Distributed leader–follower flocking control for multi-agent dynamical systems with time-varying velocities

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ABSTRACT

Using tools from algebraic graph theory and nonsmooth analysis in combination with ideas of collective potential functions, velocity consensus and navigation feedback, a distributed leader–follower flocking algorithm for multi-agent dynamical systems with time-varying velocities is developed where each agent is governed by second-order dynamics. The distributed leader–follower algorithm considers the case in which the group has one virtual leader with time-varying velocity. For each agent i, this algorithm consists of four terms: the first term is the self nonlinear dynamics which determines the final time-varying velocity, the second term is determined by the gradient of the collective potential between agent i and all of its neighbors, the third term is the velocity consensus term, and the fourth term is the navigation feedback from the leader. To avoid an impractical assumption that the informed agents sense all the states of the leader, the new designed distributed algorithm is developed by making use of observer-based pinning navigation feedback. In this case, each informed agent only has partial information about the leader, yet the velocity of the whole group can still converge to that of the leader and the centroid of those informed agents, having the leader's position information, follows the trajectory of the leader asymptotically. Finally, simulation results are presented to demonstrate the validity and effectiveness of the theoretical analysis. Surprisingly, it is found that the local minimum of the potential function may not form a commonly believed α lattice.

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1. Introduction

Flocking behavior in groups of autonomous mobile agents has attracted increasing attention in recent years due to extensive studies and wide interests in biological systems and its applications to sensor networks, UAV (Unmanned Air Vehicles) formations, robotic teams and so on. The focus of the study of flocking behavior is to analyze how coordinated grouping behavior emerges as a result of local interactions among mobile individuals. This line of research is partly motivated by scenarios in practical settings where a group of agents only share information locally and at the same time try to agree on certain global criteria of interest, e.g. the value of some measurement in a sensor network, the heading of a UAV formation, or the target position of a robotic team. As validated by simulations and experiments, flocking can be achieved in multi-agent dynamical systems by various distributed algorithms.

Understanding the basic principles of flocking algorithms can help engineers in better implementing distributed cooperative control strategies in networked multi-agent systems.

Recently, substantial effort has been made in studying collective behaviors of multi-agent dynamical systems, such as consensus [1–12], synchronization [13–15], swarming and flocking [16–26]. The consensus problem usually refers to as the problem of how to reach an agreement among a group of autonomous agents in a dynamically changing environment. Based on the algebraic graph theory [27,28], it has been shown that network connectivity is the key in reaching consensus [1,2,4,5,7,8]. It has also been shown that the consensus in a network with dynamically changing topologies can be reached if the union of the time-varying network topology contains a spanning tree frequently enough as the networked system evolves in time.

In [29], Vicsek et al. proposed a simple discrete-time model to study groups of self-propelled particles. Vicsek's model and its variations can be considered as a special class of the consensus problem, which turn out to be a simplified version of the flocking model introduced earlier by Reynolds [30]. Three heuristic rules were considered by Reynolds to animate flocking behavior: (1) velocity consensus, (2) center cohesion, and (3) collision

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avoidance. Using a simplified Reynolds' model, stability properties for swarm aggregation were discussed, and an asymptotic bound for the spatial size of the swarm was computed in [16,17]. In order to embody the three Reynolds' rules, Tanner et al. designed flocking algorithms in [24,25], where a collective potential function and a velocity consensus term were introduced. Later, in [20], Olfati-Saber proposed a general framework to investigate distributed flocking algorithms where, in particular, three algorithms were discussed for free and constrained flocking.

In [24,25], it was pointed out that due to the time-varying network topology, the set of differential equations describing the flocking behavior of a multi-agent dynamical system is in general nonsmooth; therefore, several techniques from nonsmooth analysis, such as Filippov solutions [31], generalized gradient [32], and differential inclusion [33], should be used. The work in this paper is motivated by the distributed flocking algorithms discussed in [20]. It is found that the asymptotic configuration of the network structure is not the so-called α-lattice [20], however, opposing the claim therein. Adopting nonsmooth analysis, some technical problems in [24,25] will also be pointed out in Section 2 of this paper. Some new flocking algorithms will be proposed by utilizing results regarding pinning control in the study of synchronization in complex systems.

The rest of the paper is organized as follows. In Section 2, some preliminaries about graph theory, model formulation, collective potential function, velocity consensus, navigation feedback, and nonsmooth analysis are given. A distributed flocking algorithm for multi-agent dynamical systems with pinning observer-based navigation feedback is proposed in Section 3. In Section 4, numerical examples are simulated to verify the theoretical analysis. Conclusions are finally drawn in Section 5.

2. Preliminaries

To keep this paper self-sustained, some basic concepts and results in graph theory, potential function, consensus, navigation feedback control, and nonsmooth analysis, are briefly introduced.

2.1. Graph theory

Let $G = (\mathcal{V}, \mathcal{E}, A)$ be an unweighted undirected graph of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, the set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an unweighted adjacency matrix $A = (a_{ij})_{N \times N}$. An edge in graph G is denoted by $e_{ij} = (v_i, v_j)$. If there is an edge between node v_i and node v_j, then $a_{ij} = a_{ji} > 0$; otherwise, $a_{ij} = a_{ji} = 0$. As usual, assume there are no self-loops in G.

The Laplacian matrix $L = (L_{ij})_{N \times N}$ of graph G is defined as $L_{ij} = -a_{ij}$ for $i \neq j$, $i, j \in \{1, \ldots, N\}$ and $L_{ii} = k_i$ for $i \in \{1, \ldots, N\}$, where $k_i = -\sum_{j=1, j \neq i}^{N} L_{ij}$ is the degree of node v_i. Clearly, $\sum_{j=1}^{N} L_{ij} = 0$ for all $i = 1, 2, \ldots, N$. The Laplacian matrix $L = (l_{ij})_{N \times N}$ of this graph G has the following properties:

Lemma 1 ([34]). *Assume that graph G is connected. Then, the Laplacian matrix L of G has eigenvalue 0 with algebraic multiplicity one, and the real parts of all the other eigenvalues are positive, i.e., the eigenvalues of L satisfy $0 = \lambda_1(G) < \lambda_2(G) \leq \cdots \leq \lambda_N(G)$.*

Lemma 2 ([13]). *If the network is connected, $L_{ij} = L_{ji} \leq 0$ for $i \neq j$, and $\sum_{j=1}^{N} L_{ij} = 0$, for all $i = 1, 2, \ldots, N$, then all eigenvalues of the matrix*

$$\begin{pmatrix} L_{11} + \varepsilon & L_{12} & \ldots & L_{1N} \\ L_{21} & L_{22} & \ldots & L_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \ldots & L_{NN} \end{pmatrix}$$

are positive for any positive constant ε.

2.2. Model formulation

Suppose that there is a leader, which contains the motion information that the whole group of N agents need to follow. The leader moves according to the following dynamical model:

$$\begin{aligned} \dot{r}_o(t) &= v_o, \\ \dot{v}_o(t) &= f(r_o, v_o), \end{aligned} \tag{1}$$

where $r_o \in R^n$ and $v_o \in R^n$ are its position vector and velocity vector, respectively, and $f(r_o, v_o)$ is the input vector which governs the dynamics of the leader.

Consider the dynamics of a multi-agent system consisting of the followers:

$$\begin{aligned} \dot{r}_i(t) &= v_i, \\ \dot{v}_i(t) &= u_i, \end{aligned} \tag{2}$$

where $r_i \in R^n$ is the position vector, $v_i \in R^n$ is the corresponding velocity vector, and $u_i \in R^n$ is the control input, for $i = 1, 2, \ldots, N$. Let $r = (r_1^T, r_2^T, \ldots, r_N^T)^T$, $v = (v_1^T, v_2^T, \ldots, v_N^T)^T$, and $u = (u_1^T, u_2^T, \ldots, u_N^T)^T$.

In distributed flocking algorithms, the following control input is of particular interest [20–22,24–26]:

$$u_i(t) = f(r_i, v_i) - \nabla_{r_i} V(r) + c \sum_{j \in \mathcal{N}_i} a_{ij}(\|r_j - r_i\|)(v_j - v_i) + \widetilde{f}_i, \tag{3}$$

where the first term is a nonlinear dynamical term, the second term is a gradient-based term, V is the collective potential function to be defined, the third term is the velocity consensus term, $\widetilde{f}_i$ is the navigation feedback based on information about the leader, $\mathcal{N}_i = \{j \in \mathcal{V} : \|r_j - r_i\| \leq r_s, j \neq i\}$ denotes the neighbors of agent i, r_s is the interaction range, and c is the coupling strength of velocity. Here, if $j \in \mathcal{N}_i$, then $a_{ij}(\|r_j - r_i\|) > 0$; otherwise, $a_{ij}(\|r_j - r_i\|) = 0$.

The collective potential function $V(r)$ is a nonnegative function with additional properties that are related to the overall geometric shape and graphical connectivity of system (2). In [20], a smooth collective potential function is given. It was pointed out [20] that the local minimum of $V(r)$ is an α-lattice and vice versa, which is responsible for collision avoidance and cohesion in the group. However, we will show that an α-lattice is a local minimum of $V(r)$ but the local minimum is not necessarily an α-lattice. In this paper, the following potential function is considered.

Definition 1. The potential function is defined to be

$$V(r) = \frac{1}{2} \sum_i \sum_{j \neq i} \psi(\|r_{ij}\|), \tag{4}$$

where the function ψ is a continuously differentiable nonnegative function of the distance $\|r_{ij}\|$ between nodes i and j except at $\|r_{ij}\| = r_s$, such that

1. $\psi(\|r_{ij}\|)$ reaches its maximum at $\|r_{ij}\| = 0$ and attains its unique minimum at d;
2. ψ is nonincreasing when $\|r_{ij}\| \in [0, d]$, nondecreasing when $\|r_{ij}\| \in [d, r_s]$, and constant when $\|r_{ij}\| \in [r_s, \infty)$;
3. ψ is increasing and continuous at $\|r_{ij}\| = r_s$.

In this paper, the following gradient-based term is considered:

$$-\nabla_{r_i} V(r) = \sum_{j \in \mathcal{N}_i} \phi(\|r_j - r_i\|) \frac{r_j - r_i}{\|r_j - r_i\|}, \tag{5}$$

where

$$\phi(\|r_j - r_i\|) = \begin{cases} \psi'((\|r_j - r_i\|)) & \|r_j - r_i\| < r_s, \\ \lim_{\|r_j - r_i\| \to r_s^-} \psi'((\|r_j - r_i\|)) & \|r_j - r_i\| = r_s, \\ 0 & \|r_j - r_i\| > r_s. \end{cases}$$

Let $l_{ij}(\|r_j - r_i\|) = -a_{ij}(\|r_j - r_i\|)$, for $i \neq j$, $l_{ii}(r) = \sum_{j=1, j\neq i} a_{ij}(\|r_j - r_i\|)$, $A \otimes B$ denote the Kroneker product [35] of matrices A and B, I_n be the n-dimensional identity matrix, $\mathbf{1}_N$ be the N-dimensional column vector with all entries being 1, and $\bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$ and $\bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j$ be the center position and the average velocity of the group, respectively.

In many cases, it is literally impossible for all agents to obtain the information of the leader. To reduce the number of informed agents, some local feedback injections are applied to only a small percentage of agents, which is known as pinning feedback [13,36]. Here, the pinning strategy is applied to a small fraction δ ($0 < \delta < 1$) of the agents. Without loss of generality, assume the first $l = \lfloor \delta N \rfloor$ agents can sense the leader and get its information. Moreover, in the real situations, it is impractical to assume that an informed agent can observe all the state components of the leader. Thus, assume that the first $l = \lfloor \delta N \rfloor$ agents can only measure partial state components of the leader by

$$
\begin{aligned}
x_i(t) &= H_i r_o(t), \quad i = 1, 2, \ldots, l, \\
\widetilde{x}_i(t) &= \widetilde{H}_i v_o(t), \quad i = 1, 2, \ldots, l,
\end{aligned}
\tag{6}
$$

where $x_i(t) \in R^m$ and $\widetilde{x}_i(t) \in R^m$ are the measurements of agent i by observing the leader to get the information about $r_o(t)$ and $v_o(t)$ with $H_i \in R^{m\times n}$ and $\widetilde{H}_i \in R^{m\times n}$ for $i = 1, 2, \ldots, l$. The designed navigation feedback is described by

$$
\begin{aligned}
\widetilde{f}_i &= -D_i(H_i r_i - x_i) - \widetilde{D}_i(\widetilde{H}_i v_i - \widetilde{x}_i), \quad i = 1, 2, \ldots, l, \\
\widetilde{f}_i &= 0, \quad\quad\quad\quad\quad\quad\quad\quad i = l+1, l+2, \ldots, N,
\end{aligned}
\tag{7}
$$

where $D_i \in R^{n\times m}$ and $\widetilde{D}_i \in R^{n\times m}$ are the feedback control gain matrices.

In order to derive the main results, the following assumption is needed.

Assumption 1. For all $x, y, r, s \in R^n$, there exist constants θ_1 and θ_2 such that

$$
(y - s)^T (f(x, y) - f(r, s)) \leq \theta_1 (y - s)^T (x - r) + \theta_2 \|y - s\|^2.
$$

2.3. Nonsmooth analysis

Due to the gradient-based term and the velocity consensus term in (3), the right-hand side of system (2) is discontinuous at r_s, so one cannot proceed by using classical methods (continuously differential solutions). In addition, the collective potential function in (4) is not differentiable. Therefore, nonsmooth analysis and differential inclusion [37,32,31,38] will be applied. For more detail, one is referred to a recent tutorial article [39] and the references therein.

Consider the following differential equation, in which the right-hand side may be discontinuous:

$$
\dot{x} = F(x),
\tag{8}
$$

where $F : R^m \to R^m$ is measurable. For a discontinuous vector field $F(x)$, the existence of a continuously differentiable solution is not guaranteed. Therefore, a generalized solution, i.e., the Filippov solution is introduced.

Definition 2 ([37]). Suppose $E \subset R^m$. A map $x \to F(x)$ is called a set-valued map from $E \to R^m$, if each point x in a set $E \subset R^m$ corresponds to a non-empty set $F(x) \subset R^m$.

In comparison, a standard map takes a point in its domain to a point in its image, but set-valued map takes a point to a set of points instead.

Definition 3 ([31–40], Filippov Solution). A vector function $x(\cdot)$ is called a Filippov solution of system (8) on $[t_0, t_1]$, if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and satisfies the following differential inclusion:

$$
\dot{x}(t) \in \mathcal{F}(F(x)), \quad \text{a.e. } t \in [t_0, t_1].
$$

$\mathcal{F}(F(x))$ is a set-valued map defined by

$$
\mathcal{F}(F(x)) \equiv \bigcap_{\delta > 0} \bigcap_{\mu(\bar{N})=0} \overline{co}[\varphi(B(x, \delta) - \bar{N})],
$$

where $\bar{co}(E)$ is the closure of the convex hull of set E, $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(\bar{N})$ is the Lebesgue measure of set $\bar{N}$. An equivalent definition is given by [33]: there exists N_f satisfying $\mu N_f = 0$, such that $\forall N_g \subset R^m$ with $\mu(N_g) = 0$,

$$
\mathcal{F}(F(x)) \equiv \overline{co}\{\lim F(x_i)|x_i \to x, \ x \notin N_f \cup N_g\}.
$$

The concept of Filippov solution is very important in engineering applications. At a given state, instead of using the value of the vector field at x, the idea underlying Filippov solution is to use a set of directions that are computed by the values of the vector field around x. All sets of measure zero are disregarded, which allows solutions to be defined at points where the vector field $\psi(x)$ may be discontinuous. In addition, an arbitrary set of measure zero in $B(x, \delta)$ is excluded when evaluating x such that the result is the same for any two vector fields that differ on a set of measure zero. The existence of a Filippov solution is established in [41,39].

Proposition 1 ([41,39]). Let $F : R^m \to R^m$ be measurable and locally essentially bounded, namely, bounded on a bounded neighborhood of every point, excluding sets of zero measure. Then, for any $x_0 \in R^m$, there exists a Filippov solution of (8) satisfying initial condition $x(0) = x_0$.

By Definition 3 and the calculus for computing Filippov's differential inclusion [33], the concept of Filippov solution in velocity consensus term is extended to the following:

$$
c\sum_{j\in\mathcal{N}_i} a_{ij}(\|r_j - r_i\|)(v_j - v_i) \in c\sum_{j\in\mathcal{N}_i} \mathcal{F}(a_{ij}(\|r_{ij}(t)\|))(v_j - v_i),
$$

$$
\text{a.e. } t,
\tag{9}
$$

where $\mathcal{F}(a_{ij}(\|r_{ij}(t)\|)) = \begin{cases} 1 & \|r_{ij}(t)\| < r_s \\ 0 & \|r_{ij}(t)\| > r_s \\ [0, 1] & \|r_i(t) - r_j(t)\| = r_s \end{cases}$ for $i \neq j$.

Since the collective potential function in (4) is not differentiable, a generalized derivative is defined as follows.

Definition 4 (Generalized Directional Derivative [32]). Let g be Lipschitz near a given point $x \in Y$ and let w be any vector in X. The generalized directional derivative of g at x in the direction w, denoted by $g^0(x; w)$, is defined as

$$
g^0(x; w) \equiv \lim_{y\to x, t\downarrow 0} \sup \frac{g(y + tw) - g(y)}{t}.
$$

Note that this definition does not presuppose the existence of any limit, and it reduces to the classical directional derivative in the case where the function is differentiable.

Definition 5 (Generalized Gradient [32]). The generalized gradient of g at x, denoted by $\partial g(x)$, is the subset of X^* given by

$$
\partial g(x) = \{\zeta \in X^* | g^0(x; w) \geq <\zeta, w>, \ \forall w \in X\}.
$$

The generalized gradient has the following basic properties:

Lemma 3 ([32]). *Let g be Lipschitz near x. Then, for every w in X, one has*

$$g^0(x; w) = \max\{< \zeta, w > | \zeta \in \partial g(x)\}.$$

An additional condition to turn the inclusions to equalities is given below.

Definition 6 ([32]). A function g is said to be regular at x, if

 (i) for all w, the classical one-sided directional derivative $g'(x; w)$ exists;
(ii) for all w, $g'(x; w) = g^0(x; w)$.

Lemma 4 ([32], *Calculus for Generalized Gradients of Regular Functions*).

 (i) *Let g_i $(i = 1, 2, \ldots, m)$ be a finite family of regular functions, each of which is Lipschitz near x, and assume that for any nonnegative scalars s_i,*

$$\partial \left(\sum_{i=1}^{m} s_i g_i \right)(x) = \sum_{i=1}^{m} s_i \partial g_i(x).$$

 (ii) *If g is strictly differentiable at x, then g is regular at x.*
(iii) *If g_i $(i = 1, 2, \ldots, m)$ is a finite family of regular functions, each of which is regular at x, then for any nonnegative scalars s_i, $\sum_{i=1}^{m} s_i g_i(x)$ is regular.*

Next, let

$$V(r) = \frac{1}{2} \sum_{i} \sum_{j \neq i} V_{ij}(\|r_{ij}\|), \tag{10}$$

where $V_{ij}(\|r_{ij}\|) = \psi(\|r_{ij}\|)$.

Lemma 5. *The function $-V_{ij}$ is regular everywhere in its domain.*

Proof. From the definition of V in Definition 1, it is clear that $-V_{ij}$ is strictly differentiable at $\|r_{ij}\| \neq r_s$. By Lemma 4, the function $-V_{ij}$ is regular at $\|r_{ij}\| \neq r_s$. In order to prove the regularity of $-V_{ij}$ in its domain, one only needs to show that $-V_{ij}$ is regular at r_s. In view of Definition 6, it is needed to establish the equality between the generalized directional derivative and the classical one-sided directional derivative of $-V_{ij}$ at r_s for any direction w.

The classical directional derivative of $-V_{ij}$ at r_s is given by

$$-V'_{ij}(r_s; w) = \lim_{t \downarrow 0} \frac{-V_{ij}(r_s + tw) - (-V_{ij}(r_s))}{t}.$$

If $w \geq 0$, then

$$-V'_{ij}(r_s; w) = \lim_{t \downarrow 0} \frac{V_{ij}(r_s) - V_{ij}(r_s + tw)}{t} = 0.$$

If $w < 0$, then

$$-V'_{ij}(r_s; w) = \lim_{t \downarrow 0} \frac{V_{ij}(r_s) - V_{ij}(r_s + tw)}{t} \equiv \kappa > 0.$$

The above inequalities are obtained based on the property of V in Definition 1. For the generalized directional derivative, one has the same two cases as follows.
If $w \geq 0$, then

$$-V_{ij}^0(r_s; w) = \lim_{y \to r_s, t \downarrow 0} \sup \frac{V_{ij}(y) - V_{ij}(y + tw)}{t} = 0.$$

If $w < 0$, then

$$-V_{ij}^0(r_s; w) = \lim_{y \to r_s, t \downarrow 0} \sup \frac{V_{ij}(y) - V_{ij}(y + tw)}{t} = \kappa > 0.$$

Therefore, for any direction w, one has $-V'_{ij}(r_s; w) = -V_{ij}^0(r_s; w)$, so $-V_{ij}$ is regular at r_s. The proof is completed. □

Remark 1. In [24,25], it was claimed that V_{ij} is regular. However, it is shown above that $-V_{ij}$ should be regular instead.

To proceed further, a modified chain rule is first described in the following.

Lemma 6. *Let $x(\cdot)$ be a Filippov solution of $\dot{x} = F(x)$ on an interval containing t, and let $-V : R^m \to R$ be a Lipschitz and regular function. Then, $V(x(t))$ is absolutely continuous and, in addition, $\frac{d}{dt} V(x(t))$ exists almost everywhere:*

$$\frac{d}{dt} V(x(t)) \in \tilde{\tilde{V}}(x), \quad a.e.\ t, \tag{11}$$

where

$$\tilde{\tilde{V}}(x) = \bigcap_{\xi \in \partial V(x(t))} \xi^T \mathcal{F}(F(x)). \tag{12}$$

Proof. By Lemmas 4 and 5, the function $-V$ is regular everywhere in its domain. Since V is Lipschitz and $x(\cdot)$ is absolutely continuous, $V(x(t))$ is absolutely continuous [40]. From the argument of [42], it follows that $V(x(t))$ is differentiable almost everywhere. Since V is Lipschitz and at a point where $V(x(t))$ and $x(t)$ are both differentiable, one has

$$\frac{d}{dt} V(x(t)) = \lim_{h \to 0} \frac{V(x(t + h)) - V(x(t))}{h}$$
$$= \lim_{h \to 0} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}.$$

Because of the regularity of $-V$ and Lemma 5, by letting h tend to 0 from the right, one obtains

$$\frac{d}{dt} V(x(t)) = \lim_{h \to 0^+} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}$$
$$= -\lim_{h \to 0^+} \frac{(-V(x(t) + h\dot{x}(t))) - (-V(x(t)))}{h}$$
$$= -(-V'(x(t); \dot{x}(t))) = -(-V^0(x(t); \dot{x}(t)))$$
$$= -\max\{< \zeta, \dot{x}(t) > | \zeta \in \partial(-V)(x)\}$$
$$= -\max\{< -\xi, \dot{x}(t) > | \xi \in \partial V(x)\}$$
$$= \min\{< \xi, \dot{x}(t) > | \xi \in \partial V(x)\}.$$

Similarly, by letting h tend to 0 from the left, one obtains

$$\frac{d}{dt} V(x(t)) = \lim_{h \to 0^-} \frac{V(x(t) + h\dot{x}(t)) - V(x(t))}{h}$$
$$= \lim_{h1 \to 0^+} \frac{-V(x(t) - h1\dot{x}(t)) - (-V(x(t)))}{h1}$$
$$= -V'(x(t); -\dot{x}(t)) = -V^0(x(t); -\dot{x}(t))$$
$$= \max\{< \zeta, -\dot{x}(t) > | \zeta \in \partial(-V)(x)\}$$
$$= \max\{< \xi, \dot{x}(t) > | \xi \in \partial V(x)\}.$$

Thus, one has

$$\frac{d}{dt} V(x(t)) = \{< \xi, \dot{x}(t) > | \forall \xi \in \partial V(x)\}.$$

Since $x(\cdot)$ is a Filippov solution satisfying

$$\dot{x}(t) \in \mathcal{F}(F(x)), \quad a.e.\ t.$$

It follows that V is almost differentiable everywhere, and $\frac{d}{dt} V(x(t)) = \xi^T \eta$ for all $\xi \in \partial V(x(t))$ and some $\eta \in \mathcal{F}(F(x))$, equivalently,

$$\frac{d}{dt} V(x(t)) \in \bigcap_{\xi \in \partial V(x(t))} \xi^T \mathcal{F}(F(x)), \quad a.e.\ t.$$

This completes the proof. □

Remark 2. In [40], only if V is regular, the chain rule is obtained. In this paper, however, based on the regularity of $-V$, the modified chain rule can still be derived as shown above, and moreover, the Lyapunov and LaSalle theorems can also be proved by the similar approach in [40].

3. Distributed leader–follower control with pinning observer-based navigation feedback

In this section, the leader–follower control protocol of the multi-agent system (2) with pinning observer-based navigation feedback (7) is studied.

Let $\widehat{r}_i = r_i - r_o$ and $\widehat{v}_i = v_i - v_o$ represent the relative position and velocity vectors to the leader, $\widehat{r}_{ij} = \widehat{r}_i - \widehat{r}_j = r_{ij}$, $\widehat{v}_{ij} = \widehat{v}_i - \widehat{v}_j = v_{ij}$, $\widehat{r} = (\widehat{r}_1^T, \widehat{r}_2^T, \ldots, \widehat{r}_N^T)^T$, and $\widehat{v} = (\widehat{v}_1^T, \widehat{v}_2^T, \ldots, \widehat{v}_N^T)^T$. By Definition 3 and the calculation of Filippov's differential inclusion [33], the Filippov solution is extended to the following:

$$\dot{\widehat{r}}_i(t) = \widehat{v}_i,$$

$$
\begin{aligned}
\dot{\widehat{v}}_i(t) \in{} & f(r_i, v_i) - f(r_o, v_o) - \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \\
& + \sum_{j \in \mathcal{N}_i} \mathcal{F}(\phi(\|\widehat{r}_{ji}\|)) \frac{\widehat{r}_{ji}}{\|\widehat{r}_{ji}\|} + c \sum_{j \in \mathcal{N}_i} \mathcal{F}(a_{ij}(\|\widehat{r}_{ji}\|))(\widehat{v}_{ji}), \\
& \text{a.e. } t,
\end{aligned}
\tag{13}
$$

or equivalently,

$$
\begin{aligned}
\dot{\widehat{v}}_i(t) ={} & f(r_i, v_i) - f(r_o, v_o) - \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \\
& + \sum_{j \in \mathcal{N}_i} \widehat{\phi}_{ij} \frac{\widehat{r}_{ji}}{\|\widehat{r}_{ji}\|} + c \sum_{j \in \mathcal{N}_i} \widehat{a}_{ij} \widehat{v}_{ji}, \quad \text{a.e. } t,
\end{aligned}
\tag{14}
$$

where $\widehat{D}_i = D_i$ and $\widehat{\widetilde{D}}_i = \widetilde{D}_i$ for $i = 1, 2, \ldots, l$, $\widehat{D}_i = \widehat{\widetilde{D}}_i = 0$ for $i = l+1, l+2, \ldots, N$,

$$
\widehat{\phi}_{ij} =
\begin{cases}
\phi(\|\widehat{r}_{ij}(t)\|) & \|\widehat{r}_{ij}(t)\| \neq r_s \\
\in [0, \phi(\|r_s^-\|)] & \|\widehat{r}_{ij}(t)\| = r_s
\end{cases}
\quad \text{and}
$$

$$
\widehat{a}_{ij} =
\begin{cases}
1 & \|\widehat{r}_{ij}(t)\| < r_s \\
0 & \|\widehat{r}_{ij}(t)\| > r_s \quad \text{for } i \neq j. \\
\in [0, 1] & \|\widehat{r}_{ij}(t)\| = r_s
\end{cases}
$$

Let $(B)^\eta = \frac{B + B^T}{2}$.

Definition 7. The position state component j $(1 \leq j \leq n)$ of the leader is said to be observable by an agent i $(1 \leq i \leq l)$, if there are a gain matrix D_i and a positive constant ε_j, such that

$$-s(D_i H_i)^\eta \widetilde{s} \leq -\varepsilon_j s_j \widetilde{s}_j,$$

for all $s = (s_1, s_2, \ldots, s_n)^T \in R^n$ and $\widetilde{s} = (\widetilde{s}_1, \widetilde{s}_2, \ldots, \widetilde{s}_n)^T \in R^n$.

Definition 8. The velocity state component j $(1 \leq j \leq n)$ of the leader is said to be observable by an agent i $(1 \leq i \leq l)$, if there are a gain matrix $\widetilde{D}_i$ and a positive constant $\widetilde{\varepsilon}_j$, such that

$$-\widetilde{s}^T (\widetilde{D}_i \widetilde{H}_i)^\eta \widetilde{s} \leq -\widetilde{\varepsilon}_j \widetilde{s}_j^2,$$

for all $\widetilde{s} = (\widetilde{s}_1, \widetilde{s}_2, \ldots, \widetilde{s}_n)^T \in R^n$.

Definition 9. The position (velocity) state component j $(1 \leq j \leq n)$ of the leader is said to be observable by a group of N agents if each of its position (velocity) state component is observable by an agent k $(1 \leq k \leq l)$.

Definition 10. The position (velocity) state of the leader is said to be observable by a group of N agents, if each of its position (velocity) state component is observable by the group of N agents.

Assumption 2. Suppose that each network in $\mathcal{F}(A)$ is connected at all times.

Assumption 3. Suppose that the position and velocity of the leader are both observable by a group of N agents and, without loss of generality, suppose that this position is observable by agents $i_1, i_2, \ldots, i_p$ $(i_p \leq l)$ and this velocity is observable by agents $j_1, j_2, \ldots, j_q$ $(j_q \leq l)$. Then, there exist matrices D_{i_k} and $\widetilde{D}_{j_{\widetilde{k}}}$, $k = 1, 2, \ldots, p, \widetilde{k} = 1, 2, \ldots, q$, such that

$$-\sum_{k=1}^p \widehat{r}_{i_k}^T(t)(D_{i_k} H_{i_k})^\eta \widehat{v}_{i_k} \leq -\sum_{m=1}^n \sum_{k \in \mathcal{M}_m} \varepsilon_{mk} \widehat{r}_{i_k m} \widehat{v}_{i_k m}, \tag{15}$$

and

$$-\sum_{\widetilde{k}=1}^q \widehat{v}_{j_{\widetilde{k}}}^T(t)(\widetilde{D}_{j_{\widetilde{k}}} \widetilde{H}_{j_{\widetilde{k}}})^\eta \widehat{v}_{j_{\widetilde{k}}} \leq -\sum_{\widetilde{m}=1}^n \sum_{\widetilde{k} \in \widetilde{\mathcal{M}}_{\widetilde{m}}} \widetilde{\varepsilon}_{\widetilde{m}\widetilde{k}} \widehat{v}_{j_{\widetilde{k}}\widetilde{m}}^2, \tag{16}$$

where the position state component m is observable by agents i_k $(1 \leq k \leq p)$ and the velocity state component $\widetilde{m}$ is observable by agents $j_{\widetilde{k}}$ $(1 \leq \widetilde{k} \leq q)$, respectively. Here, $\mathcal{M}_m$ and $\widetilde{\mathcal{M}}_{\widetilde{m}}$ denote the sets of all agents that can observe the position state component m and the velocity state component $\widetilde{m}$. If any position or velocity state component of the leader cannot be observed by agent k, then one can simply let $D_k = 0$ or $\widetilde{D}_k = 0$ for $1 \leq k \leq l$.

Define the sum of the total artificial potential function and the total relative kinetic energy function as follows:

$$\overline{U}(\widehat{r}, \widehat{v}) = V(\widehat{r}) + W(\widehat{v}) + \overline{W}(\widehat{r}), \tag{17}$$

where $V(\widehat{r}) = \frac{1}{2} \sum_i \sum_{j \neq i} \psi(\|\widehat{r}_{ij}\|)$, $W(\widehat{v}) = \frac{1}{2} \widehat{v}^T \widehat{v}$, and $\overline{W}(\widehat{r}) = \frac{1}{2} \sum_{m=1}^n \sum_{k \in \mathcal{M}_m} \varepsilon_{mk} \widehat{r}_{i_k m}^2$. Let $U = V + W$.

Theorem 1. Consider a group of N agents, which satisfies Assumptions 1–3. If

$$\theta_2 I_N - \Xi_{\widetilde{m}} - c\mathcal{F}(\widehat{L}) < 0, \tag{18}$$

for all $\widehat{L} \in \mathcal{F}(L)$ and $1 \leq \widetilde{m} \leq n$, where $\Xi_{\widetilde{m}} = \mathrm{diag}(0, \ldots, 0, \underbrace{\varepsilon_{\widetilde{m}\widetilde{k}_1}}_{j_{\widetilde{k}_1}}, 0, \ldots, 0, \underbrace{\varepsilon_{\widetilde{m}\widetilde{k}_2}}_{j_{\widetilde{k}_2}}, 0, \ldots, 0, \underbrace{\varepsilon_{\widetilde{m}\widetilde{k}_r}}_{j_{\widetilde{k}_r}}, 0) \in R^{N \times N}$, $\widetilde{k}_1, \ldots, \widetilde{k}_r \in \widetilde{\mathcal{M}}_{\widetilde{m}}$, then

(i) The states of dynamical system (13) converge to an equilibrium $(r^*, 0)$, where r^* is a local minimum of $\overline{U}(\widehat{r}, 0)$ in (17).

(ii) All the agents move with the same velocity as the leader asymptotically.

(iii) If the initial artificial potential energy is less than $(k+1)\psi(0)$ for some $k \geq 0$, then at most k agents may collide.

(iv) If f is independent of the position term,

$$\sum_{k=1}^p D_{i_k} H_{i_k} \widehat{r}_{i_k} \to 0, \quad t \to \infty. \tag{19}$$

In addition, if for any $k \in \mathcal{M}_m$, $D_{i_k m} H_{i_k m}$ are all equal, then

$$\frac{1}{\mu(m)} \sum_{k \in \mathcal{M}_m} r_{i_k m}(t) \to r_{om}(t), \quad t \to \infty,$$

where $\mu(m)$ is the number of agents in $\mathcal{M}_m$.

Proof. Since $-W$ and $-V$ are regular functions in Lemma 5, one knows that $-(W+V)$ is regular. Noting the following fact:

$$\frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \widehat{r}_{ij}} = \frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \widehat{r}_i} = -\frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \widehat{r}_j}, \tag{20}$$

and using Chain Rule II in [32], one has

$$
\frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \widehat{r}_i} \subset \frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \|\widehat{r}_{ij}\|} \frac{\partial \|\widehat{r}_{ij}\|}{\partial \widehat{r}_i}
$$

$$
= \begin{cases} \phi(\|\widehat{r}_{ij}(t)\|) \dfrac{\widehat{r}_{ij}}{\|\widehat{r}_{ij}\|} & \|\widehat{r}_{ij}(t)\| \neq r_s, \\ \left\{ \chi \dfrac{\widehat{r}_{ij}}{\|\widehat{r}_{ij}\|} : \chi \in [0, \phi(\|r_s^-\|)] \right\} & \|\widehat{r}_{ij}(t)\| = r_s. \end{cases} \tag{21}
$$

Then

$$
\frac{\partial U(\widehat{r}, \widehat{v})}{\partial \widehat{r}_i} = \sum_{j \in \mathcal{N}_i} \frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \widehat{r}_i}, \tag{22}
$$

and

$$
\frac{\partial U(\widehat{r}, \widehat{v})}{\partial \widehat{v}_i} = \widehat{v}_i. \tag{23}
$$

It follows that

$$
\partial U(\widehat{r}, \widehat{v}) = \left(\left(\frac{\partial U(\widehat{r}, \widehat{v})}{\partial \widehat{r}_1} \right)^T, \dots, \right.
$$

$$
\left. \left(\frac{\partial U(\widehat{r}, \widehat{v})}{\partial \widehat{r}_N} \right)^T, \widehat{v}_1^T, \dots, \widehat{v}_N^T \right)^T. \tag{24}
$$

Let

$$
\mathcal{F}(\dot{\widehat{v}}_i(t)) \equiv f(r_i, v_i) - f(r_o, v_o) - \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i
$$

$$
+ \sum_{j \in \mathcal{N}_i} \mathcal{F}(\phi(\|\widehat{r}_{ji}\|)) \frac{\widehat{r}_{ji}}{\|\widehat{r}_{ji}\|}
$$

$$
+ c \sum_{j \in \mathcal{N}_i} \mathcal{F}(a_{ij}(\|\widehat{r}_{ji}\|))(\widehat{v}_j - \widehat{v}_i). \tag{25}
$$

By Lemma 6, one obtains

$$
\frac{\mathrm{d}}{\mathrm{d}t} U(\widehat{r}, \widehat{v}) \in \bigcap_{\xi \in \partial U(\widehat{r}, \widehat{v})} \xi^T \left(\widehat{v}_1^T, \dots, \widehat{v}_N^T, \mathcal{F}^T(\dot{\widehat{v}}_1(t)), \dots, \right.
$$

$$
\left. \mathcal{F}^T(\dot{\widehat{v}}_N(t)) \right)^T, \quad \text{a.e. } t.
$$

Let $\Xi(\widehat{r}, \widehat{v}, \xi) = \xi^T \left(\widehat{v}_1^T, \dots, \widehat{v}_N^T, \mathcal{F}^T(\dot{\widehat{v}}_1(t)), \dots, \mathcal{F}^T(\dot{\widehat{v}}_N(t)) \right)^T$.
If $\|\widehat{r}_{ij}\| < r_s$, then by (20)–(25), one has

$$
\Xi(\widehat{r}, \widehat{v}, \xi) = \sum_{i=1}^N \left(\frac{\partial U(\widehat{r}, \widehat{v})}{\partial \widehat{r}_i} \right)^T \widehat{v}_i + \sum_{i=1}^N \mathcal{F}^T(\dot{\widehat{v}}_i(t)) \widehat{v}_i
$$

$$
= \sum_{i=1}^N \left(\sum_{j \in \mathcal{N}_i} \frac{\partial \psi(\|\widehat{r}_{ij}\|)}{\partial \widehat{r}_i} \right)^T \widehat{v}_i + \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \phi(\|\widehat{r}_{ij}\|) \frac{\widehat{r}_{ji}^T}{\|\widehat{r}_{ji}\|} \widehat{v}_i
$$

$$
+ c \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} a_{ij}(\widehat{v}_j - \widehat{v}_i)^T \widehat{v}_i + \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o)
$$

$$
- \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i
$$

$$
= \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \phi(\|\widehat{r}_{ji}\|) \frac{\widehat{r}_{ij}^T + \widehat{r}_{ji}^T}{\|\widehat{r}_{ji}\|} \widehat{v}_i + c \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} a_{ij}(\widehat{v}_j - \widehat{v}_i)^T \widehat{v}_i
$$

$$
= -c \widehat{v}^T (L \otimes I_n) \widehat{v} + \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o)
$$

$$
- \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i. \tag{26}
$$

If $\|\widehat{r}_{ij}\| = r_s$, then

$$
\frac{\mathrm{d}}{\mathrm{d}t} U(\widehat{r}, \widehat{v}) \in \bigcap_{\xi_i \in \partial_{\widehat{r}_i} U(\widehat{r}, \widehat{v})} \left(\sum_{i=1}^N \xi_i^T \widehat{v}_i + \sum_{i=1}^N \mathcal{F}^T(\dot{\widehat{v}}_i(t)) \widehat{v}_i \right)
$$

$$
= \bigcap_{\xi_{ij} \in [0, \phi(\|r_s^-\|)]} \left(\sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \xi_{ij} \frac{\widehat{r}_{ij}^T}{\|\widehat{r}_{ij}\|} \widehat{v}_i \right.
$$

$$
+ \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} [0, \phi(\|r_s^-\|)] \frac{\widehat{r}_{ji}^T}{\|\widehat{r}_{ji}\|} \widehat{v}_i
$$

$$
+ c \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \mathcal{F}(a_{ij})(\widehat{v}_j - \widehat{v}_i)^T \widehat{v}_i \right)
$$

$$
+ \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o) - \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i
$$

$$
= \bigcap_{\xi_{ij} \in [0, \phi(\|r_s^-\|)]} \left(\sum_{i=1}^N \sum_{j \in \mathcal{N}_i} [-\phi(\|r_s^-\|) + \xi_{ij}, \xi_{ij}] \frac{\widehat{r}_{ij}^T}{\|\widehat{r}_{ij}\|} \widehat{v}_i \right.
$$

$$
+ c \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \mathcal{F}(a_{ij})(\widehat{v}_j - \widehat{v}_i)^T \widehat{v}_i \right)
$$

$$
+ \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o)
$$

$$
- \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i. \tag{27}
$$

Since

$$
\bigcap_{\xi_{ij} \in [0, \phi(\|r_s^-\|)]} [-\phi(\|r_s^-\|) + \xi_{ij}, \xi_{ij}] = \{0\}, \tag{28}
$$

it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t} U(\widehat{r}, \widehat{v}) = c \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \widehat{a}_{ij}(\widehat{v}_j - \widehat{v}_i)^T \widehat{v}_i
$$

$$
+ \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o) - \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i
$$

$$
= -c \widehat{v}^T (\widehat{L} \otimes I_n) \widehat{v} + \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o)
$$

$$
- \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i, \quad \text{a.e. } t \tag{29}
$$

where $\widehat{L}_{ij} = -\widehat{a}_{ij}$ for $i \neq j$, and $\widehat{L}_{ii} = \sum_{j \in \mathcal{N}_i} \widehat{a}_{ij}$.
If $\|\widehat{r}_{ij}\| > r_s$, similarly, by (26) one has

$$
\Xi(\widehat{r}, \widehat{v}, \xi) = -c \widehat{v}^T (L \otimes I_n) \widehat{v} + \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o)
$$

$$
- \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i. \tag{30}
$$

Therefore, from (26), (29) and (30), one obtains

$$
\frac{\mathrm{d}}{\mathrm{d}t} U(\widehat{r}, \widehat{v}) = -c \widehat{v}^T (\widehat{L} \otimes I_n) \widehat{v} + \sum_{i=1}^N \Big(f(r_i, v_i) - f(r_o, v_o)
$$

$$
- \widehat{D}_i H_i \widehat{r}_i - \widehat{\widetilde{D}}_i \widetilde{H}_i \widehat{v}_i \Big)^T \widehat{v}_i
$$

$$\in -c\widehat{v}^T(\mathcal{F}(L) \otimes I_n)\widehat{v} + \sum_{i=1}^{N}\Big(f(r_i, v_i) - f(r_o, v_o)$$

$$-\widehat{D_iH_i}\widehat{r}_i - \widehat{\widetilde{D}_i\widetilde{H}_i}\widehat{v}_i\Big)^T\widehat{v}_i, \quad \text{a.e. } t. \tag{31}$$

By Assumptions 1–3, one obtains

$$\frac{d}{dt}\overline{U}(\widehat{r}, \widehat{v}) = \sum_{i=1}^{N}\widehat{v}_i^T[f(r_i, v_i) - f(r_o, v_o)]$$

$$+ \sum_{i=1}^{l}\widehat{v}_i^T[-(D_iH_i)^{\eta}\widehat{r}_i - (\widetilde{D}_i\widetilde{H}_i)^{\eta}\widehat{v}_i]$$

$$- c\widehat{v}^T(\widehat{L} \otimes I_n)\widehat{v} + \sum_{m=1}^{n}\sum_{k\in\mathcal{M}_m}\varepsilon_{mk}\widehat{r}_{i_km}\widehat{v}_{i_km}$$

$$\leq \widehat{v}^T[(\theta_2 I_N - c\widehat{L}) \otimes I_n]\widehat{v} - \sum_{k=1}^{p}\widehat{r}_{i_k}^T(t)(D_{i_k}H_{i_k})^{\eta}\widehat{v}_{i_k}$$

$$- \sum_{\widetilde{k}=1}^{q}\widehat{v}_{\widetilde{j}_{\widetilde{k}}}^T(t)(\widetilde{D}_{\widetilde{j}_{\widetilde{k}}}\widetilde{H}_{\widetilde{j}_{\widetilde{k}}})^{\eta}\widehat{v}_{\widetilde{j}_{\widetilde{k}}} + \sum_{m=1}^{n}\sum_{k\in\mathcal{M}_m}\varepsilon_{mk}\widehat{r}_{i_km}\widehat{v}_{i_km}$$

$$\leq \widehat{v}^T[(\theta_2 I_N - c\widehat{L}) \otimes I_n]\widehat{v}$$

$$- \sum_{\widetilde{m}=1}^{n}\sum_{\widetilde{k}\in\widetilde{\mathcal{M}}_{\widetilde{m}}}\widetilde{\varepsilon}_{\widetilde{m}k}\widehat{v}_{\widetilde{j}_{\widetilde{k}}\widetilde{m}}^2, \quad \text{a.e. } t. \tag{32}$$

Let $\acute{v}_{\widetilde{m}} = (\widehat{v}_{1\widetilde{m}}, \widehat{v}_{2\widetilde{m}}, \ldots, \widehat{v}_{N\widetilde{m}})$. Then, one has

$$\frac{d}{dt}\overline{U}(\widehat{r}, \widehat{v}) \leq \sum_{\widetilde{m}=1}^{n}\acute{v}_{\widetilde{m}}^T(\theta_2 I_N - c\widehat{L})\acute{v}_{\widetilde{m}} - \sum_{\widetilde{m}=1}^{n}\acute{v}_{\widetilde{m}}^T\Xi_{\widetilde{m}}\acute{v}_{\widetilde{m}}$$

$$\in \sum_{\widetilde{m}=1}^{n}\acute{v}_{\widetilde{m}}^T(\theta_2 I_N - \Xi_{\widetilde{m}} - c\mathcal{F}(\widehat{L}))\acute{v}_{\widetilde{m}}, \quad \text{a.e. } t. \tag{33}$$

From (17) and by Assumption 2, one knows that $\frac{d}{dt}\overline{U}(\widehat{r}, \widehat{v}) \leq 0$, a.e. t, which indicates that $\widehat{r}$ and $\widehat{v}$ are bounded. Then, by applying Lemma 2, one has $\Xi_{\widetilde{m}} + c\widehat{L} > 0$, and it follows from LaSalle invariance principle in [39] that all the solutions converge to the largest invariant set in $S = \{(\widehat{r}, \widehat{v})| \widehat{v} = 0\}$. The proof of (ii) is completed. Note that every solution of the system converges to an equilibrium point $(r^*, 0)$, where r^* is a local minimum of $\overline{U}(\widehat{r}, 0)$. This completes the proof of (i).

Next, suppose that at least $k + 1$ agents collide at time t_1. Then, one has

$$V(\widehat{r}(t_1)) \geq (k + 1)\psi(0).$$

Since $\overline{U}$ is absolutely continuous and nonincreasing almost everywhere, one obtains

$$V(\widehat{r}(t_1)) \leq \overline{U}(\widehat{r}(0), \widehat{v}(0)) < (k + 1)\psi(0).$$

This is a contradiction.

Finally, if f is independent of the position term, on the largest invariant set $S = \{(\widehat{r}, \widehat{v}) \mid \widehat{v} = 0\}$, one has

$$\widehat{\dot{r}}_i(t) = 0,$$

$$\sum_{i=1}^{N}\widehat{D_iH_i}\widehat{r}_i = \sum_{k=1}^{p}D_{i_k}H_{i_k}\widehat{r}_{i_k} = 0. \tag{34}$$

It follows that (19) is satisfied. □

Remark 11. Even if the states of the leader cannot be observed by only one agent in the group, many agents can share their information with the neighbors and the position of the leader can still be followed in a distributed way. In addition, as long as the velocity vector of the leader can be observed, all the agents in the group can move with the same velocity as the leader. Here, it is noted that the average position of the informed agents, with only partial information about leader's position, can still converge to that of the leader as shown in (iv).

If the network structure is changing very slowly, Assumption 3 can also be investigated and is omitted here. In addition, even if Assumption 2 is not satisfied and the network is not connected, the above theoretical results can be used to study the flocking problem for the connected components.

4. Simulation examples

4.1. Leader–follower control with pinning observer-based navigation feedback

Suppose that the leader moves in the following periodic manner:

$$\dot{r}_o(t) = v_o,$$
$$\dot{v}_o(t) = Av_o, \tag{35}$$

where $r_o(t), v_o(t) \in R^2$ and $A = 0.1\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Consider the same multi-agent dynamical system as follows:

$$\dot{r}_i(t) = v_i,$$
$$\dot{v}_i(t) = Av_i - D_iH_i(r_i - r_o) - \widetilde{D}_i\widetilde{H}_i(v_i - v_o)$$
$$+ \sum_{j\in\mathcal{N}_i}a_{ij}(\|r_j - r_i\|)\left[a - be^{-\frac{\|r_{ij}\|^2}{c}}\right](r_j - r_i)$$
$$+ c\sum_{j\in\mathcal{N}_i}a_{ij}(\|r_j - r_i\|)(v_j - v_i), \quad i = 1, 2, \ldots, l,$$

$$\dot{v}_i(t) = f(r_i, v_i) + \sum_{j\in\mathcal{N}_i}a_{ij}(\|r_j - r_i\|)\left[a - be^{-\frac{\|r_{ij}\|^2}{c}}\right](r_j - r_i)$$
$$+ c\sum_{j\in\mathcal{N}_i}a_{ij}(\|r_j - r_i\|)(v_j - v_i),$$
$$i = l + 1, l + 2, \ldots, N, \tag{36}$$

where $N = 20$, $l = 6$, $a = 1$, $b = 20$, $\bar{c} = 0.2$, $r_s = 1$, $c = 5$, and $d = \sqrt{c\ln(b/a)} = 0.7740$, $H_i = \widetilde{H}_i = \begin{pmatrix} 1 & 0 \end{pmatrix}$ for $i = 1, 2, 3$, and $H_i = \widetilde{H}_i = \begin{pmatrix} 0 & 1 \end{pmatrix}$ for $i = 4, 5, 6$. Choose $D_i = \widetilde{D}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $i = 1, 2, 3$, and $D_i = \widetilde{D}_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $i = 4, 5, 6$. It is easy to see that 3 agents can observe the first position (v_{o1}) and velocity (r_{o1}) state components of the leader and other 3 agents can observe the second position (v_{o2}) and velocity (r_{o2}) state components of the leader.

In the simulation, the initial positions and velocities of the 20 agents are chosen randomly from $[-1, 1] \times [-1, 1]$ and $[0, 2] \times [0, 2]$, respectively. From Theorem 1, one knows that all agents move with the same velocity, as shown in Fig. 1. The positions of agents converge to a local minimum of $\overline{U}(r^*, 0)$ in (17), as illustrated in Fig. 2. The figure in greater detail at time 30 s is shown in Fig. 3, where the circle represents the agent without navigation feedback, the diamond denotes the agent ($i = 1, 2, 3$) with pinning navigation feedback of the first state component of the leader, the star ($i = 4, 5, 6$) denotes the agent with pinning navigation feedback of the second state component of the leader, and the square is the leader. By condition (iv) in Theorem 1, one knows that

$$\frac{1}{3}\sum_{i=1}^{3}r_{l1}(t) \to r_{o1}(t), \qquad \frac{1}{3}\sum_{i=4}^{6}r_{l2}(t) \to r_{o2}(t), \quad t \to \infty.$$
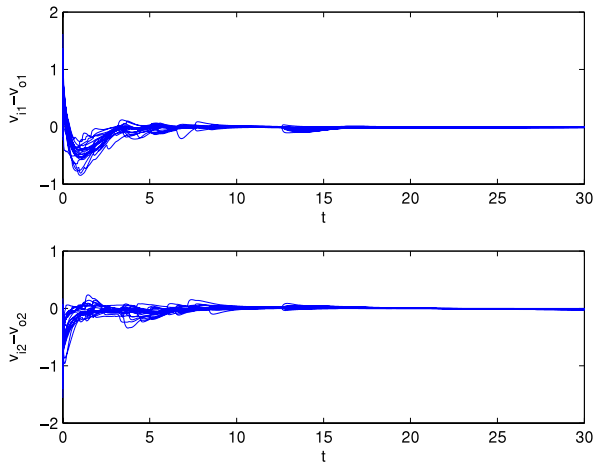
Fig. 1. Relative velocities of all agents to the leader, $i = 1, 2, \ldots, 20$.
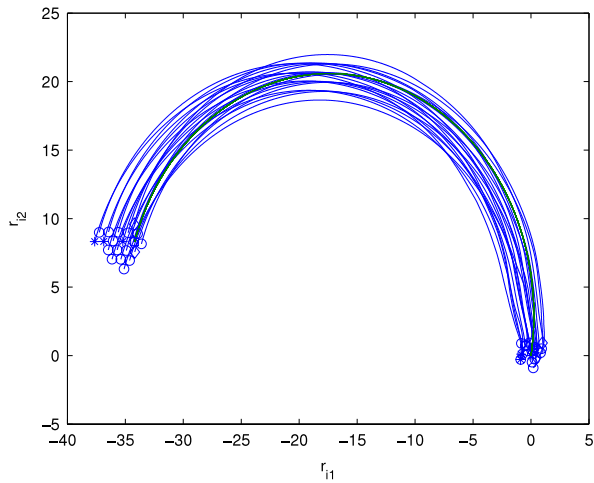

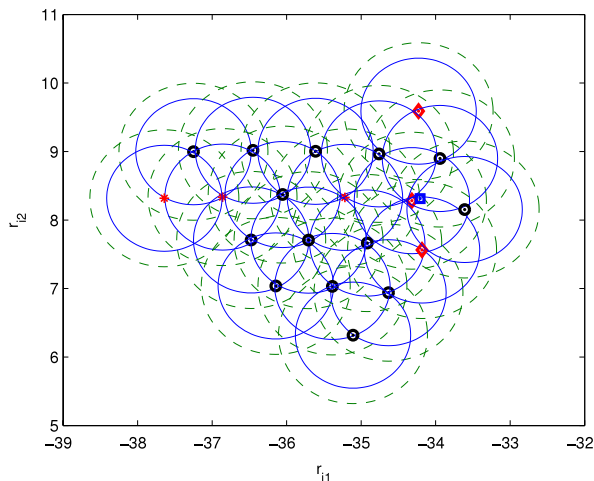**Fig. 2.** Positions of all agents, $i = 1, 2, \ldots, 20$.

Fig. 3. Positions of all the agents at time 30 s, $i = 1, 2, \ldots, 20$.

With $\frac{1}{3}\sum_{i=1}^{3} r_{l1}(t) = r_{a1}(t)$ and $\frac{1}{3}\sum_{i=4}^{6} r_{l2}(t) = r_{a2}(t)$, the error state $r_a(t) - r_o(t)$ is shown in Fig. 4.

In this example, the agents 1–3 and 4–6 have the information of the first and second state components of the leader, respectively. All agents share this information and communicate with their neighbors, so that the whole group can follow the state of the leader. Note that only some agents in the group have partial
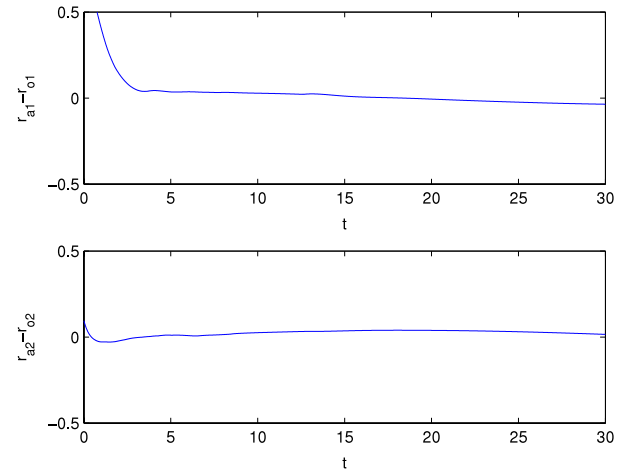
Fig. 4. Error state between the average position of agents, with pinning observer-based navigation feedback, and the leader.
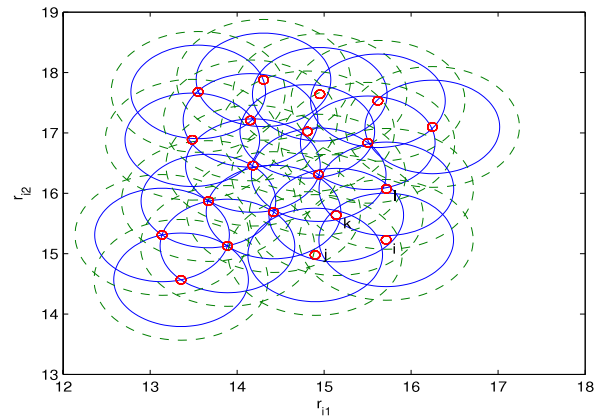
Fig. 5. Positions of all the agents at time 15 s, $i = 1, 2, \ldots, 20$.

information about the leader, but all agents move with the same velocity as the leader and the average position of the informed agents follows the position of the leader.
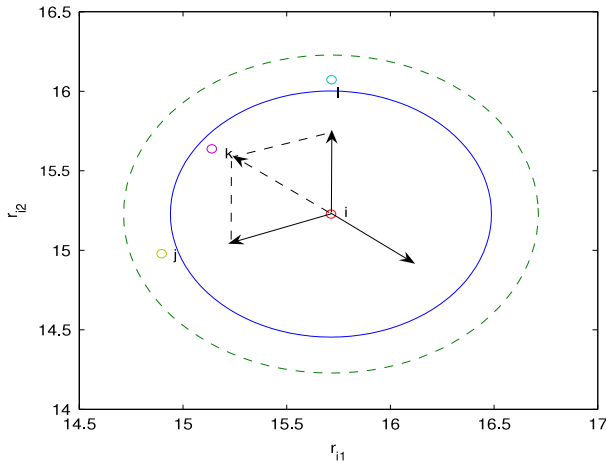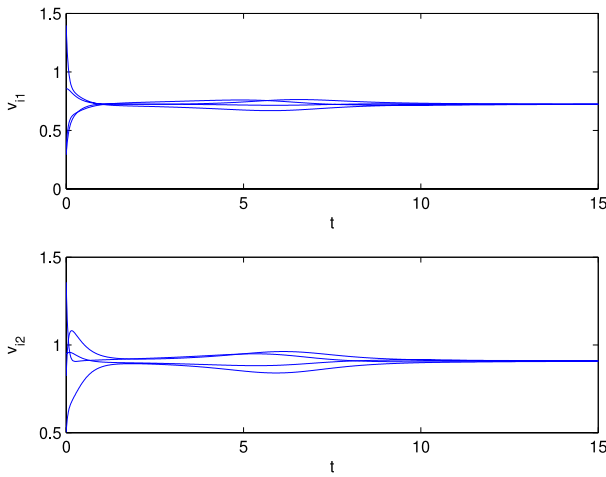
4.2. The local minimum of the potential function does not necessarily form an α-lattice

Let $f_i = 0$ and $\widetilde{f_i} = 0$ in model (36). In the simulation, the initial positions and velocities of the 20 agents are chosen randomly from $[-1, 1] \times [-1, 1]$ and $[0, 1.5] \times [0, 1.5]$, respectively. From similar analysis as in Theorem 1, all agents move with the same velocity asymptotically.

Of particular interest is the final position state, where $t = 15$ s. The position state at $t = 15$ s is shown in higher resolution in Fig. 5. The solid circles are the positions where the attraction and repulsion are balanced ($d = 0.7740$) and the dashed circles represent the sensing ranges ($r_s = 1$) of the agents. If an agent i is in the solid circle of agent j, then the repulsion plays a key role; if it is between the solid and dashed curves, then the attraction dominates; otherwise, they do not have influence on each other. The configuration of agents i, j, k, and l in Fig. 5 are amplified in Fig. 6. It is easy to see that agent i is attracted to agents j and l, and repulsed from agent k. These attractions and repulsion are balanced as illustrated by Fig. 6, which means that the position state may not form an α-lattice but reach the local minimum of the potential function.

It should also be pointed out that the configuration of α-lattice is a local minimum of the potential function, but it does not

Fig. 6. External forces around agent i.

Fig. 7. Velocities of all agents, $i = 1, 2, \ldots, 5$.

imply that the local minimum of the potential function forms necessarily an α-lattice. Therefore, Lemma 3 in [20] is incorrect. Mathematically, as it stands, one can only have

$$f_i^\alpha = \sum_{j \in \mathcal{N}_i} \phi(\|r_j - r_i\|) \frac{r_j - r_i}{\|r_j - r_i\|} = 0,$$

which does not imply that $\phi(\|r_j - r_i\|) = 0$ if there is a connection between agents i and j. The correct conclusion is that if all the external forces on each agent are balanced, then the formed configuration attains a local minimum of the potential function.

In order to show that the local minimum of the potential function does not necessarily form an α-lattice, a flocking model with 4 agents is considered as follows

$$\dot{r}_i(t) = v_i,$$

$$\dot{v}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(\|r_j - r_i\|) \left[a - b \, e^{-\frac{\|r_{ij}\|^2}{\bar{c}}} \right] (r_j - r_i)$$
$$+ c \sum_{j \in \mathcal{N}_i} a_{ij}(\|r_j - r_i\|)(v_j - v_i), \quad i = 1, 2, \ldots, N, \quad (37)$$

where $N = 5$, $a = 1$, $b = 20$, $\bar{c} = 0.2$, $r_s = 1.2$, $c = 5$, and $d = \sqrt{c \ln(b/a)} = 0.7740$. The initial positions and velocities of the 5 agents are chosen randomly from $[-1, 1] \times [-1, 1]$ and $[0, 1.5] \times [0, 1.5]$, respectively. From similar analysis, all agents move with the same velocity asymptotically as shown in Fig. 7.

Of particular interest is the final position state in Fig. 8, where $t = 15$ s. Similarly, the solid circles are the positions where the attraction and repulsion are balanced ($d = 0.7740$) and the dashed
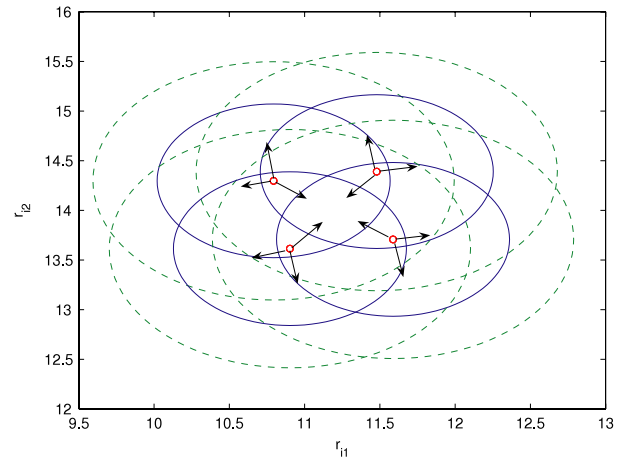
Fig. 8. Positions of all agents at time 15 s, $i = 1, 2, \ldots, 5$.

circles represent the sensing ranges ($r_s = 1.2$) of the agents. If an agent i is in the solid circle of agent j, then the repulsion plays a key role; if it is between the solid and dashed curves, then the attraction dominates; otherwise, they do not have influence on each other. It is easy to see from Fig. 8 that each agent receives attractions from two nearest agents and repulsion from one farther agent, where all the forces affecting on this agent are balanced. Therefore, the position state may not form an α-lattice but reach the local minimum of the potential function, which indicates that the results for the configuration of α lattice in [20] is incorrect.

5. Conclusions

In this paper, a distributed leader–follower flocking algorithm for multi-agent dynamical systems has been developed and analyzed, which considers the case in which the group has one virtual leader and the asymptotic velocity is time-varying. In addition, observer-based pinning navigation feedback is derived, where each informed agent only senses partial states of the leader. It has been proved that although each informed agent can only obtain partial information about the leader, the velocity of the whole group converges to that of the leader and the centroid of those informed agents, having the leader's position information, follows the trajectory of the leader asymptotically.

It has been the goal of this paper to analyze different flocking algorithms by using tools from nonsmooth analysis in combination with ideas from the study of synchronization in complex systems. The leader–follower algorithm considered in this paper fall into the category of free-flocking where no obstacles are considered. We are currently investigating constrained-flocking with obstacle avoidance because it may lead to a more challenging scenario where the group splits.

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