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FLOCKING OF MULTI-AGENT DYNAMIC SYSTEMS WITH GUARANTEED GROUP CONNECTIVITY*

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Abstract This paper investigates distributed flocking problem where the information exchange among agents is modeled by the communication topology changing with time. Previous research on this problem establishes group stabilization by assuming that the dynamic topology is connected all the time, which however cannot be guaranteed by most proposed distributed control laws. In this paper, a distributed algorithm to distill a necessary subgraph of the initial communication topology is presented. This subgraph covers all the vertices of the communication topology and is proved to be connected as long as the initial communication topology is connected. A distributed control law is then designed to pursue the flocking motion while preserving all the edges in this subgraph. In this way, connectivity can be preserved all the time, and flocking problem is thus solved only provided the initial communication topology of multi-agent system is connected.

Key words Connectivity, distributed control, flocking, multi-agent system.

1 Introduction

Flocking has fascinated many researchers in the latest few decades[1−4]. It is the common subject on how to understand the cooperative motion patterns of a large group of agents such as birds, fish, and crowds without a centralized scheme and how to develop various cooperative control capabilities of the engineering groups, for example, unmanned aerial vehicles and mobile sensor networks. Flocking is often characterized by a group of agents with similar dynamic to achieve a common motion velocity and a desired inter-agent distance in a future moment through interaction. At the same time, collisions among agents should be avoided during the group evolution.

The theoretical research of flocking can refer to [3-4]. Some distributed coordination algorithms are provided and aim at steering the agents to achieve their consensus velocities and desired inter-agent distances. However, the fragmentation of group into multiple components may take place during maneuvers under these distributed algorithms, which hinders an agent group from forming a flock. Therefore, group stabilization is always established by assuming that the topology of the group communication network is connected for all time. As is known, connectivity is an intrinsically global property and relies on the actual distances between agents. It is difficult to judge whether the topology of a group is connected or not from the agent's local

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perspective. Hence, efforts should be made to preserve the connectivity of the communication topology while achieving the desired flocking objective.

Some researchers focused on connectivity preservation, such as the geometric connectivity robustness and its applications in [5]. In [6–7], connectivity preservation of rendezvous problems of single integrator systems was solved by transforming the condition of group connectivity into the constraint on the motion range of agents. Similarly, the admissible set that allows the double integrator to remain inside disks, and the corresponding double-integrator disk graph were investigated in [8] to guarantee connectivity preservation of double integrator systems. Ji and Egerstedt[9−10] investigated the rendezvous and formation problems of multi-agent systems with single integrator dynamics while preserving the group connectivity. Some centralized approaches with regard to connectivity preservation were also developed in the literature such as [11].

This paper aims at solving flocking problem with a relaxed assumption on connectivity. After investigating the desired geometry configurations of the agents in a flock, we provide a sufficient condition to judge whether a graph is a feasible flocking topology. A distributed algorithm is then developed to achieve a feasible flocking subgraph from any initial connected communication topology. Based on two designed artificial potential functions, a distributed control law is presented for each agent, which can ensure the preservation of connectivity, and the convergence of the multi-agent system to the desired flocking motion is also proved.

2 Model and Preliminaries

2.1 Problem Formulation

Consider a multi-agent system of N mobile agents operating in $\mathbb{R}^2$. Let q_i and p_i denote the position and velocity vectors of agent i, respectively. Suppose that the dynamics of each agent is described by the double integrator as follows:

$$\begin{cases} \dot{q}_i = p_i, \\ \dot{p}_i = u_i, \end{cases} \tag{1}$$

where u_i is the distributed control law of agent i and is determined only based on the information obtained by its communication instruments. For the sake of clarity, we describe the group state with $q = \mathrm{col}\,(q_1, q_2, \cdots, q_N)$, $p = \mathrm{col}\,(p_1, p_2, \cdots, p_N)$, and denote $u = \mathrm{col}\,(u_1, u_2, \cdots, u_N)$.

The purpose of this paper is to design a proper distributed control law u_i for each agent i such that 1) the steered multi-agent system maintains connectivity for all time as long as it moves from any initial connected configuration, 2) all the agents asymptotically achieve velocity synchronization and the desired inter-agent distance, and 3) collision avoidance among agents is guaranteed during maneuvers.

2.2 The Communication Topology and the Laplacian

Let r_c denote the communication radius of agents, then the communication links of the group exist only between the agents whose distance is less than r_c. To describe the intercommunication of the total group, we establish the undirected communication topology $G = (\mathcal{V}, \mathcal{E})$. The set of vertices $\mathcal{V}$ corresponds to the N agents. The edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the communication relations between agents, and is described by

$$\mathcal{E} = \mathcal{E}(G) = \left\{(i,j) \mid \|q_j - q_i\| < r_c, i, j \in \mathcal{V}, j \neq i\right\}. \tag{2}$$

We denote the set of these edges connected with agent i by $\mathcal{E}_i$. Then, the communication neighbor set of agent i is given by

$$N_i = N_i(G) = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}_i\}. \tag{3}$$

The cardinality of N_i indicates how many edges are connected to agent i and is defined as the degree of agent i, denoted by d_i. Since the communication connections among agents depend on their relative positions, the communication topology G may be time-varying due to the motions of the agents.

It is worth mentioning that some important structure characteristics of the undirected dynamic communication topology G can be captured by analyzing its algebraic representations[3−4]. Define the adjacency matrix of G as $A = A(G) = [a_{ij}]$ with $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The degree matrix of G is a diagonal matrix $D = [d_i]$ with $d_i = \sum\limits_{j \in N_i} a_{ij}$. The Laplacian of G is then defined by

$$L = L(G) = D - A.$$

It is known that $L(G)$ is always symmetric and positive semi-definite for undirected graph G and satisfies

$$z^{\mathrm{T}} L z = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in N_i} a_{ij} (z_j - z_i)^2, \tag{4}$$

where $z = \mathrm{col}(z_1, z_2, \cdots, z_N)$. Particularly, for a connected undirected G, the Laplacian has a single zero eigenvalue and the corresponding eigenvector is ones, i.e., 1_N.

2.3 The Feasible Flocking Topology

For real flocking motions, two main characteristics are often emphasized in the literature. By the definition in [3], a group of agents is called a flock over $[t_0, t_f]$ if the corresponding communication topology G is connected over $[t_0, t_f]$. At the same time, the geometry configurations of a real-life flock is captured by defining the α-lattice structure, whose edges have the same length. Clearly, connectivity and geometry configurations are both critical to completely describe the desired flocking motions.

Consider a multi-agent system which moves from any initial connected configuration and is steered to form/keep a connected α-lattice configuration, the geometry configurations of group during maneuvers may be out of α-lattices before a flock is finally formed. To catch more characteristics of a moving group toward a flock, in what follows we extend the definition of α-lattice without considering the length of edge. Given two graphs G_1, G_2 with the same sets of $\mathcal{V}$ and $\mathcal{E}$, if it is feasible to adjust the length of each edge in G_1 to the corresponding value of G_2 without removing/adding any edges, G_1 is said to have the same topological characteristics as G_2. With this definition, let us denote $\mathbb{G}_{\mathrm{Lattice}}$ as the set of these graphs with the same topological characteristics as some graph of α-lattices, which is illustrated in Fig.1. The following lemma can be used to judge whether a graph satisfies $G \in \mathbb{G}_{\mathrm{Lattice}}$ from the agent's local perspective.

Lemma 1 *Given a graph G, $G \in \mathbb{G}_{\mathrm{Lattice}}$, if the following three conditions are simultaneously satisfied for G:*

1) *$d_i \leq 6$ for each agent i;*

2) *For each agent i with $d_i < 6$, there is no loop formed by the edges with vertexes in N_i;*

3) *For each agent i with $d_i = 6$, there is at most one loop formed by all the edges with vertexes in N_i.*

Lemma 1 is obvious via some simple deductions. The proof is omitted here.

As discussed above, connectivity is always expected for a group moving towards a flock, even if the group communication topology is time-variant due to the motion of the agents. For

(a) An α-lattice (b) A configuration with the same connections as (a)

Figure 1 Examples of $\mathbb{G}_{\text{Lattice}}$

the sake of simplicity, we denote the set of connected graphs by $\mathbb{C}$. Then, by considering both the topological characteristics of $G \in \mathbb{G}_{\text{Lattice}}$ and connectivity, any graph G is called a feasible flocking topology if $G \in \mathbb{G}_{\text{Lattice}} \cap \mathbb{C}$. Based on this definition and Lemma 1, we can give a sufficient condition to judge whether a graph is a feasible flocking topology in the following Lemma, where $\mathbb{G}_{\text{FFT}}$ represents the set of feasible flocking topologies.

Lemma 2 *Given a graph G, $G \in \mathbb{G}_{\text{FFT}}$, if the following conditions are simultaneously satisfied for G:*

1) $G \in \mathbb{C}$*;*

2) $d_i \leq 6$ *for each agent* i*;*

3) *For each agent* i *with* $d_i < 6$*, there is no loop formed by the edges with vertexes in* N_i*;*

4) *For each agent* i *with* $d_i = 6$*, there is at most one loop formed by all the edges with vertexes in* N_i*.*

Remark 1 The set $\mathbb{G}_{\text{FFT}}$ depicts the main structure characteristics of the desired flock, that is, under any connected communication topology $G \in \mathbb{G}_{\text{FFT}}$, all the agents can be steered and equally distanced from their neighbors. This guarantees that the manipulations of preserving all the edges of $G \in \mathbb{G}_{\text{FFT}}$ and adjusting their lengths to the desired value can be carried simultaneously, and thus a connected α-lattice configuration is achieved.

However, in most situations maintaining all the edges of communication topology helps to preserve connectivity but may hinder the realization of α-lattice configuration, because the actual connected communication topology may be highly intricate and does not belong to $\mathbb{G}_{\text{FFT}}$. Following this argument, we will first present a distributed algorithm to obtain a feasible flocking topology $\widehat{G} \in \mathbb{G}_{\text{FFT}}$ from any connected initial communication topology $G(0)$. A distributed control law is then designed in order to maintain all the edges in $\widehat{G}$, and simultaneously to adjust the length of edge in the communication topology $G(t)$ to the desired value. With this control law, connectivity can be preserved for all time and the steered group will converge to a stable α-lattice configuration.

3 Obtain a Feasible Flocking Topology

In this section, a distributed algorithm is developed to distill a necessary subgraph $\widehat{G} \in \mathbb{G}_{\text{FFT}}$ from any connected topology G. Since an individual agent can only obtain information from itself and from the neighbor agents within its communication range, the developed algorithm should only utilize such local information and thus the connected subgraph $\widehat{G}$ will be achieved in a completely distributed style. Denote $w_{ij} = \|q_i - q_j\|_2$ as the relative distance between the neighbor agents i, j. We say that edge (i,j) preponderates over edge (g,h) (denoted by $P_{ij} \succ P_{gh}$), if one of the following three conditions is satisfied:

1) $w_{ij} > w_{gh}$;

2) $w_{ij} = w_{gh}$ and $\max\{i,j\} > \max\{g,h\}$;

3) $w_{ij} = w_{gh}$, $\max\{i,j\} = \max\{g,h\}$ and $\min\{i,j\} > \min\{g,h\}$.

It is reasonable to say that the preponderance relation of any two edges is definite and not ambiguous. With these definitions, we propose the following distributed algorithm for each agent i to obtain its neighbor set $\widehat{N}_i$ associated with $\widehat{G}$.

Algorithm 1 (A distributed algorithm for agent i to obtain $\widehat{N}_i$)

1. Obtain N_i (the neighbor set of agent i associated with G, described by (3)), initialize $N_{temp} = N_i$, $\widehat{N}_i = \emptyset$ (empty set).

2. While $N_{\text{temp}} \neq \emptyset$ do

3. For any $j \in N_{\text{temp}}$, $N_{\text{temp}} = N_{\text{temp}} \backslash \{j\}$.

4. If each h which satisfies $h \in N_i$ and $\|x_h - x_j\| \leq r_c$ (it indicates $h \in N_j$) has the property of $P_{ij} \prec \max\{P_{ih}, P_{hj}\}$, then $\widehat{N}_i = \widehat{N}_i \cup \{j\}$.

5: End while. ∎

After executing Algorithm 1, $\widehat{G}$ is essentially constructed by $\widehat{\mathcal{V}} = \mathcal{V}(\widehat{G}) = \mathcal{V}(G)$ and $\mathcal{E} = \mathcal{E}(\widehat{G}) = \{(i,j)|j \in \widehat{N}_i, i \in \widehat{N}_j\}$. Let us denote the degree of agent i in $\widehat{G}$ by $\hat{d}_i$. It can be found that $\widehat{N}_i \subseteq N_i$, $\widehat{\mathcal{E}} \subseteq \mathcal{E}$, and $\widehat{d}_i \leq d_i$. In what follows, we will show that the subgraph $\widehat{G}$ obtained by Algorithm 1 is a feasible flocking topology if $G \in \mathbb{C}$, i.e., $\widehat{G} \in \mathbb{G}_{\text{FFT}}$.

Lemma 3 *$\widehat{G} \in \mathbb{C}$, if $G \in \mathbb{C}$, where $\widehat{G}$ is a subgraph of G and is obtained by utilizing Algorithm* 1.

Proof Given any $G \in \mathbb{C}$, its subgraphs G_{MST} (the minimum spanning tree of G) and $\widehat{G}$ are both unique according to the preponderance relation of the edges. Since the minimum spanning tree G_{MST} is the sparsest connected subgraph of G, we can prove the fact of $\widehat{G} \in \mathbb{C}$ by showing that $G_{\text{MST}} \subseteq \widehat{G}$. Consider any edge $(i,j) \in \mathcal{E}(G)$, if there is $h \in N_i \cap N_j$ satisfying $P_{ij} \succ \max\{P_{ih}, P_{hj}\}$, it must be true that $(i,j) \notin G_{\text{MST}}$. Therefore, the condition of all $h \in N_i \cap N_j$ satisfying $P_{ij} \prec \max\{P_{ih}, P_{hj}\}$ is a necessary but not sufficient condition for $(i,j) \in G_{\text{MST}}$, which however is necessary and sufficient for $(i,j) \in \widehat{G}$ based on Algorithm 1. Hence, we have $G_{\text{MST}} \subseteq \widehat{G}$, which follows that $\widehat{G} \in \mathbb{C}$ if $G \in \mathbb{C}$. ∎

From Algorithm 1, if $(i,j) \notin \widehat{G}$, there exists at least another agent $h \in N_i \cap N_j$ in G. In this case, any two of the three agents i, j, and h are mutual neighbors in G and agents i, j, and h construct a triangle connection, denoted by $\Delta(i,j,h)$. Thus, from Algorithm 1 the manipulations to obtain $\widehat{G}$ imply to remain edge $(i,j) \in \mathcal{E}(G)$ whose weight is not maximal in each triangle connection containing it. The following lemma shows that if a connected graph G contains no triangle connection and satisfies that $\widehat{d}_i \leq 6$ for each i, the graph G must belong to $\mathbb{G}_{\text{FFT}}$.

Lemma 4 *Consider any $G \in \mathbb{C}$ with properties of $\widehat{d}_i \leq 6$ for any i and containing no triangle connection, we have $G \in \mathbb{G}_{\text{FFT}}$.*

Proof Given any i with $\widehat{d}_i \leq 6$, since no triangle connection is formed by the edges in G, there is no edge between any $j, h \in N_i(G)$ (Otherwise, a triangle connection is formed by i, j, and h). Furthermore, no loop is formed by the edges with vertexes in N_i. According to Lemma 2, such a connected graph must be a feasible flocking topology. ∎

Reviewing Algorithm 1, it is clear that the edge, which is contained in a triangle connection and has the maximal weight among the corresponding three edges, must be excluded from the subgraph $\widehat{\mathcal{G}}$. We provide the formal statement of this fact by the following lemma.

Lemma 5 *There is no triangle connection in $\widehat{G}$, where $\widehat{G}$ is obtained by utilizing Algorithm* 1.

Proof To prove by contradiction, let us assume that there is a triangle connection in $\widehat{G}$. Without loss of generality, the corresponding three agents of this triangle connection are denoted by i, j, and h according to $P_{ij} \succ P_{ih} \succ P_{jh}$. From Algorithm 1, edge (i,j) will never be preserved in $\widehat{G}$, which contradicts that the triangle connection $\Delta(i,j,h)$ is contained in $\widehat{G}$. ∎

Lemma 6 *The maximum degree of agents in $\widehat{G}$ is no more than 6, i.e., $\widehat{d}_i \leq 6$ for any $i \in \widehat{\mathcal{V}}$.*

Proof To prove by contradiction, let us assume $\widehat{d}_i > 6$ for agent i and denote $\widehat{\mathcal{E}}_i = \{(i,j)|j \in \widehat{N}_i\}$ as the set of all the edges between agent i and its neighbors in $\widehat{N}_i$. We take β as the average angle between any two adjacent edges in set $\widehat{\mathcal{E}}_i$, thus, $\beta = \frac{2\pi}{\widehat{d}_i} < \frac{\pi}{3}$. Therefore, at least one angle $\beta_{\min}$ between two adjacent edges in $\widehat{\mathcal{E}}_i$, for example (i,j) and (i,h), satisfies $\beta_{\min} < \frac{\pi}{3}$. According to Sine Theorem, it holds that $w_{jh} < \max\{w_{ij}, w_{ih}\}$, which follows that $j \in N_h$ and $h \in N_j$. Therefore, agents i, j, and h construct a triangle connection $\Delta(i,j,h)$, which contradicts Lemma 5. ∎

Applying Lemmas 3, 5, and 6 to Lemma 4, we can easily deduce the following result.

Theorem 1 *If $G \in \mathbb{C}$, it holds that $\widehat{G} \in \mathbb{G}_{\mathrm{FFT}}$, where $\widehat{G}$ is the subgraph of G obtained by Algorithm 1.*

4 Flocking Control with Remained Connectivity

In this section, we provide a distributed control law u_i for each agent i described by (1) such that the driven group can achieve the desired flocking objective. The available information for the i-th agent is the relative position and velocity differences $q_j - q_i$ and $p_j - p_i$ for all $j \in N_i(t)$, with which u_i can be designed by the following form:

$$u_i = -\sum_{j\in\widehat{N}_i} \nabla_{q_i}\widehat{V}_{ij} - \sum_{j\in N_i(t)\backslash\widehat{N}_i} \nabla_{q_i}V_{ij} + \sum_{j\in\widehat{N}_i}(p_j - p_i), \tag{5}$$

where $N_i(t)$ is given by (3) and denotes the time-varying communication neighbour set of agent i, $\widehat{N}_i$ denotes the neighbour set of agent i with respect to $\widehat{G}$, which depends on $G(0)$ and is obtained by Algorithm 1. Potential function $\widehat{V}_{ij}$ in the first term aims at preserving all the connections between agent i and its neighbour $j \in \widehat{N}_i$, avoiding the collisions between them and steering agent i with desired distance from its neighbours in $\widehat{N}_i$. While potential function V_{ij} in the second term is mainly used to avoid collisions between agent i and any agent $j \in N_i(t)\backslash\widehat{N}_i$, and steering agent i with desired distance from its neighbour $j \in N_i(t)\backslash\widehat{N}_i$. The third term of (5) is the velocity consensus term and is used to promote the velocities of the agents to a common value.

With different tasks, various $\widehat{V}_{ij}$ and V_{ij} are constructed, respectively. To flocking problem discussed here, $\widehat{V}_{ij}$ is designed for all the edges (i,j), $j \in \widehat{N}_i$, according to the following principles:

1) $\widehat{V}_{ij}$ is nonnegative and everywhere continuously differentiable;

2) $\widehat{V}_{ij}(q_i, q_j)$ has a single minimal value zero at $\|q_j - q_i\| = \rho$, where ρ is the desired inter-agent distance and satisfies $\frac{r_c}{\rho} = 1 + \varepsilon\,(\varepsilon \ll 1)$;

3) $\widehat{V}_{ij} \to +\infty$, if $\|q_j - q_i\| \to 0$;

4) $\widehat{V}_{ij} \to +\infty$, if $\|q_j - q_i\| \to r_c$.

One example of such a potential function is given by

$$\widehat{V}_{ij} = \left\|\frac{1}{\|q_j - q_i\|} - \frac{1}{\rho}\right\|^a \frac{1}{(\|r_c\| - \|q_j - q_i\|)^b}, \quad (i,j) \in \widehat{\mathcal{E}}, \tag{6}$$

where $a \geq 2$, $b \geq 1$. Formula (6) implies that the desired relative position between agents i and j can be uniquely determined by minimizing the corresponding potential function $\widehat{V}_{ij}$. Besides, if $\widehat{V}_{ij}$ is always finite, it is ensured that edge (i,j) will neither be lost nor be of length zero.

Correspondingly, V_{ij} is also a function of $\|q_j - q_i\|$ and is designed with the following characteristics:

1) V_{ij} is nonnegative and everywhere continuously differentiable;

2) V_{ij} has a single minimal value zero at $\|q_j - q_i\| = \rho$, where ρ is the desired inter-agent distance and satisfies $\frac{r_c}{\rho} = 1 + \varepsilon\,(\varepsilon \ll 1)$;

3) $\widehat{V}_{ij} \to +\infty$, if $\|q_j - q_i\| \to 0$;

4) $\nabla_{q_i} V_{ij} = 0$ and $V_{ij} = k$ whenever $\|q_j - q_i\| \geq r_c$, where k is a little positive constant.

In this paper, we select potential function V_{ij} with form

$$V_{ij} = \begin{cases} \left\| \dfrac{1}{\|q_j - q_i\|} - \dfrac{1}{\rho} \right\|^a \dfrac{1}{\left(\|r_c\| - \|q_j - q_i\|\right)^b}, & \|q_j - q_i\| \leq \rho, \\ \dfrac{k}{2}\left[1 - \cos\left(\pi \dfrac{\|q_j - q_i\| - \rho}{r_c - \rho}\right)\right], & \rho < \|q_j - q_i\| \leq r_c, \\ k, & \|q_j - q_i\| > r_c, \end{cases} \tag{7}$$

where $a \geq 2$, $b \geq 1$. It can be found that V_{ij} has a single minimal zero value at $\|q_j - q_i\| = \rho$ and increases to infinity whenever $\|q_j - q_i\| \to 0$. Therefore, the finite V_{ij} guarantees that agents i and j will never collide. Besides, the characteristics 1) and 4) guarantee that the collective potential function remains continuously differentiable if a new edge is added into $\mathcal{E}(t)$ or an existing edge in $\mathcal{E}(t)$ is lost due to the motions of the agents.

5 Convergence

With $\widehat{V}_{ij}$ and V_{ij} given above, the continuously differentiable collective potential function can be given by

$$V = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j\in\widehat{N}_i} \widehat{V}_{ij} + \sum_{j\neq i, j\in N_i(t)\setminus\widehat{N}_i} V_{ij}\right). \tag{8}$$

We also define the kinetic energy of the group as

$$K = \frac{1}{2}\sum_{i=1}^{N}\|p_i\|^2. \tag{9}$$

Consider the following positive semi-define Hamiltonian function

$$H = V + K. \tag{10}$$

With the fact that

$$\nabla_{q_i} V = \nabla_{q_i}\left(\sum_{j\in\widehat{N}_i} \widehat{V}_{ij} + \sum_{j\neq i, j\in N_i(t)\setminus\widehat{N}_i} V_{ij}\right),$$

it can be deduced that

$$\begin{aligned} \dot{H} &= \dot{V} + \dot{K} \\ &= \sum_{i=1}^{N}\left(p_i^{\mathrm{T}} \cdot \left(\sum_{j\in\widehat{N}_i} \nabla_{q_i}\widehat{V}_{ij} + \sum_{j\neq i, j\in N_i(t)\setminus\widehat{N}_i} \nabla_{q_i} V_{ij}\right)\right) + \sum_{i=1}^{N}\left(p_i^{\mathrm{T}} \cdot \dot{p}_i\right) \\ &= -p^{\mathrm{T}}\left(L_{\widehat{G}} \otimes I_2\right)p, \end{aligned} \tag{11}$$

where $L_{\widehat{G}}$ is the Laplacian associated with $\widehat{G}$, and $\otimes$ denotes the Kronecker production. Since $L_{\widehat{G}}$ is always symmetric and positive semi-definite, we have

$$\dot{H} \leq 0. \tag{12}$$

Thus, the following theorem can be deduced.

Theorem 2 *Consider a multi-agent system steered by the distributed control law* (5)*. If the initial Hamiltonian H_0 is finite, then all the edges of $\widehat{G}$ are preserved in $G(t)$ during the whole evolution and the collision avoidance can be always guaranteed.*

Proof Since the nonnegative Hamiltonian function H is differentiable and nonincreasing for all time based on (12), the potential function is bounded by H_0:

$$V \leq H \leq H_0 \ll \infty.$$

However, if $\|q_j - q_i\| \to r_c$ for any $(i, j) \in \widehat{\mathcal{E}}$ or $\|q_j - q_i\| \to 0$ for any $(i, j) \in \mathcal{E}(t)$, it will occur that $V \to \infty$ according to (6)–(8). Therefore, the finite V implies that $\|q_j - q_i\|$ will never converge to r_c for any $(i, j) \in \widehat{\mathcal{E}}$ nor decrease to zero for any $(i, j) \in \mathcal{E}(t)$. It follows that no edge in $\widehat{G}$ will be lost during maneuvers, i.e., $\widehat{\mathcal{E}} \subseteq \mathcal{E}(t)$. At the same time the collision between any two communication neighbors is always avoided. ∎

Remark 2 Note that $\widehat{G} \in \mathbb{C}$ if $G(0) \in \mathbb{C}$, and furthermore, each edge of $\widehat{G}$ is preserved in $G(t)$ for all time. Hence, it holds that $G(t) \in \mathbb{C}$ for all $t \geq 0$ as long as $G(0) \in \mathbb{C}$. Additionally, Theorem 1 ensures that all the edges in $\widehat{G}$ can be adjusted to the length of ρ without resulting in conflict during the realization of connected α-lattice configurations.

The following lemma bridges a gap between the local minima of the collective potential V described by (8) and the α-lattice configurations. The proof is rather similar to Lemma 3 of [3] and is omitted here.

Lemma 7 *Every local minima of V described by* (8) *is an α-lattice and vice versa.*

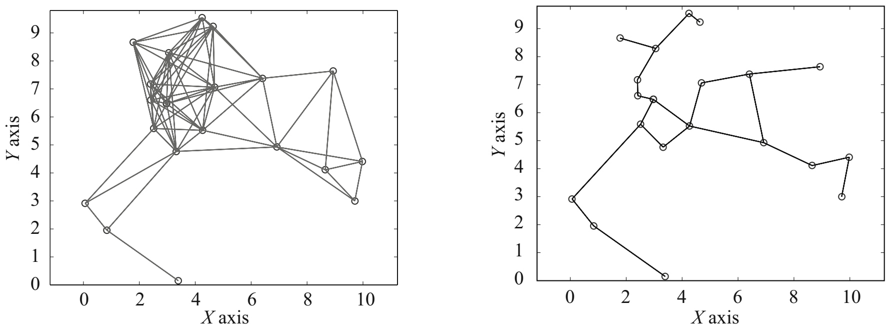
With the results shown above, we will prove the convergence of the systems by using LaSalle's invariance principle.

Theorem 3 *Consider a multi-agent system steered by the distributed control law* (5)*. If the initial topology satisfies $G(0) \in \mathbb{C}$ and $0 < H_0 < \infty$, then almost each solution of the system dynamics converges to an equilibrium with the expected equal distance between each pair of neighbors, all agent velocities asymptotically become the same and collisions among agents are avoided.*

Proof As indicated in (11) and (12), if H_0 is finite, the nonnegative H keeps finite for all time. Thus, we have $K \leq H \leq H_0 \ll \infty$ and $\widehat{V}_{ij} \leq V \leq H \leq H_0 \ll \infty$. The former guarantees $\|p_i\| \leq \sqrt{2H_0}$ and the latter implies that $\|q_j - q_i\| \leq \widehat{V}_{ij}^{-1}(H_0) < r_c$ for any $(i, j) \in \widehat{\mathcal{E}}$. From the assumption of $G(0) \in \mathbb{C}$ and Lemma 3, it can be deduced that $\widehat{G} \in \mathbb{C}$, which follows that $G(t) \in \mathbb{C}$ according to Theorem 2. Hence, any two agents i, j are connected by a path through at most $N - 1$ edges and with a distance no more than $(N-1)\,(\widehat{V}_{ij}^{-1}(H_0))$. Denote $q_j - q_i$ as q_{ji}. Thus, the set $\Omega = \{(q_{ji}, p_i)\,|H \leq H_0, t \geq 0\}$ is a compact invariant set of the multi-agent system. From LaSalle's invariance principle, system state starting from Ω will converge to the largest invariant set in $E = \{(q_{ji}, p_i) \in \Omega | \dot{H} = 0\}$ such that

$$\dot{H} = -p^{\mathrm{T}}\left(L_{\widehat{G}} \otimes I_n\right) p = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\widehat{N}_i}(p_j - p_i)^2 = 0. \tag{13}$$

Since $\widehat{G} \in \mathbb{C}$, Equation (13) holds only if $p_1 = p_2 = \cdots = p_N = p^*$. This means that the



(a) $G(0)$ (b) $\widehat{G}$

Figure 2 The Initial Communication topology $G(0)$ and its subgraph $\widehat{G}$

velocities of all agents asymptotically become the same. Furthermore,

$$\dot{p} = u = -\nabla V(q) - \left(L_{\widehat{G}} \otimes I_n\right) p = -\nabla V(q) = 0. \tag{14}$$

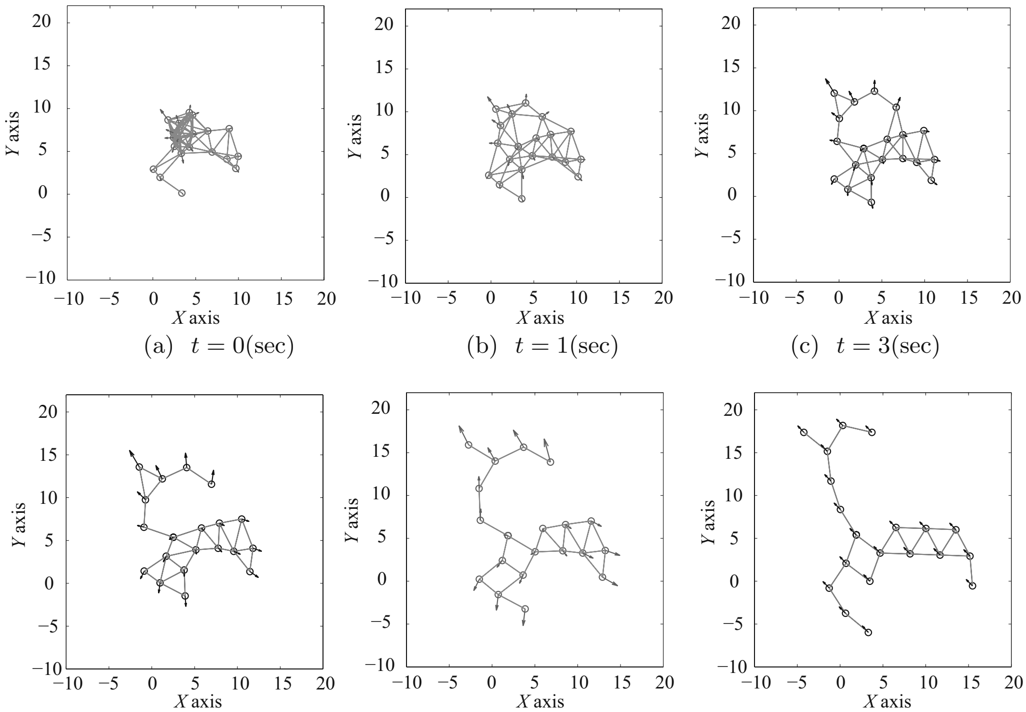
From (14), almost any solution of the system dynamics converges to a local minimum of V, which is a stable equilibrium and corresponds to an α-lattice configuration according to Lemma 7. Then all the agents asymptotically converge to the desired inter-agent distance. At the same time, collision avoidance during maneuvers is guaranteed according to Theorem 2. ▮

6 Simulation

In this section, we give a simulation to verify the above results. The initial connected group communication topology $G(0)$ is described in Fig.2 (a). By using Algorithm 1, its subgraph $\widehat{G}$ is obtained in a completely distributed style and is shown in Fig.2 (b). We illustrate the flocking motion of a multi-agent system by Fig. 3. Note that the communication topology $G(t)$ of the driven multi-agent system keeps connected for all time, which is consistent with Theorem 2. The evolution of the system in Fig.3 (a) through Fig.3 (f) shows that all agent velocities asymptotically become the same and the group converges to an α-lattice configuration. Collision avoidance among agents is also guaranteed during maneuvers. All these imply that the group forms a flock.

7 Conclusion

In this paper, flocking problem is studied with a relaxed assumption of connectivity. A distributed algorithm is at first proposed to distill a special set of edges from the initial connected topology. Preserving this set of edges can guarantee group connectivity without hindering the corresponding neighbor agents located with an expected equal distance. Two potential functions are designed to steer the neighbor agents to the desired distance while avoiding collision among them. The distributed control law is then presented to solve the flocking problem for any initial connected multi-agent system and the group convergence is analyzed. Simulation results verify the effectiveness of the proposed method.



(a) $t = 0$(sec) (b) $t = 1$(sec) (c) $t = 3$(sec)

(d) $t = 3$(sec) (e) $t = 5$(sec) (f) $t = 20$(sec)

Figure 3 Flocking of twenty agents

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