1 Double Integrator Model

1.1 Problem Formulation

We consider a group of N vehicles, each of them denoted Q_i, $i = 1, \cdots, N$. Each vehicle with dynamics described by

$$\dot{p_i} = v_i,\, \dot{v_i} = u_i, \quad \|v_i\| \leq s_{max},\, \|(u_{i,x}, u_{i,y})\| \leq u_{Qmax}, \tag{1}$$

here $p_i = (p_{i,x}, p_{i,y})$, and $v_i = (v_{i,x}, v_{i,y})$ are the x and y positions and velocities of Q_i respectively, and $u_{i,Q} := (u_{i,x}, u_{i,y})$ is the control force applied to this mobile agent.

We consider a vehicle safe if there is no other vehicle closer than a predefined collision radius c_r, in other words, if

$$\|p_i - p_j\| > c_r, \qquad \text{for any } j \neq i. \tag{2}$$

Definition *(r-Subcover).* A group of agents is an r-**subcover** for a compact domain $\Omega \subseteq \mathbb{R}^2$ if:

1. The distance between any two vehicles is at least r.
2. The signed distance from any vehicle to Ω is less than equal to $-\frac{r}{2}$.

Definition *(r-Cover).* An r-subcover for Ω is an r-**cover** for Ω if its size is maximal (i.e., no larger number of agents can be an r-subcover for Ω).

The r-subcover definition is closely related to finding a way to pack circular objects of radius $\frac{r}{2}$ inside of a container with shape Ω. Having an r-cover implies the container is full and there is no room for more of such objects.

Definition (flocking around a moving target domain) [based on Ref: A simple proof of CS] A group of vehicles with dynamics 1 has a time-asymptotic flocking around a moving target domain following the trajectory $p_d(t)$, with velocity $v_d(t)$ with states if and only if and only if its solutions $\{p_i, v_i\}$, $i = 1, \cdots, N$ satisfy the following two conditions:

1. The relative velocity fluctuation respect the domain go to zero time-asymptotically (veloctiy alignement):

$$\lim_{t \to +\infty} \sum_{i=1}^{N} \|v_i(t) - v_d(t)\|^2 = 0$$

2. The position fluctuations respect the domain are uniformly bounded in time t (forming a group):

$$\sup_{0 \leq t < \infty} \sum_{i=1}^{N} \|p_i(t) - p_d(t)\|^2 < \infty$$

We are interested in the following safe domain coverage problem.

(Safe-domain-coverage) Consider a compact domain Ω in the plane and N vehicles each with dynamics described by 1, starting from safe initial conditions. Find the maximal $r > 0$ and a control policy that leads to a stable steady state which is an r-cover for Ω, while satisfying the safety condition \eqref{eq:safety} at any time.

1.2 Metodology

1.2.1 Coverage Controller

We consider a group of N vehicles, with unconstrained dynamics

$$\dot{p_i} = v_i, \quad \dot{v_i} = u_i.$$

Let $\Omega \subseteq \mathbb{R}^2$ be a compact domain containing zero, and define $\Omega(t) = \Omega + p_d(t)$, where $p_d(t)$ is the solution for the system

$$\begin{cases} \dot{p_d} &= v_d \\ \dot{v_d} &= 0, \end{cases}$$

we call $p_d(t)$ the marker point of the moving domain $\Omega(t)$.

Define $p_{ij} := p_i - p_j$, and $v_{ij} := v_i - v_j$ and denote by $P_{\partial\Omega(t)}(p_i)$ the closest point of $\partial\Omega(t)$ to p_i (i.e., the projection of p_i on $\partial\Omega(t)$). Also, define $h_i := p_i - P_{\partial\Omega(t)}(p_i)$, and denote by $[[h_i]]$ the signed distance of p_i from $\partial\Omega(t)$.

The proposed control force is given by:

$$u_i = \underbrace{-\sum_{j\neq i}^{N} f_I(\|p_{ij}\|)\frac{p_{ij}}{\|p_{ij}\|}}_{\text{Inter Vehicle}} - \underbrace{\frac{1}{N}\sum_{j\neq i}^{N} f_{al}(\|p_{ij}\|)\, v_{ij}}_{\text{Velocity Alignment}} - \overbrace{\underbrace{f_h([[h_i]])\frac{h_i}{[[h_i]]}}_{\text{Domain Vehicle}} - \underbrace{a(v_i - v_d)}_{\text{Speed Alignment}}}^{\text{Navigational feedback}} \tag{3}$$

The position and velocity of the i th vehicle relative to the marker of the moving domain are given by:

$$\begin{cases} \tilde{p}_i & := p_i - p_d \\ v_i & := v_i - v_d. \end{cases}$$

Note that the inter-vehicle position and velocity in this new framework satisfy:

$$\begin{aligned} \tilde{p}_{ij} &:= \tilde{p}_i - \tilde{p}_j = p_i - p_d - (p_j - p_d) = p_{ij}, \\ v_{ij} &:= v_i - v_j = v_i - v_d - (v_j - v_d) = v_{ij}, \end{aligned}$$

it means the relative positions are invariant to the change of coordinates. Moreover, the vehicle domain distance satisfies $h_i = p_i - P_{\partial\Omega(t)}(p_i) = (p_i - p_d) - P_{\partial\Omega(t)-p_d}(p_i - p_d) = \tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)$. This allow us to rewrite (3) as

$$u_i = -\sum_{j\neq i}^{N} f_I(\|\tilde{p}_{ij}\|)\frac{\tilde{p}_{ij}}{\|\tilde{p}_{ij}\|} - \frac{1}{N}\sum_{j\neq i}^{N} f_{al}(\|\tilde{p}_{ij}\|)\, v_{ij} - f_h([[h_i]])\frac{h_i}{[[h_i]]} - a v_i \tag{4}$$

Let us consider the potential

$$V_h(\tilde{p}_i) = \int_{-\frac{r_d}{2}}^{[[\tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)]]} f_h(s)\, ds$$

which satisfies

$$\nabla_{\tilde{p}_i} V_h(\tilde{p}_i) = f_h\left([[\tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)]]\right)\nabla_{\tilde{p}_i}\left([[\tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)]]\right) = f_h([[h_i]])\frac{h_i}{[[h_i]]}$$

where we have used the identity $\nabla_{\tilde{p}_i}\left([[\tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)]]\right) = \frac{\tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)}{[[\tilde{p}_i - P_{\partial\Omega(0)}(\tilde{p}_i)]]}$.

Similarly, it can be shown that the inter-vehicle force is the negative gradient of the potential

$$V_I(\tilde{p}_{ij}) = \int_{r_d}^{\|\tilde{p}_{ij}\|} f_I(s)\, ds,$$

to finally get:

$$u_i = \underbrace{-\sum_{j\neq i}^{N} \nabla_{\tilde{p}_i} V_I(\tilde{p}_{ij})}_{\text{Inter Vehicle}} - \underbrace{\frac{1}{N}\sum_{j\neq i}^{N} f_{al}(\|\tilde{p}_{ij}\|)\, v_{ij}}_{\text{Velocity Alignment}} - \overbrace{\underbrace{\nabla_{\tilde{p}_i} V_h(\tilde{p}_i)}_{\text{Domain Vehicle}} - \underbrace{a v_i}_{\text{Speed Alignment}}}^{\text{Navigational feedback}} \tag{5}$$

Consider the candidate for Lyapunov function consisting in kinetic plus (artificial) potential energy:

$$\Phi = \frac{1}{2}\sum_{i=1}^{N}\Big(\dot{\tilde{p}}_i \cdot \dot{\tilde{p}}_i + \sum_{j\neq i}^{N} V_I(\tilde{p}_{ij}) + V_h(\tilde{p}_i)\Big).$$

Note that each term in Φ is non-negative, and Φ reaches its absolute minimum value when the vehicles are totally stopped.

The derivative of Φ with respect to time can be calculated as:

$$\begin{aligned}\dot{\Phi} &= \sum_{i=1}^{N} \dot{\tilde{p}}_i \cdot \left(u_i + \sum_{j\neq i}^{N} \nabla_{\tilde{p}_i} V_I\left(\tilde{p}_{ij}\right) + \nabla_{\tilde{p}_i} V_h\left(\tilde{p}_i\right) \right) \\ &= \sum_{i=1}^{N} \dot{\tilde{p}}_i \cdot \left(-\frac{1}{N} \sum_{j\neq i}^{N} f_{al}\left(\|\tilde{p}_{ij}\|\right) v_{ij} - a v_i \right)\end{aligned}$$

For the (extra) alignment term, write

$$\sum_{i=1}^{N} v_i \cdot \sum_{j\neq i}^{N} f_{al}\left(\|\tilde{p}_{ij}\|\right)\left(v_i - v_j\right) = \frac{1}{2}\sum_{i=1}^{N} v_i \cdot \sum_{j\neq i}^{N} f_{al}\left(\|\tilde{p}_{ij}\|\right)\left(v_i - v_j\right) + \frac{1}{2}\sum_{j=1}^{N} v_j \cdot \sum_{i\neq j}^{N} f_{al}\left(\|\tilde{p}_{ij}\|\right)\left(v_j - v_i\right),$$

where in the second term in the right-hand-side we simply rename $i \leftrightarrow j$ as indices of summation. From there, use that $\|\tilde{p}_{ij}\| = \|\tilde{p}_{ji}\|$ to get:

$$\sum_{i=1}^{N} v_i \cdot \sum_{j\neq i}^{N} f_{al}\left(\|\tilde{p}_{ij}\|\right)\left(v_i - v_j\right) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} f_{al}\left(\|\tilde{p}_{ij}\|\right)\|v_i - v_j\|^2.$$

With the minus sign in front this gives a negative-definite term. Conclude that $\dot{\Phi}$ is negative semidefinite and equal to zero if and only if $\dot{\tilde{p}}_i = 0$ for all i (i.e., all vehicles are at equilibrium in the relative framework).

Theorem Consider a group of N vehicles with dynamics defined by 1, and the control law given by 3. Let the equilibrium of interest be of the form $\dot{\tilde{p}}_i = 0$, $\|\tilde{p}_{ij}\| \geq r_d$ and $[[h_i]] \leq -\frac{r_d}{2}$ for $i, j = 1, \cdots, N$ (see Definitions \ref{defn:subcover} and \ref{defn:cover}), and assume that this equilibrium configuration is isolated. Also assume that there is a neighborhood about the equilibrium in which the control law remains smooth. Then, the following statements hold:

1. The group of agents has a time-asymptotic flocking around a moving target $\Omega(t)$.
2. Almost every solution of the relative dynamical system (to be defined) asymptotically converges to an equilibrium point $(\tilde{p}^*, 0)$ where $\tilde{p}^*$ is a local minima of $V_I(\tilde{p}) + V_h(\tilde{p})$.
3. Assume the initial structural energy of the particle system is less than $(k+1)c^*$ with $c^* = V_I(0)$ and $k \in \mathbb{Z}^+$. Then, at most k distinct pairs of vehicles could possibly collide ($k = 0$ guarantees a collision-free motion). (I have not checked this stronger result yet).

proof. To be done.

□

It is clear that thresholding the force the theoretical guarantees may not necessary hold anymore, however, when close to the desired operation point the coverage input forces are small enough to not be thresholded and it is feasible for the controller apply the required coverage force, implying the theoretical results are locally valid.

Choosing the Adequate Cucker Smale Parameters

We assume as premise that the major velocity alignment effects should be for those vehicles within an r_d radius neighborhood. It seems wide enough to guarantee flocking behavior without causing group inertia that may slow down the domain coverage aim. In order to so, we impose a tenth decay on the alignment strength every r_d, i.e. $l_{al} = -\frac{r_d}{\ln(0.1)}$.

1.2.2 Collision Avoidance via Analytic HJI PDE Solution

(I think we should call the result from the paper for the analytical solution of the HJ eq instead of redoing it here).

2 Fixed-wing model

Similarly as before we consider a group of N vehicles, each of them denoted W_i, $i = 1, \cdots, N$, but this time with dynamics described by

$$\dot{p}_i = s_i \left(\cos\left(\theta_i\right), \sin\left(\theta_i\right)\right), \, \dot{\theta}_i = u_{i,\theta}, \, \dot{s}_i = u_{i,s}, \quad |s_i| \leq s_{max} \, |u_{i,\theta}| \leq u_{\theta\, max}, \, |u_{i,s}| \leq u_{s\, max} \tag{6}$$

here $p_i = (p_{i,x}, p_{i,y})$, and $v_i = (v_{i,x}, v_{i,y})$ are the x and y positions and velocities respectively, θ_i is the heading angle and s_i the speed of the vehicle W_i. Additionally, the acceleration is $u_{i,W} := (u_{i,\theta}, u_{i,s})$.

2.1 Thresholding Coverage Control Force

As we assume constrained input forces, we to need modify the proposed coverage control force when necessary. For the double integrator model the given coverage control force $u = (u_x, u_y)$ is projected onto the set of admissible forces using the mapping,

$$\hat{u} = \begin{cases} u & \text{if } \|u\| \leq u_{max}, \\ u_{max} \frac{u}{\|u\|} & \text{otherwise.} \end{cases}$$

On the other hand, in order to get the appropriate Dubins car control force we use the relation

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = R\left(v, \theta\right)\begin{pmatrix} u_\theta \\ u_v \end{pmatrix}; \, R\left(v, \theta\right) := \begin{pmatrix} v\sin\left(\theta\right) & \cos\left(\theta\right) \\ v\cos\left(\theta\right) & -\sin\left(\theta\right) \end{pmatrix}, \tag{7}$$

which allow us to represent the set of admissible forces from the xy perspective as the region

$$S = \left\{ R\left(v, \theta\right)\begin{pmatrix} u_\theta \\ u_v \end{pmatrix} : \begin{pmatrix} u_\theta \\ u_v \end{pmatrix} \in \left[-u_{\theta max}, u_{\theta max}\right] \times \left[-u_{vmax}, u_{vmax}\right] \right\}$$

we set $\begin{pmatrix} \hat{u}_x \\ \hat{u}_y \end{pmatrix} = \sup\left\{ t \in \mathbb{R} : t\begin{pmatrix} u_x \\ u_y \end{pmatrix} \in S \right\}\begin{pmatrix} u_x \\ u_y \end{pmatrix}$, see Fig. 1. Finally, we get the associated Dubins car input force by inverting (7) as $\begin{pmatrix} \hat{u}_\theta \\ \hat{u}_v \end{pmatrix} = R^{-1}\left(v, \theta\right)\begin{pmatrix} \hat{u}_x \\ \hat{u}_y \end{pmatrix}$.

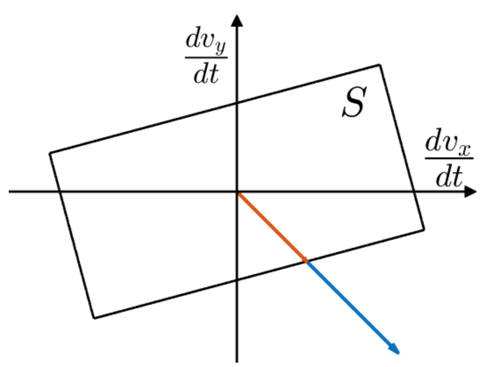

Figure 1: Thresholding Dubins car force in rectangular coordinates.

2.2 Collision avoidance

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\left(\theta_1\right) & \sin\left(\theta_1\right) \\ -\sin\left(\theta_1\right) & \cos\left(\theta_1\right) \end{pmatrix}\begin{pmatrix} p_{x,2} - p_{x,1} \\ p_{y,2} - p_{y,1} \end{pmatrix} = \begin{pmatrix} \cos\left(\theta_1\right)\left(p_{x,2} - p_{x,1}\right) + \sin\left(\theta_1\right)\left(p_{y,2} - p_{y,1}\right) \\ -\sin\left(\theta_1\right)\left(p_{x,2} - p_{x,1}\right) + \cos\left(\theta_1\right)\left(p_{y,2} - p_{y,1}\right) \end{pmatrix}$$

$$
\begin{aligned}
\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} &= \begin{pmatrix} -\dot{\theta}_1 \sin(\theta_1)(p_{x,2} - p_{x,1}) + \cos(\theta_1)(\dot{p}_{x,2} - \dot{p}_{x,1}) + \dot{\theta}_1 \cos(\theta_1)(p_{y,2} - p_{y,1}) + \sin(\theta_1)(\dot{p}_{y,2} - \dot{p}_{y,1}) \\ -\dot{\theta}_1 \cos(\theta_1)(p_{x,2} - p_{x,1}) - \sin(\theta_1)(\dot{p}_{x,2} - \dot{p}_{x,1}) - \dot{\theta}_1 \sin(\theta_1)(p_{y,2} - p_{y,1}) + \cos(\theta_1)(\dot{p}_{y,2} - \dot{p}_{y,1}) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1)(v_2\cos(\theta_2) - v_1\cos(\theta_1)) + \sin(\theta_1)(v_2\sin(\theta_2) - v_1\sin(\theta_1)) + \dot{\theta}_1 x_2 \\ -\sin(\theta_1)(v_2\cos(\theta_2) - v_1\cos(\theta_1)) + \cos(\theta_1)(v_2\sin(\theta_2) - v_1\sin(\theta_1)) - \dot{\theta}_1 x_1 \end{pmatrix} \\
&= \begin{pmatrix} v_2\cos(\theta_2 - \theta_1) - v_1 + \dot{\theta}_1 x_2 \\ v_2\sin(\theta_2 - \theta_1) - \dot{\theta}_1 x_1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1)(p_{x,2} - p_{x,1}) + \sin(\theta_1)(p_{y,2} - p_{y,1}) \\ -\sin(\theta_1)(p_{x,2} - p_{x,1}) + \cos(\theta_1)(p_{y,2} - p_{y,1}) \\ \theta_2 - \theta_1 \\ v_1 \\ v_2 \end{pmatrix}
$$

$$
\begin{pmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_5\cos(x_3) - x_4 + u_\theta x_2 \\ x_5\sin(x_3) - u_\theta x_1 \\ d_\theta - u_\theta \\ u_v \\ d_v \end{pmatrix}
$$

Optimal controller: $u_\theta^* = \text{sgn}\left(\frac{\partial V}{\partial x_1}x_2 - \frac{\partial V}{\partial x_2}x_1 - \frac{\partial V}{\partial x_3}\right) u_{\theta max}$, $u_v^* = \text{sgn}\left(\frac{\partial V}{\partial x_4}\right) u_{vmax}$

Optimal disturbance: $d_\theta^* = -\text{sgn}\left(\frac{\partial V}{\partial x_3}\right) d_{\theta max}$, $d_v^* = -\text{sgn}\left(\frac{\partial V}{\partial x_5}\right) d_{vmax}$