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# Shape Analysis via Oriented Distance Functions

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The object of this paper is twofold. We first present constructions which induce topologies on subsets of a fixed domain or hold-all $D$ in $\mathbb{R}^N$ by using set parameterized functions in an appropriate function space. Second, we study the role of the family of *oriented distance functions* (also known as algebraic or signed distance functions) in the analysis of shape optimization problems. They play an important role in the introduction of topologies which retain the classical geometric properties associated with sets: convexity, exterior normals, mean curvature, $C^k$ boundaries, etc. 

## 1. INTRODUCTION

In shape optimization problems the independent variable is a geometric domain in the Euclidean space $\mathbb{R}^N$. This class of problems is significantly different from problems where the independent variable is a vector of parameters or functions in a vector space setup. To obtain existence theorems the choice of a topology is very critical and strongly dependent on the problem at hand. In an optimization problem the topology must simultaneously provide some semicontinuity of the shape functional to be

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optimized and some compactness of the family of domains over which it is to be optimized.

The fundamental difficulty is that the classical geometry with Lipschitzian, $C^k$ or $C^\infty$ domains is not very well suited to induce a topology for this class of problems. Moreover, for sets of domains the convenience of the vector space structure is lost. When dealing with a family of compact subsets of a compact hold-all $D$, one of the topologies which naturally comes to mind is the Hausdorff topology. With that topology the set of all nonempty compact subsets of the compact hold-all $D$ is compact. However, this topology is so weak that very few interesting shape functionals will be semicontinuous.

A more fruitful approach has been provided by *geometric measure theory* where the topology of the domains is obtained from the $L^p(D)$-topology on the corresponding characteristic functions. This cumulated in the theory of *finite perimeter or Caccioppoli sets* which provided a complete solution to the celebrated Plateau's[1] problem. The reader is referred to the book by Giusti [16] for a modern treatment of this topic.

The underlying technique behind this approach is to embed equivalence classes of measurable subsets $\Omega$ of the measurable hold-all $D$ in the space $L^p(D)$ for some $p$, $1 \leqslant p < \infty$, via the characteristic function $\chi_\Omega$. This identification makes it possible to define a metric on the set of equivalence classes of measurable domains. In that sense we could say that Lebesque has introduced one of the fundamental ideas to put a topology not on sets but rather on equivalence classes of sets.

The Hausdorff metric is the result of a similar construction. The equivalence classes are made up of subsets of $D$ with the same closure and the distance function embeds this set of equivalence classes in the space $C(D)$ of continuous functions on $D$. It is also known (cf. Dellacherie [11]) that the set of such distance functions is closed in $C(D)$. The distance function plays a very important role in *set-valued analysis and nonsmooth analysis* where it arises in the definition of the various topologies and cones associated with nonsmooth domains (cf. [3, 6]).

This general idea of embedding equivalence classes of domains into a function space via a domain dependent function is very appealing. Another such function is the *support function of convex analysis* (cf. [24]) which provides a topology for equivalence classes of subsets of $D$ with the same closed convex hull.

In this paper we consider *oriented distance functions* which play an important role in the description of the geometric properties of domains. They are also known in the literature as *algebraic* or *signed distance*

[1] In honor of the Belgian physicist and professor J. Plateau [1] (1801–1883) who did experimental observations on the geometry of soap films.

*functions* (cf., for instance, [4; 27, p. 268]). This choice of terminology emphasizes the fact that, for a smooth domain, the associated oriented distance function defines an *orientation* of the normal to its boundary. We shall see that it enjoys many interesting properties. The smoothness of the oriented boundary functions in a neighbourhood of the boundary of a given set is equivalent to the smoothness of the boundary of that set. Similarly the convexity of the function is equivalent to the convexity of the corresponding set. In addition their gradient and their Laplacian are directly related to the *normal* and the *mean curvature* on the boundary of the set.

In Sections 2, 3, and 4 we present the basic constructions and the general approach we shall follow, and review and sharpen currently available results. When the elements of the Hessian matrix of second-order partial derivatives of the distance function are measures, we show that the corresponding set is a Caccioppoli set and new compactness theorems are obtained. In Section 5 we investigate the *oriented boundary distance functions* which retain the nice properties of the distance functions but also generate the standard geometric properties associated with sets: convexity, exterior normals, mean curvature, $C^k$ boundaries, .... For instance, a set is convex if and only if its oriented distance function is convex; it has a boundary of class $C^k$, $k \geqslant 2$, if and only if its oriented distance function is of class $C^k$ in a neighborhood of its boundary. For $D$ open such functions are closed in $C(\bar{D})$ but also in $W^{1,p}(D)$, $1 \leqslant p < \infty$, with the weak or the strong topology. For a convergent sequence in the strong topology the corresponding sequences of characteristic functions for the closure of the sets, for the closure of their complement, and for their boundary all converge in $L^p(D)$. Another side result is that the family of equivalence classes of subsets of $D$ whose boundary has a zero Lebesque measure in $\mathbb{R}^N$ is closed in $W^{1,p}(D)$ when $D$ is bounded and in $W^{1,p}_{\mathrm{loc}}(D)$ otherwise. This provides a link with geometric measure theory. When the domain is sufficiently smooth then the trace of the Laplacian of the oriented distance function on the boundary coincides with the *mean curvature* up to a normalization factor. In fact the Hessian matrix of second-order partial derivatives contains the *principal curvatures* and the *Gauss curvature* [15].

Finally, the following interesting fact comes out of our analysis: the set of equivalence classes of convex subsets of a compact hold-all $D$ is relatively compact in all the topologies we consider in this paper (Theorem 2.2(ii), Corollary to Theorem 2.4, Theorems 3.5(ii), 4.1(i), and 5.4(ii)).

Some of the results in this paper have been announced in [10] and an original application of the oriented boundary distance function to the theory of shells can be found in [10a].

## 2. Characteristic Functions in $L^p$ Spaces and Measurable Domains

In this section the variable domains $\Omega$ are (Lebesgue) measurable subsets of a fixed hold-all $D$ which is a measurable domain in $\mathbb{R}^N$. The *characteristic function* of a measurable subset $\Omega$ of $D$ is the function $\chi$ from $D$ to $\mathbb{R}$ which is equal to 1 in $\Omega$ and 0 outside. A measurable function $\chi$ is a characteristic function if and only if the following identity is verified

$$\chi(x)[1-\chi(x)]=0 \qquad \text{in } D. \tag{1}$$

The measurable set $\Omega$ associated with such a $\chi$ is given by $\{x \in D: \chi(x)=1\}$. These considerations also extend to equivalence classes of measurable subsets of $D$ and condition (1) must be verified almost everywhere in $D$. Many *shape optimization* problems such as the *transmission problem* can be relaxed to the set of equivalence classes of characteristic functions

$$X(D)=\{\chi: \chi \text{ is measurable and } \chi(x)[1-\chi(x)]=0, \text{ a.e. in } D\}. \tag{2}$$

However, this set is not convex and weak $L^2$-limits converge in the larger set

$$\operatorname{co} X(D)=\{\chi: \chi \text{ is measurable and } \chi(x) \in [0,1], \text{ a.e. in } D\}. \tag{3}$$

This type of solution is usually related to the appearance of a microstructure and arises from the fact that $X(D)$ is not closed in the weak topology of $L^2(D)$.

The following basic lemma will be useful.

Lemma 2.1. *Let $D$ be a measurable subset of $\mathbb{R}^N$ with finite measure, $K$ be a bounded subset of $\mathbb{R}^N$, and*

$$\mathscr{K}=\{k: k \text{ is measurable and } k(x) \in K \text{ a.e. in } D\}.$$

(i) *For any $p$, $1 \leqslant p < \infty$, and any sequence $\{k_n\} \subset \mathscr{K}$ the following statements are equivalent*

(a) *$\{k_n\}$ converges in $L^\infty(D)$-weak$^\star$,*

(b) *$\{k_n\}$ converges in $L^p(D)$-weak,*

(c) *$\{k_n\}$ converges in $\mathscr{D}(D)'$,*

*where $\mathscr{D}(D)'$ is the space of scalar distributions on $D$. For (c) it is assumed that $D$ is open.*

(ii) *If* $\{k_n\}$ *is a convergent sequence in* $L^p(D)$*-strong for some* $p$, $1 \leqslant p < \infty$, *then it is a convergent sequence in* $L^p(D)$*-strong for all* $p$, $1 \leqslant p < \infty$.

*Proof.* (i) It is clear that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). To prove that (c) $\Rightarrow$ (a) recall that since $K$ is bounded there exists a constant $c > 0$ such that $K \subset cB$ where $B$ is the unit ball in $\mathbb{R}^N$. By density of $\mathscr{D}(D)$ in $L^1(D)$, given any $\varphi$ in $L^1(D)$, there exists a sequence $\{\varphi_m\} \subset \mathscr{D}(D)$ such that $\varphi_m \to \varphi$ in $L^1(D)$. So for each $\varepsilon > 0$ there exists $M > 0$ such that

$$\forall m \geqslant M, \qquad \|\varphi_m - \varphi\|_{L^1(D)} \leqslant \frac{\varepsilon}{4c}.$$

Moreover, there exists $N > 0$ such that

$$\forall n \geqslant N,\ \forall l \geqslant N, \qquad \left|\int_D \varphi_M (k_n - k_l)\, dx\right| \leqslant \frac{\varepsilon}{2}.$$

Hence for each $\varepsilon > 0$, there exists $N$ such that for all $n \geqslant N$ and $l \geqslant N$,

$$\left|\int_D \varphi(k_n - k_l)\, dx\right| \leqslant \left|\int_D \varphi_M (k_n - k_l)\, dx\right| + \left|\int_D (\varphi - \varphi_M)(k_n - k_l)\, dx\right|$$

$$\leqslant \frac{\varepsilon}{2} + 2c\, \|\varphi - \varphi_M\|_{L^1(D)} \leqslant \varepsilon.$$

(ii) Assume that $\{k_n\}$ converges in $L^2(D)$. For $p > 2$ and all $n \geqslant N$ and $l \geqslant N$

$$\int_D |k_n - k_l|^p\, dx = \int_D |k_n - k_l|^2\, |k_n - k_l|^{p-2}\, dx \leqslant (2c)^{p-2} \int_D |k_n - k_l|^2\, dx$$

and the $L^p(D)$-convergence follows. For $1 \leqslant p < 2$,

$$\int_D |k_n - k_l|^p\, dx = \left\{\int_D |k_n - k_l|^2\, dx\right\}^{p/2} m(D)^{1-p/2}$$

and again the $L^p(D)$-convergence follows. Conversely if the $L^p(D)$-convergence is true for some $p$, $1 \leqslant p < \infty$, then for $1 \leqslant p \leqslant 2$,

$$\int_D |k_n - k_l|^2\, dx = \int_D |k_n - k_l|^p\, |k_n - k_l|^{2-p}\, dx \leqslant (2c)^{2-p} \int_D |k_n - k_l|^p\, dx$$

and the $L^2(D)$-convergence follows. For $p > 2$,

$$\int_D |k_n - k_l|^2\, dx = \left\{\int_D |k_n - k_l|^p\, dx\right\}^{2/p} m(D)^{1-2/p}$$

and again the $L^2(D)$-convergence follows. ▮

If $K$ is convex and compact, then the set $\mathscr{K}$ is closed for all topologies between the $L^\infty(D)$-weak$^\star$ topology and the $\mathscr{D}(D)'$ topology. In view of this equivalence we adopt the following terminology for the set $X(D)$ of characteristic functions of all measurable subsets of $D$.

DEFINITION 2.1. Let $D$ be a measurable subset of $\mathbb{R}^N$ with finite measure. We say that a sequence $\{\chi_n\}$ in $X(D)$ is weakly convergent if it converges for some topology between the $L^\infty(D)$-weak$^\star$ and the $\mathscr{D}(D)'$.

THEOREM 2.1. *Let $D$ be a measurable (resp., bounded measurable) subset of $\mathbb{R}^N$. Then $X(D)$ is closed in $L^p_{\text{loc}}(D)$-strong (resp., $L^p(D)$-strong) for all $p$, $1 \leqslant p < \infty$.*

*Proof.* It is sufficient to prove the result for $D$ with finite measure.

(i) We first prove the result in $L^2(D)$. Given any strongly convergent sequence $\{\chi_n\}$ of functions in $X(D)$, there exists a strong limit $\chi$ in $L^2(D)$:

$$\lim_{n\to\infty} \int_D |\chi_n - \chi|^2\, dx = 0.$$

But

$$\int_D |\chi_n - \chi|^2\, dx = \int_D \chi_n^2 + \chi^2 - 2\chi\chi_n\, dx = \int_D \chi_n + \chi^2 - 2\chi\chi_n\, dx$$

since $\chi_n(1-\chi_n) = 0$ a.e. in $D$. Moreover, since $D$ has finite measure

$$\int_D \chi_n\, dx \to \int_D \chi\, dx \qquad \text{and} \qquad \int_D \chi\chi_n\, dx \to \int_D \chi^2\, dx.$$

Therefore

$$0 = \lim_{n\to\infty} \int_D |\chi_n - \chi|^2\, dx = \int_D \chi(1-\chi)\, dx$$

$$\Rightarrow \chi(1-\chi) = 0 \qquad \text{a.e. in} \quad D \Rightarrow \chi \in X(D).$$

(ii) For $p$, $1 \leqslant p < \infty$, let $\{\chi_n\}$ be a sequence of characteristic functions which converges in $L^p(D)$. Note that

$$\|\chi_n - \chi_m\|^p_{L^p(D)} = \|\chi_n - \chi_m\|^2_{L^2(D)}$$

since the integrand is either 0 or 1 for all $x$ in $D$. Therefore $\{\chi_n\}$ is a Cauchy sequence in $L^2(D)$. From part (i) it converges to a unique characteristic function $\chi$ in the $L^2(D)$-norm. Since $\chi$ is a characteristic function we again use the identity

$$\|\chi_n - \chi\|^p_{L^p(D)} = \|\chi_n - \chi\|^2_{L^2(D)}$$

and conclude that $\{\chi_n\}$ converges to $\chi$ in the $L^p(D)$-norm. This completes the proof of the theorem. ∎

By analogy with Definition 2.1 we introduce the strong convergence.

DEFINITION 2.2. (i) Let $D$ be a measurable subset of $\mathbb{R}^N$ with finite measure. We say that a sequence $\{\chi_n\}$ in $X(D)$ is *strongly convergent* if it converges in $L^p(D)$-strong for some $p$, $1 \leqslant p < \infty$.

(ii) Assume that the measurable subset $D$ of $\mathbb{R}^N$ does not have finite measure. We say that a sequence $\{\chi_n\}$ in $X(D)$ is *locally strongly convergent* if it converges in $L^p_{\text{loc}}(D)$-strong for some $p$, $1 \leqslant p < \infty$.

Theorem 2.1 says that if we are interested in distinguishing between equivalent classes of Lebesgue measurable domains the strong topology in $L^p(D)$ is the natural topology where $X(D)$ is closed. For $D$ with finite measure, this induces the metric

$$\rho(\Omega_1, \Omega_2) = \|\chi_{\Omega_1} - \chi_{\Omega_2}\|_{L^1(D)}$$

which is the measure of the symmetric difference

$$\Omega_1 \Delta \Omega_2 = (\Omega_1 \backslash \Omega_2) \cup (\Omega_2 \backslash \Omega_1),$$

where $A \backslash B = \{x \in A : x \notin B\}$. Depending on the problem other appropriate measures can be constructed to "see" specific features of equivalence classes of measurable sets.

One consequence of the above considerations is that the weak convergence of characteristic functions is not quite appropriate if we are looking for maximizers or minimizers which are characteristic functions. So it seems reasonable to work with stronger topologies which lead to the strong $L^p(D)$-convergence of minimizing or maximizing sequences of characteristic functions.

LEMMA 2.2. *Assume that $D$ is a measurable (resp., bounded measurable) subset of $\mathbb{R}^N$. Let $\{\chi_n\}$ and $\chi$ be elements of $X(D)$ such that $\chi_n \rightharpoonup \chi$ weakly in $L^2_{\text{loc}}(D)$ (resp., $L^2(D)$). Then for all $p$, $1 \leqslant p < +\infty$,*

$$\chi_n \to \chi \textit{ strongly in } L^p_{\text{loc}}(D) \qquad (\textit{resp. } L^p(D)).$$

*Proof.* It is sufficient to show the result for $p=2$ and $D$ bounded and use Theorem 2.1. The strong $L^2(D)$-convergence follows from the property

$$\|\chi_n\|^2_{L^2(D)} = \int_D \chi_n 1\, dx \to \int_D \chi 1\, dx = \|\chi\|^2_{L^2(D)}. \quad \blacksquare$$

The following theorem will also be useful.

THEOREM 2.2. *Let $\Omega$ be an arbitrary Borel set in $\mathbb{R}^N$.*

(i) *There exists a sequence $\{\Omega_n\}$ of open $C^\infty$ domains in $\mathbb{R}^N$ such that*

$$\chi_{\Omega_n} \to \chi_\Omega \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^N).$$

(ii) *The set $\mathscr{C}(D)$ of equivalence classes $[\Omega]$ of convex measurable domains $\Omega$ in $D$ is closed in $X(D)$ for the strong $L^p$-topology, $1 \leqslant p < \infty$.*

*Proof.* (i) The construction of the family of $C^\infty$ domains $\{\Omega_n\}$ can be found in Giusti [16, §1.14, p. 10, and Lemma 1.25, p. 23].

(ii) Consider a sequence of convex measurable domains of $\{\Omega_n\}$ in $D$ whose characteristic functions converge to $\chi_\Omega$ in $L^1_{\text{loc}(D)}$ for some measurable domain $\Omega$ in $D$. There exists a subsequence (still denoted $\{\Omega_n\}$) which converges almost everywhere in $D$. Fix $x$ and $y$ in $\Omega$ for which the convergence takes place. There exists $N>0$ such that

$$\forall n \geqslant N, \qquad \chi_{\Omega_n}(x) = \chi_{\Omega_n}(y) = 1$$

and since $\Omega_n$ is convex for all $\lambda \in [0, 1]$,

$$\forall n \geqslant N, \qquad \chi_{\Omega_n}(\lambda x + (1-\lambda)\, y) = 1.$$

Therefore

$$\chi_\Omega(\lambda x + (1+\lambda)\, y) = 1 \quad \Rightarrow \quad \lambda x + (1-\lambda)\, y \in \Omega$$

and $\Omega$ is almost everywhere equal to a convex set. This concludes the proof. $\blacksquare$

The lack of a priori smoothness on $\Omega$ may introduce technical difficulties in the formulation of some boundary value problems. However, it is possible to relax such boundary value problems for a smooth simply connected domain $\Omega$ to a measurable domain (cf. [36]). For instance consider the homogeneous Dirichlet problem

$$\begin{cases} -\Delta y = f & \text{in } \Omega \\ \quad y = 0 & \text{on } \Gamma \end{cases} \tag{4}$$

over a simply connected domain $\Omega$ with a boundary $\Gamma$ of class $C^1$ and associate with its solution $y = y(\Omega)$ the cost functional

$$J(\Omega) = \tfrac{1}{2} \int_\Omega |y - g|^2 \, dx \tag{5}$$

with the additional constraints on the volume and perimeter of $\Omega$:

$$\int_\Omega dx = \pi, \qquad \text{and} \qquad \int_\Gamma d\gamma \leqslant 4\pi. \tag{6}$$

It was convenient to assume that $\Omega$ was a smooth simply connected domain. However there is a priori no reason to assume that an eventually optimal (minimizing) domain $\Omega^*$ be of class $C^1$. From the analysis point of view it is usually more convenient to relax constraints on the smoothness of $\Omega$ and work with the largest possible class of domains. Of course this relaxation must be done in a way which preserves the meaning of the underlying function spaces and the well-posedness of the original problem.

In problems (4)–(6) we first have to make sense of the Sobolev space for measurable subsets $\Omega$ of $D$. Assume that $D$ is a fixed bounded smooth open domain in $\mathbb{R}^N$. So the Sobolev space $H_0^1(D)$ is well defined in the usual way and we can introduce for each measurable $\Omega$ in $D$ the linear subspace

$$H_0^1(\Omega; D) = \{\varphi \in H_0^1(D) : (1 - \chi_\Omega) \nabla\varphi = 0 \text{ a.e in } D\}. \tag{7}$$

THEOREM 2.3. *$H_0^1(\Omega; D)$ as defined in* (7) *is a closed subspace of $H_0^1(D)$ and hence a Hilbert space.*

*Proof.* Let $\{\varphi_n\}$ in $H_0^1(\Omega; D)$ be a Cauchy sequence. It converges to an element $\varphi$ in the $H_0^1(D)$ topology. Hence $\{\nabla\varphi_n\}$ converges to $\nabla\varphi$ in $L^2(D)$. But for all $n$,

$$(1 - \chi_\Omega) \nabla\varphi_n = 0 \qquad \text{in } L^2(D).$$

Therefore by continuity

$$(1 - \chi_\Omega) \nabla\varphi = 0 \qquad \text{in } L^2(D).$$

So finally $\varphi \in H_0^1(\Omega; D)$ and $H_0^1(\Omega; D)$ is a closed subspace of $H_0^1(D)$. ∎

Given a measurable subset $\Omega$ of $D$ we can now consider the variantional problem

$$\begin{cases} \text{to find } y(\Omega) \in H_0^1(\Omega; D) \text{ such that} \\ \forall \varphi \in H_0^1(\Omega; D), \qquad \int_D \nabla y \cdot \nabla\varphi \, dx = \int_D \chi_\Omega f \varphi \, dx \end{cases} \tag{8}$$

and the associated cost function

$$J(\Omega) = h(\chi_\Omega, y(\Omega)), \tag{9}$$

where

$$h(\chi_\Omega, \varphi) = \tfrac{1}{2} \int_D \chi_\Omega \, |y(\Omega) - g|^2 \, dx. \tag{10}$$

This problem is now well-defined and its restriction to smooth simply connected domains coincides with the original problem (4)–(6). Indeed if $\Omega$ is a simply connected domain with a boundary $\Gamma$ of class $C^1$ then $\Gamma$ has a zero measure in $\mathbb{R}^N$ and the definition (7) of $H_0^1(\Omega; D)$ coincides with the usual one,

$$H_0^1(\Omega) = \{\varphi \in H_0^1(D) \colon (1 - \chi_\Omega)\, \varphi(x) = 0 \text{ a.e. in } D\}, \tag{11}$$

where $H_0^1(\Omega)$ is identified with the subspace of functions in $H_0^1(D)$ with support in $\Omega$. From Stampacchia [29] we know that

$$\nabla\varphi(x) = 0 \qquad \text{a.e. in} \quad D\backslash\bar{\Omega} \qquad \text{and} \qquad \varphi \in H^1(D\backslash\bar{\Omega})$$

is equivalent to

$$\varphi\,|_{D\backslash\bar{\Omega}} = \text{constant}.$$

Then $\Omega$ being simply connected, $\Omega \subset D$, and $\varphi\,|_{\partial D} = 0$ imply that the constant is equal to zero. Therefore the two definitions are equivalent for $C^1$ simply connected domains and problem (8) to (10) is a well-defined extension of problem (4)–(6). In general when $\Omega$ contains holes, this constant can be different from hole to hole.

We complete this section by quoting some results for *finite perimeter sets* which have been introduced by Caccioppoli [5] and De Giorgi [8] in the context of Plateau's Problem. Consider measurable subsets $\Omega$ of a fixed bounded open domain $D$ in $\mathbb{R}^N$ and consider their characteristic functions $\chi_\Omega$ which belong to $X(D)$. Then $\chi_\Omega$ is an $L^1(D)$-function

$$\|\chi_\Omega\|_{L^1(D)} = \int_D \chi_\Omega \, dx = \text{meas } \Omega \leqslant \text{meas } D < \infty \tag{12}$$

with a distributional gradient

$$\langle \nabla\chi_\Omega, \boldsymbol{\varphi} \rangle_{\mathscr{D}} = -\int_D \chi_\Omega \operatorname{div} \boldsymbol{\varphi} \, dx, \qquad \forall \boldsymbol{\varphi} \in (\mathscr{D}(D))^N. \tag{13}$$

We consider measurable subsets $\Omega$ of $D$ whose characteristic function belongs to the space $BV(D)$ of $L^1$-functions $f$ on $D$ with distributional gradient $\nabla f$ in the space of (vectorial) bounded measures, that is

$$\varphi \mapsto \langle \nabla f, \varphi \rangle_{\mathscr{D}} = -\int_D f \operatorname{div} \varphi \, dx : \mathscr{D}^0(D; \mathbb{R}^N) \to \mathbb{R} \tag{14}$$

is continuous with respect to the space of continuous functions in $D$ with compact support in $D$. Introduce the seminorm (cf. [12])

$$\|\nabla f\|_{M_1(D)^N} = \sup_{\substack{\varphi \in \mathscr{D}^0(D;\, \mathbb{R}^N) \\ \|\varphi\|_C \leq 1}} \langle \nabla f, \varphi \rangle_{\mathscr{D}^0},$$

where the space $\mathscr{D}^0(D; \mathbb{R}^N)$ is endowed with the norm

$$\|\varphi\|_C = \sup_{x \in D} |\varphi(x)|_{\mathbb{R}^N},$$

$M^0(D) = \mathscr{D}^0(D)'$, the topological dual of $\mathscr{D}^0(D)$, and

$$\nabla f \in \mathscr{L}(\mathscr{D}^0(D; \mathbb{R}^N), \mathbb{R}) \cong \mathscr{L}(\mathscr{D}^0(D), \mathbb{R})^N = M_1(D)^N.$$

We denote by $BV(D)$ the space

$$BV(D) = \{f \in L^1(D) : \nabla f \in M_1(D)^N\} \tag{16}$$

endowed with the norm

$$\|f\|_{BV(D)} = \|f\|_{L^1(D)} + \|\nabla f\|_{M_1(D)^N}. \tag{17}$$

For more details and properties see also [20, p. 117] and Federer [14, §4.5.9].

When the characteristic function of a measurable subset of $D$ belongs to $BV(D)$, the *perimeter of* $\Omega$ (relative to $D$) is defined as

$$P_D(\Omega) = \|\nabla \chi_\Omega\|_{M_1(D)^N} \tag{18}$$

and the $(N-1)$-dimensional Hausdorff measure $H_{N-1}$ of $\Gamma$ is given by

$$H_{N-1}(\Gamma) = P_D(\Omega) + H_{N-1}(\Gamma \cap \partial D). \tag{19}$$

The interest behind this construction is twofold. First, it extends the notion of perimeter of a set to measurable sets; second, this framework provides a compactness result which is useful in obtaining existence of optimal domains.

THEOREM 2.4. *Assume that D is a bounded open domain in* $\mathbb{R}^N$ *with a Lipschitzian boundary* $\partial D$. *Let* $\{\Omega_n\}$ *be a sequence of measurable domains in D with finite perimeter with respect to D. If there exists a constant* $c > 0$ *such that*

$$\forall n, \quad P_D(\Omega_n) \leqslant c, \tag{20}$$

*then there exists a subsequence* $\{\Omega_{n_k}\}$ *and a measurable set* $\Omega$ *in D such that*

$$P_D(\Omega) \leqslant \liminf_{k \to \infty} P_D(\Omega_{n_k}) \leqslant c. \tag{21}$$

*Moreover,*

$$\chi_{\Omega_{n_k}} \to \chi_\Omega \qquad \text{in } L^1(D) \quad \text{as} \quad k \to \infty, \tag{22}$$

*and* $\nabla \chi_{\Omega_{n_k}}$ *"converges in measure" to* $\nabla \chi_\Omega$ *in* $M_1(D)^N$, *that is, for all* $\varphi$ *in* $\mathscr{D}^0(D, \mathbb{R}^N)$,

$$\lim_{k \to \infty} \langle \nabla \chi_{\Omega_{n_k}}, \varphi \rangle_{M_1(D)^N} \to \langle \nabla \chi_\Omega, \varphi \rangle_{M_1(D)^N}. \tag{23}$$

*Proof.* This follows from the fact that the injection of the space $BV(D)$ endowed with the norm (17) is continuous and compact into $L^1(D)$ (cf. Giusti [16, Theorem 1.19, p. 17], Maz'ja [19, Theorem 6.1.4, p. 300, and Lemma 1.4.6, p. 62], and Morrey [21, Theorem 3.3.4, p. 75]). ∎

COROLLARY. *Let D verify the assumptions of Theorem* 2.4. *If* $\{\Omega_n\}$ *is a sequence of convex measurable domains in D, then there exists a constant* $c > 0$ *such that*

$$\forall n, \quad P_D(\Omega_n) \leqslant c, \quad m(\Omega_n) \leqslant c$$

*and we can find a convex measurable set* $\Omega$ *in D for which the conclusions of Theorem* 2.4 *are verified.*

*Proof.* If $\Omega_n$ is a convex set with zero volume, $m(\Omega_n) = 0$, then its perimeter $P_D(\Omega_n) = 0$ (cf. [16, Remark 1.7(iii), p. 6]). If $\Omega_n$ has a non-zero volume, then its interior $\operatorname{int} \Omega_n$ is not empty. For convex sets with a nonempty interior the perimeter and the volume enjoy a very nice monotonicity property. If $\mathscr{C}^\bullet$ denotes the set of nonempty closed convex subsets of $\mathbb{R}^N$ then the maps

$$\Omega \mapsto m(\Omega) : \mathscr{C}^\bullet \to \mathbb{R}$$

is strictly increasing

$$\forall A,\quad B\in\mathscr{C}^{\bullet},\quad A\subsetneq B\Rightarrow m(A)<m(B)$$

(cf. [4, Vol. 3, Prop. 12.9.3.3, p. 141]). Similarly if $P(\Omega)$ denotes the perimeter of $\Omega$ in $\mathbb{R}^N$, the map

$$\Omega\mapsto P(\Omega):\mathscr{C}^{\bullet}\to\mathbb{R}$$

is strictly increasing (cf. [4, Vol. 3, Prop. 12.10.2, p. 144]). As a result

$$\forall n,\quad m(\Omega_n)\leqslant m(D)<\infty,$$
$$\forall n,\quad P_D(\Omega_n)\leqslant P(\Omega_n)\leqslant P(\text{co } D)<\infty,$$

since the perimeter of a bounded convex set is finite. Then the conclusions of the theorem follow for some measurable domain $\Omega$ in $D$. In particular,

$$\chi_{\Omega_k}\to\chi_\Omega \qquad \text{in } L^1(D) \quad \text{as} \quad k\to\infty.$$

But we have seen that the set of characteristic functions of convex sets in $L^1(D)$ is closed (cf. Theorem 2.2(ii)). Therefore $\Omega$ can be chosen convex in the equivalence class. This proves the corollary. ∎

*Remark* 2.1. This corollary also says that the set of characteristic functions of convex measurable domains in a bounded open subset $D$ of $\mathbb{R}^N$ with Lipschitzian boundary is compact in $L^p(D)$, $1\leqslant p<\infty$, since $X(D)$ is already closed.

This theorem and the lower (resp., upper) semicontinuity of the shape functional $\Omega\mapsto J(\Omega)$ given by (9) could provide existence results for domains in the class of finite perimeter sets in $D$.

It is important to recall that even if a set $\Omega$ in $D$ has a finite perimeter $P_D(\Omega)$, its *relative boundary* could have a nonzero $N$-dimensional Lebesgue measure $m(\Omega)$. To illustrate this point consider the following modification of Example 1.10 in [16, p. 7].

EXAMPLE 2.1. Let $D=B(0,1)$ in $\mathbb{R}^2$ be the open ball in 0 of radius 1 and consider the open balls

$$B_i=B(x_i,\rho_i)=\{x\in D: |x-x_i|<\rho_i\},\qquad 0<\rho_i<\min\{2^{-i}, 1-|x_i|\},$$

for $i\geqslant 1$ and $\{x_i\}$ an ordered sequence of all points in $D$ with rational coordinates. Define the new sequence

$$A_n=\bigcup_{i=1}^{n} B_i$$

and note that for all $n \geqslant 1$,

$$m(\partial A_n) = 0, \qquad P_D(A_n) \leqslant 2\pi,$$

where $\partial A_n$ is the *boundary* of $A_n$. Moreover, since the sequence of sets $\{A_n\}$ is increasing,

$$\chi_{A_n} \to \chi_A \qquad \text{in } L^1(D), \quad A = \bigcup_{i=1}^{\infty} B_i$$

$$P_D(A) \leqslant \liminf_{n \to \infty} P_D(A_n) \leqslant 2\pi.$$

Now observe that $\bar{A} = \bar{D}$ and $\partial A = \bar{A} \cap \overline{\complement A} = \overline{\complement A} \cap \bar{D} = \complement A \cap \bar{D} = \complement_{\bar{D}} A$

$$m(\partial A) = m\left(\complement_{\bar{D}} A\right) \geqslant m\left(\complement_D A\right) = m(D) - m(A) \geqslant \frac{2\pi}{3}$$

since

$$m(D) = \pi \qquad \text{and} \qquad m(A) \leqslant \sum_{1=1}^{\infty} \pi 2^{-2i} = \pi\left[\frac{4}{3} - 1\right] = \frac{\pi}{3}.$$

For $p$, $1 \leqslant p < \infty$, the sequence of characteristic functions $\{\chi_{A_n}\}$ converge to $\chi_A$ in $L^p(D)$-strong and by Theorem 2.4 there exists a subsequence such that the relative perimeters converge to $P_D(A)$. However, for all $n$, $m(\partial A_n) = 0$, but $m(\partial A) > 0$.

## 3. Distance Functions and Associated Topologies

In Section 2 we have seen that the characteristic function can be used to embed the equivalence classes of measurable subsets of $D$ into $L^p(D)$ or $L^p_{\text{loc}}(D)$, $1 \leqslant p < \infty$, and that they induce a metric on the equivalence classes of measurable sets. However, the first metric that usually comes to mind is the Hausdorff metric. It turns out that the distance function plays a role similar to the characteristic function. The distance function embeds equivalence classes of subsets $A$ of a closed hold-all $D$ with the same closure $\bar{A}$ into the space $C(D)$ of continuous functions. In addition the metric induced by this construction is equal to the Hausdorff metric when $D$ is compact. The Hausdorff topology has many much desired properties. In particular for $D$ compact the set of equivalence classes of nonempty subsets $A$ of $D$ is compact. Yet the volume functional is not continuous with respect to that topology. However, distance functions also belong to $W^{1,p}(D)$ which provides a more interesting topology. In fact, we shall see that when the elements of the Hessian matrix of second-order derivatives are measures the closure of the corresponding domain is a set of locally finite perimeter. We shall also see in Section 5 how the distance function can be modified to obtain more interesting geometric properties. In this section we review some properties of distance functions and present the general philosophy which is adopted in the next sections.

Associate with each nonempty subset $A$ of $\mathbb{R}^N$ the distance function

$$d_A(x) = \inf_{y \in A} |y - x|, \qquad x \in \mathbb{R}^N.$$

If $d_A$ is finite in $\mathbb{R}^N$, then $\bar{A} \neq \varnothing$ and a fortiori $A \neq \varnothing$. We recall the following properties of distance functions (cf. [13, p. 185, §4.2, and Theorem 4.3]):

(i) $d_A(x) = 0 \Leftrightarrow x \in \bar{A}$;

(ii) $\bar{A} = \{x \in \mathbb{R}^N : d_A(x) = 0\}$;

(iii) the map $x \mapsto d_A(x)$ is uniformly Lipschitz continuous and

$$\forall x, y, \qquad |d_A(y) - d_A(x)| \leqslant |y - x|;$$

(iv) $d_A$ is differentiable almost everywhere and

$$|\nabla d_A(x)| \leqslant 1, \qquad \text{a.e. in } \mathbb{R}^N,$$

(cf. [14, Rademaker's theorem, §3.16]);

(v) $\varnothing \neq \bar{A} \subset \bar{B} \Leftrightarrow d_A \geqslant d_B$.

(vi) $d_{A \cup B} = \min\{d_A, d_B\}$.

From the above properties the distance function induces a map $A \mapsto d_A : \mathscr{P}_+(\mathbb{R}^N) \to C(\mathbb{R}^N)$ from the set $\mathscr{P}_+(\mathbb{R}^N)$ of nonempty subsets of $\mathbb{R}^N$ to the space $C(\mathbb{R}^N)$ of continuous functions on $\mathbb{R}^N$ endowed with the Fréchet topology of uniform convergence on compact subsets of $\mathbb{R}^N$. Moreover,

$$\forall B \text{ such that } A \subset B \subset \bar{A}, \text{ then } \forall x \in \mathbb{R}^N, \qquad d_A(x) = d_B(x) = d_{\bar{A}}(x)$$

(cf. [13, Problem 4.3c]), and in particular

$$\forall A, \quad \forall B \text{ such that } \bar{A} = \bar{B}, \quad d_A = d_{\bar{A}} = d_B = d_{\bar{B}}.$$

3.1. *The Set of Distance Functions in* $C(D)$

Let $D$ be a closed nonempty subset of $\mathbb{R}^N$. We can associate with each nonempty subset $A$ of $D$ the equivalence class

$$[A] = \{B : \forall B,\ B \subset D \text{ and } \bar{B} = \bar{A}\}$$

and consider the set

$$\mathscr{F}(D)=\{[A]:\forall A,\ \varnothing\neq A\subset D\}.$$

The map

$$[A]\mapsto d_A:\mathscr{F}(D)\to C(D)$$

is now injective since

$$d_A=d_B\Rightarrow \bar{A}=\bar{B}\Rightarrow [A]=[B].$$

Here $C(D)$ is endowed with the Fréchet topology of uniform convergence on compact subset of $D$. So $\mathscr{F}(D)$ can be identified with the subset of distance functions

$$C_d(D)=\{d_A:\forall A,\ \varnothing\neq A\subset D\}\subset C(D).$$

Therefore the distance function plays the same role as the characteristic function for equivalence classes of measurable sets. To complete the construction it remains to prove that $C_d(D)$ is closed in $C(D)$. For compact $D$'s this result can be found in Dellacherie [11, p. 42, Theorem 2, and p. 43, Remark 1]. Moreover, in that case we can introduce the metric

$$\rho(A,B)=\sup_{x\in D}|d_A(x)-d_B(x)|$$

on $\mathscr{F}(D)$ which turns out to be equal to the usual Hausdorff metric

$$\rho_H(A,B)=\max\{\sup_{x\in B} d_A(x),\ \sup_{y\in A} d_B(y)\}$$

(cf. [13, p. 205, Chap. IX, Problem 4.8] for the definition of $\rho_H$).

When $D$ is closed but not necessarily compact, the space $C(D)$ of continuous functions on $D$ is endowed with the Fréchet topology of uniform convergence on compact subsets $K$ of $D$. This topology is defined by the family of seminorms

$$q_K(f)=\max_{x\in K}|f(x)|,\qquad \forall K \text{ compact}\subset D.$$

It is metrizable since the topology induced by the family of seminorms $\{q_K\}$ is equivalent to the one generated by the subfamily $\{q_{K_k}\}_{k\geqslant 1}$, where the compact sets $\{K_k\}_{k\geqslant 1}$ are chosen as follows:

$$K_k=\{x\in D: d_{\complement D}(x)\geqslant \frac{1}{k} \text{ and } |x|\leqslant k\},\qquad k\geqslant 1$$

(cf. [17, Example 3, p. 116]). Moreover the topology on $C(D)$ is equivalent to the topology defined by the metric

$$\delta(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(f-g)}{1+q_{K_k}(f-g)}.$$

We see below that $C_d(D)$ is a closed subset of $C(D)$. This justifies the introduction of the following metric on $\mathscr{F}(D)$

$$\rho_\delta(A, B) = \delta(d_A, d_B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(d_A - d_B)}{1+q_{K_k}(d_A - d_B)}$$

which can be seen as a generalization of the Hausdorff metric for subsets of a closed unbounded domain $D$.

THEOREM 3.1. *Let $D$ be a nonempty closed (resp., compact) subset of $\mathbb{R}^N$.*

(i) *The set $C_d(D)$ is closed in $C(D)$ (resp., $C^0(D)$),*

(ii) *and $\rho_\delta$ (resp., $\rho$) defines a metric on $\mathscr{F}(D)$. Moreover, when $D$ is compact the metrics $\rho$ and $p_H$ are equal.*

*Proof.* (i) It is useful to go over the proof given by Dellacherie [11] since some of the constructions are used in Section 5. Consider a sequence $\{A_n\}$ of nonempty subsets of $D$ such that $d_{A_n}$ converges to some element $f$ of $C(D)$. He proves that $f = d_A$ for

$$A = \{x \in D : f(x) = 0\}$$

and that the closed subset $A$ of $D$ is nonempty.

(ii) By construction $\rho$ and $\rho_\delta$ are metrics on $\mathscr{F}(D)$. For $D$ compact and $A$ and $B$ in $D$ by definition

$$\begin{aligned}\rho(A, B) &= \max_{x \in D} |d_A(x) - d_B(x)| \\ &\geq \max\{\max_{x \in B} |d_A(x) - d_B(x)|, \max_{x \in A} |d_A(x) - d_B(x)|\} \\ &\geq \max\{\max_{x \in B} d_A(x), \max_{x \in A} d_B(x)\} = \rho_H(A, B).\end{aligned}$$

Conversely, for any $x \in D$ and $y$ in $A$,

$$d_A(x) - d_B(x) \leq |y - x| - d_B(x)$$

and there exists $x_B \in \bar{B}$ such that $d_B(x) = |x - x_B|$. Therefore

$$\forall y \in A, \qquad d_A(x) - d_B(x) \leqslant |y - x| - |x - x_B| \leqslant |y - x_B|$$
$$\Rightarrow d_A(x) - d_B(x) \leqslant \inf_{y \in A} |y - x_B| = d_A(x_B) \leqslant \max_{x \in B} d_A(x).$$

Similarly

$$d_B(x) - d_A(x) \leqslant \max_{x \in A} d_B(x)$$

and

$$\forall x \in D, \qquad |d_B(x) - d_A(x)| \leqslant \max\{\max_{x \in B} d_A(x), \max_{x \in A} d_B(x)\}$$
$$\Rightarrow \rho(A, B) \leqslant \rho_H(A, B). \quad \blacksquare$$

When $D$ is compact the set $\mathscr{F}(D)$ enjoys many more interesting properties.

THEOREM 3.2. *Let $D$ be a nonempty compact subset of $\mathbb{R}^N$ and $\mathscr{F}(D)$ the corresponding set of equivalence classes of nonempty subsets of $D$. Then*

(i) *For any subset $F$ of $D$ the set*

$$\{[A] \in \mathscr{F}(D) : \forall A,\ F \subset \bar{A}\}$$

*is closed.*

(ii) *If $S$ is open (resp., closed), then the sets*

$$I(S) = \{[A] \in \mathscr{F}(D) : \forall A,\ \varnothing \neq \bar{A} \subset S\}$$
$$J(S) = \{[A] \in \mathscr{F}(D) : \forall A,\ \bar{A} \cap S \neq \varnothing\}$$

*are open (resp., closed).*

(iii) *$C_d(D)$ (and hence $\mathscr{F}(D)$) is compact in $C^0(D)$.*

*Proof.* Cf. [13, Chap. XI, Prob. 4.6, p. 253] or [12, p. 61, Prob. 3]. $\blacksquare$

This theorem has many interesting corollaries. For instance, it can be used to say something about the function which gives the number of connected components of $\bar{A}$ (cf. [23] for an application to image segmentation).

COROLLARY. *Let $D$ be a nonempty compact subset of $\mathbb{R}^N$ and $\mathscr{F}(D)$ the corresponding set of equivalence classes of nonempty subsets of $D$.*

*Associate with an equivalence class $[A]$ the number*

$$\#([A]) = \textit{number of connected components of } \bar{A}.$$

*Then the map*

$$[A] \mapsto \#([A]) : \mathscr{F}(D) \to \mathbb{R}$$

*is lowersemicontinuous. In particular, for a fixed number $c \geqslant 0$ the subset*

$$\{d_A \in C_d(D) : \#([A]) \leqslant c\}$$

*is also compact in $C^0(D)$.*

*Proof.* Let $\{A_n\}$ and $A$ be nonempty subsets of $D$ such that $d_{A_n}$ converges to $d_A$ in $C^0(D)$. Assume that $\#([A]) = k$ is finite. Then there exists a family of disjoint open sets $G_1, ..., G_k$ such that

$$\bar{A} \subset G = \bigcup_{i=1}^{k} G_i, \qquad \forall i, \quad \bar{A} \cap G_i \neq \varnothing.$$

In view of Theorem 3.2(ii),

$$\bar{A} \in \mathscr{U} = \bigcap_{i=1}^{k} J(G_i) \cap I(G).$$

But $\mathscr{U}$ is not empty and open as the finite intersection of $k+1$ open sets. As a result there exists $\varepsilon > 0$ and an open neighborhood of $[A]$,

$$N_\varepsilon([A]) = \{[B] : \|d_B - d_A\| < \varepsilon\} \subset \mathscr{U}.$$

Hence since $d_{A_n}$ converges to $d_A$, there exists $\bar{n} > 0$ such that

$$\forall n \geqslant \bar{n}, \qquad [A_n] \in \mathscr{U},$$

and necessarily

$$\forall n \geqslant \bar{n}, \qquad \bar{A}_n \subset G, \quad \bar{A}_n \cap G_i \neq \varnothing, \quad \forall i$$

and

$$\#([A_n]) \geqslant \#([A]),$$

which means that $[A] \to \#([A])$ is lower semicontinuous. Now for $\#([A]) = +\infty$, we repeat the above procedure and refine the open covering. ∎

*Remark* 3.1. In this section we deal with a theory of closed sets since the equivalence class of the subsets of $D$ is completely determined by their unique closure. In partial differential equation literature one usually deals with open sets. To accomodate this point of view, consider the set of relatively open subsets $O$ of a closed hold-all $D$ and associate with each $O$ the distance function of its complement $\complement_D O$ with respect to $D$. Then we can repeat the above and the following considerations with obvious changes. This approach was suggested by Zolésio [37, §1.3, p. 405] in the context of free boundary problems. It is the *Hausdorff complementary topology*. We shall see in Section 5 that the oriented distance function implicitly contains both points of view. Also, recent results by Šverák [30] precisely use the set of distance functions of the complement $\complement_D O$ with respect to $D$ of open sets $O$ in $D$ such that $\#([\complement_D O]) \leqslant c$ for some fixed $c > 0$. His main result is that in dimension 2 the convergence of a sequence $\{O_n\}$ to $O$ of such sets implies the convergence of the corresponding projection operators $\{P_{O_n}: H_0^1(D) \to H_0^1(O_n)\}$ to $P_O: H_0^1(D) \to H_0^1(O)$, where the projection operators are directly related to an homogeneous linear boundary problem on the corresponding domains $\{O_n\}$ and $O$. In dimension 1 the constraint on the number of components can be dropped. This result is useful in situations where the shape optimization problem can be relaxed to the distance function $d_{\complement_D O}$.

### 3.2. *Projection Operators and Differentiability of $d_A$*

DEFINITION 3.1. Given a nonempty subset $A$ of $\mathbb{R}^N$ and a point $x$ in $\mathbb{R}^N$, denote by $\Pi_A(x)$ the set of projections of $x$ onto $\bar{A}$,

$$\Pi_A(x) \stackrel{\text{def}}{=} \{z \in \bar{A}: |z - x| = d_A(x)\}.$$

The elements of $\Pi_A(x)$ are called *projections* and denoted by $P_A(x)$. The subscript $A$ is dropped whenever no confusion arises.

The directional derivative of the distance function can be computed by using a theorem on the differentiability of the min with respect to a parameter. It is directly related to the support function of the set $\Pi_A(x)$.

LEMMA 3.1. *Let $A$ be a nonempty subset of $\mathbb{R}^N$ and $x$ a point in $\mathbb{R}^N$. Define*

$$f(x) = \tfrac{1}{2}\,[|x|^2 - d_A^2(x)].$$

(i) *The set $\Pi_A(x)$ is nonempty, compact, and for all $x$ in $\overline{\complement A}$, $\Pi_A(x) \subset \partial A$.*

(ii) *For all $v$ in $\mathbb{R}^N$*

$$dd_A^2(x;v)=\lim_{t\searrow 0}\frac{d_A^2(x+tv)-d_A^2(x)}{t}=\inf_{z\in\Pi_A(x)} 2(x-z)\cdot v$$

$$df(x;v)=\lim_{t\searrow 0}\frac{f(x+tv)-f(x)}{t}=\sup_{z\in\Pi_A(x)} z\cdot v=\sigma_{\Pi_A(x)}(v)=\sigma_{\operatorname{co}\Pi_A(x)}(v),$$

*where $\sigma_{\Pi_A(x)}$ is the support function of the set $\Pi_A(x)$ and* $\operatorname{co}\Pi_A(x)$ *is the convex hull of $\Pi_A(x)$.*

*Proof.* (i) Obvious. (ii) Differentiability of $f$. We use a theorem on the differentiability of a Min with respect to a parameter (cf. Correa and Seeger [7] or Delfour and Zolésio [9, Theorem 5, p. 845]). ∎

For each $v\in\mathbb{R}^N$

$$\exists P_A(x)\in\Pi_A(x),\qquad df(x;v)=2(x-P_A(x))\cdot v,$$

but the choice of this element depends on the direction $v$ and is not necessarily unique.

The distance function is uniformly Lipschitzian. Hence it is differentiable almost everywhere with respect to the Lebesque measure on $\mathbb{R}^N$. So for almost all $x$ in $\mathbb{R}^N$, the set $\Pi_A(x)$ contains a unique element $P_A(x)\in\bar{A}$. In general the smoothness of the boundary $\partial A$ does not imply that $\Pi_A(x)$ be a singleton everywhere in $\mathbb{R}^N$. For instance, for the unit sphere $A=\{x\in\mathbb{R}^N\colon |x|=1\}$ the boundary $\partial A$ is $C^\infty$ but $\Pi_A(0)=A$. However, for convex sets $A$, $\Pi_A(x)$ is always a singleton (cf. Valentine [32]).

THEOREM 3.3. *Let $A$ be a nonempty subset of $\mathbb{R}^N$.* (i) *If $\nabla d_A(x)$ exists at a point $x$ in $\mathbb{R}^N$, then there exists a unique $Px\in\bar{A}$ such that*

$$d_A(x)=|Px-x| \qquad \textit{and} \qquad \nabla d_A(x)=\begin{cases}0, & \textit{if}\quad x\in\bar{A}\\ \dfrac{x-Px}{|x-Px|}, & \textit{if}\quad x\notin\bar{A}.\end{cases}$$

(ii) *The function $d_A^2(x)$ is Gâteaux differentiable at $x$ if and only if $\Pi_A(x)$ is a singleton. In that case*

$$\nabla d_A^2(x)=2(x-Px).$$

*If, in addition, $Px\neq x$, then $d_A(x)$ is Gâteaux differentiable at $x$.*

*Proof.* (i) Adaptation of the proof given in Clarke [6, Prop. 2.5.4, p. 66].

(ii) If $d_A^2(x)$ is Gâteaux differentiable, there exists $c \in \mathbb{R}^N$ such that

$$\forall v \in \mathbb{R}^N, \qquad \lim_{t \searrow 0} \frac{d_A^2(x+tv) - d_A^2(x)}{t} = c \cdot v.$$

Now

$$\frac{d_A^2(x+tv) - d_A^2(x)}{t} = [d_A(x+tv) + d_A(x)] \frac{d_A(x+tv) - d_A(x)}{t}$$

and since $d_A$ is Lipschitzian of constant 1

$$\left| \frac{d_A^2(x+tv) - d_A^2(x)}{t} \right| \leqslant [d_A(x+tv) + d_A(x)] \, |v|.$$

As $t$ goes to zero,

$$\forall v, \qquad |c \cdot v| \leqslant 2d_A(x) \, |v| \Rightarrow |c| \leqslant 2d_A(x).$$

If $d_A(x) = 0$, then $Px = x$ and $\Pi_A(x)$ is a singleton. If $d_A(x) > 0$, then for all $Px \in \Pi_A(x)$, $d_A(x) = |x - Px|$ and

$$\forall t \in ]0, 1[, \qquad d_A(x + t(Px - x)) = (1-t) \, |x - Px|.$$

Therefore

$$d_A^2(x + t(Px - x)) - d_A^2(x) = [(1-t)^2 - 1] \, |x - Px|^2.$$

By dividing by $t > 0$ and going to zero,

$$c \cdot (Px - x) = -2 \, |x - Px|^2,$$

and since $Px \neq x$

$$c \cdot \frac{x - Px}{|x - Px|} = 2d_A(x).$$

Since $|c| \leqslant 2d_A(x)$, $c = 2(x - Px)$, $Px$ is unique,

$$Px = x - \frac{c}{2},$$

and $\Pi_A(x)$ is a singleton. Conversely if $\Pi_A(x)$ is a singleton, we conclude from Lemma 3.1(ii) that for all $v \in \mathbb{R}^N$

$$d(d_A^2)(x; v) = 2(x - Px) \cdot v.$$

Hence $d_A^2$ is Gâteaux differentiable and its gradient is

$$\nabla d_A^2(x) = 2(x - Px).$$

Finally, if $\Pi_A(x)$ is a singleton and if $Px \neq x$, then $d_A^2$ is Gâteaux differentiable and for all $v \in \mathbb{R}^N$, and $t > 0$ sufficiently small,

$$\frac{d_A(x+tv) - d_A(x)}{t} = \frac{1}{d_A(x+tv) + d_A(x)} \frac{d_A^2(x+tv) - d_A^2(x)}{t}$$

$$\Rightarrow \nabla d_A(x) = \frac{1}{2d_A(x)} \nabla d_A^2(x).$$

This completes the proof of the theorem. ∎

*Remark* 3.2. If $\nabla d_A(x)$ exists, then

$$\nabla d_A^2(x) = \begin{Bmatrix} 0, & x \in \bar{A} \\ 2(x - P_A x), & x \notin \bar{A} \end{Bmatrix}$$

and, in view of Theorem 3.3, $|\nabla d_A(x)| = 0$ in $\bar{A}$ and 1 outside of $\bar{A}$. So $\nabla d_A(x)$ is *always discontinuous* across the boundary $\partial A$. However, $\nabla d_A^2(x)$ can be continuous if the projection $P(x)$ is continuous. We shall see that we can get around this difficulty by considering the oriented distance function in Section 5.

This theorem also shows that $\nabla d_A(x)$ is related to the outward unit normal to $\bar{A}$ at the point $Px$. Obviously this normal is not always unique, as can be seen in Fig. 1. Yet, intuitively, the smoothness of the boundary $\partial A$ of $A$ seems to be related to the smoothness of $\nabla d_A(x)$ in a small exterior neighbourhood of the boundary $\partial A$ of $A$. In Fig. 2 $\nabla d_A(x)$ is continuous outside $\bar{A}$ except on a semi-infinite line $L$.

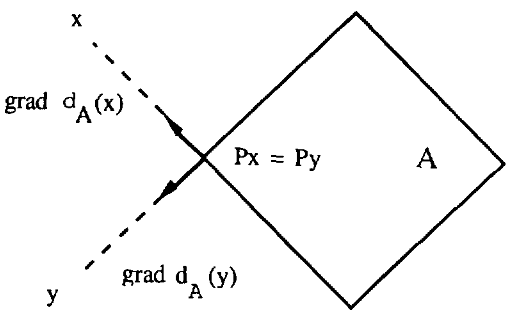



FIGURE 1

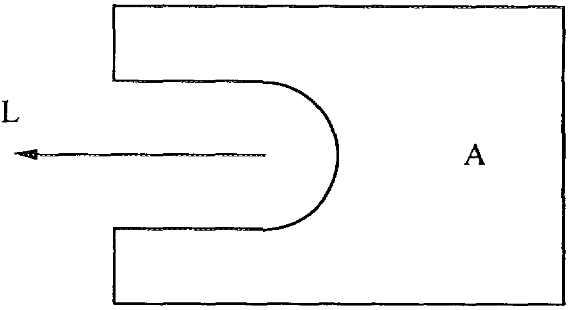



FIGURE 2

### 3.3. *Characterization of Convex Sets*

In the convex case the distance function is differentiable everywhere and this property can be used to characterize the convexity of a set.

THEOREM 3.4. *Let $A$ be a nonempty convex subset of $\mathbb{R}^N$. Then*

(i) *for each $x \in \mathbb{R}^N$ there exists a unique $P_A(x)$ in $\bar{A}$ such that $d_A(x) = |P_A(x) - x|$, $\Pi_A(x) = \{P_A(x)\}$ is a singleton, and*

$$\forall x, \forall y \in \mathbb{R}^N, \quad |P_A(y) - P_A(x)| \leqslant |y - x|.$$

(ii) *$d_A^2$ belongs to $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ and even to $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$.*

*Proof.* (i) By definition,

$$d_A^2(x) = \inf_{z \in A} |z - x|^2$$

and since $A$ is convex, there exists a unique $P_A(x)$ in $\bar{A}$ such that

$$\forall x \in \bar{A}, \qquad 2(P_A(x) - x) \cdot (z - P_A(x)) \geqslant 0$$

(cf. for instance, Zarantonello [33, pp. 237–246] or Aubin [2a, p. 24, Example 4.3]). So for any two points $x$ and $y$

$$(P_A(x) - x) \cdot (P_A(y) - P_A(x)) \geqslant 0, \qquad (P_A(y) - y) \cdot (P_A(x) - P_A(y)) \geqslant 0.$$

By adding up

$$|P_A(y) - P_A(x)|^2 \leqslant (y - x) \cdot (P_A(y) - P_A(x)) \leqslant |y - x|\ |P_A(y) - P_A(x)|.$$

(ii) From part (i) the map $x \mapsto P_A(x)$ is Lipschitz continuous and hence the map $x \mapsto \nabla d_A^2(x) = 2[x - P_A(x)]$ is also Lipschitz continuous. Therefore $d_A^2$ belongs to $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ and even to $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$. ∎

THEOREM 3.5. (i) *Let $A$ be a nonempty subset of $\mathbb{R}^N$. Then the following statements are equivalent*:

$$\text{(a)} \quad \bar{A} \text{ is convex;} \qquad \text{(b)} \quad d_A \text{ is convex.}$$

(ii) *The subset of all distance functions of nonempty convex subsets of the closed hold-all $D$ is closed in $C(D)$ and compact in $C^0(D)$ if, in addition, $D$ is bounded.*

*Proof.* (i) (a) $\Rightarrow$ (b) From Clarke [6, p. 53, lemma]. (b) $\Leftarrow$ (a) If $d_A$ is convex, then

$$\forall \lambda \in [0,1], \quad \forall x, y \in \bar{A}, \quad d_A(\lambda x + (1-\lambda)\,y) \leqslant \lambda d_A(x) + (1-\lambda)\,d_A(y).$$

But

$$x, y \in \bar{A} \Rightarrow d_A(x) = d_A(y) = 0 \qquad \Rightarrow \forall \lambda \in [0,1], \qquad d_A(\lambda x + (1-\lambda)\,y) = 0$$

and necessarily $\lambda x + (1-\lambda)\,y \in \bar{A}$. (ii) The set of all convex functions in $C(D)$ is a closed convex cone with vertex at 0. Hence its intersection with $C_d(D)$ is closed. ∎

### 3.4. $W^{1,p}$-*Topology for Distance Functions and Relation with Characteristic Functions*

An important question which arises when defining a new topology for equivalence classes of subsets of a hold-all $D$ is its relationship with the topology induced by the $L^p$-norm on characteristic functions. For the Hausdorff convergence it is already known that we do not necessarily get this $L^p$-convergence as can easily be seen from the following example.

EXAMPLE 3.1. Denote by $D$ the square $[-1,2] \times [-1,2]$ in $\mathbb{R}^2$ and for each $n \geqslant 1$ define the sequence of closed sets

$$A_n = \left\{ (x_1, x_2) \in D : \frac{2k}{2n} \leqslant x_1 \leqslant \frac{2k+1}{2n}, 0 \leqslant k < n \right\}.$$

This defines $n$ vertical stripes of equal width $1/2n$ each distant of $1/2n$ (Fig. 3). Clearly for all $n \geqslant 1$,

$$m(A_n) = \tfrac{1}{2}, \qquad P_D(A_n) = 2n+1,$$

$$d_{A_n}(x) \leqslant \frac{1}{4n}, \qquad \forall x \in S,$$

$$\|\nabla d_{A_n}\|_{L^p(S)} \geqslant 2^{-1/p},$$

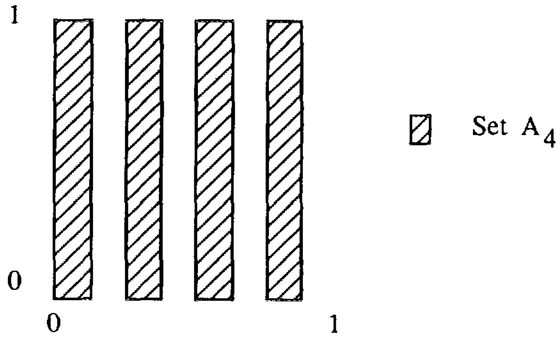



FIGURE 3

where $S = [0, 1] \times [0, 1]$ and $P_D(A_n)$ is the perimeter of $A_n$. Hence

$$d_{A_n} \to d_S \qquad \text{in } C^0(D).$$

The sequence $\{\nabla d_{A_n}\}$ does not converge in $L^p(D)$, and

$$\begin{aligned} & m(S) = 1, \qquad P_D(S) = 4, \\ & m(\bar{A}_n) = m(A_n) = \tfrac{1}{2} \nrightarrow 1 = m(S) \quad \Rightarrow \quad \chi_{A_n} \nrightarrow \chi_S \qquad \text{in } L^2(D) \\ & P_D(A_n) \nrightarrow P_D(S), \end{aligned}$$

but

$$\chi_{A_n} \rightharpoonup \tfrac{1}{2}\chi_S \qquad \text{in } L^2(D)\text{-weak}.$$

In view of Example 3.1 it is clear that Hausdorff convergence is not sufficient to get the $L^p$-convergence of the characteristic functions of the closure of the corresponding sets in the sequence. The volume functional is only upper semicontinuous with respect to the Hausdorff topology. If $d_{A_n} \to d_A$, then for all $x \notin \bar{A}$, $d_A(x) > 0$ and

$$\begin{aligned} & \exists N_x > 0, \qquad \forall n \geqslant N_x, \qquad \|d_A - d_{A_n}\| \leqslant \tfrac{1}{2} d_A(x), \\ \Rightarrow & \exists N_x > 0, \qquad \forall n \geqslant N_x, \qquad d_{A_n}(x) \geqslant \tfrac{1}{2} d_A(x) > 0, \\ \Rightarrow & \exists N_x > 0, \qquad \forall n \geqslant N_x, \qquad x \notin \bar{A}_n \quad \text{and} \quad \chi_{\bar{A}_n}(x) = 0. \end{aligned}$$

So for all $x \notin \bar{A}$,

$$\lim_{n \to \infty} \chi_{\bar{A}_n}(x) = \chi_{\bar{A}}(x) = 0.$$

Finally, for all $x \in \bar{A}$ and all $n \geqslant 1$,

$$\chi_{\bar{A}_n}(x) \leqslant 1 = \chi_{\bar{A}}(x)$$

and the result follows trivially by taking the lim sup of each term.

We conclude that

$$\limsup_{n \to \infty} \chi_{\bar{A}_n} \leqslant \chi_{\bar{A}}$$

and by using the analogue of Fatou's lemma for the lim sup we get for all compact subsets $K$ of $D$

$$\limsup_{n \to \infty} \int_K \chi_{\bar{A}_n}\, dx \leqslant \int_K \limsup_{n \to \infty} \chi_{\bar{A}_n}\, dx \leqslant \int_K \chi_{\bar{A}}\, dx.$$

To get the full continuity we need the $L^p$-convergence of the gradients of the distance functions which are related to the characteristic functions of the closure of the associated sets.

*Notation* 3.1. In the sequel the closed hold-all $D$ is a subset of $\mathbb{R}^N$ with a nonempty interior int $D$ such that

$$\overline{\operatorname{int} D} = D \qquad \text{and} \qquad \chi_{\operatorname{int} D} = \chi_D.$$

Some of the function spaces involved are defined on either $D$ or int $D$. However, for simplicity we use $D$ in both cases unless it is not clear from the context. Recall that a distance function defined on an open set continuously extends to its closure since it is Lipschitzian and hence uniformly continuous.

THEOREM 3.6. *Let $D$ be a closed domain in $\mathbb{R}^N$ with a nonempty interior and a locally Lipschitz boundary $\partial D$. Fix $p$, $1 \leqslant p < \infty$.*

(i) *For each nonempty set $A$ in $D$,*

$$\chi_{\bar{A}} = 1 - |\nabla d_A|, \qquad \text{a.e. } in\ D,$$

*and the map*

$$d_A \mapsto \chi_{\bar{A}} : W^{1,p}_{\mathrm{loc}}(D) \to L^p_{\mathrm{loc}}(D)$$

*is "Lipschitz continuous": for all compact subsets $K$ of $D$ and nonempty subsets $A_1$ and $A_2$ of $D$*

$$\|\chi_{\bar{A}_2} - \chi_{\bar{A}_1}\|_{L^p(K)} \leqslant \|\nabla d_{A_2} - \nabla d_{A_1}\|_{L^p(K)} \leqslant \|d_{A_2} - d_{A_1}\|_{W^{1,p}(K)}.$$

(ii) *The set* $C_d(D)$ *is closed in* $W^{1,p}_{\text{loc}}(D)$.

(iii) *The set* $C_d(D)$ *is closed in* $W^{1,p}_{\text{loc}}(D)$-weak. *If D is bounded, then* $C_d(D)$ *is compact in* $W^{1,p}(D)$-weak.

(iv) *Given a sequence* $\{d_{A_n}\}$ *and* $d_A$ *in* $C_d(D)$, *then*

$$d_{A_n} \to d_A \qquad \text{in } W^{1,p}_{\text{loc}}(D)\text{-strong} \quad (\text{resp., } W^{1,p}(D)\text{-strong})$$

*if and only if*

$$d_{A_n} \rightharpoonup d_A \qquad \text{in } W^{1,p}_{\text{loc}}(D)\text{-weak} \quad (\text{resp., } W^{1,p}(D)\text{-weak})$$

$$\chi_{\bar{A}_n} \to \chi_{\bar{A}} \qquad \text{in } L^p_{\text{loc}}(D)\text{-strong} \quad (\text{resp., } L^p(D)\text{-strong}).$$

(v) *Assume that, in addition, D is compact. Let* $\{d_{A_n}\}$ *be a sequence in* $C_d(D)$ *such that* $\nabla d_{A_n} \in BV(D)^N$ *and*

$$\exists c > 0, \qquad \forall n \geqslant 1, \quad \|\partial_{ij} d_{A_n}\|_{M^1(D)} \leqslant c, \quad 1 \leqslant i, j \leqslant N.$$

*Then there exists a subsequence, still denoted* $\{d_{A_n}\}$, *and* $d_A$ *in* $C_d(D)$ *such that*

$$d_{A_n} \to d_A \qquad \text{in } W^{1,p}(D)\text{-strong}$$

*and for all* $\varphi \in \mathscr{D}^0(D)$

$$\lim_{n \to \infty} \langle \partial_{ij} d_{A_n}, \varphi \rangle = \langle \partial_{ij} d_A, \varphi \rangle, \qquad 1 \leqslant i, j \leqslant N.$$

*In particular,*

$$\|\partial_{ij} d_A\|_{M^1(D)} \leqslant c, \qquad 1 \leqslant i, j \leqslant N.$$

*Remark* 3.3. As we shall see in the proof of Theorem 3.6, if a sequence of elements $\{d_{A_n}\}$ converges in $W^{1,p}$-strong for some $p$, $1 \leqslant p < \infty$, it converges in $W^{1,p}$-strong for all $p$, $1 \leqslant p < \infty$. So we say that $\{d_{A_n}\}$ *converges strongly*. Similarly, if a sequence of elements $\{d_{A_n}\}$ converges in $W^{1,p}$-weak for some $p$, $1 \leqslant p < \infty$, it converges in $W^{1,p}$-weak for all $p$, $1 \leqslant p < \infty$. This is also equivalent to the convergence in the $W^{1,\infty}$-weak$^\star$ (when $W^{1,\infty}$ is considered as a subspace of $L^\infty \times L^\infty$) and the $(\mathscr{D}(D)^N)'$ topologies. This situation is analogous to what we have seen for characteristic functions and we say that $\{d_{A_n}\}$ *converges weakly*.

*Remark* 3.4. In view of part (iv) of Theorem 3.6 an optimization problem with respect to the characteristic functions $\chi_A$ of sets $A$ in $D$ for which we have the continuity with respect to $\chi_A$ can be transformed into an optimization problem with respect to $d_{\complement_D A}$ in $W^{1,1}(D)$ since

$$\chi_{D \cap \operatorname{int} A} = |\nabla d_{\complement_D A}|.$$

This is the class of relatively open sets $A$ in $D$. For instance, this would apply to the transmission problem (8)–(10) in Section 2.

*Proof.* (i) Given any nonempty subset $A$ of $D$, the distance function $d_A$ is differentiable almost everywhere in $D$. In view of Theorem 3.3, when it is differentiable

$$|\nabla d_A(x)| = \begin{cases} 0, & \text{if } \ x \in \bar{A}, \\ 1, & \text{if } \ x \notin \bar{A}. \end{cases}$$

As a result

$$\chi_{\bar{A}} = 1 - |\nabla d_A(x)|, \qquad \text{a.e. in } D.$$

Given two nonempty subsets $A_1$ and $A_2$ of $D$,

$$\begin{aligned} &|\nabla d_{A_2}| \leqslant |\nabla d_{A_1}| + |\nabla d_{A_2} - \nabla d_{A_1}| \\ &\Rightarrow \chi_{A_1} \leqslant \chi_{A_2} + |\nabla d_{A_2} - \nabla d_{A_1}| \\ &\Rightarrow \int_D |\chi_{A_1} - \chi_{A_2}|^p \, dx \leqslant \|d_{A_2} - d_{A_1}\|^p_{W^{1,p}(D)} \end{aligned}$$

for $1 \leqslant p < \infty$ and with the ess-sup norm for $p = \infty$.

(ii) It is sufficient to prove it for $D$ compact. There exists a constant $c > 0$ such that for all $x$ in $D$, $|x| \leqslant c$. Therefore

$$d_A(x) = \inf_{y \in A} |x - y| \leqslant \inf_{y \in A \cap D} |x - y| \leqslant 2c.$$

Moreover, from part (i)

$$|\nabla d_A(x)| \leqslant 1, \qquad \forall x \in D.$$

Therefore if there exists a sequence $\{A_n\}$ of sets such that

$$d_{A_n} \to f \qquad \text{in } W^{1,p}(D)$$

it also converges for all finite $r \geqslant 1$ in $W^{1,r}(D)$. In particular, for $r > N$, $W^{1,r}(D)$ is continuously embedded in $C^0(D)$ (cf. Adams [1, Theorem 5.4, Part II, p. 98]) and $d_{A_n} \to f$ in $C^0(D)$. But from Theorem 3.1(i), $C_d(D)$ is closed in $C^0(D)$ and there exists a nonempty set $A$ in $D$ such that $f = d_A$. Hence

$$d_{A_n} \to d_A \qquad \text{in } W^{1,p}(D)$$

and $C_d(D)$ is closed in $W^{1,p}(D)$.

(iii) It is sufficient to prove the result for $D$ compact. First note that for any sequence $\{d_{A_n}\}$ in $C_d(D)$ all weak convergences of the gradient are equivalent: for any $p$, $1 \leqslant p < \infty$,

$$\{\nabla d_{A_n}\} \text{ convergent in } L^\infty(D)^N\text{-weak*}$$

$$\{\nabla d_{A_n}\} \text{ convergent in } L^p(D)^N\text{-weak}$$

$$\{\nabla d_{A_n}\} \text{ convergent in } (\mathscr{D}(D)^N)'.$$

The proof is analogous to the one of Lemma 2.1 in view of the fact that for almost all $x$ in $D$, the $\mathbb{R}^N$-norm of the gradient of a distance function is bounded by one.

To prove the weak closure of $C_d(D)$, consider a weakly convergent sequence $\{d_{A_n}\}$ in $W^{1,p}$ to some $f$ in $W^{1,p}$. In view of our previous remarks, if it is weakly convergent for some $p$, $1 \leqslant p < \infty$, it is weakly convergent for all $p$, $1 \leqslant p < \infty$. So for $p > N$, the sequence is bounded and since the embedding of $W^{1,p}(D)$ into $C^0(D)$ is compact (cf. Adams [1, Theorem 6.2, p. 144]), there exist $d_A \in C_d(D)$ and a subsequence $\{d_{A_{n_k}}\}$ such that

$$d_{A_{n_k}} \to d_A \qquad \text{in } C^0(D).$$

Therefore $f = d_A$ since

$$d_{A_n} \rightharpoonup f \qquad \text{in } W^{1,p}(D)\text{-weak} \quad \Rightarrow \quad d_{A_n} \to f \qquad \text{in } L^p(D)\text{-strong}.$$

This proves that $C_d(D)$ is closed in $W^{1,p}(D)$-weak for $p > N$ and hence for all $p$, $1 \leqslant p < N$.

The proof of the compactness in $W^{1,p}(D)$-weak is similar to the proof of the closure. Pick any sequence in $W^{1,p}(D)$, then it is bounded for all $p$, $1 \leqslant p < \infty$, since for all $x \in D$

$$d_{A_n}(x) = \inf_{y \in A_n} |y - x| \leqslant 2 \sup_{x \in D} |x|$$

$$|\nabla b_{A_n}(x)| \leqslant 1, \qquad \text{a.e. in } D.$$

Choose $p > N$. Then $W^{1,p}(D)$ is reflexive and there exists $f \in W^{1,p}(D)$ and a subsequence such that

$$d_{A_{n_k}} \rightharpoonup f \qquad \text{in } W^{1,p}(D)\text{-weak}.$$

But $C_d(D)$ is closed in $W^{1,p}(D)$-weak. So there exists $d_A$ in $C_d(D)$ such that $f = d_A$ and this proves the compactness.

(iv) If $\{d_{A_n}\}$ converges to $d_A$ in $W^{1,p}(D)$-strong, it converges in $W^{1,p}(D)$-weak and from part (i) $\chi_{\bar{A}_n}$ converges to $\chi_{\bar{A}}$ in $L^p(D)$-strong. Conversely for $D$ compact the weak convergence for some $p$, $1 \leqslant p < \infty$, implies the weak convergence for all $p$, $1 \leqslant p < \infty$, in particular for $p = 2$. Therefore

$$\begin{aligned}
&\int_D |\nabla d_{A_n} - \nabla d_A|^2 \, dx \\
&\quad = \int_D |\nabla d_{A_n}|^2 \, dx + \int_D |\nabla d_A|^2 \, dx - 2 \int_D \nabla d_{A_n} \cdot \nabla d_A \, dx \\
&\quad = \int_D 1 - \chi_{\bar{A}_n} + |\nabla d_A|^2 - 2\nabla d_{A_n} \cdot \nabla d_A \, dx \\
&\quad = \int_D \chi_{\bar{A}} - \chi_{\bar{A}_n} \, dx + 2 \int_D |\nabla d_A|^2 - \nabla d_{A_n} \cdot \nabla d_A \, dx.
\end{aligned}$$

The first term converges to 0 by strong $L^p$-convergence of the $\chi_{\bar{A}_n}$'s and the second term also converges to 0 by weak $W^{1,p}$ convergence of the $d_{A_n}$'s. Hence $d_{A_n} \to d_A$ in $W^{1,2}(D)$-strong. By Lemma 2.1(ii) and the fact that both $\{d_{A_n}(x)\}$ and $\{\nabla d_{A_n}(x)\}$ are uniformly bounded in $D$, then the sequence converges in $W^{1,p}(D)$ for all $p$, $1 \leqslant p < \infty$.

(v) Again it is sufficient to prove the result for $D$ compact. The sequence $\{d_{A_n}\}$ belongs to the space

$$BV^2(D) = \{ f \in W^{1,1}(D) : \nabla f \in BV(D)^N \}.$$

The injection of $BV^2(D)$ into $W^{1,1}(D)$ is compact for a compact domain $D$ with a Lipschitzian boundary. The proof is similar to the one for the compactness of the embedding of $BV(D)$ into $L^1(D)$. It is sufficient to use Rellich's theorem (cf. Morrey [21, Def. 3.4.1, p. 72, and Theorem 3.4.4, p. 75]). The convergence of a subsequence of $\{d_{A_n}\}$ to some $d_A$ in $W^{1,1}(D)$-strong now follows. But since both $\{d_{A_n}(x)\}$ and $\{\nabla d_{A_n}(x)\}$ are uniformly bounded, the convergence is also true in $W^{1,p}(D)$-strong for all $p$, $1 \leqslant p < \infty$. ∎

For $D$ compact and any sequence $\{A_n\}$ in $D$ there exists a subsequence (still denoted $\{A_n\}$), a set $B$ in $D$, and a function $\chi$ in $L^2(D)$ with values in $[0, 1]$ such that

$$d_{A_n} \rightharpoonup d_B \qquad \text{in } W^{1,2}(D)\text{-weak}$$

$$\chi_{\bar{A}_n} \rightharpoonup \chi \qquad \text{in } L^2(D)\text{-weak.}$$

Then it is easy to show that

$$\lim_{n \to \infty} \int_D |\nabla d_{A_n} - \nabla d_B|^2 \, dx = \int_D \chi_B - \chi \, dx.$$

In particular,

$$\chi_B - \chi \geqslant 0 \qquad \text{in } D$$

and

$$\forall x \in \complement_D \bar{B}, \qquad \chi(x) = 0.$$

Therefore the strong convergence in $W^{1,2}(D)$ is achieved if and only if $\chi = \chi_B$, which in turn implies that

$$\chi_{\bar{A}_n} \to \chi = \chi_B \qquad \text{in } L^2(D)\text{-strong.}$$

We have seen in Example 3.1 that the weak convergence of the characteristic functions is not sufficient to obtain the strong convergence of the $d_{A_n}$'s in $W^{1,2}(D)$. However, if we assume that $\{\chi_{\bar{A}_n}\}$ is strongly convergent, then, by Lemma 2.2, $\chi$ is a characteristic function of some measurable subset $A$ of $D$ and necessarily $A \subset \bar{B}$. Is this sufficient to conclude that $\chi_A = \chi_B$? The answer is negative. The counterexample is provided by Example 2.1 where

$$d_{A_n} \rightharpoonup d_A \qquad \text{in } W^{1,2}(D)\text{-weak} \quad \text{and} \quad \chi_{A_n} \to \chi_A \quad \text{in } L^2(D)\text{-strong}$$

for some $A \subset D$ such that

$$m(\bar{A}) = \pi > \frac{\pi}{3} \geqslant m(A) \Rightarrow \chi_{\bar{A}} \neq \chi_A.$$

Part (v) of Theorem 3.6 provides a sufficient condition to obtain the strong $W^{1,p}(D)$ convergence. Is the existence of the Hessian matrix of second-order partial derivatives of $d_A$ a realistic property and what are those second derivatives? In order to answer this question, it is useful to consider a few simple examples. This is followed by a theorem which summarizes some of our observations.

EXAMPLE 3.2: Half plane in $\mathbb{R}^2$. Consider the domain

$$A = \{(x_1, x_2) : x_1 \leqslant 0\}, \qquad \partial A = \{(x_1, x_2) : x_1 = 0\}.$$

It is readily seen that

$$d_A(x_1, x_2) = \max\{x_1, 0\}$$

$$\nabla d_A(x_1, x_2) = \begin{cases} (0, 0), & x_1 < 0 \\ (1, 0), & x_1 > 0 \end{cases}$$

$$\langle \partial_{11} d_A, \varphi \rangle = \int_{\partial A} \varphi \, dx$$

$$\partial_{12} d_A = \partial_{21} d_A = \partial_{22} d_A = 0$$

$$\langle \Delta d_A, \varphi \rangle = \int_{\partial A} \varphi \, dx.$$

Thus $\Delta d_A$ is the surface measure on the boundary $\partial A$.

EXAMPLE 3.3: Ball of radius $R > 0$ in $\mathbb{R}^2$. Consider the domain

$$A = \{x \in \mathbb{R}^2 : |x| \leqslant R\}, \qquad \partial A = \{x \in \mathbb{R}^2 : |x| = R\}.$$

Clearly

$$d_A(x) = \min\{0, |x| - R\}, \qquad \nabla d_A(x) = \begin{cases} \dfrac{x}{|x|}, & |x| > R \\ (0, 0), & |x| < R \end{cases}$$

$$\langle \partial_{11} d_A, \varphi \rangle = \int_0^{2\pi} R \cos^2(\theta) \, \varphi \, d\theta + \int_{\complement A} \frac{x_2^2}{(x_1^2 + x_2^2)^{3/2}} \, dx$$

$$\langle \partial_{22} d_A, \varphi \rangle = \int_0^{2\pi} R \sin^2(\theta) \, \varphi \, d\theta + \int_{\complement A} \frac{x_1^2}{(x_1^2 + x_2^2)^{3/2}} \, dx$$

$$\begin{aligned} \langle \partial_{12} d_A, \varphi \rangle &= \langle \partial_{21} d_A, \varphi \rangle \\ &= \int_0^{2\pi} R \cos(\theta) \sin(\theta) \, \varphi \, d\theta + \int_{\complement A} \frac{x_2 x_1}{(x_1^2 + x_2^2)^{3/2}} \, dx \end{aligned}$$

$$\langle \Delta d_A, \varphi \rangle = \int_{\partial A} \varphi \, dx + \int_{\complement A} \frac{1}{(x_1^2 + x_2^2)^{1/2}} \, dx.$$

We see that $\Delta d_A$ contains the boundary measure on $\partial A$ plus a term which looks like the volume integral of the mean curvature.

EXAMPLE 3.4: Unit square in $\mathbb{R}^2$. Consider the domain

$$A=\{x=(x_1,x_2)\colon |x_1|\leqslant 1, |x_2|\leqslant 1\}.$$

Since $A$ is symmetrical with respect to both axes, it is sufficient to specify $d_A$ in the first quadrant. We use the notation $Q_1$, $Q_2$, $Q_3$, and $Q_4$ for the four quadrants in the counterclockwise order and $c_1$, $c_2$, $c_3$, and $c_4$ for the four corners of the square in the same order. We also divide the plane in three regions

$$D_1=\{(x_1,x_2)\colon |x_2|\leqslant \min\{1,|x_1|\}\}$$
$$D_2=\{(x_1,x_2)\colon |x_1|\leqslant \min\{1,|x_2|\}\}$$
$$D_3=\{(x_1,x_2)\colon |x_1|\geqslant 1 \text{ and } |x_2|\geqslant 1\}.$$

Hence

$$d_A(x)=\begin{cases}\min\{x_2-1,0\}, & x\in D_2\cap Q_1\\ |x-c_1|, & x\in D_3\cap Q_1\\ \min\{x_1-1,0\}, & x\in D_1\cap Q_1\end{cases}$$

$$\nabla d_A(x)=\begin{cases}(0,1), & x\in D_2\cap Q_1 \quad\text{and}\quad x_2>1\\ \dfrac{x-c_1}{|x-c_1|}, & x\in D_3\cap Q_1\\ (1,0), & x\in D_1\cap Q_1 \quad\text{and}\quad x_1>1\\ (0,0), & x\in Q_1,\quad x_1<1 \quad\text{and}\quad x_2<1.\end{cases}$$

$$\langle \partial_{11} d_A,\varphi\rangle=\sum_{i=1}^{4}\int_{D_3\cap Q_i}\frac{(x_2-c_{i,2})^2}{|x-c_i|^2}\varphi\,dx+\int_{\partial A\cap Q_i\cap D_1}\varphi\,dx$$

$$\langle \partial_{22} d_A,\varphi\rangle=\sum_{i=1}^{4}\int_{D_3\cap Q_i}\frac{(x_1-c_{i,1})^2}{|x-c_i|^2}\varphi\,dx+\int_{\partial A\cap Q_i\cap D_2}\varphi\,dx$$

$$\langle \partial_{12} d_A,\varphi\rangle=\langle \partial_{21} d_A,\varphi\rangle=\sum_{i=1}^{4}\int_{D_3\cap Q_i}\frac{(x_2-c_{i,2})(x_1-c_{i,1})}{|x-c_i|^2}\varphi\,dx$$

$$\langle \Delta d_A,\varphi\rangle=\sum_{i=1}^{4}\int_{D_3\cap Q_i}\frac{1}{|x-c_i|}\varphi\,dx+\int_{\partial A}\varphi\,dx.$$

Note that the structure of the Laplacian is similar to the ones observed in the previous examples.

For locally piecewise $C^2$-domains, we can conjecture that the surface measure on the boundary $\partial A$ is contained in $\Delta d_A$. Recall that this surface measure was obtained from the gradient of the characteristic function of the corresponding set in the context of finite perimeter sets. Here we know that $\chi_{\bar{A}} = 1 - |\nabla d_{\bar{A}}|$ and we have the following result which provides the first connection between distance functions and finite perimeter sets.

THEOREM 3.7. *Assume that $D$ is a compact domain with nonempty interior such that $\overline{\operatorname{int} D} = D$ and a locally Lipschitzian boundary. For each domain $A \subset D$ such that $d_A$ belongs to the space*

$$BV^2(D) = \{f \in W^{1,1}(D) : \nabla f \in BV(D)^N\},$$

*$\bar{A}$ is a finite perimeter set, that is*

$$\nabla \chi_{\bar{A}} \in M_1(D)^N.$$

*Proof.* Given $d_A$ in $BV^2(D)$, there exists a sequence $\{u_k\}$ in $C^\infty(D)^N$ such that

$$u_k \to \nabla d_A \qquad \text{in } L^1(D)^N$$

$$\|\nabla (u_k)_i\|_{M_1(D)^N} \to \|\nabla \partial_i d_A\|_{M_1(D)^N}, \qquad 1 \leqslant i \leqslant N,$$

as $k$ goes to infinity and since $|\nabla d_A(x)| \leqslant 1$, this sequence can be chosen in such a way that

$$|u_k(x)| \leqslant 1, \qquad \forall k \geqslant 1.$$

This follows from the use of mollifiers (cf. [16, Theorem 1.17, p. 15]). For all $V$ in $\mathscr{D}(\operatorname{int} D)^N$

$$-\int_D \chi_{\bar{A}} \operatorname{div} V\, dx = \int_D [|\nabla d_A|^2 - 1] \operatorname{div} V\, dx = \int_D |\nabla d_A|^2 \operatorname{div} V\, dx.$$

For each $u_k$

$$\int_D |u_k|^2 \operatorname{div} V\, dx = -2 \int_D {}^{\top}[Du_k]\, u_k \cdot V\, dx = -2 \int_D u_k \cdot [Du_k]\, V\, dx,$$

where ${}^{\top}[Du_k]$ is the transposed of the matrix $[Du_k]$ and

$$\begin{aligned} \left| \int_D |u_k|^2 \operatorname{div} V\, dx \right| &\leqslant 2 \int_D |u_k|\, |Du_k|\, |V|\, dx \\ &\leqslant 2\, \|Du_k\|_{L_1}\, \|V\|_{C(D)} \\ &\leqslant 2\, \|Du_k\|_{M_1}\, \|V\|_{C(D)} \end{aligned}$$

since for $W^{1,1}(D)$-functions $\|\nabla f\|_{L^1(D)} = \|\nabla f\|_{M_1(D)^N}$. Therefore as $k$ goes to infinity

$$\left| \int_D |\nabla d_A|^2 \operatorname{div} V \, dx \right| \leqslant 2 \, \|H d_A\|_{M_1} \, \|V\|_{C(D)},$$

where $Hd_A$ is the Hessian matrix of second-order partial derivatives of $d_A$. Therefore $\nabla \chi_{\bar{A}} \in M_1(D)^N$. ∎

## 4. Support Function

Another interesting embedding is provided by the *support function* of *convex analysis*. Here the set of equivalence classes of domains consists in nonempty bounded domains with the same closed convex hull.

Let $A$ be a nonempty subset of $\mathbb{R}^N$. The *indicator function* of $A$ is

$$\psi_A(x) = \begin{Bmatrix} 0, & \text{if } x \in A \\ +\infty, & \text{if } x \notin A \end{Bmatrix}.$$

The *support function* $\sigma_A$ of $A$ is the conjugate $\psi_A^*$ of $\psi_A$

$$\sigma_A(x^*) = \psi_A^*(x^*) = \sup_{x \in \mathbb{R}^N} [x \cdot x^* - \psi_A(x)], \qquad x^* \in \mathbb{R}^N,$$

or simply

$$\sigma_A(x^*) = \sup_{x \in A} x \cdot x^*.$$

The support functions of subsets in $\mathbb{R}^N$ form a very special class of functions from $\mathbb{R}^N$ to $\mathbb{R} \cup \{+\infty\}$.

Lemma 4.1 (cf. [2, p. 40, Prop 38]).

(i) *Given any nonempty subset $A$ of $\mathbb{R}^N$, the support function $\sigma_A$ is convex, lower semicontinuous, and positively homogeneous on $\mathbb{R}^N$.*

(ii) *Conversely, given any convex, lower semicontinuous, and positively homogeneous function $\sigma$ on $\mathbb{R}^N$, $\sigma$ is the support function of the closed convex set*

$$A_\sigma = \{x \in \mathbb{R}^N : \forall x^* \in \mathbb{R}^N, x \cdot x^* \leqslant \sigma(x^*)\}.$$

(iii) *$\sigma_A$ is uniformly Lipschitz continuous if and only if $A$ is bounded in $\mathbb{R}^N$.*

(iv) *For any* $\alpha \geqslant 0$, $\beta \geqslant 0$, *and nonempty sets* $A$ *and* $B$,

$$\alpha\sigma_A + \beta\sigma_B = \sigma_{\alpha A + \beta B}.$$

*Proof.* (i) and (ii). Cf. [2, p. 40, Prop. 38].

(iii) If $A$ is bounded, then for all $x_1^\star$ and $x_2^\star$ and for $x$ in $A$,

$$x \cdot x_2^\star = x \cdot x_1^\star + x \cdot [x_2^\star - x_1^\star]$$

and if $c > 0$ is the upper bound on the norm of elements in $A$,

$$\sigma_A(x_2^\star) \leqslant \sigma_A(x_1^\star) + c\,|x_2^\star - x_1^\star|,$$

and by interchanging $x_1^\star$ and $x_2^\star$

$$|\sigma_A(x_2^\star) - \sigma_A(x_1^\star)| \leqslant c\,|x_2^\star - x_1^\star|.$$

Conversely, if $\sigma_A$ is uniformly Lipschitz continuous

$$\begin{aligned}\sup_{x\in A} |x| &= \sup_{x \in A} \sup_{|x^\star| \leqslant 1} x \cdot x^\star = \sup_{|x^\star| \leqslant 1} \sigma_A(x^\star)\\ &= \sup_{|x^\star| \leqslant 1} [\sigma_A(x^\star) - \sigma_A(0)] \leqslant \sup_{|x^\star| \leqslant 1} c\,|x^\star - 0| = c.\end{aligned}$$

(iv) Obvious. ∎

Denote by $C_+(\mathbb{R}^N)$ the set of all positively homogeneous and continuous functions on $\mathbb{R}^N$ and consider the embedding

$$f \mapsto f|_B : C_+(\mathbb{R}^N) \to C^0(B),$$

where $B$ is the closed unit ball. Let $C_+(B)$ be the image of $C_+(\mathbb{R}^N)$. By construction this map is bijective from $C_+(\mathbb{R}^N)$ onto $C_+(B)$. To see this assume that there exist $f_1$ and $f_2$ such that

$$f_1|_B = f_2|_B.$$

Then by positive homogeneity for all $x$ in $\mathbb{R}^N$

$$f_1(x) = |x|\, f_1\left(\frac{x}{|x|}\right) = |x|\, f_2\left(\frac{x}{|x|}\right) = f_2(x).$$

To prove that it is surjective construct the following extension $\tilde{g}$ of $g$ in $C_+(B)$: $\tilde{g}(0) = 0$ and if $x \neq 0$,

$$\tilde{g}(x) = |x|\; g\left(\frac{x}{|x|}\right).$$

By definition $\tilde{g}(0)=0=g(0)$ and for $0<|x|\leqslant 1$

$$\tilde{g}(x)=|x|\, g\left(\frac{x}{|x|}\right)=g(x).$$

Thus $\tilde{g}$ is an extension of $g$ from $B$ to $\mathbb{R}^N$. For $\alpha>0$ and $x\neq 0$,

$$\tilde{g}(\alpha x)=|\alpha x|\, g\left(\frac{\alpha x}{|\alpha x|}\right)=\alpha\, |x|\, g\left(\frac{x}{|x|}\right)=\alpha\tilde{g}(x)$$

and for $\alpha\geqslant 0$ and $x$ such that $\alpha x=0$,

$$\tilde{g}(\alpha x)=0=\alpha\tilde{g}(x)$$

since either $\alpha=0$ or $x=0$. Thus $\tilde{g}$ is positively homogeneous. Moreover, by construction $\tilde{g}$ is continuous on $B$ and hence on all $\mathbb{R}^N$. So there is a one-to-one correspondence between $C_+(\mathbb{R}^N)$ and the subset $C_+(B)$ of $C^0(B)$ endowed with the norm

$$\|f\|=\sup_{|x^*|\leqslant 1}|f(x^*)|.$$

In particular,

$$\|\sigma_A\|=\sup_A |x|.$$

So for $A$ bounded this defines a map

$$A\mapsto\sigma_A\mapsto\sigma_A|_B\colon \mathscr{B}(\mathbb{R}^N)\to C_+(\mathbb{R}^N)\to C^0(B),$$

where $\mathscr{B}(\mathbb{R}^N)$ denotes the set of all bounded nonempty subsets of $\mathbb{R}^N$. For a convex lower semicontinuous function $f\colon\mathbb{R}^N\to\mathbb{R}\cup\{+\infty\}$, the biconjugate function $f^{**}$ is equal to $f$ (cf. [2, Theorem 3.1, p. 30]). By convexity of $\sigma_A$

$$\sigma_A^*=\psi_A^{**}=\psi_{\overline{\text{co}}\,A}\Rightarrow\sigma_A=\sigma_{\overline{\text{co}}\,A}$$

(cf. Ekeland-Temam [13a, Examples 4.3, p. 18]). This induces on $\mathscr{B}(\mathbb{R}^N)$ equivalence classes $[A]$ of domains with the same support function $\sigma_A$

$$[A]=\{B\subset\mathbb{R}^N\colon \overline{\text{co}}\,B=\overline{\text{co}}\,A,\ B \text{ bounded and nonempty}\}.$$

If we denote by $\mathscr{K}(\mathbb{R}^N)$ the set of all such equivalence classes we obtain the following embedding

$$[A]\mapsto\sigma_A\mapsto\sigma_A|_B\colon\mathscr{K}(\mathbb{R}^N)\to C_+(\mathbb{R}^N)\to C^0(B).$$

Denote by $\Sigma(\mathbb{R}^N)$ the subset of all convex functions in $C_+(\mathbb{R}^N)$. In view of Lemma 4.1(ii), it coincides with all the support functions of nonempty bounded subsets of $\mathbb{R}^N$. If we denote by $\Sigma(B)$ the image of $\Sigma(\mathbb{R}^N)$ in $C^0(B)$, then $\Sigma(B)$ is the subset of all bounded convex, positively homogeneous, and continuous functions on $B$.

THEOREM 4.1. (i) *$\Sigma(B)$ is a closed convex cone with vertex 0 in $C^0(B)$.*

(ii) *Any bounded subset of $\Sigma(B)$ is relatively compact in $C^0(B)$. In particular, for a sequence of nonempty subsets of a fixed bounded domain $D$, there exists a nonempty set $A \subset \overline{\text{co}}\, D$ and a subsequence of these subsets such that the corresponding subsequence of support functions converges to $\sigma_A$ in $C^0(B)$. If the sets of the sequence are convex, then $A$ can be chosen convex.*

*Proof.* (i) $\Sigma(B)$ is obviously a cone in 0. From Lemma 4.1(iv) for any nonempty sets $A_1$ and $A_2$ and $\lambda \in [0, 1]$,

$$\lambda\sigma_{A_1} + (1-\lambda)\,\sigma_{A_2} = \sigma_{\lambda A_1 + (1-\lambda) A_2}$$

and $\Sigma(B)$ is convex. To prove the closure let $\{\sigma_n\} \subset \Sigma(\mathbb{R}^N)$ be a sequence such that $\{\sigma_n|_B\}$ is Cauchy in $C^0(B)$. Then there exists $f$ in $C^0(B)$ such that

$$\sigma_n|_B \to f \qquad \text{in } C^0(B).$$

The function $f$ is positively homogeneous: for all $\alpha \geqslant 0$ and $x \in B$,

$$\sigma_n(\alpha x) = \alpha\sigma_n(x) \quad \Rightarrow \quad f(\alpha x) = \alpha f(x).$$

It is convex since for each $n$, $\sigma_n$ is convex:

$$\forall \alpha \in [0, 1], \qquad \forall x, y \in B, \quad \sigma_n(\alpha x + (1-\alpha)\,y) \leqslant \alpha\sigma_n(x) + (1-\alpha)\,\sigma_n(y),$$

and by going to limit we get the convexity of $f$. In view of this, $f$ has a unique positive homogeneous continuous extension $\tilde{f}$ to $\mathbb{R}^N$. This extension is also convex. By positive homogeneity

$$\tilde{f}(\lambda x + (1-\lambda)\,y) = [|x| + |y|]\, f\left(\lambda \frac{x}{|x| + |y|} + (1-\lambda)\frac{y}{|x| + |y|}\right)$$

and by convexity of $f$ on $B$

$$\tilde{f}(\lambda x + (1-\lambda)\,y) \leqslant [|x| + |y|] \left\{\lambda f\left(\frac{x}{|x| + |y|}\right) + (1-\lambda)\, f\left(\frac{y}{|x| + |y|}\right)\right\}.$$

By Lemma 4.1(ii) there exists a set $A$ such that $\tilde{f} = \sigma_A$. Moreover

$$\|\sigma_A|_B\| = \sup_{|x^*| \leq 1} \sup_{x \in A} x \cdot x^* = \sup_{x \in A} |x|$$

is finite. Thus $A$ is bounded, $[A] \in \mathscr{K}(\mathbb{R}^N)$, and $\Sigma(B)$ is closed in $C^0(B)$.

(ii) By Lemma 4.1(iii) the elements of $\Sigma(\mathbb{R}^N)$ are uniformly Lipschitz continuous on $\mathbb{R}^N$ and a fortiori on $B$. Therefore

$$\Sigma(B) \subset \mathrm{Lip}(B) \subset C^0(B)$$

and the result follows by compactness of Lip$(B)$ into $C^0(B)$. ∎

## 5. Oriented Distance Functions and Geometry of Domains

The distance function $d_A$ provides a good description of a domain $A$ from the "outside," but does not say much about its "inside" where it is identically zero. To get a complete description of a nonempty set $A$ in a fixed closed domain $D$ of $\mathbb{R}^N$, it is natural to take into account its complement

$$\complement A = \{ y \in \mathbb{R}^N : y \notin A \}.$$

When both $A$ and $\complement A$ are nonempty, we introduce the *oriented distance function*

$$b_A(x) = d_A(x) - d_{\complement A}(x), \qquad \forall x \in \mathbb{R}^N. \tag{1}$$

This can also be seen as an "algebraic distance" to the boundary. Noting that $b_{\complement A} = -b_A$, it means that we have chosen the negative sign for the interior of $A$ and the positive sign for the interior of its complement.

By construction,

$$b_A(x) = \begin{cases} d_A(x) > 0, & x \in \mathbb{R}^N \backslash \bar{A} = \mathrm{int}\, \complement A \\ 0, & x \in \bar{A} \cap \overline{\complement A} = \partial A \\ -d_{\complement A}(x) < 0, & x \in \mathbb{R}^N \backslash \overline{\complement A} = \mathrm{int}\, A. \end{cases} \tag{2}$$

Lemma 5.1. *Let $A$ be a nonempty subset of $\mathbb{R}^N$. Then*

(i) $A \neq \emptyset$ *and* $\complement A \neq \emptyset \Leftrightarrow \partial A \neq \emptyset$.

(ii) $|b_A| = \max\{d_A, d_{\complement A}\} = d_{\partial A}$, *where $d_{\partial A}$ is the function which is equal to the minimum distance from $x$ to $\partial A$.*

*Proof.* (i) Obvious. (ii) For $x$ in $\bar{A}$,

$$b_A(x) = -d_{\complement A}(x) \quad \Rightarrow \quad |b_A(x)| = d_{\complement A}(x) \leqslant d_{\partial A}(x)$$

since $\overline{\complement A} \supset \partial A$ and the inf over $\overline{\complement A}$ is smaller than the inf over its subset $\partial A$. Similarly, for $x$ in $\overline{\complement A}$,

$$b_A(x) = d_A(x) \quad \Rightarrow \quad |b_A(x)| = d_A(x) \leqslant d_{\partial A}(x)$$

and finally

$$|b_A(x)| \leqslant \max\{d_{\complement A}(x), d_A(x)\} \leqslant d_{\partial A}(x).$$

Conversely, for each $x$ in $\overline{\complement A}$, the set of projections $\Pi_A(x) \subset \bar{A} \cap \overline{\complement A} = \partial A$ is not empty. Hence

$$|b_A(x)| = d_A(x) = \min_{y \in \Pi_A(x)} |x - y| \geqslant \inf_{y \in \partial A} |x - y| = d_{\partial A}(x),$$

and similarly for all $x$ in $\bar{A}$

$$|b_A(x)| = d_{\complement A}(x) \geqslant d_{\partial A}(x).$$

Therefore $|b_A(x)| \geqslant d_{\partial A}(x)$ and this concludes the proof. ∎

So the nonemptyness hypothesis is equivalent to assuming that the boundary $\partial A$ is nonempty. This excludes $\varnothing$ and $\mathbb{R}^N$. The zero function $b_A(x) = 0$, $\forall x \in \mathbb{R}^N$, corresponds to the equivalence class of sets $A$ such that

$$b_A = 0 \Leftrightarrow d_A = d_{\complement A} \quad \text{in} \quad \mathbb{R}^N \Leftrightarrow \bar{A} = \overline{\complement A}.$$

But

$$\operatorname{int} \complement A = \complement \bar{A} = \complement \overline{\complement A} = \operatorname{int} A \qquad \text{and} \qquad A \cap \complement A = \varnothing$$

$$\Rightarrow \operatorname{int} \complement A = \operatorname{int} A = \varnothing \quad \Rightarrow \quad \partial A = \overline{\complement A} = \bar{A} = \mathbb{R}^N.$$

This class of sets is not empty. For instance, choose the subset of points of $\mathbb{R}^N$ with rational coordinates.

### 5.1. *Basic Properties*

This oriented distance function enjoys properties similar to those of $d_A$.

THEOREM 5.1. *Let $A$ be a subset of $\mathbb{R}^N$ with nonempty boundary $\partial A$.*

(i) *The function $b_A$ is Lipschitz continuous and*

$$\forall x, y \in \mathbb{R}^N, \qquad |b_A(y) - b_A(x)| \leqslant |y - x|. \tag{3}$$

(ii) *For all $x \in \partial A$, $\nabla b_A^2(x)$ exists and is equal to 0. For all $x \notin \partial A$, $\nabla b_A^2(x)$ exists if and only if $\nabla b_A(x)$ exists.*

(iii) *If $\nabla b_A(x)$ exists, then there exists a unique $P_{\partial A}(x) \in \partial A$ such that*

$$b_A(x) = \begin{Bmatrix} |P_{\partial A}(x) - x|, & \text{if } x \in \operatorname{int} \complement A \\ 0, & \text{if } x \in \partial A \\ -|P_{\partial A}(x) - x|, & \text{if } x \in \operatorname{int} A \end{Bmatrix}, \tag{4}$$

$$\nabla b_A(x) = \frac{x - P_{\partial A}(x)}{b_A(x)}, \qquad \text{if} \quad x \notin \partial A, \tag{5}$$

*and*

$$\tfrac{1}{2} \nabla b_A^2(x) = x - P_{\partial A}(x). \tag{6}$$

*Moreover, for almost all $x \in \partial A$, $\nabla b_A(x) = \nabla d_{\partial A}(x) = 0$.*

(iv) *Given an integer $k \geqslant 1$ and a neighbourhood $U(x)$ of a point $x \in \partial A$, then $b_A^2 \in C^k(U(x))$ if and only if for all $y \in U(x)$ the projection $P_{\partial A}(y)$ is unique and $P_{\partial A} \in C^{k-1}(U(x))$.*

(v) *For all subsets $A$ and $B$ of $\mathbb{R}^N$ with nonempty boundaries $\partial A$ and $\partial B$*

$$\bar{A} \subset \bar{B} \quad \text{and} \quad \operatorname{int} A \subset \operatorname{int} B \quad \Leftrightarrow \quad b_B \leqslant b_A.$$

*Proof.* (i) Clearly

$$\forall x, y \in \bar{A}, \qquad |b_A(y) - b_A(x)| = |d_{\complement A}(y) - d_{\complement A}(x)| \leqslant |y - x|$$

$$\forall x, y \in \overline{\complement A}, \qquad |b_A(y) - b_A(x)| = |d_A(y) - d_A(x)| \leqslant |y - x|.$$

For $x \in \bar{A}$ and $y \in \operatorname{int} \complement A$

$$b_A(y) - b_A(x) = d_A(y) + d_{\complement A}(x) > 0.$$

There exists $\lambda \in ]0, 1]$ such that $x_\lambda = \lambda x + (1 - \lambda)\, y \in \partial A$ and

$$0 \leqslant b_A(y) - b_A(x) \leqslant |x_\lambda - y| + |x_\lambda - x| = |x - y|$$
$$\Rightarrow |b_A(y) - b_A(x)| \leqslant |x - y|.$$

The argument is similar for $x \in \operatorname{int} A$ and $y \in \overline{\complement A}$.

(ii) For $x \in \partial A$ and any $y \in \mathbb{R}^N$, consider the differential quotient

$$\Delta(y) = \frac{b_A^2(y) - b_A^2(x)}{|y-x|} = [b_A(y) + b_A(x)] \frac{[b_A(y) - b_A(x)]}{|y-x|}.$$

Then

$$|\Delta(y)| \leqslant |b_A(y)| \frac{|y-x|}{|y-x|} \to 0 \qquad \text{as} \quad y \to x.$$

Hence

$$\forall x \in \partial A, \qquad \nabla b_A^2(x) = 0 = \tfrac{1}{2}(x-x) = \tfrac{1}{2}(x - P_{\partial A}(x)).$$

If $x \notin \partial A$ and $\nabla b_A^2(x)$ exists, then $b_A(x) \neq 0$ and

$$\nabla b_A(x) = \frac{1}{2} \frac{\nabla b_A^2(x)}{b_A(x)}.$$

To see this consider the new differential quotient

$$q(y) = \left[ b_A(y) - b_A(x) - \frac{1}{2} \frac{\nabla b_A^2(x)}{b_A(x)} \cdot (y-x) \right] \frac{1}{|y-x|}.$$

Then

$$\begin{aligned}
&(b_A(y) + b_A(x))\, q(y) \\
&\quad = \left[ b_A^2(y) - b_A^2(x) - \frac{b_A(x) + b_A(y)}{2b_A(x)} \nabla b_A^2(x) \cdot (y-x) \right] \frac{1}{|y-x|} \\
&\quad = [b_A^2(y) - b_A^2(x) - \nabla b_A^2(x) \cdot (y-x)] \frac{1}{|y-x|} \\
&\qquad + \frac{b_A(x) - b_A(y)}{2b_A(x)} \nabla b_A^2(x) \cdot \frac{(y-x)}{|y-x|}.
\end{aligned}$$

Since $b_A^2$ is differentiable at $x$, the first term goes to zero as $y$ goes to $x$. As for the second term it is bounded by

$$\frac{|\nabla b_A^2(x)|}{2\,|b_A(x)|} |x-y|$$

and hence goes to zero as $y$ goes to $x$. So the term

$$[b_A(y) + b_A(x)]\, q(y) \to 0 \qquad \text{as} \quad y \to x$$

and since $b_A(x) \neq 0$, $q(y) \to 0$ as $y \to x$ and $b_A$ is differentiable at $x$. In the other direction the result is trivial.

(iii) If $x \in \operatorname{int} A$, use the fact that $b_A = -d_{\complement A}$ in $\operatorname{int} A$ and repeat the proof of Theorem 3.3(ii) to obtain

$$-\nabla b_A(x) = \nabla d_{\complement A}(x) = \frac{x - P_{\partial A}(x)}{|x - P_{\partial A}(x)|}$$

and $P_{\partial A}(x) \in \partial A$ is unique. Similarly for $x \in \operatorname{int} \complement A$, $b_A = d_A$,

$$\nabla b_A(x) = \nabla d_A(x) = \frac{x - P_{\partial A}(x)}{|x - P_{\partial A}(x)|}$$

and $P_{\partial A}(x) \in \partial A$ is unique. Finally when $x \in \partial A$, $P_{\partial A}(x) = x$. The distance function $d_{\partial A}$ is differentiable almost everywhere. From Theorem 3.3 whenever it is differentiable at a point $x \in \partial A$, $\nabla d_{\partial A}(x) = 0$. But for all $v$ in $\mathbb{R}^N$ and $t > 0$

$$\left| \frac{b_A(x+tv) - b_A(x)}{t} \right| = \frac{|b_A(x+tv)|}{t} = \frac{d_{\partial A}(x+tv) - d_{\partial A}(x)}{t}$$

since $|b_A| = d_{\partial A}$. Therefore for almost all $x \in \partial A$, $\nabla b_A(x) = \nabla d_{\partial A}(x) = 0$.

(iv) is obvious.

(v) Let $b_B \leqslant b_A$. Then for all $x$ in $\operatorname{int} \complement B$

$$0 < d_B(x) = b_B(x) \leqslant b_A(x) \quad \Rightarrow \quad b_A(x) = d_A(x) > 0$$

and $x$ belongs to $\operatorname{int} \complement A$. Therefore by taking complements $\bar{A} \subset \bar{B}$. Since $b_{\complement B} = -b_B$ we repeat the above argument for $b_{\complement A} \leqslant b_{\complement B}$ and obtain $\overline{\operatorname{int} A} \subset \overline{\operatorname{int} B}$ and equivalently $\overline{\complement B} \subset \overline{\complement A}$. Conversely, $\bar{A} \subset \bar{B}$ (resp., $\overline{\complement B} \subset \overline{\complement A}$) imply $d_B \leqslant d_A$ (resp., $d_{\complement A} \leqslant d_{\complement B}$). ∎

*Remark* 5.1. In general $\nabla d_A(x)$ and $\nabla d_{\complement A}(x)$ do not exist for $x \in \partial A$. This is readily seen by constructing the directional derivatives for the half space

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leqslant 0\} \tag{7}$$

at the point $(0, 0)$. Nevertheless $\nabla b_A(0, 0)$ exists and is equal to $(1, 0)$ which is the outward unit normal at $(0, 0) \in \partial A$ to $A$. Note also that for all $x \in \partial A$, $|\nabla b_A(x)| = 1$. This is possible since $m(\partial A) = 0$. If $\nabla b_A(x)$ exists, then it is easy to verify that $|\nabla b_A(x)| \leqslant 1$.

*Remark* 5.2. The uniqueness of the projection $P_{\partial A}(x)$ at $x$ is directly related to the existence $\nabla b_A^2(x)$ since they are related by

$$P_{\partial A}(x) = x - \tfrac{1}{2}\nabla b_A^2(x). \tag{8}$$

In other words, $P_{\partial A}(x)$ is the gradient of the functional

$$f(x) = \tfrac{1}{2}\left[|x|^2 - b_A^2(x)\right] \tag{9}$$

and all the properties of $P_{\partial A}$ can be obtained from those of $\nabla b_A^2$.

DEFINITION 5.1. Given a subset $A$ of $\mathbb{R}^N$ with a nonempty boundary $\partial A$ and a point $x$ in $\mathbb{R}^N$, the set of projections of $x$ onto $\partial A$ associated with $b_A(x)$ is defined as

$$\Pi_{\partial A}(x) = \begin{cases} \Pi_A(x), & \text{if } x \in \operatorname{int} \complement A \\ \{x\}, & \text{if } x \in \partial A \\ \Pi_{\complement A}(x), & \text{if } x \in \operatorname{int} A. \end{cases} \tag{10}$$

The choice of notation is compatible with the fact that $\Pi_{\partial A}(x)$ coincides with the set of projections of $x$ onto $\partial A$ for the distance function $d_{\partial A}$ (cf. Lemma 5.1). By definition

$$\forall P_{\partial A}(x) \in \Pi_{\partial A}(x), \qquad |x - P_{\partial A}(x)| = |b_A(x)| = d_{\partial A}(x). \tag{11}$$

In fact, the oriented distance function coincides with the signed distance (cf. [27, p. 268]) up to a change of sign since

$$b_A(x) = \begin{cases} d_{\partial A(x)}, & \text{if } x \in \overline{\complement A}, \\ -d_{\partial A(x)}, & \text{if } x \in \bar{A}. \end{cases}$$

We have the analogue of Lemma 3.1 in Section 3.

LEMMA 5.2. *Let $A$ be a subset of $\mathbb{R}^N$ with a nonempty boundary $\partial A$ and $x \in \mathbb{R}^N$. Then for all $v$ in $\mathbb{R}^N$*

$$db_A^2(x; v) = \inf_{z \in \Pi_{\partial A}(x)} 2(x - z)\cdot v = 2[x \cdot v - \sigma_{\Pi_{\partial A}(x)}(v)], \tag{12}$$

*where*

$$\sigma_{\Pi_{\partial A}(x)}(v) = \sup_{z \in \Pi_{\partial A}(x)} z \cdot v \; (= \sigma_{\operatorname{co} \Pi_{\partial A}(x)}(v) = df(x; v)) \tag{13}$$

*is the support function to the set $\Pi_{\partial A}(x)$, $\operatorname{co} \Pi_{\partial A}(x)$ its convex hull and $f$ is defined by* (9).

5.2. *Oriented Boundary Distance Functions in $C(D)$ and $W^{1,p}(D)$*

Let $D$ be a fixed nonempty closed domain in $\mathbb{R}^N$. Given two domains $A$ and $B$ in $D$ with nonempty boundaries $\partial A$ and $\partial B$, then

$$b_A = b_B \quad \Leftrightarrow \quad \bar{A} = \bar{B} \quad \text{and} \quad \overline{\complement A} = \overline{\complement B} \quad \Leftrightarrow \quad \bar{A} = \bar{B} \quad \text{and} \quad \partial A = \partial B.$$

To see this observe that

$$b_A = b_B \quad \text{on } D \quad \Leftrightarrow \quad d_A = d_B \quad \text{and} \quad d_{\complement A} = d_{\complement B} \quad \text{on } D.$$

But $\bar{A}$ and $\bar{B}$ are subsets of $D$ and

$$\bar{A} = \{x \in D : d_A(x) = 0\} = \{x \in D : d_B(x) = 0\} = \bar{B}.$$

Also,

$$A \subset D \quad \text{and} \quad B \subset D \quad \Rightarrow \quad \complement A \supset \complement D \quad \text{and} \quad \complement B \supset \complement D$$

and

$$d_{\complement A}(x) = 0 = d_{\complement B}(x) \qquad \text{on } \complement D$$

$$\Rightarrow d_{\complement A} = d_{\complement B} \qquad \text{on } \mathbb{R}^N \quad \Rightarrow \quad \overline{\complement A} = \overline{\complement B}.$$

Associate with each subset $A$ of $D$ which has a nonempty boundary $\partial A$ the equivalence class

$$[A]_b = \{B : \forall B,\, B \subset D,\, \bar{B} = \bar{A} \text{ and } \partial A = \partial B\}$$

and define the family of equivalence classes

$$\mathscr{F}_b(D) = \{[A]_b : \forall A,\, A \subset D \text{ and } \partial A \neq \emptyset\}.$$

Note that the equivalence classes induced by $b_A$ are finer than those induced by $d_A$ since both the closures and the boundaries of the respective sets must coincide. As in the case of $d_A$ we identify $\mathscr{F}_b(D)$ with the set

$$C_b(D) = \{b_A : \forall A,\, A \subset D \text{ and } \partial A \neq \emptyset\}$$

of oriented boundary distances in $C(D)$ through the embedding

$$[A]_b \mapsto b_A : \mathscr{F}_b(D) \to C_b(D).$$

We see below that $C_b(D)$ is closed in $C(D)$. Again this justifies the introduction of the following metrics on $\mathscr{F}_b(D)$

$$\rho(A, B) = \sup_{x \in D} |b_A(x) - b_B(x)|, \qquad \text{for } D \text{ compact}$$

$$\rho_\delta(A, B) = \delta(b_A, b_B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_{K_k}(b_A - b_B)}{1 + q_{K_k}(b_A - b_B)}, \qquad \text{for } D \text{ unbounded.}$$

We now have the equivalent of Theorem 3.1.

THEOREM 5.2. *Let $D$ be a nonempty closed (resp., compact) subset of $\mathbb{R}^N$.*

(i) *The set $C_b(D)$ is closed in $C(D)$ and $W^{1,p}_{\text{loc}}(D)$ (resp., $C^0(D)$ and $W^{1,p}(D)$), for $p$, $1 \leqslant p < \infty$.*

(ii) *The function $\rho_\delta$ (resp., $\rho$) defines a metric on $\mathscr{F}_b(D)$.*

(iii) *If $D$ is compact, then $C_b(D)$ (and hence $\mathscr{F}_b(D)$) is compact in $C^0(D)$.*

(iv) *The set $C_b(D)$ is closed (resp., compact) in $W^{1,p}_{\text{loc}}(D)$-weak (resp., $W^{1,p}(D)$-weak).*

*Proof.* (i) Consider a sequence $\{A_n\}$, $A_n \subset D$, $\partial A_n \neq \varnothing$ such that $b_{A_n} \to f$ in $C(D)$ for some $f$ in $C(D)$. Associate with each $g$ in $C(D)$ its positive and negative parts

$$g^+(x) = \max\{g(x), 0\}, \qquad g^-(x) = \max\{-g(x), 0\}.$$

Then by continuity of this operation

$$d_{A_n} = b^+_{A_n} \to f^+ \qquad \text{and} \qquad d_{\complement A_n} = b^-_{A_n} \to f^- \qquad \text{in } C(D),$$

and

$$d_{\partial A_n} = |b_{A_n}| \to |f| \qquad \text{in } C(D).$$

Let

$$A^{\pm} = \{x \in \mathbb{R}^N : \liminf_{n \to \infty} b_{A_n}(x) \gtrless 0\}, \qquad A^0 = \{x \in \mathbb{R}^N : \liminf_{n \to \infty} b_{A_n}(x) = 0\}.$$

By construction for all $n \geqslant 1$, $A_n \subset D$ and $\complement D \subset \complement A_n$. So for all $x \in \complement D$,

$$d_{\complement A_n}(x) = 0 \qquad \text{and} \qquad \liminf_{n \to \infty} d_{A_n}(x) \geqslant d_D(x) > 0.$$

Therefore

$$\complement D \subset A^+ \qquad \text{and} \qquad D \supset \complement A^+ = A^0 \cup A^-.$$

By assumption $b^+_{A_n} = d_{A_n} \to f^+$ on $D$ and

$$A^- = \{x \in D: \lim_{n\to\infty} b_{A_n}(x) < 0\}$$

$$A^0 = \{x \in D: \lim_{n\to\infty} b_{A_n}(x) = 0\}$$

$$A^+ \cap D = \{x \in D: \lim_{n\to\infty} b_{A_n}(x) > 0\}.$$

From Theorem 3.1(i) applied to each component of $f$ on $D$,

$$f^+ = d_{A^0 \cup A^-}, \qquad A^0 \cup A^- \neq \varnothing \quad f^- = d_{A^0 \cup (A^+ \cap D)}, \quad A^0 \cup (A^+ \cap D) \neq \varnothing.$$

*Construction of* $A$. We now construct a set $A \subset D$, $\varnothing = A \neq \mathbb{R}^N$, such that $f = b_A$. By construction $A^-$ is open, $A^0$ and $A^0 \cup A^-$ are closed, and hence $A^+$ is open. Define the open set

$$B^0 = \complement\, [\overline{A^+} \cup \overline{A^-}]$$

and note that

$$B^0 \subset A^0 \quad \text{and} \quad \overline{B^0} \subset A^0.$$

Let $\mathbb{Q}$ be the subset of points in $\mathbb{R}^N$ with rational coordinates. Define

$$B^0_+ = B^0 \cap \mathbb{Q}, \qquad B^0_- = B^0 \cap \complement\, \mathbb{Q},$$

and note that

$$\overline{B^0_+} = \overline{B^0} = \overline{B^0_-}.$$

Finally define

$$A = A^- \cup \partial A^+ \cup B^0_-.$$

By construction

$$\complement\, A = A^+ \cup \partial A^- \cup B^0_+$$

and

$$\partial A^+ \cup \partial A^- \cup \overline{B^0} \subset A^0.$$

Now

$$A = A^- \cup \partial A^+ \cup B^0_- \subset A^- \cup A^0 \subset D$$

$$\complement\, A = A^+ \cup \partial A^- \cup B^0_+ \subset A^+ \cup A^0.$$

To see that $\emptyset \neq A$, assume that $A = \emptyset$. Then

$$A^- = \emptyset, \quad \partial A^+ = \emptyset, \quad B^0_- = \emptyset \quad \Rightarrow \quad \overline{A^-} = \emptyset, \quad \partial A^+ = \emptyset, \quad \overline{B^0} = \emptyset$$
$$\Rightarrow \quad A^0 = \partial A^+ \cup \partial A^- \cup B^0 = \emptyset \quad \Rightarrow \quad A^0 \cup A^- = \emptyset,$$

and this is a contradiction with the construction of $A^0 \cup A^- \neq \emptyset$. Similarly if $A = \mathbb{R}^N$, then $\complement A = \emptyset$ and

$$A^+ = \emptyset, \quad \partial A^- = \emptyset, \quad B^0_+ = \emptyset \quad \Rightarrow \quad \overline{A^+} = \emptyset, \quad \partial A^- = \emptyset, \quad \overline{B^0} = \emptyset$$
$$\Rightarrow \quad A^0 = \partial A^+ \cup \partial A^- \cup B^0 = \emptyset \quad \Rightarrow \quad A^0 \cup A^+ = \emptyset,$$

which is a contradiction with the construction of $A^0 \cup (A^+ \cap D) \neq \emptyset$.

*Proof that $f = b_A$ on $D$.* The last part of the proof is to show that $f = b_A$ on $D$. First note that

$$\bar{A} = \overline{A^- \cup \partial A^+ \cup B^0_-} = \overline{A^-} \cup \partial A^+ \cup \overline{B^0_-} = \overline{A^-} \cup \partial A^+ \cup \overline{B^0}$$
$$= A^- \cup \partial A^- \cup \partial A^+ \cup \overline{B^0}.$$

But since $A^0$ is closed

$$A^0 = \partial A^- \cup \partial A^+ \cup B^0 \subset \partial A^- \cup \partial A^+ \cup \overline{B^0} \subset A^0$$

and necessarily

$$\bar{A} = A^- \cup A^0 \quad \Rightarrow \quad d_A = d_{\bar{A}} = d_{A^- \cup A^0} = f^+.$$

Similarly,

$$\overline{\complement A} = \overline{A^+ \cup \partial A^- \cup B^0_+} = A^+ \cup \partial A^+ \cup \partial A^- \cup \overline{B^0} = A^+ \cup A^0.$$

In general

$$f^- = d_{A^0 \cup (D \cap A^+)} \geqslant d_{A^0 \cup A^+} = d_{\overline{\complement A}} = d_{\complement A} \qquad \text{in } \mathbb{R}^N,$$

but they coincide on $D$. Clearly on $A^0 \cup (A^+ \cap D)$

$$f^-(x) = 0 = d_{\complement A}(x).$$

For all points $x \in A^-$,

$$\exists \bar{y} \in \partial(A^+ \cup A^0) \subset \partial A^+ \cup A^0 \subset A^0 \qquad \text{such that} \quad d_{A^+ \cup A^0}(x) = |\bar{y} - x|$$

and

$$d_{A^+ \cup A^0}(x) = |\bar{y} - x| \geqslant \inf_{y \in (D \cap A^+) \cup A^0} |y - x| = f^-(x)$$
$$\geqslant \inf_{y \in A^+ \cup A^0} |y - x| = d_{A^+ \cup A^0}(x).$$

Hence

$$f^- = d_{(D \cap A^+) \cup A^0} = d_{A^+ \cup A^0} = b_A^- \qquad \text{on } D.$$

For $W^{1,p}$ we use the result in $C(D)$ and the same proof as in Theorem 3.6(ii).

(ii) Proved by construction of the embedding.

(iii) Proved by the same proof as for $\mathscr{F}(D)$ and $d_A$ in Theorem 3.2(iii).

(iv) Proved from part (i) and arguments similar to the ones used in the proof of Theorem 3.6(iii). ▮

We now have the analogue of Theorem 3.2 and its corollary for $A$, $\complement A$, and $\partial A$. In the last case it takes the following form (to be compared with Richardson [23, Lemma 3.2, p. 44] and Kulbarni *et al.* [18] for an application to image segmentation).

COROLLARY. *Let $D$ be a nonempty compact subset of $\mathbb{R}^N$ and $\mathscr{F}_b(D)$ the corresponding set of equivalence classes of subsets of $D$ with a nonempty boundary. Then*

(i) *For each subset $F$ of $D$ the set*

$$\{[A]_b \in \mathscr{F}_b(D) : \forall A,\ F \subset \partial A\}$$

*is closed.*

(ii) *If $S$ is open (resp., closed), then the sets*

$$I_b(S) = \{[A]_b \in \mathscr{F}_b(D) : \forall A,\ \varnothing \neq \partial A \subset S\}$$

$$J_b(S) = \{[A]_b \in \mathscr{F}_b(D) : \forall A,\ \partial A \cap S \neq \varnothing\}$$

*are open (resp., closed).*

(iii) *Associate with an equivalent class $[A]_b$ the number*

$$\#_b([A]_b) = \text{number of connected components of } \partial A.$$

*Then the map*

$$[A]_b \mapsto \#_b([A]_b) : \mathscr{F}_b(D) \to \mathbb{R}$$

*is lower semicontinuous.*

*Proof.* (i) If $F$ is empty there is nothing to prove. If not then let $b_{A_n}$ be Cauchy sequence which converges to $b_A$ in $C^0(D)$. Then $d_{\partial A_n} = |b_{A_n}|$ converges to $d_{\partial_A} = |b_A|$ in $C^0(D)$. Moreover,

$$\forall n, \quad d_F(x) \geqslant d_{\partial A_n} = |b_{A_n}| \quad \Rightarrow \quad d_F(x) \geqslant d_{\partial A} = |b_A| \quad \Rightarrow \quad F \subset \bar{F} \subset \partial A.$$

(ii) When $S$ is closed we use the same technique as in (i) to prove that $I_b(S)$ is closed. When $S$ is open fix $A \in I_b(S)$ and notice that $\partial A \cap \complement S = \varnothing$. By the separation theorem there exists $\delta > 0$ such that

$$\forall x \in \partial A, \ \forall y \in \complement S, |x - y| > \delta.$$

So for all $[B]_b$ in the neighbourhood

$$N_\delta = \{[B]_b : \|b_B - b_A\| < \delta/2\},$$

we have for all $z \in \partial B, x \in \partial A$, and $y \in \complement S$

$$|z - y| \geqslant |x - y| - \|d_{\partial B} - d_{\partial A}\| \geqslant \delta - \|d_{\partial B} - d_{\partial A}\| \geqslant \delta/2.$$

As a result

$$\partial B \cap \complement S = \varnothing \quad \Rightarrow \quad \partial B \subset S.$$

Therefore $I_b(S)$ is open.

Assume that $S$ is closed and consider the set $J_b(S)$. Let $b_{A_n}$ be a Cauchy sequence which converges to $b_A$ in $C^0(D)$. Then, for all $n$, $\partial A_n \cap S \neq \varnothing$, and there exist $x_n \in \partial A_n \cap S$. But $S$ is a closed subset of the compact set $D$ and there exists a subsequence and a point $x \in S$ such that $x_{n_k} \to x$. But we also know that $d_{\partial A_n} = |b_{A_n}|$ converges to $d_{\partial A} = |b_A|$ in $C^0(D)$. Hence

$$\begin{aligned}
d_{\partial A}(x) &= d_{\partial A}(x) - d_{\partial A_{n_k}}(x_{n_k}) \\
&= d_{\partial A}(x) - d_{\partial A_{n_k}}(x) + d_{\partial A_{n_k}}(x) - d_{\partial A_{n_k}}(x_{n_k}) \\
&\leqslant \|d_{\partial A} - d_{\partial A_{n_k}}\| + |x_{n_k} - x| \\
&\leqslant \|b_A - b_{A_{n_k}}\| + \|x_{n_k} - x| \to 0 \\
&\Rightarrow \quad x \in \partial A \cap S \quad \Rightarrow \quad A \in J_b(S).
\end{aligned}$$

Finally, consider the case when $S$ is open. Pick any $A \in J_b(S)$. Assume that $\forall \varepsilon > 0$, $\exists B_\varepsilon \subset \complement S$. From the previous results $I_b(\complement S)$ is closed since $\complement S$ is closed. Therefore

$$\lim_{\varepsilon \to 0} b_{B_\varepsilon} = b_A \quad \Rightarrow \quad \partial A \subset \complement S \quad \Rightarrow \quad \partial A \cap S = \varnothing.$$

From this contradiction we conclude that $J_b(S)$ is open.

(iii) Let $\{A_n\}$ and $A$ be nonempty subsets of $D$ such that $b_{A_n}$ converges to $b_A$ in $C^0(D)$. Then $d_{\partial A_n} = |b_{A_n}|$ converges to $d_{\partial A} = |b_A|$ in $C^0(D)$. Assume that $\#_b([A]_b) = k$ is finite. Then there exists a family of disjoint open sets $G_1, ..., G_k$ such that

$$\partial A \subset G = \bigcup_{i=1}^{k} G_i, \qquad \forall i, \quad \partial A \cap G_i \neq \varnothing.$$

In view of Theorem 5.2(ii),

$$\partial A \in \mathscr{U} = \bigcap_{i=1}^{k} J_b(G_i) \cap I_b(G).$$

But $\mathscr{U}$ is not empty and open as the finite intersection of $k+1$ open sets. As a result there exists $\varepsilon > 0$ and an open neighourhood of $[A]_b$,

$$N_\varepsilon([A]_b) = \{[B]_b \colon \|b_B - b_A\| < \varepsilon\} \subset \mathscr{U}.$$

Hence since $b_{A_n}$ converges to $b_A$, there exists $\bar{n} > 0$ such that

$$\forall n \geqslant \bar{n}, \qquad [A_n]_b \in \mathscr{U},$$

and necessarily

$$\forall n \geqslant \bar{n}, \qquad \partial A_n \subset G, \quad \partial A_n \cap G_i \neq \varnothing, \quad \forall i,$$

and

$$\#_b([A_n]_b) \geqslant \#_b([A]_b)$$

which means that $[A]_b \to \#_b([A]_b)$ is lower semicontinuous. Now for $\#_b([A]_b) = +\infty$, we repeat the above procedure and refine the open covering. ∎

*Remark* 5.3. Note that if $b_{A_n} \to b_A$ in $C(D)$, then

$$d_{A_n} = \max\{0, b_{A_n}\}, \qquad d_{\complement A_n} = \max\{0, -b_{A_n}\}, \qquad d_{\partial A_n} = |b_{A_n}|$$

converge to $d_A$, $d_{\complement A}$, and $d_{\partial A}$ in $C(D)$.

We now introduce an important subfamily of equivalence classes of domains. They are the domains whose boundary has a zero Lebesgue measure in $\mathbb{R}^N$. Define the following subsets of $\mathscr{F}_b(D)$ and $C_b(D)$:

$$\mathscr{F}_b^0(D) = \{[A]_b : \forall A,\ A \subset D,\ \partial A \neq \emptyset,\ m(\partial A) = 0\}$$

$$C_b^0(D) = \{b_A : \forall A,\ A \subset D,\ \partial A \neq \emptyset,\ m(\partial A) = 0\}.$$

The next theorem says that for a sequence of nonempty sets $\{A_n\}$ for which $\{b_{A_n}\}$ converges in the $W^{1,p}$-topology there exists a set $A$ with nonempty boundary for which the characteristic functions of $\bar{A}_n$, $\overline{\complement A_n}$, and $\partial A_n$ converge to the characteristic functions of $\bar{A}$, $\overline{\complement A}$, and $\partial A$. In particular, sequences of sets whose boundary has a zero measure converge to sets whose boundary has a zero measure.

*Notation* 5.1. As in Section 3 the closed hold-all $D$ is a subset of $\mathbb{R}^N$ with a nonempty interior int $D$ such that

$$\overline{\operatorname{int} D} = D \qquad \text{and} \qquad \chi_{\operatorname{int} D} = \chi_D.$$

Some of the function spaces involved are defined on either $D$ or int $D$. However, for simplicity we use $D$ in both cases unless it is not clear from the context.

THEOREM 5.3. *Let $D$ be a closed (resp., compact) domain in $\mathbb{R}^N$ with nonempty interior such that $\overline{\operatorname{int} D} = D$ and a locally Lipschitzian boundary.*

(i) *For each set $A$ in $D$ with nonempty boundary*

$$\chi_{\partial A} = 1 - |\nabla b_A|, \qquad \text{a.e. } in\ D,$$

*and the set*

$$\partial^* A = \{x \in \partial A : \nabla b_A(x) \text{ exists and } |\nabla b_A(x)| = 1\}$$

*has zero $N$-dimensional Lebesgue measure. Moreover, for all $p$, $1 \leqslant p \leqslant \infty$, the maps*

$$b_A \mapsto \chi_{\partial A} : W^{1,p}_{\text{loc}}(D) \to L^p_{\text{loc}}(D) \qquad (\textit{resp.},\ W^{1,p}(D) \to L^p(D))$$

$$b_A \mapsto \chi_{\bar{A}} : W^{1,p}_{\text{loc}}(D) \to L^p_{\text{loc}}(D) \qquad (\textit{resp.},\ W^{1,p}(D) \to L^p(D))$$

$$b_A \mapsto \chi_{\overline{\complement A}} : W^{1,p}_{\text{loc}}(D) \to L^p_{\text{loc}}(D) \qquad (\textit{resp.},\ W^{1,p}(D) \to L^p(D))$$

*are "Lipschitz continuous": for all compact subsets $K$ of $D$ and nonempty subsets $A_1$ and $A_2$ of $D$*

$$\|\chi_{\partial A_2} - \chi_{\partial A_1}\|_{L^p(K)} \leqslant \|\nabla b_{A_2} - \nabla b_{A_1}\|_{L^p(K)} \leqslant \|b_{A_2} - b_{A_1}\|_{W^{1,p}(K)}$$

$$\|\chi_{\bar{A}_2} - \chi_{\bar{A}_1}\|_{L^p(K)} \leqslant \|\nabla b_{A_2} - \nabla b_{A_1}\|_{L^p(K)} \leqslant \|b_{A_2} - b_{A_1}\|_{W^{1,p}(K)}$$

$$\|\chi_{\overline{\complement A_2}} - \chi_{\overline{\complement A_1}}\|_{L^p(K)} \leqslant \|\nabla b_{A_2} - \nabla b_{A_1}\|_{L^p(K)} \leqslant \|b_{A_2} - b_{A_1}\|_{W^{1,p}(K)}.$$

(ii) *The set $C_b^0(D)$ is closed in $W^{1,p}_{\mathrm{loc}}(D)$ (resp., $W^{1,p}(D)$) for all $p$, $1 \leqslant p < \infty$.*

(iii) *For any $b_A$ and any sequence $\{b_{A_n}\}$ in $C_b(D)$*

$$b_{A_n} \to b_A \qquad \text{in } W^{1,p}_{\mathrm{loc}}(D)\text{-strong} \quad (\text{resp., } W^{1,p}(D)\text{-strong})$$

*if and only if*

$$\begin{aligned} b_{A_n} &\rightharpoonup b_A && \text{in } W^{1,p}_{\mathrm{loc}}(D)\text{-weak} && (\text{resp., } W^{1,p}(D)\text{-weak}) \\ \chi_{\bar{A}_n} &\to \chi_{\bar{A}} && \text{in } L^p_{\mathrm{loc}}(D)\text{-strong} && (\text{resp., } L^p(D)\text{-strong}) \\ \chi_{\overline{\complement A_n}} &\to \chi_{\overline{\complement A}} && \text{in } L^p_{\mathrm{loc}}(D)\text{-strong} && (\text{resp., } L^p(D)\text{-strong}). \end{aligned}$$

(iv) *Assume that $D$ is compact. Let $\{b_{A_n}\}$ be a sequence in $C_b(D)$ such that $\nabla b_{A_n} \in BV(D)^N$ and*

$$\exists c > 0, \qquad \forall n \geqslant 1, \quad \|\partial_{ij} b_{A_n}\|_{M_1(D)} \leqslant c, \quad 1 \leqslant i, j \leqslant N,$$

*where $\partial_{ij} b_{A_n}$ is the second-order partial derivative with respect to $x_i$ and $x_j$. Then there exists a subsequence, still denoted $\{b_{A_n}\}$, and $b_A$ in $C_b(D)$ such that*

$$b_{A_n} \to b_A \qquad \text{in } W^{1,p}(D)\text{-strong}$$

*and for all $\varphi \in \mathscr{D}^0(D)$*

$$\lim_{n \to \infty} \langle \partial_{ij} b_{A_n}, \varphi \rangle = \langle \partial_{ij} b_A, \varphi \rangle, \quad 1 \leqslant i, j \leqslant N.$$

*In particular,*

$$\|\partial_{ij} b_A\|_{M_1(D)} \leqslant c, \qquad 1 \leqslant i, j \leqslant N.$$

*Proof.* (i) Since $b_A$ is a Lipschitzian function, it is differentiable almost everywhere in $D$ and in view of Theorem 5.1(iii) when it is differentiable

$$|\nabla b_A(x)| = \begin{cases} 1, & x \notin \partial A, \\ 0, & \text{a.e. in } \partial A. \end{cases}$$

As a result,

$$\chi_{\partial A}(x) = 1 - |\nabla_A(x)|, \qquad \text{a.e. in } D.$$

Therefore the set $\partial^* A$ has at most a zero measure. Moreover, for any two subsets $A_1$ and $A_2$ of $D$ with nonempty boundaries

$$|\nabla b_{A_2}| \leqslant |\nabla b_{A_1}| + |\nabla b_{A_2} - \nabla b_{A_1}|$$

$$\Rightarrow \chi_{\partial A_1} \leqslant \chi_{\partial A_2} + |\nabla b_{A_2} - \nabla b_{A_1}|$$

$$\Rightarrow \int_D |\chi_{\partial A_2} - \chi_{\partial A_1}|^p \, dx \leqslant \|\nabla b_{A_2} - \nabla b_{A_1}\|^p_{L^p(D)} \leqslant \|b_{A_2} - b_{A_1}\|^p_{W^{1,p}(D)}$$

for $p$, $1 \leqslant p < \infty$, and with the ess sup for $p = \infty$. The remaining part of the results follows from the fact that

$$b_A^+ = d_A, \qquad b_A^- = d_{\complement A},$$

the continuity of the maps $b_A \mapsto b_A^+$, $b_A \mapsto b_A^- : W^{1,p} \to W^{1,p}$ and Theorem 3.6.

(ii) It is sufficient to give the proof for $D$ compact. Let $\{b_{A_n}\}$ in $C_b^0(D)$ be a Cauchy sequence for the $W^{1,p}(D)$-topology. From Theorem 5.2(i) we know that there exists a set $A$ in $D$ with nonempty boundary such that the sequence converges to $b_A$ in $W^{1,p}(D)$. In view of part (i),

$$m(\partial A_n) = 0 \quad \Rightarrow \quad \chi_{\partial A_n} = 0 \quad \Rightarrow \quad \chi_{\partial A} = 0 \quad \Rightarrow \quad m(\partial A) = 0$$

and $b_A$ belongs to $C_b^0(D)$.

(iii) We apply Theorem 3.6(iv) to the positive and negative parts of the functions $b_{A_n}$'s and $b_A$.

(iv) The proof is similar to the proof of Theorem 3.6(v). ∎

*Remark* 5.4. Part (iii) of Theorem 5.3 is to be interpreted carefully. If we only know that $\{b_{A_n}\}$ converges in $W^{1,p}(D)$-weak and that the characteristic functions $\{\chi_{\overline{A_n}}\}$ and $\{\chi_{\overline{\complement A_n}}\}$ converge in $L^p$-strong, it does not mean that there exists a $b_A$ such that $\{b_{A_n}\}$ converges to $b_A$ in $W^{1,p}(D)$-strong. This is readily seen from Example 2.1 where the sequence $\{\chi_{\overline{A_n}}\}$ converges strongly in $L^p$. In addition, $m(\partial A_n) = 0$ and $\{\chi_{\overline{\complement A_n}}\}$ also converges strongly. Now $D$ is bounded and we can always find a subsequence of $\{b_{A_n}\}$ which weakly converges in $W^{1,p}(D)$. If that subsequence was strongly convergent in $W^{1,p}(D)$, then we would have, according to Theorem 5.3(i), the strong convergence of $\{\chi_{\partial A_n}\}$ to $\{\chi_{\partial A}\}$. But we have seen in Example 2.1 that for all $n \geqslant 1$,

$$m(\partial A_n) = 0 < m(\partial A).$$

Therefore the $L^p$-convergence of the characteristic functions and the convergence in measure of their gradients do not imply the strong convergence in $W^{1,p}(D)$. This leads to the same discussion as the one following the proof of Theorem 3.6 in Section 3.3 about part (iv) of Theorem 3.6. For $D$ compact and any sequence $\{A_n\}$ in $D$ there exists a subsequence (still denoted $\{A_n\}$), a set $A$ in $D$, and functions $\chi^+$ and $\chi^-$ in $L^2(D)$ with values in $[0, 1]$ such that

$$d_{A_n} \rightharpoonup d_A \quad \text{and} \quad d_{\complement A_n} \rightharpoonup d_{\complement A} \quad \text{in } W^{1,2}(D)\text{-weak}$$

$$\chi_{\overline{A_n}} \rightharpoonup \chi^+ \quad \text{and} \quad \chi_{\overline{\complement A_n}} \rightharpoonup \chi^- \quad \text{in } L^2(D)\text{-weak.}$$

Then it is easy to show that

$$\lim_{n \to \infty} \int |\nabla d_{A_n} - \nabla d_A|^2 \, dx = \int \chi_{\bar{A}} - \chi^+ \, dx$$

$$\lim_{n \to \infty} \int |\nabla d_{\complement A_n} - \nabla d_{\complement A}|^2 \, dx = \int \chi_{\overline{\complement A}} - \chi^- \, dx.$$

In particular,

$$\chi_{\bar{A}} - \chi^+ \geqslant 0 \quad \text{in } D \quad \text{and} \quad \chi_{\overline{\complement A}} - \chi^- \geqslant 0 \quad \text{in } D.$$

Even if $\chi^+ = \chi_B$ and $\chi^- = \chi_C$ for some measurable subsets $B$ and $C$ of $D$ (the convergences of $\{\chi_{\overline{A_n}}\}$ and $\{\chi_{\overline{\complement A_n}}\}$ are strong), the sets $B$ and $C$ cannot in general be modified on a set of zero measure in such a way that

$$\chi_{\overline{\complement A}} = \chi_C \quad \text{and} \quad \chi_{\bar{A}} = \chi_B \quad \text{on } D.$$

*Remark* 5.5. It is readily seen that for sufficiently smooth domains, the subset $\partial^* A$ of the boundary $\partial A$ coincides with the reduced boundary of finite perimeter sets.

### 5.3. *Characterization of Convex Sets*

The next interesting property which was completely characterized by the distance function $d_A$ was the convexity of $\bar{A}$. This characterization remains true with $b_A$ in place of $d_A$.

THEOREM 5.4. (i) *Let $A$ be a subset of $\mathbb{R}^N$ with a nonempty boundary $\partial A$. Then*

$$\bar{A} \text{ convex} \quad \Leftrightarrow \quad b_A \text{ convex in } \mathbb{R}^N. \tag{14}$$

(ii) *Let $D$ be a closed (resp., compact) domain in $\mathbb{R}^N$. The subset of all oriented boundary distance functions for nonempty bounded convex subsets of $D$ is closed in $C(D)$ and $W^{1,p}_{\mathrm{loc}}(D)$ (resp., $W^{1,p}(D)$), $1 \leqslant p < \infty$. If, in addition, $D$ is bounded, then it is also compact.*

*Proof.* (i) ($\Leftarrow$) Denote by $x_\lambda$ the convex combination $\lambda x + (1-\lambda) y$ of any two points $x$ and $y$ in $\bar{A}$ and $\lambda \in [0, 1]$. Then $b_A(x) \leqslant 0$, $b_A(y) \leqslant 0$, and by convexity of $b_A$,

$$b_A(x_\lambda) \leqslant \lambda b_A(x) + (1-\lambda)\, b_A(y) \leqslant 0$$

$$\Rightarrow x_\lambda \in \operatorname{int} A \cup \partial A = \bar{A} \quad \Rightarrow \bar{A} \text{ is convex.}$$

($\Rightarrow$) For $x$ and $y$ in $\overline{\complement A}$, $b_A(x) = d_A(x)$ and $b_A(y) = d_A(y)$. But from Theorem 3.5(i); $d_A$ is convex and for all $\lambda$ in $[0, 1]$,

$$b_A(x_\lambda) \leqslant d_A(x_\lambda) \leqslant \lambda d_A(x) + (1-\lambda)\, d_A(y) = \lambda b_A(x) + (1-\lambda)\, b_A(y).$$

Therefore $b_A$ is convex in $\operatorname{co} \overline{\complement A}$. By convexity of $A$ for all $x$ and $y$ in $\operatorname{int} A$ and $\lambda$ in $[0, 1]$,

$$x_\lambda = \lambda x + (1-\lambda)\, y \in \operatorname{int} A.$$

Let $P_{\partial A}(x_\lambda) \in \partial A$ be a projection of $x_\lambda$ onto $\overline{\complement A}$. Let $H$ be the tangent plane to $\bar{A}$ at the point $P_{\partial A}(x_\lambda)$ orthogonal to $x_\lambda - P_{\partial A}(x_\lambda)$. Denote by $x_H$ and $y_H$ the respective projections of $x$ and $y$ on $H$. Then

$$d_{\complement A}(x_\lambda) = |P_{\partial A}(x_\lambda) - x_\lambda| = (1-\lambda)\, |y - y_H| + \lambda\, |x - x_H|.$$

Since $y_H$ and $x_H$ belong to $\overline{\complement A}$

$$|y - y_H| \geqslant d_{\complement A}(y) \qquad \text{and} \qquad |x - x_H| \geqslant d_{\complement A}(x)$$

and

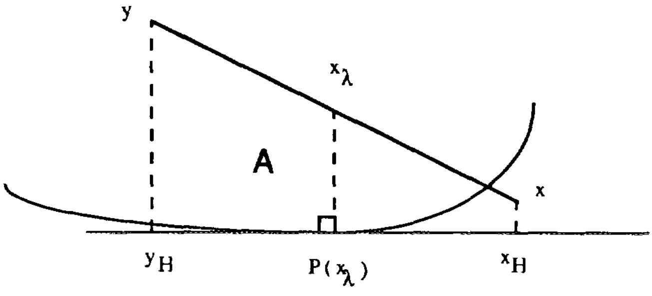
$$d_{\complement A}(x_\lambda) \geqslant (1-\lambda)\, d_{\complement A}(y) + \lambda d_{\complement A}(x).$$

But $d_A(x_\lambda) = d_A(x) = d_A(y) = 0$ and necessarily

$$b_A(x_\lambda) \leqslant \lambda b_A(x) + (1-\lambda)\, b_A(y).$$

The last case is $y \in \operatorname{int} A$ and $x \in \overline{\complement A}$. Fix $\lambda$ in $[0, 1]$, $x_\lambda = \lambda x + (1-\lambda)\, y$, and a projection $P_{\partial A}(x_\lambda)$ of $x_\lambda$ on $\partial A$. Denote by $H$ an hyperplane tangent to $\bar{A}$ in $P_{\partial A}(x_\lambda)$ and normal to $x_\lambda - P_{\partial A}(x_\lambda)$ if $x_\lambda \neq P_{\partial A}(x_\lambda)$. Let $x_H$ and $y_H$ be the respective orthogonal projections of $x$ and $y$ on $H$.

(a) If $x$ and $y$ are on the same side of $H$, then $x_\lambda$ is also on that side and necessarily $x_\lambda \in A$. Moreover,

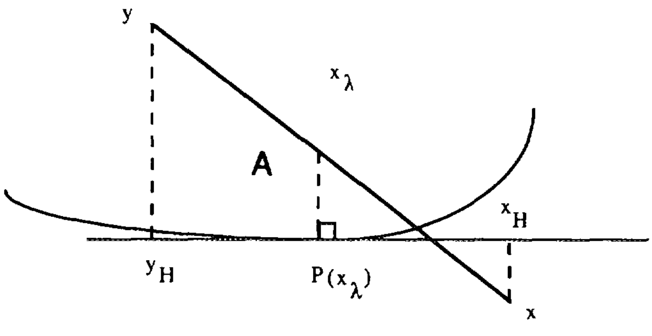



$$|P_{\partial A}(x_\lambda) - x_\lambda| = \lambda\, |x - x_H| + (1-\lambda)\, |y - y_H|$$

$$\Rightarrow\ d_{\complement A}(x_\lambda) \geqslant (1-\lambda)\, d_{\complement A}(y) + \lambda\, |x - x_H| \geqslant (1-\lambda)\, d_{\complement A}(y) - \lambda d_A(x)$$

$$\Rightarrow\ b_A(x_\lambda) \leqslant \lambda b_A(x) + (1-\lambda)\, b_A(y).$$

(b) If $x$ and $y$ are on different sides of $H$ we must consider two cases: $x_\lambda \in \bar{A}$ and $x_\lambda \in \overline{\complement A}$. When $x_\lambda \in \bar{A}$,

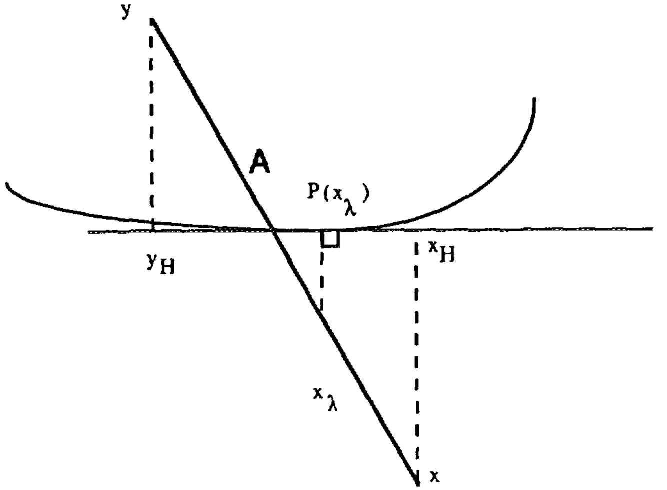



$$\frac{|x_\lambda - P_{\partial A}(x_\lambda)| + |x - x_H|}{(1-\lambda)\, |x - y|} = \frac{|y - y_H| + |x - x_H|}{|x - y|}$$

$$|x_\lambda - P_{\partial A}(x_\lambda)| = (1-\lambda)\, |y - y_H| - \lambda\, |x - x_H|$$

$$d_{\complement A}(x_\lambda) \geqslant (1-\lambda)\, d_{\complement A}(y) - \lambda d_A(x) \quad \Rightarrow \quad b_A(x_\lambda) \leqslant \lambda b_A(x) + (1-\lambda)\, b_A(y).$$

Now when $x_\lambda \in \overline{\complement A}$,




$$\frac{|P_{\partial A}(x_\lambda)| + |y - y_H|}{\lambda\,|x-y|} = \frac{|y-y_H| + |x - x_H|}{|x-y|}$$

$$|P_{\partial A}(x_\lambda) - x_\lambda| = \lambda\,|x - x_H| - (1-\lambda)\,|y - y_H|$$

$$\Rightarrow\ d_A(x_\lambda) \leqslant \lambda d_A(c) - (1-\lambda)\,d_{\complement A}(y),$$

and again

$$b_A(x_\lambda) \leqslant \lambda b_A(x) + (1-\lambda)\,b_A(y).$$

(ii) The set of continuous convex functions is closed in $C(D)$ (resp., $C^0(D)$). Thus it is closed in $C_b(D)$ and a fortiori in $W^{1,p}_{\mathrm{loc}}(D)$ (resp., $W^{1,p}(D)$). This completes the proof. ∎

5.4. *Characterization of the Smoothness of the boundary $\partial A$*

When the gradient of the oriented boundary distance function exists in a neighbourhood of a point $x$ of the boundary $\partial A$ of a smooth domain $A$, then for $t$ small enough

$$\nabla b_A(x + tn(x)) = n(x) = \nabla b_A(x - tn(x)), \tag{15}$$

where $n(x)$ is the unit outward normal to $A$ in $x$. This suggests that $\nabla b_A(x) = n(x)$ and that $\nabla b_A$ is an extension field of the outward unit normal from $\partial A$ to a neighbourhood of $\partial A$. Another consequence of this observation is that

$$\Delta b_A(x) = \operatorname{div}(\nabla b_A(x)) \tag{16}$$

becomes a candidate for the extension of the "mean curvature" from $\partial A$ to that same neighbourhood of $\partial A$ (cf. Zolésio [34, 35], Gilbarg and Trudinger [15], and Section 5.5). It turns out that the smoothness of the boundary $\partial A$ is directly related to the smoothness of the function $b_A$ in a neighbourhood of $\partial A$. In the next two theorems we establish the equivalence for domains with a $C^k$ boundary, $k \geqslant 2$, and a partial result for $k=1$ when the normal is Lipschitzian. This link between $b_A$ and classical geometric concepts clearly shows its advantages over the one-sided distance function $d_A$. In particular, a domain $A$ can now be described by the level curves of $b_A$ (cf. [10a] for an application to the theory of shells).

THEOREM 5.5. *Let $k \geqslant 1$ be an integer and $A$ be a domain in $\mathbb{R}^N$ with a nonempty boundary $\partial A$. If for each $x$ in $\partial A$ there exists a neighourhood $V(x)$ of $x$ where $b_A$ is of class $C^k$, then the boundary $\partial A$ is of class $C^k$ and for each $x$ in $\partial A$ there exists a neighbourhood $U(x)$ of $x$ where*

$$\Pi_{\partial A}(y) = \{P_{\partial A}(y)\} \tag{17}$$

*is a singleton and*

$$y \mapsto P_{\partial A}(y) \colon V(x) \to \mathbb{R}^N \tag{18}$$

*is a $C^{k-1}$ mapping.*

*Proof.* Fix $x \in \partial A$ and let $V(x)$ be the neighbourhood of $x$ where $b_A$ is of class $C^k$. So from Theorem 5.1,

$$\tfrac{1}{2}\nabla b_A^2(y) = y - P_{\partial A}(y), \qquad \forall y \in V(x) \tag{19}$$

and since $b_A^2$ is $C^k$, (18) is well-defined and $C^{k-1}$. Moreover, $|\nabla b_A(y)| = 1$ and we can associate with $x$ a local orthogonal basis $\{a_1, ..., a_N\}$ of $\mathbb{R}^N$ such that $a_N = \nabla b_A(x)$. Now construct the mapping

$$y \mapsto m_x(y) = (\{(y-x)\cdot a_i\}_{i=1}^{N-1}, b_A(y)) \colon \mathbb{R}^N \to \mathbb{R}^N. \tag{20}$$

By hypothesis on $b_A$, it is of class $C^k$ in the neighbourhood $V(x)$ of $x$. Moreover, its Jacobian matrix at $y$ is given by

$$Dm_x(y) = \begin{bmatrix} {}^{\top}a_1 \\ {}^{\top}a_1 \\ \vdots \\ {}^{\top}a_{N-1} \\ {}^{\top}\nabla b_A(y) \end{bmatrix}, \tag{21}$$

where ${}^{\top}z$ denotes the row vector corresponding to the column vector $z$. By construction $\det[Dm_x(x)] = 1$ and $Dm_x(x)$ is invertible. Thus by the inverse function theorem (cf. [25, Theorem 9.17, p. 193] for $N=1$ and [26, Vol. 1, pp. 294, Theorem 29, p. 299, Theorem 31]) there exists a neighbourhood $\mathscr{V}'(x)$ of $x$ and a neighbourhood $\mathscr{W}$ of 0 such that

$$m_x : \mathscr{V}'(x) \to \mathscr{W}$$

is bijective and its inverse is also class $C^k$. Now there exists an open ball $B_\varepsilon(0)$ of radius $\varepsilon > 0$ around 0 such that $B_\varepsilon(0) \subset \mathscr{W}$. Let $\mathscr{U}(x) = m_x^{-1}(B_\varepsilon(0))$ and define the map

$$c_x = \frac{1}{\varepsilon} m_x : \mathscr{U}(x) \to B \overset{\text{def}}{=} B_1(0).$$

It is bijective of class $C^k$, its inverse is of class $C^k$, and

$$\operatorname{int} A \cap \mathscr{U}(x) = c_x^{-1}(B_-), \qquad \partial A \cap \mathscr{U}(x) = c_x^{-1}(B_0),$$
$$\operatorname{int} \complement A \cap \mathscr{U}(x) = c_x^{-1}(B_+),$$

where

$$B_0 = \{(\zeta_1, ..., \zeta_N) \in B_1(0) : \zeta_N = 0\}$$
$$B_\pm = \{(\zeta_1, ..., \zeta_N) \in B_1(0) : \zeta_N \gtrless 0\}.$$

This is precisely the characterization of a domain with a boundary $\partial A$ of class $C^k$ in a neighourhood of $x$. ∎

The following result and its proof provide a partial converse to the previous theorem. A different proof of Theorem 5.6 can be found in Gilbarg and Trudinger [15, Lemma 14.16, p. 355]. However, $b_A$ should be used instead of the distance function $d_{\partial A}$.

THEOREM 5.6. *Let $A$ be a domain in $\mathbb{R}^N$ with a nonempty boundary $\partial A$ of class $C^k$ for $k \geqslant 2$. Then for each $x$ on $\partial A$, there exists a neighbourhood of $x$ where $b_A$ is $C^k$ and the boundary projection $P_{\partial A}$ is $C^{k-1}$.*

*Proof.* Since $\partial A$ is $C^k$ for each $x$ on $\partial A$ there exists a neighbourhood $V(x)$ of $x$ and a bijection $h: B \to V(x)$ such that $h$ and $h^{-1}$ are $C^k$ mappings. Then for $k \geqslant 1$, $Dh(\zeta)$ is continuous and invertible for all $\zeta \in B$. Recall the notation

$$B_\pm = \{\zeta \in B : \zeta_N \gtrless 0\}$$
$$B_0 = \{\zeta \in B : \zeta_N = 0\},$$

where $\zeta_N$ is the $N$th component of the vector $\zeta$ in $\mathbb{R}^N$ and the fact that

$$h(B_0) = V(x) \cap \partial A$$
$$h(B_-) = V(x) \cap \operatorname{int} A,$$
$$h(B_+) = V(x) \cap \operatorname{int} \complement A.$$

(i) Localization of the projections. We first reduce the size of $V(x)$ in such a way that the minimizers which achieve $d_\partial A$ lie in $h(B_0)$. Introduce the *neighbourhood*

$$\hat{V} = \{ y \in V : d_{h(B_0)}(y) < d_{\complement\, h(B)}(y) \}.$$

$\hat{V}$ is not empty. Indeed since $V$ is a neighbourhood of $x$, there exists $R > 0$ such that the open ball $B(x, R) \subset V$. We show that $B(x, R/4) \subset \hat{V}$. For $y \in B(x, R/4)$

$$d_{h(B_0)}(y) = d_{h(B_0)}(y) - d_{h(B_0)}(x) \leqslant |y - x| < \frac{R}{4}.$$

But

$$\forall z \in A \backslash h(B_-) \cup \complement\, A \backslash h(B_+) \subset \complement\, h(B), \qquad |z - x| \geqslant R$$
$$\Rightarrow \quad |z - y| \geqslant |z - x| - |x - y| \geqslant R - \frac{R}{4} = 3\,\frac{R}{4}$$
$$\Rightarrow \quad \min\{ d_{A \backslash h(B_-)}(y),\ d_{\complement\, A \backslash h(B_+)}(y) \} \geqslant 3\,\frac{R}{4} > \frac{R}{4} \geqslant d_{h(B_0)}(y),$$

where $X \backslash Y = \{ x \in X : x \notin Y \}$.
Therefore

$$B\left(x, \frac{R}{4}\right) \subset \hat{V}$$

and $\hat{V}$ is a nonempty neighbourhood of $x$. Next we show that for all $y$ in $\hat{V}$

$$d_{\partial A}(y) = |b_A(y)| = d_{h(B_0)}(y) \qquad \text{and} \qquad \Pi_{\partial A}(y) = \Pi_{h(B_0)} \subset h(B_0) \cap \hat{V}.$$

By definition $h(B_0) \subset \partial A$ and for all $y$

$$d_{\partial A}(y) \leqslant d_{h(B_0)}(y).$$

Assume that for some $y \in \hat{V}$ there exists $\hat{y} \in \partial A$ such that

$$d_{\partial A}(y) = |y - \hat{y}| \qquad \text{and} \qquad \hat{y} \notin h(B_0).$$

Then $\hat{y} \notin h(B)$ and we obtain the contradiction

$$d_{h(B_0)}(y) < d_{\complement h(B)}(y) \leqslant |\hat{y} - y| = d_{\partial A}(y) \leqslant d_{h(B_0)}(y).$$

As a result, for all $y$ in $\hat{V}$

$$d_{\partial A}(y) = d_{h(B_0)}(y) \qquad \text{and} \qquad \Pi_{\partial A}(y) = \Pi_{h(B_0)}(y).$$

Finally, either $\hat{y} \in \hat{V}$ or not. In the second case, by definition of $\hat{V}$,

$$0 = d_{h(B_0)}(\hat{y}) \geqslant d_{\complement h(B)}(\hat{y}) \quad \Rightarrow \quad \hat{y} \in \overline{\complement h(B)}$$

and

$$\hat{y} \in h(B_0) \cap \overline{\complement h(B)} = h(B_0) \cap \complement h(B) = \varnothing$$

since $V = h(B)$ is an open neighbourhood of $x$ by bicontinuity of the bijection $h$ between $B$ and $V$. This proves that $\hat{y} \in \hat{V}$.

(ii) Characterization of the projections. The minimizing points can be characterized as follows. For each $y \in \hat{V}$ there exist $\hat{y} \in \hat{V} \cap h(B_0)$ such that

$$d_{\partial A}(y) = \inf_{z \in h(B_0)} |z - y| = |\hat{y} - y|.$$

This is equivalent to $(y = h(\xi),\ z = h(\zeta))$

$$d^2_{\partial A}(y) = \inf_{\zeta \in \bar{B}_0} |h(\zeta) - h(\xi)|^2 \quad \Rightarrow \quad \exists \hat{\zeta} \in B_0 \cap h^{-1}(\hat{V}),$$

where $\bar{B}_0$ is compact and convex and $\zeta \mapsto |h(\zeta) - h(\xi)|^2$ is continuous. The minimizing points $\hat{\zeta}$ are characterized by the following variational inequality: $\exists \hat{\zeta} \in B_0 \cap h^{-1}(\bar{V})$ such that

$${}^{\top}Dh(\hat{\zeta})[h(\hat{\zeta}) - h(\xi)] \cdot (\zeta - \hat{\zeta}) \geqslant 0, \qquad \forall \zeta \in \bar{B}_0.$$

So $\hat{\zeta} = (\hat{\zeta}', 0)$ and we know that it is an interior point of $B$. So we can use tests of he form $\zeta = (\pm\zeta', 0)$ to conclude that

$${}^{\top}Dh(\hat{\zeta})[h(\hat{\zeta}) - h(\xi)] \cdot e_i = 0, \qquad 1 \leqslant i \leqslant N-1.$$

As a result there exists a constant $\alpha$ such that

$${}^{\top}Dh(\hat{\zeta})[h(\hat{\zeta}) - h(\xi)] = \alpha e_N.$$

But at point $\hat{y} = h(\hat{\zeta}) \in \partial A$ the outward unit normal to the boundary $\partial A$ of the $C^1$ domain is given by

$$n(\hat{y}) = -\frac{{}^{\top}Dh(\hat{\zeta})^{-1} e_N}{|{}^{\top}Dh(\hat{\zeta})^{-1} e_N|}.$$

Therefore for some constant $\beta$,

$$\hat{y} - y = h(\hat{\zeta}) - h(\xi) = \beta n(\hat{y})$$

and

$$|\beta| = |h(\hat{\zeta}) - h(\xi)| = |\hat{y} - y| = d_{\partial A}(y) = |b_A(y)|.$$

Moreover, for $y \in A$ (resp., $y \in \complement A$), $\beta \geqslant 0$ (resp., $\beta \leqslant 0$) and finally

$$\hat{y} - y = -b_A(y)\, n(\hat{y}).$$

Recall that $b_A$ is differentiable almost everywhere. By Theorem 5.1(iii) for all $y \notin \partial A$

$$\nabla b_A(y) = \frac{y - \hat{y}}{b_A(y)}$$

and since the boundary $\partial A$ of a $C^1$ domain has a zero Lebesgue measure

$$\nabla b_A(y) = n(\hat{y}) \text{ a.e.}$$

(iii) Uniqueness and continuity of the projection. We now use the implicit function theorem to show the existence of a neighbourhood of $x$ where $P_{\partial A}(y)$ is unique and $P_{\partial A}$ is a $C^{k-1}$-mapping. We construct the $C^{k-1}$ mapping $g\colon \hat{V} \times \hat{V}_0 \to \mathbb{R}^{N-1}$

$$g_i(y, \zeta') = [h(\zeta', 0) - y] \cdot Dh(\zeta', 0)\, e_i, \qquad 1 \leqslant i \leqslant N-1,$$

where

$$\hat{V}_0 = M(B_0 \cap h^{-1}(\hat{V}))$$

and $M$ is the $(N-1) \times N$ matrix

$$M = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & 0 & \ddots & & & \\ & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{bmatrix}.$$

The mapping $g$ is of class $C^{k-1}$, $g(x, 0) = 0$, and

$$\frac{\partial g_i}{\partial \zeta_j} = \sum_{l=1}^{N} \frac{\partial h_l}{\partial \zeta_i} \frac{\partial h_l}{\partial \zeta_j} + (h_l - y_l) \frac{\partial^2 h_l}{\partial \zeta_j \partial \zeta_i}$$

or in compact form

$$Dg = M \left[ {}^{\top}Dh\, Dh + \sum_{l=1}^{N} (h_l - y_l)\, H_l \right] {}^{\top}M,$$

where $Dg$ and $Dh$ are the Jacobian matrices and $H_l$ is the Hessian matrix of $h_l$. Recall that $h$ and $h^{-1}$ are $C^k$ bijections and that $Dh(\zeta)$ is invertible,

$$Dh^{-1}(h(\zeta))\, Dh(\zeta) = I, \qquad Dh(\zeta)\, Dh^{-1}(h(\zeta)) = I, \quad \forall \zeta \in B.$$

Now

$$Dg(x, 0) = M[{}^{\top}Dh(0)\, Dh(0)]\, {}^{\top}M$$

is invertible and the conclusions of the Implicit Function Theorem apply. There exists a neighbourhood $Y \subset \hat{V}$ of $x$ and a unique $C^1$-mapping

$$f\colon Y \to \mathbb{R}^{N-1} \qquad \text{such that} \quad f(x) = 0 \quad \text{and} \quad \forall y \in Y, \quad g(y, f(y)) = 0$$

(cf. for instance [25, Theorem 9.18, p. 196]).

In fact, $f$ is a $C^{k-1}$-mapping since there exists a neighbourhood $W \subset Y$ of $x$ where $Dg(y, g(y))$ is invertible. By the previous computations,

$$Dg(y, \zeta') = M \left[ {}^{\top}Dh(\zeta', 0)\, Dh(\zeta', 0) + \sum_{l=1}^{N} (h_l(\zeta', 0) - y_l)\, H_l(\zeta', 0) \right] {}^{\top}M.$$

There exists $\alpha > 0$ such that

$$M\, {}^{\top}Dh(\zeta', 0)\, Dh(\zeta', 0)\, {}^{\top}M \geqslant \alpha I_{N-1}, \qquad \forall (\zeta', 0) \in B_0.$$

It is perturbed by matrices $H_l$ which are bounded by some constant $c > 0$ in $B$ and

$$Dg(y, \zeta') - \frac{\alpha}{2} I_{N-1} \geqslant \left[ \frac{\alpha}{2} - |h(\zeta', 0) - y|\, cN \right] I.$$

So for $\zeta' = f(y)$, $|h(f(y), 0) - y| \leqslant |x - y|$ and for all $y$ in $W \overset{\text{def}}{=} Y \cap B(x, \alpha/2cN)$

$$Dg(y, y) - \frac{\alpha}{2} I_{N-1} \geqslant \left[ \frac{\alpha}{2} - |x - y|\, cN \right] I \geqslant 0.$$

Hence $f$ is a $C^k$ mapping in $W$ (cf. [26, Theorem 31, p. 299]). The equation $g(y, f(y)) = 0$ characterizes a stationary point $\hat{\zeta} = (f(y), 0)$ of the functional $|h(\zeta) - y|^2$. From part (ii) we know the existence of a minimizing point and from the above construction the point $\hat{\zeta}$ is unique. Hence

$$h(\hat{\zeta}) = h(f(y), 0) = P_{\partial A}(y)$$

and the mapping

$$y \mapsto f(y) \mapsto P_{\partial A}(y) = h(f(y), 0) \colon W \to \mathbb{R}^N$$

is of class $C^{k-1}$.

In view of the fact that the normal $n$ is $C^{k-1}$, then

$$y \mapsto n(P_{\partial A}(y)) \colon W \to \mathbb{R}^N$$

is a $C^{k-1}$-mapping. Now in view of part (ii) for almost all $y \in W$,

$$\nabla b_A(y) = n(P_{\partial A}(y))$$

and $\nabla b_A$ is almost everywhere equal to a $C^{k-1}$ function. Therefore $b_A$ is $C^k$ in $W$. ∎

*Remark* 5.6. For $k \geqslant 2$ a domain $A$ in $\mathbb{R}^N$ has a nonempty boundary $\partial A$ of class $C^k$ if and only if the function $b_A$ is $C^k$ in a neighbourhood of $\partial A$. For $k = 1$ we have half of the result but it is not clear that $b_A$ is $C^1$ in a neighbourhood of $\partial A$ when the boundary of the domain is only $C^1$.

Nevertheless we have the following additional result.

THEOREM 5.7. *Let $A$ be a domain in $\mathbb{R}^N$ with a nonempty boundary $\partial A$ of class $C^1$.*

(i) *For all $x \in \partial A$, there exists a neighbourhood $V(x)$ of $x$ such that*

$$\forall y \in V(x), \qquad \forall z \in \Pi_{\partial A}(y), \quad y - z = b_A(y)\, n(z). \tag{22}$$

(ii) *If, in addition, the normal $n$ is locally Lipschitz continuous, then for all $x \in \partial A$, there exists a neighbourhood $W(x)$ of $x$ such that*

$$\forall y \in W(x), \qquad \Pi_{\partial A}(y) = \{P_{\partial A}(y)\} \text{ is a singleton} \tag{23}$$

*and the map*

$$y \mapsto P_{\partial A}(y) \colon W(x) \to \mathbb{R}^N \tag{24}$$

*is Lipschitz continuous. Moreover,*

$$\forall y \in W(x), \qquad \nabla b_A(y) = n(P_{\partial A}(y)), \tag{25}$$

*$\nabla b_A$ is Lipschitz continuous in $W(x)$, and $b_A \in W^{2,\infty}(W(x))$.*

*Proof.* (i) As in part (ii) of the proof of Theorem 5.6. (ii) By hypothesis for each $x$, there exists a neighbourhood $V(x)$ of $x$ where $n$ is Lipschitzian with Lipschitz constant denoted by $c > 0$. If $U(x)$ is the neighbourhood of $x$ where (22) holds, we wish to construct a smaller neighbourhood $W(x)$ in $V(x) \cap U(x)$ such that for all $y$ in $W(x)$, the map

$$z \mapsto F(y, z) = y - b_A(y)\, n(z)$$

has a unique fixed point in $\Pi_{\partial A}(y)$. Let $z_1$, $z_2$ be two arbitrary points in $\Pi_{\partial A}(y)$

$$\begin{aligned} |F(y, z_2) - F(y, z_1)| &= |b_A(y)|\, |n(z_2) - n(z_1)| \\ &\leqslant c\, |b_A(y)|\, |z_2 - z_1|. \end{aligned}$$

The set

$$W(x) = \{ y \in V(x) \cap U(x) : c\, |b_A(y)| < \tfrac{1}{2} \}$$

is a neighbourhood of $x$ and for all $y$ in $W(x)$, $z \mapsto F(y, z)$ is a contraction with a unique fixed point

$$z = F(y, z) = y - b_A(y)\, n(z).$$

The fixed point necessarily coincides with $P_{\partial A}(y)$ since projections exist and necessarily verify (22). Moreover, since the fixed point is unique $\Pi_{\partial A}(y)$ is a singleton. Finally, let $z_1$ and $z_2$ be the respective fixed points corresponding to $y_1$ and $y_2$. Then

$$\begin{aligned} & z_i = y_i - b_A(y_i)\, n(z_i), \qquad i = 1, 2 \\ \Rightarrow \quad & z_2 - z_1 = y_2 - y_1 - b_A(y_2)\, n(z_2) + b_A(y_1)\, n(z_1) \\ & |z_2 - z_1| \leqslant |y_2 - y_1| + |b_A(y_2) - b_A(y_1)|\, |n(z_2)| + |b_A(y_1)|\, |n(z_2) - n(z_1)| \\ & \qquad \leqslant 2\, |y_2 - y_1| + c\, |b_A(y_1)|\, |z_2 - z_1| \leqslant 2\, |y_2 - y_1| + \tfrac{1}{2}\, |z_2 - z_1| \\ \Rightarrow \; & |z_2 - z_1| \leqslant 4\, |y_2 - y_1|. \end{aligned}$$

So for all $y_1$ and $y_2$ in $W(x)$,

$$|P_{\partial A}(y_2) - P_{\partial A}(y_1)| \leqslant |y_2 - y_1|.$$

Recall from the proof of part (ii) in Theorem 5.6 that

$$\nabla b_A = n \circ P_{\partial A}, \qquad \text{a.e. in } V(x).$$

Since $n \circ P_{\partial A}$ is Lipschitz continuous, we conclude that $\nabla b_A$ is also Lipschitz continuous and hence $b_A \in W^{2,\infty}$ in $W(x)$. ∎

*Remark* 5.7. At best we can say that $b_A$ is $C^1$ in a neighbourhood of $\partial A$ if $\partial A$ is $C^1$ and the projection $P_{\partial A}$ is $C^0$ in a neighbourhood of $\partial A$.

## 5.5. *Mean Curvature of the Boundary*

In this section we make the connection between the classical notion of *mean curvature* $H$ of the boundary $\partial A$ and the Laplacian $\Delta b_A$ of $b_A$. This question was studied independently and in different context by Zolésio [34, 35] and by Gilbarg and Trudinger [15, Appendix 14.6, pp. 354–357].

We first recall some results by Zolésio [35, p. 1103–1104]. Let $\Omega$ be a domain in $\mathbb{R}^N$ with a boundary $\Gamma = \partial\Omega$ of class $C^2$. The *tangential divergent* of $V$ in $C^1$ $(\Gamma; \mathbb{R}^N)$ is defined as

$$\operatorname{div}_\Gamma V = \operatorname{div} \tilde{V} - D\tilde{V}n \cdot n|_\Gamma,$$

where $\tilde{V}$ is a $C^1$ extension of $V$ in an open neighbourhood $U$ of $\Gamma$, $D\tilde{V}$ is the Jacobian matrix of $\tilde{V}$, and $n$ is the unit outward normal to $\Omega$ on $\Gamma$. It can be shown that this quantity is independent of the choice of the extension (see Sokolowski and Zolésio [28] for a more recent treatment of this notion). It turns out that for such a domain the mean curvature is given by

$$H = \operatorname{div}_\Gamma n \qquad \text{on } \Gamma.$$

Here our definition of mean curvature does not use the normalization factor $1/(N-1)$ which is generally present to make the radius of curvature of the unit sphere in $\mathbb{R}^N$ equal to one in all dimensions.

Recall from the previous section that for a domain $A$ with a boundary $\partial A$ of class $C^2$ $b_A \in C^2(U)$ and that $\nabla b_A$ is a unitary extension of the normal $n$ on $\partial A$. Necessarily

$$H = \operatorname{div}_\Gamma \nabla b_A = \operatorname{div} \nabla b_A - D(\nabla b_A)\, n \cdot n|_{\partial A}.$$

Now

$$|\nabla b_A|^2 = 1 \qquad \text{in } U$$

and necessarily

$$0 = \nabla\, |\nabla b_A|^2 = D(\nabla b_A)\, \nabla b_A$$

which yields the following result

$$H = \Delta b_A \qquad \text{on } \partial A.$$

This elegant result is true for a boundary $\partial A$ of class $C^2$ since $m(\partial A) = 0$ and necessarily

$$|\nabla b_A|^2 = 1 \qquad \text{everywhere in } U$$
$$\nabla\, |\nabla b_A|^2 \in C^0(U).$$

The question now is how could we extend this result to domains which are less smooth.

It is useful to compute the Laplacian of $b_A$ for a few examples.

EXAMPLE 5.1: Half plane in $\mathbb{R}^2$ of Example 3.2. Consider the domain

$$A = \{(x_1, x_2) : x_1 \leqslant 0\}, \qquad \partial A = \{(x_1, x_2) : x_1 = 0\}.$$

It is readily seen that

$$b_A(x_1, x_2) = x_1, \qquad \nabla b_A(x_1, x_2) = (1, 0), \quad \Delta b_A(x) = 0$$

and

$$b_{\partial A}(x_1, x_2) = |x_1|, \qquad \nabla b_{\partial A}(x_1, x_2) = \left(\frac{x_1}{|x_1|}, 0\right)$$

$$\langle \Delta b_{\partial A}, \varphi \rangle = 2 \int_{\partial A} \varphi \, dx.$$

EXAMPLE 5.2: Ball of radius $R > 0$ in $\mathbb{R}^2$ of Example 3.3. Consider the domain

$$A = \{x \in \mathbb{R}^2 : |x| \leqslant R\}, \qquad \partial A = \{x \in \mathbb{R}^2 : |x| = R\}.$$

Clearly

$$b_A(x) = |x| - R, \qquad \nabla b_A(x) = \frac{x}{|x|},$$

and

$$\langle \Delta b_A, \varphi \rangle = \int_{\mathbb{R}^2} \frac{1}{|x|} \varphi \, dx.$$

Also,

$$b_{\partial A}(x) = \big||x_1| - R\big|, \qquad \nabla b_{\partial A}(x) = \begin{cases} \dfrac{x}{|x|}, & |x| > R \\ -\dfrac{x}{|x|}, & |x| < R, \end{cases}$$

and

$$\langle \Delta b_{\partial A}, \varphi \rangle = 2 \int_{\partial A} \varphi \, ds - \int_A \frac{1}{|x|} \varphi \, dx + \int_{\complement A} \frac{1}{|x|} \varphi \, dx.$$

Again we see that $\Delta b_{\partial A}$ contains twice the boundary measure on $\partial A$.

EXAMPLE 5.3. Unit square in $\mathbb{R}^2$ of Example 3.4. Consider the domain

$$A = \{x = (x_1, x_2) \colon |x_1| \leqslant 1, \ |x_2| \leqslant 1\}.$$

Since $A$ is symmetrical with respect to both axes, it is sufficient to specify $b_A$ in the first quadrant. We use the notation $Q_1$, $Q_2$, $Q_3$, and $Q_4$ for the four quadrants in the counterclockwise order and $c_1$, $c_2$, $c_3$, and $c_4$ for the four corners of the square in the same order. We also divide the plane in three regions

$$D_1 = \{(x_1, x_2) \colon |x_2| \leqslant \min\{1, |x_1|\}\}$$

$$D_2 = \{(x_1, x_2) \colon |x_1| \leqslant \min\{1, |x_2|\}\}$$

$$D_3 = \{(x_1, x_2) \colon |x_1| \geqslant 1 \text{ and } |x_2| \geqslant 1\}.$$

Hence

$$b_A(x) = \begin{cases} x_2 - 1, & x \in D_2 \cap Q_1, \\ |x - c_1|, & x \in D_3 \cap Q_1, \\ x_1 - 1, & x \in D_1 \cap Q_1, \end{cases}$$

$$\nabla b_A(x) = \begin{cases} (0, 1), & x \in D_2 \cap Q_1, \\ \dfrac{x - c_1}{|x - c_1|}, & x \in D_3 \cap Q_1, \\ (1, 0), & x \in D_1 \cap Q_1, \end{cases}$$

and for the whole plane

$$\langle \Delta b_A, \varphi \rangle = \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{1}{|x - c_i|} \varphi \, dx + \sqrt{2} \int_{D_1 \cap D_2} \varphi \, dx,$$

where $D_1 \cap D_2$ is made up of the two diagonals of the square where $\nabla b_A$ has a singularity. Moreover

$$b_{\partial A}(x) = \begin{cases} |x_2 - 1|, & x \in D_2 \cap Q_1, \\ |x - c_1|, & x \in D_3 \cap Q_1, \\ |x_1 - 1|, & x \in D_1 \cap Q_1, \end{cases}$$

and

$$\langle \Delta b_{\partial A}, \varphi \rangle = \sum_{i=1}^{4} \int_{D_3 \cap Q_i} \frac{1}{|x - c_i|} \varphi \, dx - \sqrt{2} \int_{D_1 \cap D_2} \varphi \, dx + 2 \int_{\partial A} \varphi \, dx.$$

Note that the structures of the Laplacian are similar to the ones observed in the previous examples except for the presence of a singular term along the two diagonals of the square.

The set of points $x$ where the gradient $\nabla b_A(x)$ does not exist is directly related to the concept of *skeleton* in the literature on prairie fires, grass fires (for instance in [5a]), watersheds and image processing [4a, 32a]. It is part of *Mathematical Morphology* [18a] and *Computational Geometry*. Yet the skeleton is only part of the singularities of $\nabla b_A(x)$.

The above examples are quite informative. For locally piecewise $C^2$-domains, we can conjecture that the mean curvature is the regular part of $\Delta b_A$ and that the surface measure on the boundary $\partial A$ is contained in $\Delta d_A$. It would be nice to find a formulation which would filter out the regular part of $\Delta b_A$ and extract the singular part of $\Delta d_A$ with support in $\partial A$. Recall that this surface measure was obtained from the gradient of the characteristic function $\chi_{\bar{A}} = 1 - |\nabla d_{\bar{A}}|$ in the context of finite perimeter sets.

Another observation is that for the previous examples not only $\Delta b_A$ is a measure but also all second-order derivatives of $b_A$. This suggests to introduce the space

$$X^{2,p}(D) = \left\{ f \in W^{1,p}(D) \colon |\nabla f| = 1 \text{ and } \nabla \frac{\partial f}{\partial x_i} \in M_1(D),\ 1 \leqslant i \leqslant N \right\},$$

where $M_1(D) = \mathscr{D}^0(D)'$. A subset of $D$ whose oriented boundary distance function belongs to $X^{2,p}(D)$ wil be called a *Bounded Global Curvature Set*. They can be used to obtain results on the continuity of solutions of partial differential equations with respect to domains and existence theorems in Shape Optimization. For $D$ compact the embedding of $X^{2,p}(D)$ into $W^{1,p}(D)$ is now compact and if we could show that, for a sequence $\{b_{A_n}\}$, the norms of $\{\nabla \partial_i b_{A_n}\}$ are uniformly bounded with respect to $i$ and $n$, then we could extract a subsequence which converges in $W^{1,p}(D)$-strong. Of course it would be a lot nicer to reach the same conclusion only with the uniform boundedness of the Laplacians of the sequence. In that case we could use the space

$$Y^{2,p}(D)=\{f\in W^{1,p}(D)\colon |\nabla f|=1 \text{ and } \Delta f\in M_1(D)\}.$$

A subset of $D$ whose oriented boundary distance function belongs to $Y^{2,p}(D)$ will be called a *Bounded Global Mean Curvature Set.*

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