x)) = (g \circ f)(x) \neq (g \circ f)(y) = g(f(y)),$$

so it must be the case that $f(x) \neq f(y)$. We infer that f is injective.

3. Assume that f and g are bijective. First we show that $g \circ f$ is injective. Let $x, y \in A$ and assume that $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Since g is injective, it follows that $f(x) = f(y)$, and since f is injective, it follows that $x = y$. Thus $g \circ f$ is injective.

   Now we show that $g \circ f$ is surjective. Let $z \in C$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. Thus there exists $x \in A$ such that

   $$(g \circ f)(x) = g(f(x)) = g(y) = z,$$

   proving that $g \circ f$ is surjective.

   Since $g \circ f$ is both injective and surjective, it follows that it is bijective. This finishes the proof.

$\square$

Exercise 2.13: For each of the following functions, determine whether it is an injection, a surjection, a bijection, or none of these.

1. $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by $h(x) = x^2$ for every $x \in \mathbb{R}$
2. $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $k(x) = x^2$ for every $x \in \mathbb{R}_{\geq 0}$

Proof.

1. We claim that h is a surjection but not an injection. Let $y \in \mathbb{R}_{\geq 0}$. Since $y \geq 0$, the root $\sqrt{y}$ is defined as an element of $\mathbb{R}$, and we have

   $$h(\sqrt{y}) = (\sqrt{y})^2 = y.$$

   This proves that h is surjective. To see that h is not injective, consider 1 and -1 in $\mathbb{R}$. We have $1 \neq -1$, but $h(-1) = 1 = h(1)$. The proof is complete.

2. The proof that k is surjective goes through as above. To see that k is injective, let $x, y \in \mathbb{R}$ with $k(x) = k(y)$. Then $x^2 = y^2$. Taking roots of both sides gives us $|x| = |y|$. Since the domain of k is $\mathbb{R}_{\geq 0}$, we drop the absolute value bars to obtain $x = y$. The proof is complete.

$\square$

Chapter 3

Topological Space

Topology is the study of the properties of geometric objects that are invariant under the maps we call *homeomorphisms*. These maps are sometimes called "reversibly continuous" because they preserve topological properties in both directions: from domain to codomain and back. To a lesser extent, ordinary continuous functions also preserve topological properties.

Thus we have our objects, which are topological spaces, and our structure-preserving maps, which are homeomorphisms and continuous functions. But what structure is preserved by these maps? The answer is that these maps preserve *open sets*. Before we define a topological space through axioms, we should familiarize ourselves with open sets in topological spaces that you already know well.

We often view $\mathbb{R}$ as a geometric object by visualizing it as a line. In the set $\mathbb{R}$, a standard open interval has the form (a, b), where $a, b \in \mathbb{R}$. It turns out that we can combine open intervals, through union and intersection, to form other sets considered open in the standard topology on $\mathbb{R}$.

Definition 3.1: The set $\mathcal{O} \subseteq \mathbb{R}$ is *standard open* in $\mathbb{R}$ if and only if, for each $x \in \mathcal{O}$, there exists a standard open interval (a, b) such that $x \in (a, b) \subseteq \mathcal{O}$.

Exercise 3.2: Determine whether each of the following is open in $\mathbb{R}$.

1. A finite set
2. The complement of a finite set
3. $\{x \in \mathbb{R} \mid 2x + 1 > 5\}$
4. $\{x \in \mathbb{R} \mid 2x + 1 \geq 5\}$
5. $\mathbb{Z}$
6. $\mathbb{R} \setminus \mathbb{Z}$
7. $\mathbb{R}$
8. $\varnothing$

Proof.

1. A finite set is not open in $\mathbb{R}$.
2. The complement of a finite set is open in $\mathbb{R}$.
3. $\{x \in \mathbb{R} \mid 2x + 1 > 5\}$ is open in $\mathbb{R}$ because if you try to squeeze an interval between 2 and 3, you can fit another interval between these integers.
4. $\{x \in \mathbb{R} \mid 2x + 1 \geq 5\}$ is not open in $\mathbb{R}$ because if you try to squeeze an interval between 2 and 3, you can't fit another interval between these integers.
5. $\mathbb{Z}$ is not open in $\mathbb{R}$.
6. $\mathbb{R} \setminus \mathbb{Z}$ is open in $\mathbb{R}$.
7. $\mathbb{R}$ is open in $\mathbb{R}$ because by definition, the set itself is open inside itself.
8. $\varnothing$ is open in $\mathbb{R}$.

$\square$

Exercise 3.3: Let $\mathcal{T}_S$ be the collection of standard open sets in $\mathbb{R}$. Prove the following:

1. Both $\varnothing$ and $\mathbb{R}$ belong to $\mathcal{T}_S$.
2. If A and B are in $\mathcal{T}_S$, then $A \cap B$ is in $\mathcal{T}_S$.
3. If Λ is an indexing set, and if A_α is in $\mathcal{T}_S$ for each $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is in $\mathcal{T}_S$.

Proof.

1. It is vacuously true that if $x \in \varnothing$, there exists a standard open interval (a, b) such that $x \in (a, b) \subseteq \varnothing$. Thus $\varnothing \in \mathcal{T}_S$. Then it can be shown that for each $x, a \in \mathbb{R}$ with $a \neq 0$, there exists a standard open interval $(x - a, x + a)$ such that $x \in (x - a, x + a) \subseteq \mathbb{R}$. Therefore $\mathbb{R} \in \mathcal{T}_S$.
2. Let $A, B \in \mathcal{T}_S$ and let $x \in A \cap B$. Since $x \in A$, there exists a standard open interval (a, b) with $x \in (a, b) \subseteq A$, and since $x \in B$ there is also a standard open interval (c, d) with $x \in (c, d) \subseteq B$. If we then define a set $M = max\{a, c\}$ and $m = min\{b, d\}$. Since $x \in A$ and $x \in B$, with $A, B \in \mathcal{T}_S$, there exists a standard open interval (M, m) such that $x \in (M, m) \subseteq A \cap B$, and therefore $A \cap B \in \mathcal{T}_S$.
3. Let $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$. Then $x \in A_\alpha$ for some $\alpha \in \Lambda$. Therefore there exists a standard open interval (a, b) such that $x \in (a, b) \subseteq A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$. Thus we have $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathcal{T}_S$.

$\square$

We have just defined a topology on $\mathbb{R}$, called the *standard topology* on $\mathbb{R}$. Equipped with this topology, the real line becomes a topological space, written $(\mathbb{R}, \mathcal{T}_S)$. We can form different topological spaces by changing the topology on a fixed set, or by changing both the set and the topology. For the moment, we will keep the set and change the topology.

Definition 3.4: The set $\mathcal{S} \subseteq \mathbb{R}$ is *lower-limit open* in $\mathbb{R}$ if and only if, for each $x \in \mathcal{S}$, there exists a lower-limit open interval $[a, b)$ such that $x \in [a, b) \subseteq \mathcal{S}$.

Exercise 3.5: Let $\mathcal{T}_{LL}$ be the collection of lower-limit open sets in $\mathbb{R}$. Prove the following:

1. Both $\varnothing$ and $\mathbb{R}$ belong to $\mathcal{T}_{LL}$.
2. If A and B are in $\mathcal{T}_{LL}$, then $A \cap B$ is in $\mathcal{T}_{LL}$.
3. If Λ is an indexing set, and if A_α is in $\mathcal{T}_{LL}$ for each $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is in $\mathcal{T}_{LL}$.

Proof.

1. Since $\varnothing$ has no elements, it is vacuously true that if $x \in \varnothing$, then there exists a lower-limit open interval $[a, b)$ such that $x \in [a, b) \subseteq \varnothing$. Thus $\varnothing \in \mathcal{T}_{LL}$.

   For each $x \in \mathbb{R}$, we have that $[x - 1, x + 1)$ is a lower-limit interval such that $x \in [x - 1, x + 1) \subseteq \mathbb{R}$. Thus $\mathbb{R} \in \mathcal{T}_{LL}$.
2. Let $A, B \in \mathcal{T}_{LL}$, and let $x \in A \cap B$. Since $x \in A$, there exists a lower-limit open interval $[a, b)$ with $x \in [a, b) \subseteq A$. Since $x \in B$, there exists a lower-limit open interval $[c, d)$ with $x \in [c, d) \subseteq A$. Define $M := \max\{a, c\}$ and $m := \min\{b, d\}$. Then $x \in [M, m) \subseteq A \cap B$, so we have $A \cap B \in \mathcal{T}_{LL}$.
3. Assume that Λ is an indexing set and that A_α is in $\mathcal{T}_{LL}$ for each $\alpha \in \Lambda$. Let $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$. Then $x \in A_\alpha$ for some $\alpha \in \Lambda$, so there exists an interval $[a, b)$ such that $x \in [a, b) \subseteq A_\alpha$. Hence $x \in [a, b) \subseteq A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$. Therefore, $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathcal{T}_{LL}$. The proof is complete.

□

Equipped with this topology, the real line becomes a different topological space, written $(\mathbb{R}, \mathcal{T}_{LL})$. To form the next topological space, we will change the set *and* its topology, but the result will still be familiar.

Definition 3.6: The *Euclidean plane* is the set $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$.

Definition 3.7: For each $\mathbf{p} = (a, b)$ and $\epsilon > 0$, let

$$B_\epsilon(\mathbf{p}) = \{(x, y) \mid \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\}$$

be the *open disk* of radius ϵ centered at (a, b).

Then the set $\mathcal{U} \subseteq \mathbb{R}^2$ is open in the Euclidean plane if and only if, for each $\mathbf{p} \in \mathcal{U}$, there exists a positive real number ϵ such that $B_\epsilon(\mathbf{p}) \subseteq \mathcal{U}$.

Exercise 3.8: Consider the standard topology $\mathcal{T}_S$ on $\mathbb{R}$.

1. Let $A_n = (1 - 1/n, 1 + 1/n)$ for each $n \in \mathbb{Z}^+$. What is $\bigcup_{n=1}^{\infty} A_n$? Is it open? Justify your claim, giving a proof if it's open and a counterexample if it's not.
2. What is $\bigcap_{n=1}^{\infty} A_n$? Is it open? Justify your claim, giving a proof if it's open and a counterexample if it's not.
3. Is $(-\infty, 0) \cup (1, 2] \cup \{3\}$ open? If it is, give a proof; if it's not, give a counterexample.
4. Is $\mathbb{Q}$ open? What about $\mathbb{Q}^c$? If so, give a proof; if not, give a counterexample. Fact from real analysis: Between each pair of real numbers lies a rational number.

Consider the Euclidean topology on $\mathbb{R}^2$.

1. Is $A = \{(x, y) \mid x \in (1, 2) \text{ and } y \in (2, 4)\}$ open?
2. Is $A = \{(x, y) \mid x^2 + y^2 < 1 \text{ and } y \geq 0\}$ open?
3. Is $A = \{(x, y) \mid 1 < x < 2\}$ open?

Proof. 1. We claim that $\bigcup_{n=1}^{\infty} A_n = (0, 2)$. Let $x \in (0, 2)$. Then $x \in A_1 \subseteq \bigcup_{n=1}^{\infty} A_n$. Now let $x \in \bigcup_{n=1}^{\infty} A_n$ Then $x \in A_n$ for some

$n \in \mathbb{Z}^+$, and thus $1 - 1/n < x < 1 + 1/n$ for some $n \in \mathbb{Z}^+$. But $n \geq 1$, so $0 \leq 1 - 1/n < x < 1 + 1/n \leq 2$. We conclude that $x \in (0, 2)$.

Now we will prove that the set $\bigcup_{n=1}^{\infty} A_n = (0, 2)$ is open. Let $x \in (0, 2)$, and set $\varepsilon := \min\{x, 2-x\}$. We claim that $(x-\varepsilon, x+\varepsilon) \subseteq (0, 2)$. Let $y \in (x-\varepsilon, x+\varepsilon)$. Then $y > x-\varepsilon \geq 0$. On the other hand, $y < x + \varepsilon \leq 2$. It follows that $y \in (0, 2)$, and hence the inclusion follows. We conclude that the set is open as claimed.

2. We claim that $\bigcap_{n=1}^{\infty} A_n = \{1\}$. Clearly $1 \in A_n$ for each $n \in \mathbb{Z}^+$, so one inclusion is immediate. Now let $x \in \bigcap_{n=1}^{\infty} A_n$. Then for all $n \in \mathbb{Z}^+$, we have $1 - 1/n < x < 1 + 1/n$. It is an exercise in real analysis to show that

$$\sup\{1 - 1/n \mid n \in \mathbb{Z}^+\} = 1 = \inf\{1 + 1/n \mid n \in \mathbb{Z}^+\}.$$

Since $x > 1 - 1/n$ for all $n \in \mathbb{Z}^+$, we see that x is an upper bound for the left-hand set above, and hence $x \geq 1$. On the other hand, since $x < 1 + 1/n$ for all $n \in \mathbb{Z}^+$, we see that x is a lower bound for the right-hand set above, and hence $x \leq 1$. This forces $x = 1$, clinching the equality. The claim follows.

The set $\{1\}$ is closed since no open interval containing 1 is contained in $\{1\}$.

3. Put $A := (-\infty, 0) \cup (1, 2] \cup \{3\}$. Since no open interval containing 2 is contained in A, we see that A is not open.

4. We claim that neither $\mathbb{Q}$ nor $\mathbb{Q}^c$ is open. Let $x \in \mathbb{Q}$. It is a fact established in real analysis that any open interval (a, b) contains irrational numbers, and thus there is no open interval containing x that is contained in $\mathbb{Q}$. Similarly, between any two reals is a rational, so an analogous argument shows that $\mathbb{Q}^c$ is not open.

1. Yes, for any $\mathbf{x} \in A$, there exists an $\epsilon > 0$ such that $\mathbf{x} \in B_\epsilon(\mathbf{x}) \subseteq A$.

2. No. We have $\mathbf{p} = (0, 0) \in A$, but there exists no $\epsilon > 0$ such that $\mathbf{p} \in B_\epsilon(\mathbf{p}) \subseteq A$.

3. Yes, for any $\mathbf{x} \in A$, there exists an $\epsilon > 0$ such that $\mathbf{x} \in B_\epsilon(\mathbf{x}) \subseteq A$.

$\square$

Before we greet topological spaces in their full generality, it is useful to note that different topologies defined on the same set can be equal or unequal.

Exercise 3.9: Prove or disprove that $\mathcal{T}_S = \mathcal{T}_{LL}$ on the set $\mathbb{R}$.

Proof. We claim that $\mathcal{T}_S \neq \mathcal{T}_{LL}$ on the set $\mathbb{R}$. Consider the set $[0, 1) \subseteq \mathbb{R}$. First we claim that this set is open in $\mathcal{T}_{LL}$. To see this, let $x \in [0, 1)$. Then we have $x \in [0, 1) \subseteq [0, 1)$, so the set is open in $\mathcal{T}_{LL}$ as claimed.

Now we claim that $[0, 1) \notin \mathcal{T}_S$. Consider 0. Then $0 \in [0, 1)$, but there exists no standard open interval containing 0 that is contained in $[0, 1)$. We conclude that $\mathcal{T}_S \neq \mathcal{T}_{LL}$ on the set $\mathbb{R}$. $\square$

Definition 3.10: A *topological space* $(X, \mathcal{T})$ is a set X and a collection $\mathcal{T}$ of subsets of X satisfying the following:

1. Both $\varnothing$ and X belong to $\mathcal{T}$.
2. If A and B are in $\mathcal{T}$, then $A \cap B$ is in $\mathcal{T}$.
3. If Λ is an indexing set, and if A_α is in $\mathcal{T}$ for each $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is in $\mathcal{T}$.

The members of $\mathcal{T}$ are called *open sets,* and $\mathcal{T}$ is called a *topology* on X.

The sets $\mathbb{R}$ and $\mathbb{R}^2$, though most familiar, are not the only sets on which a topology can be defined; nor are the topologies so far presented the only ones that can be defined on them. Many more interesting examples of topological spaces abound, but it is often difficult to characterize the open sets in these spaces. To do this, we will require the notion of a *basis*, which is presented in the next chapter.

Exercise 3.11: Find all possible topologies on the set

$$X = \{a, b, c\}.$$

Proof. The 29 possible topologies are $\{\varnothing, X\}$, $\{\varnothing, \{a\}, X\}$, $\{\varnothing, \{b\}, X\}$, $\{\varnothing, \{c\}, X\}$, $\{\varnothing, \{a, b\}, X\}$, $\{\varnothing, \{a, c\}, X\}$, $\{\varnothing, \{b, c\}, X\}$, $\{\varnothing, \{a\}, \{a, c\}, X\}$, $\{\varnothing, \{a\}, \{a, c\}, X\}$, $\{\varnothing, \{b\}, \{a, b\}, X\}$, $\{\varnothing, \{b\}, \{b, c\}, X\}$, $\{\varnothing, \{c\}, \{a, c\}, X\}$, $\{\varnothing, \{c\}, \{b, c\}, X\}$, $\{\varnothing, \{a\}, \{b, c\}, X\}$, $\{\varnothing, \{b\}, \{a, c\}, X\}$, $\{\varnothing, \{c\}, \{a, b\}, X\}$, $\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\{\varnothing, \{c\}, \{b, c\}, \{a, c\}, X\}$, $\{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$, $\{\varnothing, \{a\}, \{c\}, \{a, c\}, X\}$, $\{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$, $\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, $\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b\}, X\}$, $\{\varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\{\varnothing, \{b\}, \{c\}, \{b, c\}, \{a, b\}, X\}$, $\{\varnothing, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$, and $\mathcal{P}(X)$. □

Exercise 3.12: Let X be any set. Show that the power set of X is a topology on X. We call this topology the *discrete topology* on X.

Proof.

1. The topology of any set must contain both the empty set and the entire set itself, and satisfy the properties of closure under finite intersections and arbitrary unions. In the case of the power set of X, denoted $\mathcal{P}(X)$, it is first always true that the first and last element of the power set is the empty set and the entire set X itself respectively. Therefore we have that $\varnothing, X \in \mathcal{P}(X)$. Then, let $A, B \in \mathcal{P}(X)$. Therefore $A, B \subseteq X$. Now, if $A \cap B = \varnothing$, then $A \cap B \in \mathcal{P}(X)$, and the proof is done. However, if $A \cap B \neq \varnothing$, then it is still true that $A \cap B \subseteq X \subseteq \mathcal{P}(X)$. Therefore $A \cap B \in \mathcal{P}(X)$. Finally, if we have a collection $\{A_\alpha\}_{\alpha\in\Lambda}$ of open subsets of $\mathcal{P}(X)$, then they are all subsets of X, or equal to X. Therefore

$$\bigcup_{\alpha\in\Lambda} A_\alpha \in X \subseteq \mathcal{P}(X).$$

Therefore the power set $\mathcal{P}(X)$ satisfies all properties of a topological space and is hence a topology on X.

$\square$

Exercise 3.13: Let X be a set, and put

$$\mathcal{T} = \{U \subseteq X \mid U = \varnothing \text{ or } X \setminus U \text{ is finite}\}.$$

Show that $\mathcal{T}$ defines a topology on X. We call this topology the *finite complement topology*.

Proof. 1. By definition, we notice that $\varnothing \in \tau$. Now, to prove that $X \in \tau$, we need to show that $X \setminus X$ is finite. We know that $X \setminus X = \varnothing$, and since $\varnothing$ is finite, it follows that $X \in \tau$.

2. We now need to prove that $A \cap B \in \tau$. Let A and B be elements of τ. We know that,

$$X \setminus A \cap B = (X \setminus A) \cup (X \setminus B)$$

And since $(X \setminus A)$ and $(X \setminus B)$ are finite, $(X \setminus A) \cup (X \setminus B)$ is also finite. Thus $A \cap B \in \tau$.

3. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in the finite complement topology τ on X. We must show that $\bigcup_{\alpha\in\Lambda} A_\alpha \in \tau$. By De Morgan's Law, we have

$$\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right)^c = \bigcap_{\alpha\in\Lambda} (A_\alpha)^c$$

Since $A_\alpha \in \tau$ for each $\alpha \in \Lambda$, it follows that $(A_\alpha)^c$ is finite for each $\alpha \in \Lambda$. Since the intersection of these sets is contained in each set, let $\alpha_0 \in \Lambda$ such that $A_{\alpha_0} \neq \varnothing$. It follows that $\bigcap_{\alpha\in\Lambda}(A_\alpha)^c \subset (A_{\alpha_0})^c$. Therefore, $(\bigcup_{\alpha\in\Lambda} A_\alpha)^c$ is finite, and $\bigcup_{\alpha\in\Lambda} A_\alpha \in \tau$. $\square$

Definition 3.14: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then A is *closed* if and only if $X \setminus A$ is open.

Exercise 3.15: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$.

1. Prove or find a counterexample: A must be either open or closed.
2. Prove or find a counterexample: A cannot be both open and closed.

Proof.

1. The claim that A must be either open or closed is false. Counterexample: Consider the Euclidean topology on $\mathbb{R}^2$. Let $A = \{(x, y) \mid 4 < x \leq 7\}$. Then A does not fit the definition of open or closed. Thus, A is neither open nor closed.
2. The claim that A cannot be both open and closed is false. Counterexample: The empty set $\varnothing$ is both open and closed. On one hand, the empty set is open since it is an element of every topology. On the other, it is closed since its complement $X \setminus \varnothing = X$ is open in every topology.

$\square$

Theorem 3.16: Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X.

1. If A and B are closed, then $A \cup B$ is closed.
2. If Λ is an indexing set, and A_α is a closed subset of X for each $\alpha \in \Lambda$, then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is closed.

Proof.

1. Let A and B be closed subsets of X, and let $\mathcal{T}$ be a topology on X. Given the definition of open and closed sets, if A and B are closed, then we have that $X \setminus A$ and $X \setminus B$ are open. Therefore,

$$(X \setminus A) \cap (X \setminus B)$$

is open. Applying DeMorgan's Law, we have that

$$X \setminus (A \cup B)$$

is open. Therefore $A \cup B$ is closed.

2. Let Λ be an indexing set, and let A_α be a closed subset of X for each $\alpha \in \Lambda$. To show that $\bigcap_{\alpha\in\Lambda} A_\alpha$ is closed, it suffices to show that $\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c$ is open. By Demorgan's Law we have that

$$\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right)^c = \bigcup_{\alpha\in\Lambda} A_\alpha^c.$$

Because A_α is a closed subset of X, we know that $X \setminus A_\alpha$ is open. Meaning that A_α^c is open. Therefore A_α^c is an open subset of X for some $\alpha \in \Lambda$. Hence $\bigcup_{\alpha\in\Lambda} A_\alpha^c$ is open, and therefore $\bigcap_{\alpha\in\Lambda} A_\alpha$ is closed.

□

Theorem 3.17: For any topological space $(X, \mathcal{T})$, the sets $\varnothing$ and X are open.

Proof. Let $(X, \mathcal{T})$ be a topological space. By the axioms for a topological space, we have $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$. Therefore $\emptyset$ and X are open. □

Theorem 3.18: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then A is open if and only if, for each $x \in A$, there is an open set U_x containing x such that $U_x \subseteq A$.

Proof. Assume A is open. We must show that for each $x \in A$ there is an open set U_x containing x such that $U_x \subseteq A$. For each $x \in A$, set $U_x = A$. Then $x \in A \subseteq A$.

Now, let $x \in A$. By assumption, we have $x \in U_x \subseteq A$ for some open set U_x containing x. Then, $A \subseteq \bigcup_{x\in A} U_x$. Since $U_x \subseteq A$ for each $x \in A$, it is immediate that $\bigcup_{x \in A} U_x \subseteq A$. This forces $A = \bigcup_{x \in A} U_x$ so that A is a union of open sets. Hence A is open. $\square$

Definition 3.19: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The *interior* of A, denoted A°, is the union of all open subsets contained in A.

Theorem 3.20: Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X. The following hold:

1. $X^\circ = X$
2. $(A^\circ)^\circ = A^\circ$
3. $(A \cap B)^\circ = A^\circ \cap B^\circ$
4. $A^\circ = A$ if and only if A is open.

Proof. Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X.

1. X is open by definition since $X \in \mathcal{T}$ and $\mathcal{T}$ is composed of open subsets. It is also clear that $X^\circ \subseteq X$ because the union of elements of open sets in X is in X. Then, since X is open, it is clear that X is one of these open sets contained in the union of open sets, thus $X \subseteq X^\circ$, and hence $X^\circ = X$.

2. We claim that for any $B \subseteq X$, we have $B = B^\circ$ if and only if B is open. Suppose $B = B^\circ$, and let $x \in B$. Since $x \in B^\circ$, there exists an open set U such that $x \in U \subseteq B$. Hence B is a union of open sets, and it is open. Now suppose B is open, and let $x \in B$. Then there exists an open set U such that $x \in U \subseteq B$. By definition, this means that $x \in B^\circ$. This establishes the claim.

   Now simply let $B = A^\circ$. Since B is a union of open sets, it is open. By the above argument, it follows that $B = B^\circ$, so that $(A^\circ)^\circ = A^\circ$.

3. Let $x \in (A \cap B)^\circ$. Then x is in the union of all open subsets contained in $A \cap B$. Therefore we have that $x \in A \cap B$, meaning $x \in A$ and $x \in B$. Since x is specifically in an open subset in the intersection of both A and B, we have that $x \in A^\circ \cap B^\circ$. Therefore $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. Now let $y \in A^\circ \cap B^\circ$. Therefore y is in the union of all open subsets contained in A, and y is in the union of all open subsets contained in B. Since $A^\circ \subseteq A$ and $B^\circ \subseteq B$, we also have that $y \in A \cap B$. Then because y is only found within open subsets contained in A and B, we have that $y \in (A \cap B)^\circ$. Therefore $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. Hence, $(A \cap B)^\circ = A^\circ \cap B^\circ$.

4. If $A^\circ = A$, then for any $x \in A^\circ$, $x \in A$ and vice versa. Since $x \in A^\circ$, there is an open set U_x such that $x \in U_x \subseteq A^\circ = A$. So any open subset U of A° is also a subset of A. Therefore A is open. Now if A is open, then there exists an open set U_x for all $x \in A$. Since A° is the union of all open subsets of A, A is one of these sets and hence $A \subseteq A^\circ$.

$\square$

Exercise 3.21: Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X. State and prove a conjecture about $(A \cup B)^\circ$.

Proof. Conjecture: We assert that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ but that the reverse inclusion does not hold.

Let $x \in A^\circ \cup B^\circ$, and without loss of generality, assume that $x \in A^\circ$. Then there exists an open set U such that $x \in U \subseteq A \subseteq A \cup B$. This proves that $x \in (A \cup B)^\circ$.

To see that the reverse inclusion does not hold, let $X = \mathbb{R}$, let $A = [0, 1/2]$, and let $B = [1/2, 1]$. Then

$$(A \cup B)^\circ = [0, 1]^\circ = (0, 1),$$

but

$$A^\circ \cup B^\circ = [0, 1/2]^\circ \cup [1/2, 1]^\circ = (0, 1/2) \cup (1/2, 1).$$

Hence $(A \cup B)^\circ \nsubseteq A^\circ \cup B^\circ$. $\square$

Definition 3.22: Let $(X, \mathcal{T})$ be a topological space, let $x \in X$, and let $A \subseteq X$. We call x a *boundary point* of A if every open set containing x has nonempty intersection with both A and $X \setminus A$. The set of all boundary points of A is called the *boundary* of A, and is denoted $\partial(A)$.

Theorem 3.23: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\partial(A) = \partial(X \setminus A)$.

Proof. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Let $x \in X$. Then $x \in \partial(A)$ if and only if every open set containing x contains a point of A and a point of A^c. This is true if and only if every open set containing x contains a point of A^c and a point of A, which in turn is true if and only if $x \in \partial(X \setminus A)$. The proof is complete. $\square$

Theorem 3.24: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then the sets A° and $\partial(A)$ are mutually disjoint.

Proof. Assume for the sake of contradiction that $A^\circ \cap \partial(A) \neq \varnothing$. Let $x \in A^\circ \cap \partial(A)$. By the definition of the interior of A, there exists an open set U such that $x \in U$ and $U \subseteq A$. By the definition of boundary of A, it follows that both $U \cap A$ and $U \cap (X \setminus A)$ are nonempty. The assertion that $U \cap (X \setminus A)$ is nonempty contradicts that $U \subseteq A$. Therefore, the sets are mutually disjoint as claimed. $\square$

Theorem 3.25: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\partial(A)$ is closed.

Proof. To show that $\partial(A)$ is closed, we can show that $\partial(A)^c$ is open. If we take $y \in \partial(A)^c$, then some open set U_y will be such that either

$$y \in U_y \subseteq A \text{ or } y \in U_y \subseteq A^c.$$

In either case, there exists an open set U_y such that $y \in U_y \subseteq \partial(A)^c$. Thus $\partial(A)^c$ is open, and therefore $\partial(A)$ is closed. □

Theorem 3.26: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then A is closed if and only if $\partial(A) \subseteq A$.

Proof. Assume that A is closed, and let $x \in \partial(A)$. If $x \in A^c$, then A^c is an open set containing x that does not intersect A, contradicting that $x \in \partial(A)$. It follows that $\partial(A) \subseteq A$.

Assume that $\partial(A) \subseteq A$, and let $x \in A^c$. Then $x \notin \partial(A)$, so there exists an open set U containing x which does not intersect A. We have $x \in U \subseteq A^c$, proving that A^c is open. Therefore, A is closed. □

Definition 3.27: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. The *closure* of A, denoted $\overline{A}$, is the intersection of all closed sets containing A.

Theorem 3.28: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then A is closed if and only if $\overline{A} = A$.

Proof. Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Assume that A is closed. By definition of closure, we know that $A \subseteq \overline{A}$. Then, since A is closed and $A \subseteq A$, it follows that $\overline{A} \subseteq A$. Conversely, assume that $\overline{A} = A$. We know that in a topological space, the intersection of arbitrarily many closed sets is closed, so $A = \overline{A}$ is closed. □

Theorem 3.29: Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $x \in X$. Then $x \in \overline{A}$ if and only if every open set containing x intersects A.

Proof. Let's assume $x \in \overline{A}$, and suppose for the sake of contradiction that there is some open set U containing x that does not intersect A. Then, by Theorem 1.16, we have that $(X \setminus U)$ is a closed set, and must contain A. Since $\overline{A}$ is the intersection of all closed sets in X containing A, then $\overline{A} \subset (X \setminus U)$. Therefore, $x \in (X \setminus U)$. But this is a contradiction since we assumed $x \in U$.

Now, we will assume every open set U containing x intersects A, but $x \notin \overline{A}$. Then, there exists a closed set C with $A \subseteq C$ and with $x \notin C$. But then $X \setminus C$ is an open set containing x that does not intersect A, contradicting our assumption. We conclude that $x \in \overline{A}$.

Hence, $x \in \overline{A}$ if and only if every open set containing x intersects A.

□

Theorem 3.30: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\overline{A} = A^\circ \cup \partial(A)$.

Proof. Let $x \in A^\circ \cup \partial(A)$. Assume first that $x \in A^\circ$. By definition of *interior*, we have $A^\circ \subseteq A$. By definition of *closure*, we have $A \subseteq \overline{A}$. Altogether, we have

$$x \in A^\circ \subseteq A \subseteq \overline{A}.$$

Now assume that $x \in \partial(A)$. By definition of *boundary*, we have that every open set containing x has nonempty intersection with A. Thus, by definition of *closure*, we see that $x \in \overline{A}$. This establishes that $A^\circ \cup \partial(A) \subseteq \overline{A}$.

Let $x \in \overline{A}$, and assume that $x \notin A^\circ$. Then no open set containing x is contained in A. Let U be an open set in X with $x \in U$. Since $x \in \overline{A}$, it follows that U has nonempty intersection with A. Since $x \notin A^\circ$, however, it follows from the above observation that U has nonempty intersection with $X \setminus A$. Thus $x \in \partial(A)$. This establishes that $\overline{A} \subseteq A^\circ \cup \partial(A)$, forcing equality. The proof is complete. □

Theorem 3.31: Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X. The closure operation satisfies the following:

1. $A \subseteq \overline{A}$
2. $\overline{\overline{A}} = \overline{A}$
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$

Proof. Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X.

1. By definition, the closure of A is the intersection of all closed sets in X containing A, so it is immediate that $A \subseteq \overline{A}$.

2. Put $B := \overline{A}$, and note that $\overline{\overline{A}} = \overline{(\overline{A})}$. We have
$$\overline{A} = B \subseteq \overline{B} = \overline{(\overline{A})} = \overline{\overline{A}},$$
using the first statement of the theorem for the inclusion.

3. First we show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Let $x \in \overline{A} \cup \overline{B}$, and assume, without loss of generality, that $x \in \overline{A}$. Let U be any open set containing x. Then $U \cap A \neq \varnothing$. But $A \subseteq A \cup B$, so in fact $U \cap (A \cup B) \neq \varnothing$. Since U was arbitrary, it follows that $x \in \overline{A \cup B}$.

   Next we show the reverse inclusion. First observe that the closure of a set is always closed, since the intersection of arbitrarily many closed sets is closed in a topological space. We have $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ by the first statement of the theorem. Hence $A \cup B \subseteq \overline{A} \cup \overline{B}$. Now this latter set is closed because it is the union of two closed sets in a topological space. Since $\overline{A \cup B}$ is the intersection of all closed sets containing $A \cup B$, it follows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. The proof is finished.

4. Assume that $A \subseteq B$. From the first statement of the theorem, we have $B \subseteq \overline{B}$, implying that $A \subseteq \overline{B}$. From the fact that $\overline{A}$ is the intersection of all closed sets containing A, and the fact that $\overline{B}$ is closed, we infer that $\overline{A} \subseteq \overline{B}$.

□

Exercise 3.32: Let $(X, \mathcal{T})$ be a topological space, and let A and B be subsets of X. State and prove a conjecture about $\overline{A \cap B}$.

Proof. Conjecture: The closure of the intersection of two sets in a topological space is equal to the intersection of their closures.

We will first show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cap B}$. This means that every open set containing x intersects $A \cap B$. By the distributive law of set operations, any open set containing x intersects both A and B. Thus $x \in \overline{A}$ and $x \in \overline{B}$ implies $x \in \overline{A} \cap \overline{B}$. Hence, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

However, $\overline{A} \cap \overline{B} \subseteq \overline{A \cap B}$ is false. One counterexample includes: $A = (0, 1)$ and $B = (1, 2)$. We have, $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$ and $A \cap B = \varnothing$ and $\overline{A \cap B} = \varnothing$ but $\overline{A} \cap \overline{B} = \{1\}$ □

Definition 3.33: Let $(X, \mathcal{T})$ be a topological space, let $x \in X$, and let $A \subseteq X$. We call x a *limit point* of A if every open set containing x contains an element of A distinct from x. We denote the set of all limit points of a set A by A'.

Theorem 3.34: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then A is closed if and only if $A' \subseteq A$.

Proof. Towards contradiction, assume that A is closed but that $A' \nsubseteq A$. Then there exists a point $x \in A'$ with $x \in A^c$. Now since A^c is open, there exists an open set U such that $x \in U \subseteq A^c$. But then U is an open set containing no point of A, contradicting that x is a limit point of A. Therefore, we have $A' \subseteq A$.

Conversely, suppose that $A' \subseteq A$. Let $x \in A^c$. Thus, x is not a limit point of A. Then, there exists an open set U such that $x \in U$ and U does not contain any point of A other than x. Since x is not an element of A, U

does not contain any point of A implying that $x \in U \subseteq A^c$. Thus, A^c is open. This implies that A is closed. Therefore, A is closed if and only if $A' \subseteq A$. $\square$

Theorem 3.35: Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\overline{A} = A \cup A'$.

Proof. Let $x \in \overline{A}$. Then by Theorem 1.32, we have $x \in A^\circ$ or $x \in \partial(A)$. If $x \in A^\circ$, then by definition of interior of A, we have $A^\circ \subseteq A$, and as a consequence $x \in A$. If $x \in \partial(A)$, then every open set containing x has nonempty intersection with both A and $X \setminus A$. We now have two cases: Either $x \in A$ or $x \notin A$. If $x \in A$, then $x \in A \cup A'$ and we are done. Suppose $x \notin A$. It follows that an open set containing x contains an element of A distinct from x. Therefore, $x \in A'$.

Now, let $x \in A \cup A'$. Then $x \in A$ or $x \in A'$. If $x \in A$, by definition of the closure of A, we know $A \subseteq \overline{A}$. Thus, $x \in \overline{A}$. If $x \in A'$, then every open set containing x contains an element of A distinct from x, and by Theorem 1.31, we have $x \in \overline{A}$. Hence, $\overline{A} = A \cup A'$. $\square$

Chapter 4

Basis and Subbasis

Definition 4.1: A collection $\mathcal{B}$ of subsets of a set X is a *basis* for a topology on X if the following hold:

1. For each $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.
2. If A and B are in $\mathcal{B}$, and if $x \in A \cap B$, then there is a $C \in \mathcal{B}$ such that $x \in C$ and $C \subseteq A \cap B$.

Theorem 4.2: Let $\mathcal{B}$ be a basis for a topology on a set X, and put

$$\mathcal{T} = \{\varnothing\} \cup \{U \mid U \text{ is the union of some members of } \mathcal{B}\}.$$

Then $\mathcal{T}$ is a topology on X.

Proof. First note that $\emptyset \in \mathcal{T}$ by definition. Since $\mathcal{B}$ is a basis for a topology on X, it follows that each element of X is contained in some element of $\mathcal{B}$, and hence $X \subseteq \bigcup_{B \in \mathcal{B}} B$. Since $\mathcal{B}$ consists of subsets of X, we get $\bigcup_{B \in \mathcal{B}} B \subseteq X$ as well. Thus $X = \bigcup_{B \in \mathcal{B}} B$, and we see that $X \in \mathcal{T}$.

Let $A, B \in \mathcal{T}$. Then $A = \bigcup_{\alpha \in \Lambda} B_\alpha$ and $B = \bigcup_{\beta \in \Gamma} B_\beta$, where $B_\alpha, B_\beta \in$

$\mathcal{B}$ for each $\alpha \in \Lambda$ and $\beta \in \Gamma$. Then

$$A \cap B = \left(\bigcup_{\alpha \in \Lambda} B_\alpha \right) \bigcap \left(\bigcup_{\beta \in \Gamma} B_\beta \right) = \bigcup_{\alpha \in \Lambda,\ \beta \in \Gamma} (B_\alpha \cap B_\beta) \, .$$

Due to the second property of a basis, each set of the form $B_\alpha \cap B_\beta$ is also a union of basis elements, and thus we conclude that $A \cap B$ is a union of basis elements. It follows that $A \cap B \in \mathcal{T}$.

Finally, let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a collection of elements in $\mathcal{T}$. Then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is a union of unions of basis elements, and thus $\bigcup_{\alpha \in \Lambda} A_\alpha \in \mathcal{T}$. The proof is finished. $\square$

Definition 4.3: The topology $\mathcal{T}$ defined above is called the *topology generated by* $\mathcal{B}$.

Exercise 4.4: Let $\mathcal{B}$ be the collection of *open intervals* in $\mathbb{R}$. We have

$$\mathcal{B} = \{(a,b) \mid a, b \in \mathbb{R} \text{ and } a < b\}.$$

Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the *standard topology on* $\mathbb{R}$.

Proof. First, let $x \in \mathbb{R}$. Then $(x-1, x+1) \in \mathcal{B}$, and we have $x \in (x-1, x+1)$.

Now let $A = (a_1, b_1)$, let $B = (a_2, b_2)$, and let $x \in A \cap B$. Set $M := \max\{a_1, a_2\}$ and set $m := \min\{b_1, b_2\}$. Then $(M, m) \in \mathcal{B}$. With $C := (M, m)$, we have

$$x \in C = A \cap B,$$

satisfying the second condition for a basis. $\square$

Exercise 4.5: Find a basis $\mathcal{D}$ for the discrete topology on $\mathbb{R}$, and show that it is a basis.

Proof. We claim that $\mathcal{D} = \{\{x\} \mid x \in \mathbb{R}\}$ is a basis for the discrete topology on $\mathbb{R}$. We must prove first that $\mathcal{D}$ is a basis, and second that it generates the discrete topology on $\mathbb{R}$.

Clearly for $x \in \mathbb{R}$ we have $x \in \{x\} \in \mathcal{D}$. Now let $\{x\}, \{y\} \in \mathcal{D}$ and let $z \in \{x\} \cap \{y\}$. Note that $z = x = y$. We have

$$z \in \{z\} = \{x\} \cap \{y\},$$

from which we conclude that $\mathcal{D}$ is a basis.

Let $\mathcal{T}_G$ consist of the empty set and all unions of the elements of $\mathcal{D}$. It has been established that this set generates a topology. We now claim that the topology it generates is the discrete topology $\mathcal{T}_D$. Let V be any set of real numbers. Then clearly $V \in \mathcal{T}_D$ if and only if $V = \bigcup_{x \in V} \{x\} \in \mathcal{T}_G$. Therefore $\mathcal{T}_G = \mathcal{T}_D$. $\square$

Exercise 4.6: Let $\mathcal{L}$ be the collection of half-open intervals in $\mathbb{R}$, given by the set

$$\mathcal{L} = \{[a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}.$$

Show that $\mathcal{L}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathcal{L}$ is called the *lower limit topology on* $\mathbb{R}$.

Proof. With minor adjustments, the proof is similar to the proof of 4.5. $\square$

Theorem 4.7: A collection $\mathcal{B}$ of subsets of a set X is a basis for a topology on X, and generates a given topology $\mathcal{T}$ on X, if and only if the following hold:

1. For each $U \in \mathcal{T}$ and each $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

2. $\mathcal{B} \subseteq \mathcal{T}$

Proof. Assume that $\mathcal{B}$ is a basis for a topology on X, and that it generates a given topology $\mathcal{T}$ on X. Since $\mathcal{B}$ is a basis for τ, for any open set U in τ

and any point x in U, there exists a basis element B such that $x \in B$ and $B \subseteq U$. Since $\mathcal{B}$ is a basis for τ, all elements of $\mathcal{B}$ are also open sets in τ. Trivially, each $B \in \mathcal{B}$ is a union consisting of itself. Therefore, $\mathcal{B} \subseteq \tau$.

Assume that conditions 1 and 2 hold. We must show that $\mathcal{B}$ is a basis for τ. Let $A, B \in \mathcal{B}$ and let $x \in A \cap B$. By Theorem 1.2, $A \cap B$ is a union of elements of τ. Put $U := A \cap B$. Given $U \in \tau$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. This satisfies the definition of a basis. Then, let $x \in U$. By condition 2, there is a $V \in \mathcal{B}$ such that $x \in V \subseteq U = A \cap B$. Hence, a collection $\mathcal{B}$ of subsets of a set X is a basis for a topology τ on X if and only if conditions 1 and 2 hold. $\square$

Definition 4.8: A collection $\mathcal{S}$ of subsets of a set X is a *subbasis* for a topology on X if, for each $x \in X$, there is an $S \in \mathcal{S}$ such that $x \in S$.

Theorem 4.9: Let $\mathcal{S}$ be a subbasis for a topology on a set X. Then the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$ is a topology on X.

Proof. Let $\mathcal{S}$ be a subbasis for a topology on a set X. Since $\mathcal{S}$, for each $x \in X$, there exists an $s \in S$ such that $x \in S$. Thus, $\mathcal{T}$ must be non-empty. Since $\mathcal{T}$ is made up of unions of finite intersecting elements from $\mathcal{S}$, any open set $\mathcal{T}$ can be expressed as such finite intersections and unions. Thus, $\mathcal{T}$ covers the entire set X. Now consider $U_1, U_2 \in \mathcal{T}$ expressed as finite intersections from $\mathcal{S}$. Since $U_1 \cap U_2$ is a finite intersection, $U_1 \cap U_2 \in \mathcal{T}$. Now, consider U_i to be a collection of sets in $\mathcal{T}$. Thus, by a similar argument as above, the union of all $U_i \in \mathcal{T}$. T satisfies all three conditions for being a topology on X. Therefore, the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$ is a topology on X.

$\square$

Definition 4.10: The topology $\mathcal{T}$ defined above is called the *topology generated by $\mathcal{S}$*.

Exercise 4.11: Let $\mathcal{S}$ be the collection of *open rays* in $\mathbb{R}$, given by

$$\mathcal{S} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}.$$

1. Show that $\mathcal{S}$ is a subbasis for a topology on X.

2. Show that the topology generated by $\mathcal{S}$ is the standard topology on $\mathbb{R}$.

Chapter 5

Subspace

Theorem 5.1: Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. Then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y.

Proof. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. First, we have $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$, so both $\emptyset$ and Y are in $\mathcal{T}_Y$.

Let $A, B \in \mathcal{T}_Y$. Then $A = Y \cap U$ and $B = Y \cap V$ for some $U, V \in \mathcal{T}$. We have

$$A \cap B = (Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V),$$

and since $U \cap V \in \mathcal{T}$, it follows that $A \cap B \in \mathcal{T}_Y$.

Finally, let $\{A_\alpha\}_{\alpha \in \Lambda}$ be a set of open sets in Y. Then for each $\alpha \in \Lambda$, we have $A_\alpha = Y \cap U_\alpha$ for some $U_\alpha \in \mathcal{T}$. Since $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}$, it follows that

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} (Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha \in \Lambda} U_\alpha \right) \in \mathcal{T}_Y.$$

The proof is complete. $\square$

Definition 5.2: The topological space $(Y, \mathcal{T}_Y)$ is called the *subspace* (or *relative* or *induced*) topology on Y. Sets in $\mathcal{T}_Y$ are called *open in* Y or *open relative to* Y. Similar terminology is used for closed sets.

Theorem 5.3: Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$ and let $A \subseteq Y$. Then

1. A is closed in Y if and only if $A = Y \cap C$, where C is a closed subset in X.
2. An element $x \in Y$ is a $\mathcal{T}_Y$-limit point of A if and only if x is a $\mathcal{T}$-limit point of X
3. The $\mathcal{T}_Y$-closure of A is the intersection of Y and the $\mathcal{T}$-closure of A.

Proof. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$ and let $A \subseteq Y$.

1. Assume that A is closed in Y. Then $Y \setminus A$ is open in Y, so we write $Y \setminus A = Y \cap U$ for some $U \in \mathcal{T}$. Since U is open in X, it follows that $X \setminus U$ is closed in X. Using the fact that

   $$Y = (Y \cap U) \cup (Y \cap (X \setminus U)),$$

   we deduce that $A = Y \cap (X \setminus U)$. Put $C = X \setminus U$. Then $A = Y \cap C$ for some closed $C \subseteq X$, as claimed.

   Assume that $A = Y \cap C$, where C is a closed subset of X. Since C is closed in X, it follows that $X \setminus C$ is open in X. Using an argument similar to that above, we infer that $Y \setminus A = Y \cap (X \setminus C)$, which is open in Y. Therefore A is closed in Y. The proof is finished.

2. Assume that $x \in Y$ is a $\mathcal{T}_Y$-limit point of A. Let $U \in \mathcal{T}$ with $x \in U$. Then $Y \cap U$ contains a point of A distinct from x, and hence so does U itself. Therefore, x is a $\mathcal{T}$-limit point of X.

   Assume that x is a $\mathcal{T}$-limit point of X. Let $Y \cap U$ be an open set in $\mathcal{T}_Y$ with $x \in Y \cap U$, where $U \in \mathcal{T}$. Then U contains a point of A distinct from x. But $A \subseteq Y$, so in fact $Y \cap U$ contains a point of A distinct from x. Therefore, $x \in Y$ is a $\mathcal{T}_Y$-limit point of A. The proof is complete.

3. First we introduce notation. Let $\overline{A}_Y$ denote the $\mathcal{T}_Y$-closure of A and let $\overline{A}_X$ denote the $\mathcal{T}$-closure of A. The claim is that $\overline{A}_Y = Y \cap \overline{A}_X$.

   First note that $\overline{A}_X$ is closed in X as an intersection of closed sets. From the first claim of this theorem, it follows that $Y \cap \overline{A}_X$ is closed in Y. Now $A \subseteq \overline{A}_X$ and $A \subseteq Y$, so we have $A \subseteq Y \cap \overline{A}_X$. Since $\overline{A}_Y$ is the intersection of all closed sets in Y containing A, it follows that $\overline{A}_Y \subseteq Y \cap \overline{A}_X$.

   Now we note that $\overline{A}_Y$ is closed in Y, so from this theorem's first claim we write $\overline{A}_Y = Y \cap C$ for some closed $C \subseteq X$. Since $A \subseteq \overline{A}_Y$, it follows that $A \subseteq C$. Thus C is a closed set in X containing A, from which we infer that $\overline{A}_X \subseteq C$. We conclude that

   $$Y \cap \overline{A}_X \subseteq Y \cap C = \overline{A}_Y.$$

   This finishes the proof.

□

Theorem 5.4: Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$, and let $A \subseteq Y$. Then

1. If A is closed in Y and Y is closed in X, then A is closed in X.
2. If A is open in Y and Y is open in X, then A is open in X.

Proof. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$, and let $A \subseteq Y$.

1. We assume that A is closed in Y and that Y is closed in X. We then have that $(Y \setminus A)$ is open in Y, and $(X \setminus Y)$ is open in X. Additionally, $A \notin (X \setminus Y)$ because $A \subseteq Y$. Therefore $A \in X \setminus (X \setminus Y)$, and since $(X \setminus Y)$ is open in X, we have that $X \setminus (X \setminus Y)$ is closed in X. Hence because $A \in X \setminus (X \setminus Y)$, it is true that A is closed in X. This concludes the proof.

2. We assume that A is open in Y and Y is open in X. Since A is open in Y, we have $A = Y \cap U$ for some open $U \subseteq X$. Since Y is open in X, it follows that A is the intersection of two open sets in X, so A is open in X.

□

Theorem 5.5: If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ on X, then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology $\mathcal{T}_Y$ on Y.

Proof. We will show that $\mathcal{B}_Y$ is a basis for the subspace topology $\mathcal{T}_Y$ on Y by demonstrating that it satisfies both properties of the basis for a topological space. First, let $y \in Y$. Because $Y \subseteq X$, we also have that $y \in X$. Therefore there is a $B \in \mathcal{B}$ such that $y \in B$, and since $y \in Y$, we have that $y \in B \cap Y$ such that $B \in \mathcal{B}$, and so $y \in \mathcal{B}_Y$. To demonstrate condition 2, let $A, D \in \mathcal{B}_Y$ and let $x \in A \cap D$. Because $A, D \in \mathcal{B}_Y$, we can let $A = B_1 \cap Y$ and $D = B_2 \cap Y$ for some $B_1, B_2 \in \mathcal{B}_Y$. Therefore we have $x \in A \cap D = (B_1 \cap Y) \cap (B_2 \cap Y) = (B_1 \cap B_2) \cap Y$. Now if we let $C = (B_1 \cap B_2) \cap Y$, we have that $x \in C \subseteq A \cap D$. Therefore $\mathcal{B}_Y$ is a basis for the subspace topology $\mathcal{T}_Y$ on Y. □

Exercise 5.6: Let $Y = [0, 1]$ be endowed with the subspace topology from $\mathbb{R}$. List the types of basic open sets in this subspace.

Proof. A *basic* open set in Y has the form $Y \cap B$, where $B = (a, b)$ with $a < b$.

The first possibilities are that B engulfs Y completely, in which case we get $[0, 1]$, and that B misses Y completely, in which case we get $\emptyset$. (These possibilities also follow from the general claim that $\emptyset, Y \in \mathcal{T}_Y$.)

The other possibilities are nonempty intersections that do not contain all of $[0, 1]$. With $B = (a, b)$ as above, we get $[0, b)$, $(a, 1]$, and (a, b). □

Exercise 5.7: Determine whether each of the following subsets of $Y = [0, 1]$ is open, closed, or neither in Y.

1. $(0, 1/2)$
2. $(0, 1/2]$
3. $(1/3, 1)$
4. $(1/3, 1]$
5. $\{1\}$
6. $\{x \in [0, 1] \mid x \notin [0, 1] \cap \mathbb{Q}\}$

Proof. Let $Y = [0, 1]$.

1. We have $(0, 1/2) = Y \cap (0, 1/2)$, so this subset is open.
2. We have that $(0, 1/2]$ is the complement in Y of $(1/2, 1]$, which is open in Y. Hence $(0, 1/2]$ is closed in Y.
3. We have $(1/3, 1) = Y \cap (1/3, 1)$, so this subset is open.
4. We have $(1/3, 1] = Y \cap (1/3, 2)$, so this subset is open.
5. We have that $\{1\}$ is the complement in Y of $[0, 1)$, which is open in Y. Hence $\{1\}$ is closed in Y.
6. Put $I = \{x \in [0, 1] \mid x \notin [0, 1] \cap \mathbb{Q}\}$. This set consists of all irrational numbers in Y. Now any open set in Y is the union of *basic* open sets in Y. If the given set were the union of basic open sets, then we would have $(a, 1] \subseteq I$ or $(a, b) \subseteq I$ or $[0, b) \subseteq I$ for some basic open set (a, b) in $\mathbb{R}$. But this is impossible. Since the rationals are *dense* in the real numbers, this would imply that there are rationals in I, contradicting that it contains no rationals. Therefore I is not open.

   Consider the complement of I, and call it Q. This is the set of all rationals in Y. A similar argument shows that if Q were the union of basic open sets, then it would contain irrationals, contradicting its definition. Since Q is not open, it follows that I is not closed. Therefore, I is neither open nor closed in Y.

$\square$

Chapter 6

Metric Space

Definition 6.1: A *metric space* (X, d) is a set X and a function $d : X \times X \to \mathbb{R}$ such that the following hold:

1. $d(x, y) \geq 0$ for all $x, y \in X$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ for all $x, y \in X$
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

We call d a *metric*.

Definition 6.2: Let (X, d) be a metric space. For $x \in X$ and $r > 0$, the set
$$B(x, r) = \{y \mid y \in X \text{ and } d(x, y) < r\}$$
is called the *r-ball centered at x*.

Theorem 6.3: Let (X, d) be a metric space. Then the collection
$$\{B(x, r) \mid x \in X \text{ and } r \in \mathbb{R}\}$$
is a basis for a topology on X.

Proof. Let $x \in X$. Then for $r = 1$, we have $x \in B(x, 1) \subseteq X$, satisfying the first condition for a basis.

Now let $B(x, r_1)$ and $B(y, r_2)$ be two r-balls in the collection. Let $z \in B(x, r_1) \cap B(y, r_2)$. Set $r = \min\{r_1 - d(x, z), r_2 - d(y, z)\}$. We claim that $B(z, r) \subseteq B(x, r_1) \cap B(y, r_2)$. Let $p \in B(z, r)$. Using the triangle inequality, we have

$$d(p, x) \leq d(p, z) + d(z, x) < r + d(z, x) \leq (r_1 - d(z, x)) + d(z, x) = r_1.$$

This proves that $p \in B(x, r_1)$. A similar argument shows that $p \in B(y, r_2)$. We conclude that

$$z \in B(z, r) \subseteq B(x, r_1) \cap B(y, r_2).$$

Therefore, the stated collection is a basis for a topology on X. □

Definition 6.4: The topology generated by the basis of r-balls, as above, is called the *metric topology* on X generated by d.

Theorem 6.5: Let (X, d) be a metric space, and let $Y \subseteq X$. Then (Y, d_Y) is a metric space, where d_Y is the restriction of d to $Y \times Y$.

Proof. We have a metric space X, and a distance function $d : X \times X \to \mathbb{R}$ where the 4 properties hold. Since $Y \subseteq X$, by the restriction of a function, d_Y inherits the properties from d. Thus,

1. $d_Y(x, y) \geq 0$ for all $x, y \in Y$
2. $d_Y(x, y) = 0$ if and only if $x = y$
3. $d_Y(x, y) = d(x, y) = d(y, x)$ for all $x, y \in Y$
4. $d(x, y) = d_Y(x, y) \leq d(x, z) + d(z, y) = d_Y(x, z) + d_Y(z, y)$ for all $x, y, z \in Y$

□

Theorem 6.6: Let $a, b \in \mathbb{R}$. Prove the Triangle Inequality: $|a + b| \leq |a| + |b|$.

Proof. Let $a, b \in \mathbb{R}$. We will use cases to prove the theorem.

Case 1: $a \geq 0$ **and** $b \geq 0$**.** In this case, $a + b$ is nonnegative and so $|a| = a$, $|b| = b$ and $|a + b| = a + b$. Then, $|a + b| = a + b = |a| + |b|$.

Case 2: $a \leq 0$ **and** $b \leq 0$**.** In this case, we let $a = -a'$ and $b = -b'$ where a' and b' are nonnegative. It follows from case 1 that $|a + b| = |-(a' + b')| = |a' + b'| = a' + b' = |a'| + |b'| = |-a'| + |-b'| = |a| + |b|$.

Case 3: One of a **or** b **is positive and the other negative.** Without loss of generality we assume $a > 0$ and $b < 0$. Again we consider cases. Note that $b < 0$ implies $a + b < a$. **Sub-case 1:** $b \geq -2a$**.** Suppose $b \geq -2a$. Then $a + b \geq -a$ and so $-a \leq a + b < a$. It follows that $|a + b| \leq a = |a| < |a| + |b|$. **Sub-case 2:** $b \leq -2a$**.** The last case is when $b \leq -2a$. In this case, $-b > 2a$ and so $|b| = -b > 2a = 2|a| > |a|$. Then $a + b < a = |a| < |b|$. Finally, $a > 0$ implies $a + b > b = -|b|$. So $-|b| < a + b < |b|$ and $|a + b| \leq |b| < |a| + |b|$.

Hence, $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.

□

Exercise 6.7: Let $x, y \in \mathbb{R}$, and set $d(x, y) = |x - y|$.

1. Prove that $(\mathbb{R}, d)$ is a metric space. We call d the *standard* metric on $\mathbb{R}$.

2. Show that the topology on $\mathbb{R}$, generated by r-balls, is the standard topology on $\mathbb{R}$.

Proof. We will show that $(\mathbb{R}, d)$ satisfies all 4 properties of a metric space. In this case we have a set, $\mathbb{R}$, and a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For the first property, $d(x, y) = |x - y|$. Note that the square root definition of the absolute value states that $|x| = \sqrt{x^2}$. Therefore let $Q = x - y$ and note that whether or not $x \geq y$ or $y \geq x$, $|x-y| = |y-x| = |Q| = |-Q| = \sqrt{Q^2} \geq 0$ by proven properties of the absolute value function. Hence $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$, so the first property is satisfied. For the second property, if $x = y$, $d(x,y) = |x-y| = |x-x| = |y-y| = 0$ to prove the reverse direction. For the forward direction, if $d(x,y) = 0$, we have $|x-y| = 0$. Now, what we could say is that $x = y$ by the identity of indiscernibles, and equivalently by positive-definiteness, which state that

$$|x-y| = 0 \iff x = y.$$

However, towards a proof by contradiction, suppose $d(x,y) = 0$ and $x \neq y$. Then $x > y$ implies, $x - y > 0$ and $|x-y| > 0$, or if $x < y$, $x - y < 0$, and $|x-y| > 0$. These are both contradictions. Hence, $x = y$, and $d(x,y) = 0$ if and only if $x = y$. Now for condition 3, we have

$$d(x,y) = |x-y|$$

and

$$d(y,x) = |y-x|.$$

Let P be the distance between x and y. Thus for $x > y$, $x - y = P$, and for $x < y$, $x - y = -P$. From a real analytic lemma, we know that $|-x| = |x|$, so without loss of generality for $x < y$ we have that

$$|x-y| = |-P| = |P| = |y-x|.$$

and vice versa for all other cases. Therefore $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{R}$.

Finally, we have that $d(x,y) = |x-y|$, $d(x,z) = |x-z|$, and $d(z,y) = |z-y|$. Thus by theorem 6.6 we have that

$$d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y|.$$

Hence the triangle inequality holds for all real numbers of $d(x,y)$, and thus d is a metric space. $\square$

Exercise 6.8: Given a set X, define d by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

1. Prove that (X, d) is a metric space.
2. Under the metric d, is the topology generated by r-balls a familiar topology on X? If not, say why. If so, prove it, as in the above exercise.

Proof. Define d on the set X as above.

1. The first three conditions for being a metric space are trivially verified using the definition of d. It remains to prove that the Triangle Inequality holds for (X, d).

   Let $x, y, z \in X$, and first suppose that $x = y$. Since $d(x, z) \geq 0$ and $d(z, y) \geq 0$, it follows that
   $$d(x, y) = 0 \leq d(x, z) + d(z, y).$$
   Now suppose that $x \neq y$. If both $d(x, z) = 0$ and $d(z, y) = 0$, then $x = z = y$, contradicting our assumption. Hence at least one of these distances equals 1, so we deduce that
   $$d(x, y) = 1 \leq d(x, z) + d(z, y).$$
   In either case, the inequality holds. This completes the proof.

2. Under the metric d, the topology generated by r-balls is the discrete topology. To prove this, let $\mathcal{T}_G$ and $\mathcal{T}_D$ be the topology generated by r-balls and the discrete topology, respectively. We must prove that $\mathcal{T}_G = \mathcal{T}_D$.

   Let $U \in \mathcal{T}_G$. Since U is open in the topology generated by a basis, it can be expressed as a union of basis elements. Now for each $x \in U$, we have $x \in B(x, 1)$. But due to the metric d, we also have $B(x, 1) = \{x\}$. Thus
   $$U \in \mathcal{T}_G \iff U = \bigcup_{x \in U} B(x, 1) \iff U = \bigcup_{x \in U} \{x\} \iff U \in \mathcal{T}_D.$$

$\square$

Chapter 7

Product Topology

Definition 7.1: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. The *box topology* on $X \times Y$ is the topology generated by the basis consisting of all sets of the form $U \times V$, where U is open in X and V is open in Y.

Theorem 7.2: If $\mathcal{B}$ is a basis for the topology of X, and $\mathcal{C}$ is a basis for the topology on Y, then the collection of sets of the form $B \times C$, where $B \in \mathcal{B}$ and $C \in \mathcal{C}$, is a basis for the box topology on $X \times Y$.

Proof. Let $\mathcal{B}$ be a basis for the topology of X and let $\mathcal{C}$ be a basis for the topology of Y. Let $(x, y) \in X \times Y$. Then there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subseteq X$ and $y \in C \subseteq Y$. Since $B \in \mathcal{T}$ and $C \in \mathcal{S}$, it follows that $B \times C$ is open in the box topology on $X \times Y$, and hence

$$(x, y) \in B \times C \subseteq X \times Y.$$

Now let $B_1 \times C_1$ and $B_2 \times C_2$ be sets in the given collection, and let $(p, q) \in (B_1 \times C_1) \cap (B_2 \times C_2)$. By elementary set theory, we have

$$(B_1 \times C_1) \cap (B_2 \times C_2) = (B_1 \cap B_2) \times (C_1 \cap C_2).$$

Thus $p \in B_1 \cap B_2$ and $q \in C_1 \cap C_2$. By the second basis condition, there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $p \in B \subseteq B_1 \cap B_2$ and $q \in C \subseteq C_1 \cap C_2$.

We conclude that there exists an element $B \times C$ in the given collection such that
$$(p, q) \in B \times C \subseteq (B_1 \times C_1) \cap (B_2 \times C_2).$$
This completes the proof that the given collection is a basis for the box topology on $X \times Y$. $\square$

Definition 7.3: Let Λ be an indexing set, and for each $\alpha \in \Lambda$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space.

1. The Cartesian product of the collection $\{X_\alpha\}_{\alpha \in \Lambda}$, denoted
$$\prod_{\alpha \in \Lambda} X_\alpha,$$
is the set of all Λ-tuples $(x_\alpha)_{\alpha \in \Lambda}$ such that $x_\alpha \in X_\alpha$ for each $\alpha \in \Lambda$.
2. The *box topology* on the Cartesian product is the topology generated by the basis of sets of the form
$$\prod_{\alpha \in \Lambda} U_\alpha,$$
where U_α is open in X_α for each $\alpha \in \Lambda$.
3. The *product topology* on the Cartesian product is the topology generated by the basis of sets of the form
$$\prod_{\alpha \in \Lambda} U_\alpha,$$
where U_α is open in X_α for each $\alpha \in \Lambda$, and $U_\alpha = X_\alpha$ except for finitely many indices α.

Exercise 7.4: Show that the collections defined in (2) and (3) above are indeed bases.

Proof. It is a trivial exercise to show that the collections satisfy the first condition for a basis. Let $\mathbf{x} = (x_\alpha)_{\alpha\in\Lambda} \in \prod_{\alpha\in\Lambda} X_\alpha$. Let $\mathcal{U} = \prod_{\alpha\in\Lambda} U_\alpha$ and $\mathcal{V} = \prod_{\alpha\in\Lambda} V_\alpha$, where U_α and V_α are open in X_α for each $\alpha \in \Lambda$. Assume that $\mathbf{x} \in \mathcal{U} \cap \mathcal{V}$. We must find $\mathcal{W}$ in the collection such that

$$\mathbf{x} \in \mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}.$$

By elementary set theory, we have

$$\mathcal{U} \cap \mathcal{V} = \left(\prod_{\alpha\in\Lambda} U_\alpha\right) \cap \left(\prod_{\alpha\in\Lambda} V_\alpha\right) = \prod_{\alpha\in\Lambda}(U_\alpha \cap V_\alpha).$$

Thus $x_\alpha \in U_\alpha \cap V_\alpha$ for each $\alpha \in \Lambda$, and there exists a basis element B_α such that $x_\alpha \in B_\alpha \subseteq U_\alpha \cap V_\alpha$ for each $\alpha \in \Lambda$. Set $\mathcal{W} = \prod_{\alpha\in\Lambda} B_\alpha$. Then clearly

$$\mathbf{x} \in \mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V},$$

which finishes the proof that the stated collection is a basis for the box topology on X.

To adapt the proof for the product topology, convince yourself that if only finitely many of the U_α's do not equal X_α, and only finitely many of the V_α's do not equal X_α, then only finitely many of the sets $U_\alpha \cap V_\alpha$ do not equal X_α. Hence for each $\alpha \in \Lambda$ for which $U_\alpha \cap V_\alpha \neq X_\alpha$, there exists a basis element $B_\alpha \subseteq U_\alpha \cap V_\alpha$. We define $\mathcal{W} = \prod_{\alpha\in\Lambda} W_\alpha$ so that for each such $\alpha \in \Lambda$, we put $W_\alpha = B_\alpha$; for all other indices we define $W_\alpha = X_\alpha$. The same conclusion now holds for the product topology on X. □

Theorem 7.5: Let A be an indexing set, and for each $\alpha \in A$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. If A is finite, then the box and product topologies on $\prod_{\alpha\in A} X_\alpha$ are the same.

Proof. Assume that A is finite, so that $A = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Let $U = \prod_{i=1}^n U_\alpha$ be open in the box topology on $X = \prod_{i=1}^n X_\alpha$. It is automatic that, for only finitely many indices $\alpha \in A$, we have $U_\alpha \neq X_\alpha$. Thus U is open in the product topology on X. Conversely, every open set in the product topology on X already satisfies the requirements to be open in the box topology on X, so we conclude that the product and box topologies on X are the same. $\square$

Theorem 7.6: For each $n \in \mathbb{N}$, let $(X_n, \mathcal{T}_n)$ be a topological space, and let $\mathcal{B}$ be the above defined basis for the product topology on $X = \prod_{n\in\mathbb{N}} X_n$. Define $\mathcal{B}'$ to be the basis for X consisting of sets of the form

$$\prod_{n\in\mathbb{N}} U_n,$$

where U_n is open in X_n for each $n \in \mathbb{N}$, and where, for some $N \in \mathbb{N}$, we have $U_n = X_n$ for all $n \geq N$. Then $\mathcal{B} = \mathcal{B}'$.

Proof. Let $B \in \mathcal{B}$, and write $B = \prod_{n\in\mathbb{N}} U_n$. Then the number of indices for which $U_n \neq X_n$ is finite. Let $\{n_1, n_2, \ldots, n_k\}$ be this set of indices. Define $N = \max\{n_1, n_2, \ldots, n_k\}$. Then for all $n \geq N$, we have $U_n = X_n$. Hence $B \in \mathcal{B}'$.

Conversely, let $B \in \mathcal{B}'$ and write $B = \prod_{n\in\mathbb{N}} U_n$. Then there exists some $N \in \mathbb{N}$ such that $U_n = X_n$ for all $n \geq N$. Since the indices leading up to N constitute a finite list, it follows that for only finitely of them does $U_n \neq X_n$. Hence $B \in \mathcal{B}$. This finishes the proof that $\mathcal{B} = \mathcal{B}'$. $\square$

Exercise 7.7: Let X be the set of all sequences with entries in $\{0, 1\}$. We have

$$X = \prod_{n\in\mathbb{N}} \{0, 1\}.$$

If we equip $\{0, 1\}$ with the discrete topology, describe the basis for the product topology on X, using both of the foregoing characterizations.

Proof. A basis element for the product topology on X has the form

$$\mathcal{U} = \prod_{i=1}^{\infty} U_i,$$

where U_i is open in X_i for each $i \in \mathbb{N}$ and $U_i = \{0, 1\}$ except for finitely many indices. Each factor space U_i thus has the form $U_i = \{0, 1\}$ or $U_i = \{0\}$ or $U_i = \{1\}$ or $U_i = \emptyset$. (If $U_i = \emptyset$ for any $i \in \mathbb{N}$, then $\mathcal{U} = \emptyset$.) $\square$

Chapter 8

Continuity and Homeomorphism

Definition 8.1: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. We say that f is *continuous* if the inverse image of each open set of Y is open in X.

Theorem 8.2: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. Then f is continuous if and only if the inverse image of each closed set of Y is closed in X.

Proof. Assume f is continuous. Let C be a closed set in Y. Since C is closed in Y, $Y \setminus C$ is open in Y. By continuity of f, $f^{-1}(Y \setminus C)$ must be open in X. But $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$. Therefore, $X \setminus f^{-1}(C)$ is open in X, implying that $f^{-1}(C)$ is closed in X.

Assume that the inverse image of each closed set of Y is closed in X. Let V be an open set in Y. Since V is open, $Y \setminus V$ is closed in Y. By the assumption, $f^{-1}(Y \setminus V)$ is closed in X. But $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Therefore, $X \setminus f^{-1}(V)$ is closed in X, implying that $f^{-1}(V)$ is open in X. Therefore, f is continuous.

Hence, the result has been proven. $\square$

Theorem 8.3: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. The following are equivalent:

1. f is continuous.
2. For every $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For every $B \subseteq Y$, we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Proof.

1. We first prove that (1) implies (2). Assume that f is continuous, and let $A \subseteq X$. Using elementary set theory, and the fact that a set is contained in its closure, we have

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

Since f is continuous, it follows that the inverse images of closed sets are closed under f. Hence the set on the far right is a closed set containing A. We infer that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Taking the image of both sides, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

2. Now assume that (2) holds, and let $B \subseteq Y$. Put $A = f^{-1}(B)$. We have

$$\overline{f^{-1}(B)} = \overline{A} \subseteq f^{-1}(f(\overline{A})) \subseteq f^{-1}(\overline{f(A)}) \subseteq f^{-1}(\overline{B}),$$

using elementary set theory, then using (2), and then using more elementary set theory.

3. Let $C \subseteq Y$ be closed, and note that $C = \overline{C}$. We have $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C}) = f^{-1}(C)$, using (3). This says that the closure of a set is contained in the set, from which it follows that the set is closed. Hence $f^{-1}(C)$ is closed, proving that f is continous.

The circle of implications is now complete.

$\square$

Theorem 8.4: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, let $f : X \to Y$, and let $\mathcal{B}$ be a basis for Y. Then f is continuous if and only if the inverse image of each element of $\mathcal{B}$ is open in X.

Proof. Assume that f is continuous, and let $B \in \mathcal{B}$. Now B, as a trivial union of a single basis element, is open in the topology on Y generated by $\mathcal{B}$. Hence $f^{-1}(B)$ is open in X.

Assume that the inverse image of each element of $\mathcal{B}$ is open in X. Let $U \subseteq Y$ be open. Then

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha,$$

where $B_\alpha \in \mathcal{B}$ for each $\alpha \in \Lambda$. Therefore,

$$f^{-1}(U) = f^{-1}\left(\bigcup_{\alpha \in \Lambda} B_\alpha\right) = \bigcup_{\alpha \in \Lambda} f^{-1}(B_\alpha),$$

which is open in X. The proof is complete. □

Theorem 8.5: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, let $f : X \to Y$, and let $\mathcal{S}$ be a subbasis for Y. Then f is continuous if and only if the inverse image of each element of $\mathcal{S}$ is open in X.

Proof. Recall that every element of $\mathcal{S}$ is a basis element, since it can be viewed trivially as a finite intersections of elements of $\mathcal{S}$. Assume that f is continuous. Then the inverse images of open sets in Y are open in X, so for $S \in \mathcal{S}$ we have that $f^{-1}(S)$ is open in X by 8.4.

Now assume that the inverse image of each element of $\mathcal{S}$ is open in X. Let $\mathcal{B}$ be a basis for Y and let $B \in \mathcal{B}$. Then $B = \cap_{i=1}^n S_i$ for some subbasis elements $S_1, \ldots, S_n$. We have

$$f^{-1}(B) = f^{-1}\left(\bigcap_{i=1}^n S_i\right) = \bigcap_{i=1}^n f^{-1}(S_i).$$

Hence the inverse image of B is an intersection of sets that are open by assumption, and so this inverse image must be open. By 8.4, we conclude that f is continuous. □

Exercise 8.6: Recall from Calculus the definition of continuity (at a point): Let $f : \mathbb{R} \to \mathbb{R}$, and let $x_0 \in \mathbb{R}$. Then f is continuous at x_0 if, for each

$\varepsilon > 0$, there is a $\delta > 0$ such that, if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Show that this definition is equivalent to the topological definition.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and assume that f is continuous at each point in $\mathbb{R}$. Let $V \subseteq \mathbb{R}$ be open in the codomain of f. If $f^{-1}(V) = \emptyset$, then $f^{-1}(V)$ is open, so let $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$. Since V is open, it follows that there exists $\varepsilon > 0$ such that $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subseteq V$. By continuity of f, there exists $\delta > 0$ such that if $x \in (x_0 - \delta, x_0 + \delta)$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Therefore

$$(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}\left((f(x_0) - \varepsilon, f(x_0) + \varepsilon)\right) \subseteq f^{-1}(V),$$

which proves that $f^{-1}(V)$ is open.

Assume that for every open $V \subseteq \mathbb{R}$, we have $f^{-1}(V)$ is open in $\mathbb{R}$. Let $\varepsilon > 0$ and let $x_0 \in \mathbb{R}$. Then $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ is open in the codomain $\mathbb{R}$, so $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ is open in the domain $\mathbb{R}$ and contains x_0. Thus there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$. Hence for any $x \in \mathbb{R}$ such that $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$. We conclude that f is continuous at each point in its domain. □

Exercise 8.7: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Define the map

$$\pi_2 : X \times Y \longrightarrow Y$$

by $\pi_2((x, y)) = y$. The map π_2 is called the *projection map* onto the second coordinate.

1. Show that π_2 is continuous.
2. Show that π_2 is an *open map*: that it takes open sets to open sets.

Of course, the analogous result holds for the projection onto any coordinate.

Proof. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Define the map

$$\pi_2 : X \times Y \longrightarrow Y$$

by $\pi_2((x,y)) = y$.

1. Let $V \subseteq Y$ be open. It is easily verified that $\pi_2^{-1}(V) = X \times V$. Since this set is open in $X \times Y$, it follows that π_2 is continuous.
2. Let $U \times V \subseteq X \times Y$ be open, where $U \subseteq X$ and $V \subseteq Y$ are open. It is easily verified that $\pi_2(U \times V) = V$, which is open in Y. It follows that π_2 is an open map.

□

Exercise 8.8: Let Λ be an indexing set, and for each $\alpha \in \Lambda$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. Fix $\beta \in \Lambda$. Define the map

$$\pi_\beta : \prod_{\alpha \in \Lambda} X_\alpha \longrightarrow X_\beta$$

by $\pi_\beta((x_\alpha)_{\alpha\in\Lambda}) = x_\beta$. The map π_β is called the β-th projection map.

1. Show that π_β is continuous with respect to the product topology.
2. Show that π_β is an open map with respect to the product topology.

Proof. We use the topological definition of continuity.

1. Let U_β be open in X_β. Then it is easy to verify that $\pi_\beta^{-1}(U_\beta) = \prod_{\alpha\in\Lambda} U_\alpha$, where $U_\alpha = X_\alpha$ unless $\alpha = \beta$, in which case $U_\alpha = U_\beta$. Since only finitely many factor spaces in this product are unequal to the entire space, it follows that this set is open with respect to the product topology. Therefore, π_β is continuous with respect to the product topology.
2. Let $U = \prod_{\alpha\in\Lambda} U_\alpha$ be open in the product topology on $\prod_{\alpha\in\Lambda} X_\alpha$. Then $\pi_\beta(U) = U_\beta$, which is open in the topology on X_β. Therefore π_β is an open map with respect to the product topology.

□

Theorem 8.9: Let Λ be an indexing set, and for each $\alpha \in \Lambda$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. Put $X = \prod_{\alpha \in \Lambda} X_\alpha$. Then the set

$$\mathcal{S} = \{\pi_\alpha^{-1}(U_\alpha) \mid \alpha \in \Lambda \text{ and } U_\alpha \in \mathcal{T}_\alpha\}$$

is a subbasis for the product topology on X.

Proof. To show that the topology generated by $\mathcal{S}$ is the *product* topology on X, it suffices to show that the topology generated by $\mathcal{S}$ equals the product topology. Let $\mathcal{T}_G$ be the former and let $\mathcal{T}_P$ be the latter.

As noted earlier, every element of $\mathcal{S}$ is open in $\mathcal{T}_P$, and hence arbitrary unions of finite intersections of elements of $\mathcal{S}$ are open in $\mathcal{T}_P$ as well. This proves that $\mathcal{T}_G \subseteq \mathcal{T}_P$.

Now let $U \in \mathcal{T}_P$. Then $U = \prod_{\alpha \in \Lambda} U_\alpha$, where $U_\alpha = X_\alpha$ except for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$. For each $1 \leq i \leq n$, define $S_{\alpha_i} = \pi_{\alpha_i}^{-1}(U_{\alpha_i})$. We claim that

$$U = \bigcap_{i=1}^{n} S_{\alpha_i}.$$

Let $\mathbf{x} \in U$. Then $x_{\alpha_i} \in U_{\alpha_i}$ if and only if $\mathbf{x} \in \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ for each i, which holds if and only if $\mathbf{x}$ belongs to the intersection on the right-hand side above. Thus U is a finite intersection of subbasis elements, and is a (trivial) union of finite intersections of such elements. Therefore $U \in \mathcal{T}_G$, which forces $\mathcal{T}_G = \mathcal{T}_P$. $\square$

Theorem 8.10: Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$, and $(Z, \mathcal{R})$, be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.

Proof. Assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. Let $W \subseteq Z$ be open. By continuity of g, we have that $g^{-1}(W) \subseteq Y$ is open. By continuity of f, we have that $f^{-1}(g^{-1}(W)) \subseteq X$ is open. Therefore,

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$$

is open in X. We conclude that $g \circ f : X \to Z$ is continuous. $\square$

Theorem 8.11: Let Λ be an indexing set, and for each $\alpha \in \Lambda$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. Let $(X, \mathcal{T})$ be the product space, and let $(Y, \mathcal{S})$ be another topological space. A function $f : Y \to X$ is continuous if and only if the composition $p_\alpha \circ f : Y \to X_\alpha$ is continuous for each $\alpha \in \Lambda$.

Proof. Assume that $f : Y \to X$ is continuous. By exercise $1.8.1$, we know that $p_\alpha : X \to X_\alpha$ is continuous for each $\alpha \in \Lambda$. Since f and p_α are continuous, by theorem 8.10, the composition $p_\alpha \circ f$ is also continuous for each $\alpha \in \Lambda$.

Assume that $p_\alpha \circ f : Y \to X_\alpha$ is continuous for each $\alpha \in \Lambda$. This means that for each open $U_\alpha \subseteq X_\alpha$, we have $f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$ is open in Y.

Let $U \subseteq X$ be open. By Theorem 8.9, there exist finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$U = \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i}),$$

where U_{α_i} is open for each $1 \leq i \leq n$. Hence we have

$$f^{-1}(U) = f^{-1}\left(\bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_{\alpha_i})\right) = \bigcap_{i=1}^{n} f^{-1}(p_{\alpha_i}^{-1}(U_{\alpha_i})).$$

Since the set on the right-hand side is a finite intersection of open sets, it follows that $f^{-1}(U)$ is open in Y, whence $f : Y \to X$ is continuous. $\square$

Definition 8.12: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. We say that f is a *homeomorphism* if the following hold.

1. f is bijective.
2. Both f and f^{-1} are continuous.

Definition 8.13: *Topology* is the study of the properties of topological spaces that are preserved under homeomorphism.

Lemma 8.1: Let $f : X \to Y$ be a bijection and let $f^{-1} : Y \to X$ be its inverse. Let $U \subseteq X$. Then $(f^{-1})^{-1}(U) = f(U)$.

Proof. Let $y \in Y$. Then

$$y \in \left(f^{-1}\right)^{-1}(U) \iff f^{-1}(y) \in U \iff y \in f(U).$$

It follows that $(f^{-1})^{-1}(U) = f(U)$. □

Theorem 8.14: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$ be a bijection. Then the following are equivalent:

1. f is a homeomorphism.
2. If $G \subseteq X$, then $f(G)$ is open in Y if and only if G is open in X.
3. If $F \subseteq Y$, then $f^{-1}(F)$ is open in X if and only if F is open in Y.
4. If $E \subseteq X$, then $f(\overline{E}) = \overline{f(E)}$.

Proof. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$ be a bijection.

1. First we prove that (1) implies (2). Assume that f is a homeomorphism, and let $G \subseteq X$.

   Assume that $f(G)$ is open in Y. Using continuity and bijectivity of f, we have that $G = f^{-1}(f(G))$ is open in X.

   Assume that G is open in X. Using continuity of f^{-1} and Lemma 8.1, we have that $f(G) = (f^{-1})^{-1}(G)$ is open in Y.

2. Now we prove that (2) implies (3). Assume that (2) holds, and let $F \subseteq Y$.

   Assume that $f^{-1}(F)$ is open in X. Using (2) and bijectivity of f, we have that $F = f(f^{-1}(F))$ is open in Y.

Assume that F is open in Y. Since f is bijective, we must have $F = f(G)$ for some $G \subseteq X$. Using (2) and bijectivity of f, we get that $f^{-1}(F) = f^{-1}(f(G)) = G$ is open in X.

3. Next we prove that (3) implies (4). Assume that (3) holds, and let $E \subseteq X$. From the second claim of Theorem 8.3, we have $f(\overline{E}) \subseteq \overline{f(E)}$. It remains to prove the other inclusion.

   Note that $\overline{E}$ is closed in X, and hence $X \setminus \overline{E}$ is open in X. Using first elementary set theory and second the fact that f is bijective, we have that

   $$f^{-1}(Y \setminus f(\overline{E})) = X \setminus f^{-1}(f(\overline{E})) = X \setminus \overline{E}$$

   is open in X. By (3), this implies that $Y \setminus f(\overline{E})$ is open in Y, and hence that $f(\overline{E})$ is closed in Y.

   Since closed sets contain their own closures, it now follows that

   $$\overline{f(E)} \subseteq \overline{f(\overline{E})} \subseteq f(\overline{E}).$$

   The implication follows.

   Finally, we prove that (4) implies (1). Assume that (4) holds. Since f is a bijection, it suffices to show that both f and f^{-1} are continuous.

   Let $C \subseteq Y$ be closed. Since f is a bijection, we see that $C = f(E)$ for some $E \subseteq X$. We have

   $$f^{-1}(C) = f^{-1}(\overline{C}) = f^{-1}(\overline{f(E)}) = f^{-1}(f(\overline{E})) = \overline{E}.$$

   Thus the inverse image of closed sets are closed under f, and we conclude that f is continuous.

   Now let $E \subseteq X$ be closed. Using Lemma 8.1, we have

   $$(f^{-1})^{-1}(E) = f(E) = f(\overline{E}) = \overline{f(E)},$$

   which is closed in Y. Therefore f^{-1} is continuous as well. This finishes the proof.

$\square$

Chapter 9

Compactness

Definition 9.1: Let Λ be an indexing set, and let $\Phi = \{A_\alpha \mid \alpha \in \Lambda\}$ be a collection of sets. Then Φ is a *cover* of the set Y if

$$Y \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha.$$

Any subcollection of Φ that also covers Y is called a *subcover*.

Definition 9.2: Let $(X, \mathcal{T})$ be a topological space, and let $Y \subseteq X$. A cover Φ of Y is called an *open cover* of Y if each member of Φ is an open subset of X.

Definition 9.3: Let $(X, \mathcal{T})$ be a topological space, and let $K \subseteq X$. We say that K is *compact* if every open cover of K has a finite subcover.

Theorem 9.4: Let $(X, \mathcal{T})$ be a topological space, and let $Y \subseteq K \subseteq X$. If K is compact and Y is closed, then Y is compact.

Proof. Assume that K is compact and Y is closed. Then $A = X \setminus Y$ is an open set with $K \setminus Y \subseteq A$. Let $\Phi = \{A_\alpha \mid \alpha \in \Lambda\}$ be an open cover of Y.

Then

$$K \subseteq A \cup \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right),$$

so $\{A\} \cup \Phi$ is an open cover of K. By compactness of K, this cover has a finite subcover $\{A, A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ (Note that A need not be in the subcover if the indexed sets cover all of K.) It follows that

$$Y \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha,$$

so we conclude that $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a finite subcover of Φ. Therefore, Y is compact. $\square$

Theorem 9.5: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. If $f : X \to Y$ is a continuous function, and $K \subseteq X$ is compact, then $f(K)$ is compact.

Proof. Assume that $f : X \to Y$ is continuous and that $K \subseteq X$ is compact. Let $\Phi = \{A_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $f(K)$. We have

$$f(K) \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha.$$

Hence

$$K \subseteq f^{-1}(f(K)) \subseteq f^{-1} \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} f^{-1}(A_\alpha).$$

By continuity of f, the set $f^{-1}(A_\alpha)$ is open for each $\alpha \in \Lambda$, and hence $\{f^{-1}(A_\alpha) \mid \alpha \in \Lambda\}$ is an open cover of K. By compactness of K, there exists a finite subcover $\{f^{-1}(A_{\alpha_i})\}_{i=1}^n$ of K. Thus

$$K \subseteq \bigcup_{i=1}^n f^{-1}(A_{\alpha_i}),$$

We deduce that

$$f(K) \subseteq f \left(\bigcup_{i=1}^n f^{-1}(A_{\alpha_i}) \right) = \bigcup_{i=1}^n f(f^{-1}(A_{\alpha_i})) \subseteq \bigcup_{i=1}^n A_{\alpha_i}.$$

Hence $\{A_{\alpha_i}\}_{i=1}^n$ is a finite subcover of Φ, from which it follows that $f(K)$ is compact. $\square$

Exercise 9.6: Let $(X, \mathcal{T})$ be a topological space, where X is any set and $\mathcal{T}$ is the discrete topology. Describe the compact subsets of X.

Proof. Let X be any set and let $\mathcal{T}$ be the discrete topology on X.

First let $A = \{x_\alpha \mid \alpha \in \Lambda\}$ be an infinite subset of X. Since $\mathcal{T}$ is the discrete topology, we see that $\{x_\alpha\}$ is open in X for each $\alpha \in \Lambda$. Hence $\{\{x_\alpha\} \mid \alpha \in \Lambda\}$ is an open cover of A. But this open cover has no finite subcover, since any finite subcollection fails to cover infinitely many elements of A. Thus infinite subsets of X are never compact under the discrete topology.

Now let $A = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of X. Let $\{O_\alpha \mid \alpha \in \Lambda\}$ be an open cover of A. Then each element of A is contained in at least one element of this open cover. Thus there exist indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x_1 \in O_{\alpha_1}$, $x_2 \in O_{\alpha_2}$, and so on. This implies that $\{O_{\alpha_i}\}_{i=1}^n$ is a finite subcover of A. Therefore, A is compact. $\square$

Definition 9.7: Let $A \subseteq \mathbb{R}$. Then M is an *upper bound* for A if $M \geq x$ for all $x \in A$. If A is bounded above, we say that ℓ is a least upper bound for A provided the following hold:

1. We have $\ell \geq x$ for all $x \in A$.
2. If M is another upper bound for A, then $\ell \leq M$.

The *completeness axiom* for $\mathbb{R}$ states that every subset of $\mathbb{R}$ that is bounded above has a least upper bound. An analogous statement holds for subsets bounded below.

Exercise 9.8: Equip $\mathbb{R}$ with the standard topology.

1. Is the open interval $(0, 1)$ compact in $\mathbb{R}$? Justify your answer.

2. Prove that the interval $[0, 1]$ is compact in $\mathbb{R}$.

Proof. Equip $\mathbb{R}$ with the standard topology.

1. The open interval $(0, 1)$ is not compact in $\mathbb{R}$. We construct an open cover that has no finite subcover.

   Define $\Phi = \{(1/n, 1) \mid n \in \mathbb{N}\}$. We claim that Φ is an open cover of $(0, 1)$. First note that each interval of the form $(1/n, 1)$ is open in $\mathbb{R}$. Now let $x \in (0, 1)$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ with $1/N < x$. Thus $x \in (1/N, 1)$, and we infer that

   $$(0, 1) \subseteq \bigcup_{n \in \mathbb{N}} (1/n, 1).$$

   Thus Φ is an open cover.

   Now we show that Φ has no finite subcover. Let $\{(1/n_1, 1), (1/n_2, 1), \ldots, (1/n_k, 1)\}$ be a finite subcollection of Φ. Set $N = \max\{n_1, n_2, \ldots, n_k\}$. Then $x = \dfrac{1}{2N}$ is positive but $x < \dfrac{1}{n_i}$ for all $1 \le i \le k$. This proves that Φ has no finite subcover. It follows that $(0, 1)$ is not compact.

2. Let $\Phi = \{A_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $[0, 1]$. Define

   $$\Psi = \{x \in [0, 1] \ : \ [0, x] \subseteq \cup_{i=1}^{n} A_{\alpha_i} \text{ for some } n \in \mathbb{N}\},$$

   the set of all numbers in $[0, 1]$ that can be covered by finitely many elements of Φ. The set Ψ is nonempty because $0 \in \Psi$.

   Since Ψ is nonempty and bounded above, $t = \sup \Psi$ exists by the Axiom of Completeness. Note that since $t \in [0, 1]$, there exists $A_\beta \in \Phi$ with $t \in A_\beta$.

   Since A_β is open, there exists $\epsilon > 0$ such that $(t - \epsilon, t + \epsilon) \subseteq A_\beta$. Now $t - \epsilon$ is not an upper bound for Ψ, so it follows that there exists $x \in \Psi$ with $x > t - \epsilon$. Hence there is a finite collection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}\}$ of sets in Φ that covers $[0, x]$. Consequently, the finite collection $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}, A_\beta\}$ covers $[0, t + \epsilon)$.

Now if $t \neq 1$, then $t < 1$ and the finite cover captures reals greater than t, contradicting that $t = \sup \Psi$. Therefore, $t = 1$ and we conclude that $[0, 1]$ has a finite subcover.

□

Theorem 9.9: Equip $\mathbb{R}$ with the standard topology. A subset K of $\mathbb{R}$ is compact if and only if K is closed and bounded.

Proof. Assume that K is closed and bounded. Since K is bounded, there exist $s, t \in \mathbb{R}$ such that $s \leq x$ for all $x \in K$ and $t \geq x$ for all $x \in K$. Thus $K \subseteq [s, t]$. Since K is a closed subset of a compact set, it follows by Theorem 9.4 that K is compact.

Assume that K is compact. For each $x \in K$, put $I_x = (x - 1, x + 1)$. Then $\Phi = \{I_x \mid x \in K\}$ is an open cover of K with finite subcover $\{I_{x_1}, \ldots, I_{x_n}\}$. Set $m = \min\{x_i - 1 \mid 1 \leq i \leq n\}$ and set $M = \max\{x_i + 1 \mid 1 \leq i \leq n\}$. Thus K is bounded below by m and above by M.

Still assuming that K is compact, let $p \in K^c$. We claim that K^c is open, from which it will follow that K is closed. For each $x \in K$, there exists open sets V_x and U_x such that $p \in U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$. (Just halve the distance between p and x and then build open intervals around them with this half-distance as the radius of each interval.) Then $\{U_x \mid x \in K\}$ is an open cover of K, and so by compactness has a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ for some $n \in \mathbb{N}$. Let $\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}$ be the corresponding intervals about p. Then $V = \cap_{i=1}^{n} V_{x_i}$ is open and $p \in V$.

We claim that $V \subseteq K^c$. Suppose that $y \in V \cap K$. Then $y \in V_{x_i}$ but $y \notin U_{x_i}$ for each $1 \leq i \leq n$. Hence $y \notin \cup_{i=1}^{n} U_{x_i}$, contradicting that these sets cover K. Therefore, K^c is open and thus K is closed. □

Theorem 9.10: Extreme value theorem Let $f : [a, b] \to \mathbb{R}$ be continuous, where $a, b \in \mathbb{R}$ and $\mathbb{R}$ has the standard topology. Then there exist points c and d in $[a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for every $x \in [a, b]$.

Proof. Assume that $f : [a, b] \to \mathbb{R}$ is continuous, where $a, b \in \mathbb{R}$ and $\mathbb{R}$ has the standard topology.

Since the continuous image of a compact set is compact, it follows that $f([a, b]) = \{f(x) \mid x \in [a, b]\}$ is compact. By Theorem 9.9, we see that $\mathcal{F} := f([a, b])$ is bounded. Hence $s = \sup \mathcal{F}$ and $t = \inf \mathcal{F}$ exist by the Axiom of Completeness.

Note that $t \leq f(x) \leq s$ for every $x \in [a, b]$, so it remains to show that there exists $c \in [a, b]$ such that $t = f(c)$. An analogous argument will show that there exists $d \in [a, b]$ such that $s = f(d)$.

We assert that t is a limit point of $\mathcal{F}$. Let $\mathcal{O} = (t - \epsilon, t + \epsilon)$ be an open interval containing t. If $\mathcal{O}$ contains no point of $\mathcal{F}$, then $t + \epsilon$ is a greater lower bound for $\mathcal{F}$ than t, contradicting that $t = \inf \mathcal{F}$. Hence t is a limit point of $\mathcal{F}$. By Theorem 9.9, we must have $t \in \mathcal{F}$. This proves that there exists $c \in [a, b]$ such that $t = f(c)$.

The proof is complete. $\square$

Chapter 10

Separability

Definition 10.1: Let $(X, \mathcal{T})$ be a topological space.

1. The space X is called a T_0-space if, for each pair of distinct elements of X, there is an open set U containing one of the elements but not the other.

2. The space X is called a T_1-space if, for each pair of distinct elements x and y of X, there is an open set U containing x but not y.

3. The space X is called a *Hausdorff space* (or a T_2-space) if, for each pair of distinct elements x and y of X, there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

4. The space X is called *regular* if, for each closed subset K of X and each point $x \in X \setminus K$, there are disjoint open sets U and V such that $K \subseteq U$ and $x \in V$.

5. The space X is called *normal* if, for each pair E and F of disjoint closed subsets of X, there are disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Theorem 10.2: Let $(X, \mathcal{T})$ be a topological space. If, for each $x \in X$, we have that $\{x\}$ is closed, then

$$X \text{ is normal} \Rightarrow X \text{ is regular} \Rightarrow X \text{ is Hausdorff} \Rightarrow X \text{ is } T_1 \Rightarrow X \text{ is } T_0.$$

Exercise 10.3: Let $X = \mathbb{R}$ with the standard topology. In the chain of implications above, what is the leftmost (strongest) condition that X satisfies?

Conjecture: $(\mathbb{R}, \mathcal{T}_s)$ satisfies up to being a normal space.

Proof. If we let

$$x - \delta < x + \delta < z$$

for $\delta, x, z \in \mathbb{R}$ such that $\delta > 0$, then we have that

$$z \notin (x - \delta, x + \delta) = U$$

but

$$x \in (x - \delta, x + \delta) = U.$$

Therefore there is an open set U containing an element x disjoint from z. Thus X is a T_0 space. Then we can define

$$x - \delta < x + \delta \leq y - \epsilon < y + \epsilon$$

for $x, y, \delta, \epsilon \in \mathbb{R}$ with $\delta, \epsilon > 0$. We then have that

$$(x - \delta, x + \delta) \cap (y - \epsilon, y + \epsilon) \neq \emptyset,$$

but $x \in U_1 = (x - \delta, x + \delta)$ and $y \notin U_1$. Thus there is an open set U_1 containing x but not y and vice versa. Thus X is a T_1 space as well. Now, given the same conditions previously, we can define our interval as such:

$$x - \delta < x + \delta < y - \epsilon < y + \epsilon,$$

and thus we have disjoint open intervals

$$U_1 = (x - \delta, x + \delta), U_2 = (y - \epsilon, y + \epsilon)$$

such that $x \in U_1$ and $y \in U_2$. Hence X is also a Hausdorff Space. Next, let $k = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Let $x \notin k$. Thus $x \in \mathbb{R} \setminus \{k\}$. Further define the open interval

$$U = (a - \epsilon, b + \epsilon),$$

for some real number $\epsilon > 0$. Thus we have $K \subseteq U$. If we then let

$$b + \epsilon < x - \delta$$

for some real number $\delta > 0$, we have that

$$x \in V = (x - \delta, x + \delta)$$

and

$$U \cap V = \emptyset.$$

This demonstrates that X is a regular space as well. Finally, let

$$E = [a, b], F = [c, d]$$

for $a, b, c, d \in \mathbb{R}$ such that $a < b < c < d$ so that $E \cap F = \emptyset$. Let

$$U = (a - \epsilon, b + \epsilon)$$

$$V = (c - \delta, d + \delta)$$

for some $\epsilon, \delta > 0$ such that $b + \epsilon < c - \delta$. Thus we have that

$$U \cap V = \emptyset$$

and

$$[a, b] \subseteq (a - \epsilon, b + \epsilon)$$

$$[c, d] \subseteq (c - \delta, d + \delta),$$

as in,

$$E \subseteq U$$

$$F \subseteq V.$$

Thus X is also a normal space. Therefore the set of real numbers under the standard topology is a T_0 space, a T_1 space, a Hausdorff space, a regular space, and a normal space. □

Exercise 10.4: Let $X = \mathbb{R}$ with the finite complement topology. In the chain of implications above, what is the strongest condition that X satisfies?

Theorem 10.5: A space X is a T_1 space if and only if singleton sets are closed.

Proof. Assume that X is a T_1 space. Let $\{z\}$ be a singleton set in X. Then for every $w \in X$ with $w \neq z$, there exists an open set U_w containing w but not z. We have that

$$U = \bigcup_{w \in X,\ w \neq z} U_w$$

is open in X. It now follows that $\{z\} = X \setminus U$ is closed in X.

Assume that singleton sets are closed in X. Let z and w be distinct elements of X. Since $\{z\}$ is closed, it follows that $X \setminus \{z\}$ is an open set containing w but not z. Therefore, X is a T_1 space. The proof is complete. $\square$

Theorem 10.6: Let X be a Hausdorff space.

1. If $Y \subseteq X$, then Y is Hausdorff in the subspace topology on Y.
2. Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a collection of Hausdorff spaces. Then

$$\prod_{\alpha \in \Lambda} X_\alpha$$

is Hausdorff with respect to the product topology.

Proof. Let X be a Hausdorff space.

1. Assume that $Y \subseteq X$, and let $x, y \in Y$ be distinct points. Then x and y are distinct in X, so by assumption there exist disjoint open sets U and V containing x and y respectively. Thus $U \cap Y$ and $V \cap Y$ are disjoint open sets in the subspace topology on Y with $x \in U \cap Y$ and

$y \in V \cap Y$. Therefore, Y is Hausdorff with the subspace topology on Y.

2. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a collection of Hausdorff spaces, and let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the Cartesian product of these spaces under the product topology.

   Let $\mathbf{x} = (x_\alpha)_{\alpha\in\Lambda}$ and $\mathbf{y} = (y_\alpha)_{\alpha\in\Lambda}$ be distinct elements of X. Let $\beta \in \Lambda$ be such that $x_\beta \neq y_\beta$. (At least one such index exists, since $\mathbf{x} \neq \mathbf{y}$.) Since X_β is Hausdorff by assumption, there exist disjoint open sets U_β and V_β with $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$.

   Now define
   $$\mathcal{U} = \prod_{\alpha\in\Lambda} U_\alpha \quad \text{and} \quad \mathcal{V} = \prod_{\alpha\in\Lambda} V_\alpha,$$
   where $U_\alpha = X_\alpha$ for all indices except β and $V_\alpha = X_\alpha$ for all indices except β. Then
   $$\mathcal{U} \cap \mathcal{V} = \prod_{\alpha\in\Lambda} (U_\alpha \cap V_\alpha).$$
   Clearly $U_\alpha \cap V_\alpha = U_\beta \cap V_\beta$ for $\alpha = \beta$ and $U_\alpha \cap V_\alpha = X_\alpha$ otherwise. It is also clear that $\mathbf{x} \in \mathcal{U}$ and $\mathbf{y} \in \mathcal{V}$. Now if $\mathcal{U} \cap \mathcal{V} \neq \emptyset$, then there exists an element in the intersection with its β-th coordinate in $U_\beta \cap V_\beta$, contradicting that $U_\beta \cap V_\beta = \emptyset$.

   Therefore, $\mathcal{U}$ and $\mathcal{V}$ are disjoint open sets containing $\mathbf{x}$ and $\mathbf{y}$ respectively. This finishes the proof that X is Hausdorff.

□

Theorem 10.7: Let X be a Hausdorff space. The following hold:

1. Finite sets are closed

2. The point x is a limit point of a subset A of X if and only if each open set containing x contains infinitely many elements of A.

Proof. Let X be a Hausdorff space.

1. A finite set is a union of finitely many singletons. By Theorem 10.2, the fact that X is Hausdorff implies that it is T_1. Thus all of these singleton sets are closed by Theorem 10.5. Since the finite union of closed sets is closed in a topological space, it follows that finite sets are closed.

2. Assume that x is a limit point of a subset A of X. Let $U \subseteq X$ be open and let $x \in U$. Suppose that the set $F = \{p \in U \mid p \in A \text{ and } p \neq x\}$ is finite. By the first part of this theorem, we have that F is closed, and hence $U \setminus F$ is an open set in X containing x but containing no point of A distinct from x. This contradicts that x is a limit point of A, so we conclude that U contains infinitely many elements of A.

   Assume that each open set in X contains infinitely many elements of A. Let U be an open set with $x \in U$. Since U contains infinitely many elements of A, it follows that one of them must be distinct from x. Hence x is a limit point of $A \subseteq X$.

□

Theorem 10.8: A space X is regular if and only if, for each $x \in X$ and each open set U containing x, there is an open set V such that $x \in V$ and $\overline{V} \subseteq U$.

Proof. Assume that X is regular. Let $U \subseteq X$ be open with $x \in U$. Since $X \setminus U$ is a closed set with $x \notin X \setminus U$, by regularity there exists an open set W containing $X \setminus U$ and an open set V containing x such that $V \cap W = \emptyset$. Since $X \setminus U \subseteq W$, it follows by elementary set theory that $X \setminus W \subseteq U$. Furthermore, $X \setminus W$ is closed and contains V, so we must have

$$x \in V \subseteq \overline{V} \subseteq X \setminus W \subseteq U.$$

Now assume that for each open set U containing x, there is an open set V such that $x \in V$ and $\overline{V} \subseteq U$. Let $C \subseteq X$ be closed with $x \notin C$. Then $X \setminus C$ is an open set containing x, so by assumption there exists an open set V such that $x \in V$ and $\overline{V} \subseteq X \setminus C$. By elementary set theory, we have that $C \subseteq X \setminus \overline{V}$, which is open. Since $V \cap (X \setminus \overline{V}) = \emptyset$, it follows that X is regular. $\square$

Theorem 10.9: A space X is normal if and only if, for each closed set K and each open set U containing K, there is an open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$.

Proof. Assume that X is normal. Let $U \subseteq X$ be open with $K \subseteq U$. Since $X \setminus U$ is a closed set with $K \cap (X \setminus U) = \emptyset$, by normality there exists an open set W containing $X \setminus U$ and an open set V containing K such that $V \cap W = \emptyset$. Since $X \setminus U \subseteq W$, it follows by elementary set theory that $X \setminus W \subseteq U$. Furthermore, $X \setminus W$ is closed and contains V, so we must have

$$K \subseteq V \subseteq \overline{V} \subseteq X \setminus W \subseteq U.$$

Now assume that for each open set U containing K, there is an open set V such that $K \subseteq V$ and $\overline{V} \subseteq U$. Let $C \subseteq X$ be closed with $K \cap C = \emptyset$. Then $X \setminus C$ is an open set containing K, so by assumption there exists an open set V such that $K \subseteq V$ and $\overline{V} \subseteq X \setminus C$. By elementary set theory, we have that $C \subseteq X \setminus \overline{V}$, which is open. Since $V \cap (X \setminus \overline{V}) = \emptyset$, it follows that X is normal. $\square$

Exercise 10.10: Show that the Hausdorff property is not preserved by continuous functions. In other words, find an example of a continuous function $f : X \to Y$ such that X is Hausdorff but Y is not.

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$, where $\mathbb{R}$ as the domain has the discrete topology and $\mathbb{R}$ as the codomain has the trivial topology. Then $\mathbb{R}$ is Hausdorff as the domain because two distinct elements of $\mathbb{R}$ can be separated by their disjoint singleton sets, but $\mathbb{R}$ as the codomain is not Hausdorff because any two distinct elements belong to the same open set. $\square$

Theorem 10.11: Let X and Y be spaces, and let $f : X \to Y$ be an open bijection. If X is Hausdorff, then Y is Hausdorff.

Proof. Let $f : X \to Y$ be an open bijection, and assume that X is Hausdorff. Let y_1 and y_2 be distinct elements of Y. Since f is bijective, we have that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ are distinct elements of X. Since X is Hausdorff, there exist disjoint open sets U and V containing x_1 and y_1 respectively.

Now f is an open bijection, so $f(U)$ and $f(V)$ are open sets containing y_1 and y_2 respectively. If $f(U) \cap f(V) \neq \emptyset$, then there exists z in the intersection. Because f is a bijection, z has a single preimage, which is in both U and V. This contradicts that U and V are disjoint. Therefore, $f(U)$ and $f(V)$ are *disjoint* open sets containing y_1 and y_2 $\square$

Theorem 10.12: A compact subset of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space and let $K \subseteq X$ be compact. We will prove that $X \setminus K$ is open.

Let $x \in X \setminus K$. Since X is Hausdorff, it follows that for each $y \in K$ there exist disjoint open sets U_y and V_y containing y and x respectively. Then $\{U_y \mid y \in K\}$ is an open cover of K, and so has a finite subcover $\{U_{y_1}, \ldots, U_{y_n}\}$ by compactness of K. Let $\{V_{y_1}, \ldots, V_{y_n}\}$ be the corresponding disjoint open sets containing x.

Thus $V = \cap_{i=1}^{n} V_{y_i}$ is an open set containing x that does not intersect the union of elements in the finite subcover of K. This proves that $V \subseteq X \setminus K$, from which it follows that K is closed. The proof is finished. $\square$

Theorem 10.13: A continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.

Proof. Let X be a compact space and let Y be a Hausdorff space. Let

$f : X \to Y$ be a continuous bijection of X onto Y. We need only show that $f^{-1} : Y \to X$ is continuous.

Let $C \subseteq X$ be closed. Since X is compact, and since closed subsets of compact spaces are compact, it follows that C is compact. Since the continuous image of a compact set is compact, we must have $f(C)$ is compact in Y. Since a compact subset of a Hausdorff space is closed, we see that $f(C)$ is closed. Hence $(f^{-1})^{-1}(C) = f(C)$ is closed, and thus f^{-1} is continuous. $\square$

Theorem 10.14: A compact Hausdorff space is regular.

Proof. Let X be a compact Hausdorff space. Let $K \subseteq X$ be closed and let $x \in X$ with $x \notin K$. Since X is Hausdorff, we have that there exist open sets U_y and V_y containing y and x respectively for each $y \in K$.

The set $\{U_y \mid y \in K\}$ is an open cover of K, and so by compactness has a finite subcover $\{U_{y_1}, \ldots, U_{y_n}\}$ for some $n \in \mathbb{N}$. Let $\{V_{y_1}, \ldots, V_{y_n}\}$ be the corresponding disjoint open sets containing x.

Then $\cap_{i=1}^{n} V_{y_i}$ is an open set containing x that does not intersect the union of the elements in the finite subcover of K. It follows that this union and the intersection above are disjoint open sets containing K and x respectively, whence X is regular. $\square$

Theorem 10.15: A compact Hausdorff space is normal.

Proof. Let X be a compact Hausdorff space. Let K and L be disjoint closed subsets of X. Since X is regular, it follows that for each $x \in L$, there exist disjoint open sets U_x and V_x with $K \subseteq U_x$ and $x \in V_x$. Then $\mathcal{V} = \{V_x \mid x \in L\}$ is an open cover of L. Since L is compact, there is a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}$ of $\mathcal{V}$ that covers L.

For each $1 \leq i \leq n$, the sets V_{x_i} and the corresponding U_{x_i} are disjoint. Therefore, $U = \cap_{i=1}^{n} U_{x_i}$ and $V = \cup_{i=1}^{n} V_{x_i}$ are disjoint open sets such that $K \subseteq U$ and $L \subseteq V$. This finishes the proof. $\square$

Chapter 11

Connectedness

Definition 11.1: If a topological space $(X, \mathcal{T})$ is not the union of two nonempty disjoint open sets, then we say that it is *connected*. A subset Y of X is connected if $(Y, \mathcal{T}_Y)$ is connected.

Theorem 11.2: The space X is connected if and only if the only subsets of X that are both open and closed are $\varnothing$ and X.

Proof. Assume that X is connected. Toward contradiction, suppose that A is a nonempty proper subset of X that is both open and closed. Then $B = X \setminus A$ is also open, and we have $X = A \cup B$, the union of two nonempty disjoint open sets. This contradicts that X is connected. Thus the only subsets of X that are both open and closed are $\emptyset$ and X.

Assume that the only subsets of X that are both open and closed are $\emptyset$ and X. Let A and B be two nonempty disjoint open sets in X. Toward contradiction, suppose that $X = A \cup B$. Since A is open, we have $B = X \setminus A$ is closed. Similarly, since B is open, we have $A = X \setminus B$ is closed. This contradicts the assumption that the only subsets of X that are both open and closed are $\emptyset$ and X. Therefore, X is connected. □

Exercise 11.3: Justify your answers to the following:

1. Is $\mathbb{R}$ connected with the lower limit topology?
2. Is $\mathbb{R}$ connected with the standard topology?

Proof.

1. No. Clearly $[0, \infty)$ is open in the lower limit topology. The set $(-\infty, \infty)$ is open as well, since it is the union of the basic open sets of the form $[x, 0)$ for $x < 0$. Since $\mathbb{R}$ is the union of these two nonempty disjoint open sets, it follows that $\mathbb{R}$ is not connected with the lower limit topology.

2. Yes. Suppose that A and B are nonempty disjoint open sets such that $\mathbb{R} = A \cup B$. Without loss of generality, we can assume there exists $a \in A$ with $a < b$ for some $b \in B$. Then the set of all elements in A that are less than b is nonempty and bounded above by b, so this set has a supremum s.

   Either $s \in A$ or $s \in B$. If $s \in A$, then there exists $\delta > 0$ such that $s + \delta < b$ and $(s - \delta, s + \delta) \subseteq A$. But then $s + \delta/2 \in A$ and $s + \delta/2 < b$, contradicting that s is an upper bound for the set of all elements less than b.

   If $s \in B$, then there exists $\delta > 0$ such that $s - \delta < b$ and $(s - \delta, s + \delta) \subseteq B$. But then $s - \delta/2 \in B$ and $s - \delta/2 < b$, contradicting that s is the *least* upper bound for the set of all elements less than b.

   Therefore, $\mathbb{R}$ is connected with the standard topology.

□

Theorem 11.4: Let X be a space, and let A and B be connected subsets of X, with $A \cap B \neq \varnothing$. Then $A \cup B$ is connected.

Proof. Assume that A and B are connected subsets of X with $A \cap B \neq \emptyset$. Toward contradiction, let $A \cup B = C \cup D$, where C and D are nonempty disjoint open sets in the subspace topology on $A \cup B$.

Since there exists $x \in A \cap B$, we can assume without loss of generality that $x \in C$. Since A and B are both connected, this forces $A, B \subseteq C$. Thus $A \cup B \subseteq C$, contradicting that D is nonempty. We conclude that $A \cup B$ is connected. □

Theorem 11.5: Let X be a space, and let $A \subseteq X$. If A is connected, and if $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof. Assume that A is connected and that $A \subseteq B \subseteq \overline{A}$. We seek a contradiction by supposing that $B = C \cup D$, where C and D are nonempty disjoint open sets in the subspace topology on B.

Since A is connected, it must lie completely in C or D, so it is no loss to assume that $A \subseteq C$. Note that, since D is open and $C = B \setminus D$, we have that C is closed in the subspace topology on B. Hence

$$\overline{A} \subseteq \overline{C} = C.$$

Since nothing in B lies outside of $\overline{A}$, it follows that all of B lies in C, which contradicts that D is nonempty. Therefore, B is connected.

□

Theorem 11.6: The continuous image of a connected space is connected.

Proof. Assume that $f : X \to Y$ be a continuous map between topological spaces X and Y, and that C is a connected subset of X. By restricting the range of f to $f(C)$, we obtain a continuous and surjective map $g : X \to f(C)$.

Toward contradiction, suppose that $f(C) = A \cup B$ is a separation of $f(C)$ into two disjoint nonempty open sets in the subspace topology on $f(C)$. We claim that $X = g^{-1}(A) \cup g^{-1}(B)$ is a separation of X into two disjoint nonempty open sets.

Since g is continuous, both sets are open. Since g is surjective, both sets are nonempty. To see that they are disjoint, let x be in their intersection.

Then $g(x) \in A \cap B$, contradicting that A and B are disjoint.

This contradicts that X is connected, so we conclude that the continuous image of a connected space is connected. □

Theorem 11.7: Intermediate Value Theorem Let $a, b \in \mathbb{R}$, and assume that $[a, b]$ is connected. If $f : [a, b] \to \mathbb{R}$ is continuous, and if r is a real number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = r$.

Proof. Let $a, b \in \mathbb{R}$ and assume that $[a, b]$ is connected. Assume that $f : [a, b] \to \mathbb{R}$ is continuous and that r is a real number with $f(a) < r < f(b)$.

Set $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Then A and B are disjoint, they are nonempty since $f(a)$ is in one and $f(b)$ is in the other, and they are open in the subspace topology on $f(X)$.

Suppose there exists no $c \in [a, b]$ such that $f(c) = r$. Then $f(X) = A \cup B$ is a separation of $f(X)$, contradicting that the image of a connected set under a continuous map is connected. Therefore, there exists $c \in [a, b]$ such that $f(c) = r$.

□

Chapter 12

The Cantor Set

Definition 12.1: Following the class presentation, we define the Cantor set in two ways, as follows:

1. The *Cantor set* is the set of all $x \in [0,1]$ that have ternary expansions $0.x_1x_2x_3\ldots$, where $x_n \in \{0,2\}$ for each $n \in \mathbb{N}$.

2. Let $C_0 = [0,1]$, and for each $n \in \mathbb{N}$, let C_{n+1} be the subset of C_n obtained by removing the open middle thirds of the 2^n components of C_n. Then $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$ is the *Cantor set*.

Theorem 12.2: For each $n \in \mathbb{N}$, let $X_n = \{0,2\}$, and endow X_n with the discrete topology. Put $X = \prod_{n\in\mathbb{N}} X_n$, and endow X with the product topology T. Endow the Cantor set, $\mathcal{C}$, with the topology S it inherits as a subspace of $[0,1]$. Then $(\mathcal{C}, S) \simeq (X, T)$.

Proof. Observe that if $x = 0.x_1x_2x_3\ldots \in \mathcal{C}$, then this number has a unique ternary expansion such that $x_k \in \{0,2\}$ for all $k \in \mathbb{N}$. Henceforth we will assume this ternary expansion for all $x \in \mathcal{C}$.

Define a function $f : \mathcal{C} \to X$ by $f(0.x_1x_2x_3\ldots) = (x_1, x_2, x_3, \ldots)$. We claim that f is a bijection. Let $f(x) = f(y)$ for some $x, y \in \mathcal{C}$. Since two sequences are equal if and only if each of their corresponding entries are equal, it follows that $x_i = y_i$ for each $i \in \mathbb{N}$. Hence

$$x = 0.x_1x_2x_3\ldots = 0.y_1y_2y_3\ldots = y.$$

Now let $y = (x_1, x_2, x_3, \ldots) \in X$. Then for $x = (0.x_1x_2x_3\ldots) \in \mathcal{C}$ we have $f(x) = y$. Thus f is bijective as claimed.

We claim that $\mathcal{C}$ is compact. At each stage of construction of the middle-thirds Cantor set, a union of open intervals is subtracted from $[0, 1]$. Thus for each $n \in \mathbb{N}$, we have that C_n is closed. Since the intersection of arbitrarily many closed sets is closed in a topological space, it follows that $\mathcal{C} = \cap_{i=1}^{\infty} C_n$ is closed. The interval $[0, 1]$ bounds $\mathcal{C}$, so we infer from the Heine-Borel Theorem that $\mathcal{C}$ is compact.

We claim that X is Hausdorff. Under the discrete topology, any two distinct points x and y can be separated by the open sets $\{x\}$ and $\{y\}$, so $X_n = \{0, 2\}$ is Hausdorff for each $n \in \mathbb{N}$. From a theorem, we see that X must be Hausdorff as well.

Finally, we claim that f is continuous. Let $n \in \mathbb{N}$ and let $p_n : X \to X_n$ be the projection onto the n-th factor space in X. By a theorem, it suffices to show that $p \circ f : \mathcal{C} \to X_n$ is continuous. Let $\epsilon > 0$ and set $\delta = \frac{1}{3^n}$. Suppose that $|x - y| < \delta$ for two elements $x, y \in X_n$. Then x and y are equal in their n-th decimal places, so we have

$$|p(f(x)) - p(f(y))| = |x_n - y_n| = 0 < \epsilon.$$

Therefore f is continuous.

Since f is a continuous bijection from a compact space into a Hausdorff space, it follows that f is a homeomorphism. Therefore, $(\mathcal{C}, S) \simeq (X, T)$.

$\square$