Spectral Multiscale Coverage: A Uniform Coverage Algorithm for Mobile Sensor Networks

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***Abstract*— In this paper we propose centralized feedback control laws for mobile sensor networks so that sensor trajectories cover a given domain as uniformly as possible. The design of the feedback law is based on a measure for uniformity of the trajectories given as a distance between a certain delta-like distribution on the trajectories and a desired probability distribution. The design of control is of Lyapunov-type and we present control laws so that the agents move with constant speed or constant forcing (in the case of second-order dynamics).**

I. Introduction

Design of algorithms to coordinate the motion of networked mobile agents is an area of intense recent research. Applications are numerous - some of them being search-and-rescue operations, surveillance and environmental monitoring. A couple of recent representative papers that deal with cooperative control are [1] and [2]. In [1], the problem is to design motion of the sensors so that they converge to an optimal stationary sensor location. This stationary sensor location is optimal in the sense that it maximizes the detection probability of some event. In [2], the authors discuss algorithms for optimal data collection. They discuss an example of a fleet of underwater gliders that move with ocean currents and sample various dynamic oceanographic signals.

In many applications of mobile sensors, it is desirable to design their dynamics so that they achieve good 'coverage' of a given domain. The term 'coverage' can mean slightly different things to different authors. For example, in [1], 'coverage' is more a static concept. i.e., it is a measure of how a static configuration of agents covers a domain or samples a probability distribution. In [3], the term 'coverage' is more of a dynamic concept and is a measure of how well the points on the trajectories of the agents cover a domain. That is, coverage gets better and better as every point in the domain is visited or is close to being visited by an agent. The notion of coverage we use in this paper is closer to that in [3]. Moreoever, we use the notion of 'uniform coverage' which we precisely quantify in the later sections. In this paper, by 'uniform coverage', we roughly mean that points on the agent trajectories must be as uniformly distributed or evenly spaced throughout the domain. To quantify this, we use a specific metric which was inspired by our previous work to quantify the degree of mixedness of material in a fluid medium. This notion of 'uniform coverage' can also be related to concepts in ergodic theory that relate time averages of functions along trajectories to spatial averages of functions. The behavior of an algorithm that attempts to achieve uniform coverage is going to be inherently multiscale. By this we mean that, features of large size are guaranteed to be detected first, followed by features of smaller and smaller size.

There are many applications of mobile sensor networks where it is useful to design their dynamics so that their trajectories are as uniformly distributed as possible in the domain of interest. A couple of scenarios where it is useful to have such dynamics are:

- For fleets of gliders in the ocean that are taking various measurements, it is useful that their trajectories are uniformly distributed so that spatial averages of various oceanographic fields like temperature, salinity and flow can be estimated accurately and efficiently.
- For military applications and search-and-rescue operations, it is desirable that there is little space between the trajectories of the mobile sensors so that it becomes difficult for a target to evade detection by the sensors.

In Section II, we setup the problem and in particular, we present the coverage measure. In Section III, we present feedback control design for both first-order dynamics and second-order dynamics and we present various numerical simulations to demonstrate the performance of the algorithm. The algorithm presented in this paper will be referred to as Spectral Multiscale Coverage (SMC).

II. Problem setup

There are N agents and we assume that they move either by first-order or second-order dynamics. First-order dynamics is described by

$$\dot{x}_j(t) = u_j(t), \tag{1}$$

and second-order dynamics is described by

$$\ddot{x}_j(t) = -c\dot{x}_j(t) + u_j(t), \tag{2}$$

where $c > 0$ is a damping coefficient to model the resistance due to air or water. The objective is to design feedback laws $u_j(t) = F_j(x)$ such that the agents uniformly sample the domain.

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A. Measure for uniformity of trajectories

Given trajectories $x_j : [0,t] \to \Re^n$, we need a measure for the uniformity of the trajectories. This is given as follows. Given the trajectories x_j, we define the distribution W_0^t whose action on functions is defined as

$$\left\langle W_0^t, f\right\rangle = \frac{1}{Nt}\sum_{j=1}^{N}\int_0^t f(x_j(\tau))d\tau = \frac{1}{Nt}\int_0^t \sum_{j=1}^{N} f(x_j(\tau))d\tau. \tag{3}$$

Note that the distribution W_0^t can be thought of as a probability distribution because we have

$$\left\langle W_0^t, 1\right\rangle = \frac{1}{Nt}\sum_{j=1}^{N}\int_0^t 1 d\tau = \frac{Nt}{Nt} = 1. \tag{4}$$

Our objective is to design trajectories so that the distribution W_0^t is as close to a desired probability distribution in some appropriate metric. Let μ be the desired probability distribution. We assume that μ is zero outside a rectangular domain U. Now we are going to require that W_0^t weakly converge to μ. i.e.,

$$\begin{aligned} & W_0^t \xrightarrow{w} \mu, \text{ or equivalently} \\ & \lim_{t\to\infty}\left\langle W_0^t, f\right\rangle = \langle \mu, f\rangle \\ & \text{ for all bounded functions } f : U \to \mathbb{R}. \end{aligned} \tag{5}$$

We compare W_0^t and μ in the weak sense because W_0^t is a delta-like distribution and it is more appropriate to compare W_0^t and μ in terms of their actions on functions. For more discussions on weak convergence, see [4] and [5]. Let f_k be the Fourier basis functions that satisfy Neumann boundary conditions on the rectangular domain U and k is the corresponding wave-number vector. For instance, on a rectangular domain $[0, L_1] \times [0, L_2]$, we have,

$$\begin{aligned} & f_k(x) = cos(k_1x_1)cos(k_2x_2), \text{ where} \\ & k_1 = \frac{K_1\pi}{L_1} \text{ and } k_2 = \frac{K_2\pi}{L_2}, \\ & \text{ for } K_1, K_2 = 0, 1, 2.... \end{aligned} \tag{6}$$

Now, computing the Fourier coefficients of the distirbution W_0^t, we have

$$c_k(t) = \frac{\langle W_0^t, f_k\rangle}{\langle f_k, f_k\rangle} = \frac{1}{Nt}\frac{\sum_{j=1}^{N}\int_0^t f_k(x_j(\tau))d\tau}{\langle f_k, f_k\rangle}. \tag{7}$$

The Fourier coefficients of μ are given as

$$\mu_k = \frac{\langle \mu, f_k\rangle}{\langle f_k, f_k\rangle}. \tag{8}$$

The measure for the uniformity of the trajectories that we are going to use is the distance between W_0^t and μ as given by the Sobolev space norm of negative index (H^{-s} for $s = (n+1)/2$). i.e.,

$$\begin{aligned} & \phi(t) = \|W_0^t - \mu\|^2_{H^{-s}} = \sum_k \Lambda_k |s_k(t)|^2, \\ & \text{where } s_k(t) = c_k(t) - \mu_k, \quad \Lambda_k = \frac{1}{(1+\|k\|^2)^s}, \\ & s = \frac{n+1}{2}, \text{ and } n \text{ is the dimension of the space.} \end{aligned} \tag{9}$$

It can be shown that $\phi(t)$ decays to zero if and only if W_0^t weakly converges to μ (see [5]). Requiring $\phi(t)$ to converge to zero is the same as requiring time-averages of the Fourier basis functions functions along trajectories to converge to the spatial averages of the basis functions. In other words $\phi(t)$ is a measure of how much the time averages deviate from the spatial averages, but with more importance given to large-scale modes than the small-scale modes. The requirement that time averages along trajectories be equal to spatial averages is a well known necessary and sufficient condition for ergodicity (See [4] and [6]). A related metric to capture deviation from ergodicity that uses wavelet basis functions is described in [7].

The metric defined above was motivated by our previous work in [5] and [8] to quantify mixing in fluid flows or the degree of uniformity of material distributed throughout a domain. In particular, this metric can be shown to be equivalent to a metric that compares the fraction of the time spent by the trajectories in an arbitrary set to the ‘size’ of that set. See Appendix A for details. A modification of this metric can be made to take into account the sensor range as in [3]. The metric and algorithm described in this paper is useful when the sensor range is very small compared to the area to be sampled.

III. Feedback control design : Spectral Multiscale Coverage (SMC)

For notational convenience, let us define the quantities

$$\begin{aligned} C_k(t) &= \frac{\sum_{j=1}^{N}\int_0^t f_k(x_j(\tau))d\tau}{\langle f_k, f_k\rangle} = Ntc_k(t) \\ M_k(t) &= Nt\mu_k \\ S_k(t) &= C_k(t) - M_k(t) = Nts_k(t) \\ \text{and } \Phi(t) &= \sum_k \Lambda_k |S_k(t)|^2 = N^2t^2\phi(t). \end{aligned} \tag{10}$$

The following limit is useful in our discussions.

$$\begin{aligned} s_k(0) := \lim_{t\to 0} s_k(t) &= \left(\lim_{t\to 0}\frac{\sum_{j=1}^{N}\int_0^t f_k(x_j(\tau))d\tau}{Nt\,\langle f_k, f_k\rangle} - \mu_k\right) \\ &= \frac{\sum_{j=1}^{N}\ f_k(x_j(0))}{N\,\langle f_k, f_k\rangle} - \mu_k. \\ &\quad \text{(by L'Hospital's rule)} \end{aligned} \tag{11}$$

And it follows that

$$\phi(0) := \lim_{t\to 0}\phi(t) = \sum_k \Lambda_k \left|\frac{\sum_{j=1}^{N} f_k(x_j(0))}{N\,\langle f_k, f_k\rangle} - \mu_k\right|^2. \tag{12}$$

1) First-order dynamics: The problem of choosing the controls $u_j(t)$ at time t such that it minimizes $\phi(t+dt)$ for some small dt is the same as that of minimizing $\Phi(t+dt)$. Assuming continuity of $u_j(.)$, from basic calculus, we know that

$$\Phi(t+dt) \approx \Phi(t) + \dot{\Phi}(t)dt + \frac{1}{2}\ddot{\Phi}(t)dt^2 := \widetilde{\Phi}(t, dt). \tag{13}$$

Let us now compute the first and second time-derivatives of $S_k(t)$.

$$\frac{dS_k(t)}{dt} = \frac{\sum_{j=1}^{N} f_k\left(x_j(t)\right)}{\langle f_k, f_k\rangle} - N\mu_k, \tag{14}$$

$$\frac{d^2S_k(t)}{dt^2} = \frac{\sum_{j=1}^{N} \nabla f_k(x_j(t)) \cdot u_j(t)}{\langle f_k, f_k\rangle}. \tag{15}$$

$\nabla f_k(.)$ is the gradient vector field of the basis functions f_k. For the basis functions with Neumann boundary conditions as in (6), we have

$$\nabla f_k(x_j(t)) = \left[\begin{array}{c} -k_1 sin(k_1 x_{(j,1)}(t))cos(k_2 x_{(j,2)}(t)) \\ -k_2 cos(k_1 x_{(j,1)}(t))sin(k_2 x_{(j,2)}(t)) \end{array}\right]. \tag{16}$$

Now, the first time-derivative of $\Phi(t)$ is given as

$$\dot{\Phi}(t) = 2\sum_k \Lambda_k S_k(t)\frac{dS_k(t)}{dt}. \tag{17}$$

Note that since the cost-function $\Phi(t)$ involves time-integrals of functions of the states, the first time-derivative of the states ($u_j(t)$) has no direct influence on the first time-derivative of Φ. Now, the second time-derivative of $\Phi(t)$ is given as

$$\ddot{\Phi}(t) = 2\sum_k \Lambda_k \left|\frac{dS_k(t)}{dt}\right|^2 + 2\sum_k \Lambda_k S_k(t)\frac{d^2S_k(t)}{dt^2}. \tag{18}$$

Substituting the expressions for the derivatives of S_k in the above expression we get,

$$\begin{aligned} \ddot{\Phi}(t) &= 2\sum_k \Lambda_k \left|\frac{dS_k(t)}{dt}\right|^2 \\ &\quad + 2\sum_k \Lambda_k S_k(t)\left[\frac{\sum_{j=1}^{N} \nabla f_k(x_j(t)) \cdot u_j(t)}{\langle f_k, f_k\rangle}\right] \\ &= 2\sum_k \Lambda_k \left|\frac{dS_k(t)}{dt}\right|^2 + 2\sum_{j=1}^{N} B_j(t)\cdot u_j(t), \\ &\text{where } B_j(t) = \left[\sum_k \frac{\Lambda_k S_k(t)\nabla f_k(x_j(t))}{\langle f_k, f_k\rangle}\right]. \end{aligned} \tag{19}$$

Clearly, the choice of $u_j(t)$ that minimizes $\widetilde{\Phi}(t, dt)$ subject to the constraint $\|u_j(t)\|_2 \leq u_{max}$ is

$$u_j(t) = -u_{max}\frac{B_j(t)}{\|B_j(t)\|_2}. \tag{20}$$

In other words, the above choice of the feedback law minimizes $\ddot{\Phi}(t)$ subject to the constraint $\|u_j(t)\|_2 \leq u_{max}$. It can be shown that

$$\begin{aligned} &\lim_{t\to 0} u_j(t) = -u_{max}\frac{b_j(0)}{\|b_j(0)\|_2}, \\ &\text{where } b_j(0) = \left[\sum_k \frac{\Lambda_k s_k(0)\nabla f_k(x_j(0))}{\langle f_k, f_k\rangle}\right], \end{aligned} \tag{21}$$

and $s_k(0)$ is as defined in (11). The feedback law (20) is well defined as long as $B_j \neq 0$.

In our first example, the objective is to uniformly cover a square domain excluding regions covered by foliage which are represented as shaded regions in Figure 1. The shaded regions can be thought of as areas where no sensor measurements can be made and therefore there is no value in the sensors spending time over these regions. The target probability distribution μ is setup as follows. First, we define a Terrain function as:

$$Ter(x) = \begin{cases} 1, & \text{if } x \text{ is outside foliage} \\ 0, & \text{if } x \text{ is inside foliage.} \end{cases} \tag{22}$$

Now, μ is defined as

$$\mu(x) = \frac{Ter(x)}{\int_U Ter(y)dy}. \tag{23}$$

The snapshots in Figure 1 were generated with a simulation of 4 agents and with the feedback law in (20). The domain is a unit square domain $[0,1]\times[0,1]$, the total simulation time is $T = 9$, and $u_{max} = 5.0$. The basis functions f_k used are as in (6) for $k_1, k_2 = 0, 1, ...50$. The initial positions of the agents are chosen randomly.

The closed-loop differential equations for the agent trajectories (1) and the coefficients S_k (14) are solved using a fixed time-step 4^{th} order Runge-Kutta method. Note that the Neumann boundary condition of the basis functions f_k guarantees that the velocity component normal to the boundary at the boundaries of the domain is always zero and therefore the agents never escape the domain U.

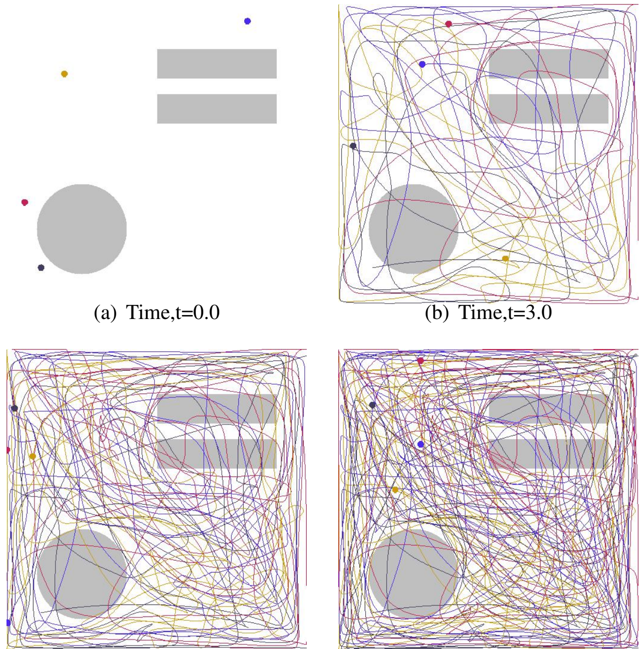
From the snapshots in Figure 1, one can see the multiscale nature of the algorithm. The spacing between the trajectories becomes smaller and smaller as time proceeds. One should note that the trajectories are not such that they completely avoid going over the foliage regions as in collision avoidance. Rather, the algorithm generates trajectories such that the fraction of the time spent over the foliage regions is close to zero.

Figure 2 shows a plot of the decay of the coverage measure $\phi(t)$ with time. The decay is not monotonic and in particular for larger times, the decay is irregular. This is partly due to the finite number of basis functions used and the fixed time-stepping used for solving the differential equations. Figure 2 also shows a plot of the fraction of the time spent by the agents outside the foliage regions. It clearly shows that this fraction approaches one, as time proceeds, confirming that the agents spend very little time in the foliage regions. Figure 3 shows a plot of the norm of the vectors $B_j(t)$. As one can see, although $\|B_j(t)\|_2$ comes close to zero often, it never stays close to zero.

2) Second-order dynamics: In this case, we assume the agent dynamics to be as in (2). $u_j(t)$ is the force applied on the agent and is subject to the constraint $\|u_j(t)\|_2 \leq F_{max}$. Following a similar argument as in the previous section for first-order dynamics, but looking at the third time-derivative $\dddot{\Phi}(t)$, we get the feedback law

$$u_j(t) = -F_{max}\frac{B_j(t)}{\|B_j(t)\|_2}. \tag{24}$$

The results showed in Figure 4 were generated with a simulation of 4 agents, with a damping coefficient $c = 20.0$



(a) Time,t=0.0 (b) Time,t=3.0

(c) Time,t=6.0 (d) Time,t=9.0

Fig. 1. Snapshots at various times of the agent trajectories generated by the SMC algorithm with first-order dynamics. One can observe the multiscale nature of the algorithm. The spacing between the trajectories becomes smaller and smaller as time proceeds.

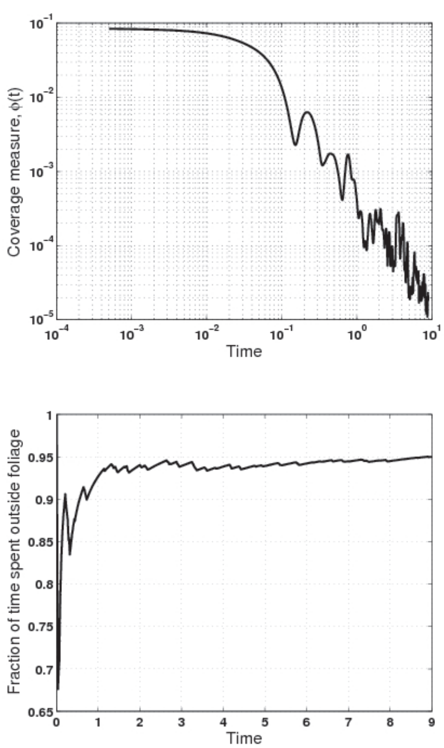


Fig. 2. Results with first-order dynamics: Plot on the top shows the decay of coverage measure $\phi(t)$ with time. The decay is not monotonic. The plot on the bottom shows the fraction of the time spent by the agents outside the foliage. This fraction gets closer and closer to one as time proceeds.

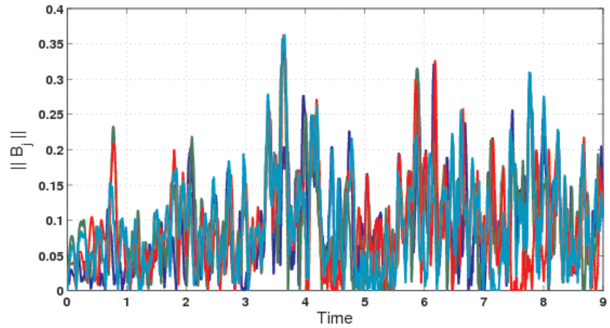


Fig. 3. Results with first-order dynamics: The plot shows the norm of the vectors $B_j(t)$ defined in (19). As one can see, $\|B_j\|_2$, comes close to zero often, but never remains identically close to zero.

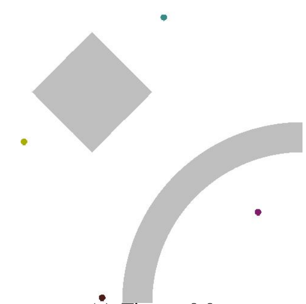
and $F_{max} = 50.0$. The total simulation time is $T = 18.0$, the domain is a unit square as before, but with a different foliage region. The basis functions used are the same as before. The initial positions of the agents are chosen randomly and the initial velocities are zero. As with first-order dynamics, it is not guaranteed that the agents are always confined in the domain eventhough the force component normal to the boundaries is zero at the boundary. To resolve this, we force the agents inward toward the domain with maximum force whenever they leave the domain.

Figure 5 shows a plot of the decay of the coverage measure $\phi(t)$ with time. Figure 5 also shows a plot of the fraction of the time spent by the agents outside the foliage regions. Here, this fraction is smaller than that in first-order dynamics because the agents occasionally leave the square domain. Figure 6 shows a plot of the norm of the vectors $B_j(t)$ and as before they never stay close to zero.

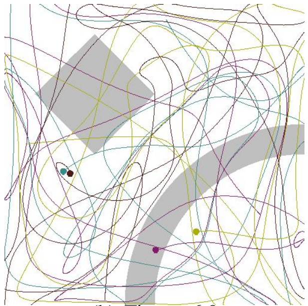
IV. Discussion

We have proposed centralized feedback control laws for mobile sensor networks to achieve uniform coverage of any given domain.The algorithm presented in this paper is fairly easy to implement and more importantly it is very easy to apply to domains of various geometries. A change in the geometry requires only defining an appropriate terrain function as in (22), computing the corresponding target probability distribution μ and its Fourier coefficients μ_k. Also, for a fixed number of basis functions, the computational complexity of the algorithm is $O(N)$. Various numerical simulations have demonstrated the effectiveness of the algorithm.

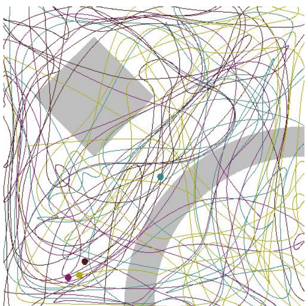
Proving asymptotic decay of the coverage measure $\phi(t)$ with the proposed feedback laws and for arbitrary probability measures μ, remains an open problem and will be the subject of future work. In particular, it needs to be shown that the vector $B_j(t)$ defined in (19) never approaches and stays at zero. However, numerical simulations suggest that this is unlikely. Rigorous proofs for this will be the subject of future work. Future work also includes modifications of the algorithm to achieve decentralization in the case when there is limited communication between agents.



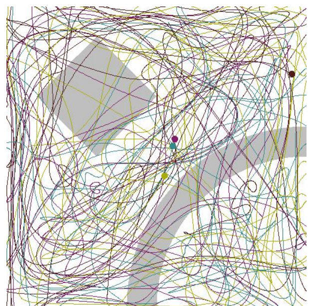
(a) Time,t=0.0



(b) Time,t=6.0



(c) Time,t=12.0



(d) Time,t=18.0

Fig. 4. Snapshots at various times of the agent trajectories generated by the SMC algorithm with second-order dynamics

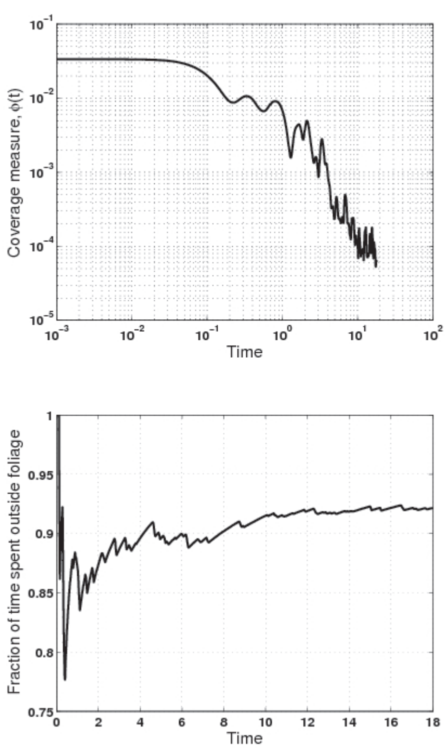


Fig. 5. Results with second-order dynamics: Plot on the tops shows decay of the coverage measure $\phi(t)$ with time. Decay is not monotonic. The plot on the bottom shows the fraction of the time spent by the agents outside the foliage. This fraction is smaller than that with first-order dynamics because agents occasionally leave the square domain.

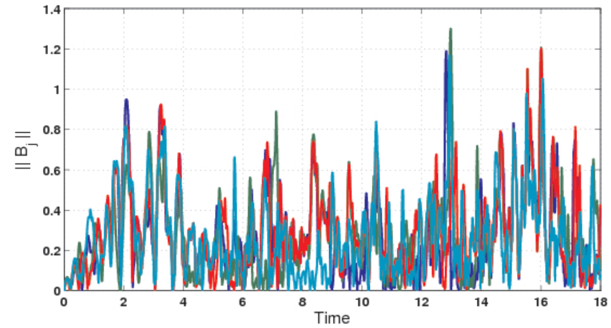


Fig. 6. Results with second-order dynamics: The plot shows the norm of the vectors $B_j(t)$ defined in (19).

APPENDIX

A. Multiscale interpretation of the measure for uniformity of trajectories

Let $x_j : [0,t] \to \mathbb{R}^n$ be sensor trajectories and as in the introduction of the paper, we have the corresponding distribution W_0^t defined by its action on functions as:

$$\left\langle W_0^t, f \right\rangle = \frac{1}{Nt} \sum_{j=1}^{N} \int_0^t f(x_j(\tau)) d\tau. \tag{25}$$

Let $B_{(p,r)} = \{x : \|x - p\| \leq r\}$ be a sphere of radius r with centre at point p. And let $\chi_{(p,r)}$ be the indicator function on the set $B_{(p,r)}$. Note that

$$d(p,r) = \left\langle W_0^t, \chi_{(p,r)} \right\rangle = \frac{1}{Nt} \sum_{j=1}^{N} \int_0^t \chi_{(p,r)}(x_j(\tau)) d\tau, \tag{26}$$

is the fraction of the total time spent by the agents in the set $B_{(p,r)}$. For the points on the trajectories $x_j(.)$ to be distributed according to the probability distribution μ on the rectangular domain U, the fraction $d(p,r)$ must be equal to $\mu(p,r) = \left\langle \mu, \chi_{(p,r)} \right\rangle$, for all points $p \in U$ and all scales $r \in (0,1)$. This motivates defining the metric

$$H(t) = \int_0^1 \left(\int_{p \in U} (d(p,r) - \mu(p,r))^2 \, dp \right) dr. \tag{27}$$

It can be shown that there exists bounded constants $C_1, C_2 > 0$ such that

$$C_1 H(t) \leq \phi(t) \leq C_2 H(t), \tag{28}$$

for all $x_j : [0,t] \to \mathbb{R}^n$. Here $\phi(t)$ is the coverage measure as defined in the introduction using the Fourier basis functions. The proof for the inequality in (28) is along the same lines as that given in [5]. We note that for the above inequality to hold, $H(t)$ is assumed to be computed by extending the distributions W_0^t and μ outside the domain U by even-reflections across the domain boundaries.

In other words, the two metrics $H(t)$ and $\phi(t)$ are equivalent and therefore the decay of one implies the decay of the other.. Thus, requiring $\phi(t)$ to decay to zero is the same as requiring that the fraction of the time spent by the agents in each and every spherical set to be equal to the 'size'of the set. Here, by 'size' of the set, we mean the quantity $\mu(p,s) = \left\langle \mu, \chi_{(p,s)} \right\rangle$, which is exactly equal to the size of the set if the probability measure μ is uniform.

REFERENCES

[1] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions of Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[2] N. Leonard, D. Paley, F. Lekien, R. Sepulchre, D. Fratantoni, and R. Davis, "Collective motion, sensor networks, and ocean sampling," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 48–74, 2007.

[3] I. Hussein and D. Stipanović, "Effective coverage control for mobile sensor networks with guaranteed collision avoidance," *IEEE Transactions on Control Systems Technology, Special Issue on Multi-Vehicle Systems Cooperative Control with Applications*, vol. 15, no. 4, pp. 642–657, 2007.

[4] A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise*, ser. Applied Mathematical Sciences. Springer, 1994.

[5] G. Mathew, I. Mezic, and L. Petzold, "A multiscale measure for mixing," *Physica D: Nonlinear Phenomena*, vol. 211, pp. 23–46, November 2005.

[6] K. Petersen, *Ergodic theory*. Cambridge University press, 1984.

[7] S. E. Scott, T. C. Redd, L. Kuznetsov, I. Mezic, and C. K. Jones, "Capturing deviation from ergodicity at different scales," *Physica D: Nonlinear Phenomena*, vol. 238, no. 16, pp. 1668 – 1679, 2009.

[8] G. Mathew, I. Mezic, S. Grivopoulos, U. Vaidya, and L. Petzold, "Optimal control of mixing in stokes fluid flows," *Journal of Fluid Mechanics*, vol. 580, pp. 261–281, 2007.