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Quantitative Local Analysis for Nonlinear Systems

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SUMMARY

This paper develops theoretical and numerical tools for quantitative local analysis of nonlinear systems. Specifically, sufficient conditions are provided for bounds on the reachable set and $\mathbf{L}_2$ gain of the nonlinear system subject to norm bounded disturbance inputs. The main theoretical results are extensions of classical dissipation inequalities but enforced only on local regions of the state and input space. Computational algorithms are derived from these local results by restricting to polynomial systems, using convex relaxations, e.g. the S-procedure, and applying sum-of-squares optimizations. Several pedagogical and realistic examples are provided to illustrate the proposed approach. Copyright © 0000 John Wiley & Sons, Ltd.

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KEY WORDS: dissipation inequalities, nonlinear systems, reachability, local L2 gain, sum-of-squares polynomial

1. INTRODUCTION

This paper focuses on dynamical systems governed by differential equations of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))w(t), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $t \in \mathbb{R}$, $x(0) = x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be Lipschitz continuous, or locally Lipschitz continuous, depending on the situation. If f and g are not Lipschitz continuous (as in the case of polynomial f and g, for example), then the differential equation may exhibit finite escape times in the presence of bounded inputs and/or initial conditions.

There is a large literature on input/output gain of nonlinear dynamical systems described by ordinary differential equations (ODEs) [1, 2]. Disturbance rejection and noise insensitivity are critical metrics of performance in a closed-loop control system, and being able to quantify such metrics allows one to discriminate among competing designs. The importance of the gain, and other general properties of dissipativeness (e.g., passivity, or more general forms) is realized in hierarchical interconnection theorems, such as small-gain, passivity theorems, and integral quadratic

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constraints, where coarse input/output properties of a collection of individual subsystems can be used to infer properties of specific interconnections of these components, [3, 4, 5, 6].

The overarching goal of the research, reported here and in related papers, [7, 8, 9] is quantitative, local analysis of nonlinear dynamical systems. By "quantitative" we mean algorithms and sufficient conditions which lead to concrete guarantees about a particular system's response. By "local" we refer to guarantees about the reachability and/or system gain which are based on assumptions concerning the magnitude of initial conditions and input signals. We extensively use, without further citation, the basic, fundamental ideas from dissipative systems theory [10], [11], barrier functions and reachability [12], and nonlinear optimal control [1], [2]. Specifically, inequalities involving the Lie derivative of a scalar function, the *storage* function, that hold throughout regions of the state and input space, which when integrated over trajectories of the system, give certificates of input/output properties of the system. The necessity of the existence of such storage functions to prove input/output properties, which leads to the most elegant results of the above mentioned works, is actually not used. Our computational approach is based on polynomial storage functions of fixed degree which can be viewed as extensions of known linear matrix inequality conditions to compute reachable sets and input/output gains for linear systems [13]. Due to the restriction to polynomial storage functions our results typically do not approach the theoretical optimal storage functions. Current work in [14] is addressing the necessity of polynomial storage functions for systems with polynomial vector fields, and should be considered of deep theoretical importance for our work.

Two recent works that are very similar in spirit to this paper are [15] and [16]. Reference [15] uses local dissipation inequalities (similar to those used in the current paper) to characterize reachability and input-output gain properties of nonlinear systems under affine, bounded disturbance inputs. For systems with vector fields rational in the states, it provides semidefinite programming based methods to search for polynomial storage functions that satisfy these dissipation inequalities. Reference [16] introduces a generalization of $\mathbf{L}_2$ gain, nonlinear $\mathbf{L}_2$ gain function that bounds the output $\mathbf{L}_2$ norm as a function of the input $\mathbf{L}_2$. It characterizes this gain function in terms of a dissipation property of an augmented system and seeks storage functions that are solutions of certain partial differential inequalities.

The contributions of the current paper are as follows: dissipation inequality formulation of local reachability and dissipativeness for uncertain systems that are not nominally globally stable are derived (Sections 3.1 and 3.3); refinements on the reachability and $\mathbf{L}_2$ gain conditions that can be used to efficiently compute improved quantitative performance bounds (Section 3.2); sum-of-squares (SOS) characterizations of the required set containment conditions in the dissipation inequalities (Section 5.1); proof of feasability of the SOS conditions for systems with stable linearizations (Section 5.2) ; development of scheme to find feasible solutions to the bilinear SOS conditions, and improve the objective through a specific iteration scheme (Section 5.3); a suite of pedagogical and realistic examples illustrating the methods (Sections 4, 6.2 and 6.3).

2. NOTATION

The set of $m \times n$ matrices whose elements are in $\mathbb{R}$ or $\mathbb{C}$ are denoted as $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$. A single superscript index denotes vectors, e.g. $\mathbb{R}^m$ is the set of $m \times 1$ vectors whose elements are in $\mathbb{R}$.

Basic system theory and functional analysis drawn from texts such as [5], [17] and [18] are used without further citation. $\mathbf{L}_2^m$ is the space of $\mathbb{R}^m$-valued functions $f : [0, \infty) \to \mathbb{R}^m$ of finite energy $\|f\|_2^2 = \int_0^\infty f(t)^T f(t)\, dt$. Define $\|r\|_{2,T}^2 := \int_0^T r^T(t) r(t) dt$. Associated with $\mathbf{L}_2^m$ is the *extended* space $\mathbf{L}_{2e}^m$, consisting of functions whose truncation $f_T(t) := f(t)$ for $t \le T$; $f_T(t) := 0$ for $t > T$, is in $\mathbf{L}_2^m$ for all $T > 0$.

For $\eta > 0$ and continuous $g : \mathbb{R}^n \to \mathbb{R}$, $\Omega_{g,\eta} := \{x \in \mathbb{R}^n : g(x) \le \eta\}$. If $g(\bar{x}) \le 0$ and $\eta > 0$, then $\Omega_{g,\eta}^{cc,\bar{x}}$ denotes the connected component of $\Omega_{g,\eta}$ which contains $\bar{x}$.

For $\xi \in \mathbb{R}^n$, $\mathbb{R}[\xi]$ represents the set of polynomials in ξ with real coefficients. The subset $\Sigma_n := \{\pi = \pi_1^2 + \pi_2^2 + \ldots + \pi_M^2 : \pi_1, \ldots, \pi_M \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of sum-of-squares (SOS) polynomials.

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In several places, a relationship between an algebraic condition on some real variables and input/output/state properties of a dynamical system is claimed. In nearly all of these types of statements, we use same symbol for a particular real variable in the algebraic statement as well as the corresponding signal in the dynamical system. This could be a source of confusion, so care on the readers part is required.

3. PERFORMANCE CHARACTERIZATIONS

3.1. Reachability

In this section, we establish conditions which guarantee invariance of certain sets under $\mathbf{L}_2$ and pointwise-in-time ($\mathbf{L}_\infty$-like) constraints on w. These are subsequently referred to as "reachability" results, since the conclusions yield outer bounds on the set of reachable states. In that vein, w is loosely interpreted as a disturbance, whose worst-case effect on the state x is being quantified. We obtain bounds on x that are tightly linked with the assumed bounds on w and x_0, and specifically allow for systems which are not well-defined on all input signals (finite escape times). Specific computational approaches based on the S-procedure and sum-of-squares, which are outlined in the Appendix, are introduced in Section 5.

A known set $\mathcal{W} \subseteq \mathbb{R}^m$ is used to express the $\mathbf{L}_\infty$-like, pointwise-in-time bound on the signal w, specifically $w(t) \in \mathcal{W}$ for all t. Note that $\mathcal{W} = \mathbb{R}^m$ is equivalent to the absence of known, pointwise-in-time bounds on w.

Consider the system (affine in input)

$$\dot{x}(t) \quad = f(x(t)) + g(x(t))w(t), \tag{2}$$

where $t \in \mathbb{R}$, $x(0) = x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$.

Theorem 3.1
[Reach:] Suppose $\mathcal{W} \subseteq \mathbb{R}^m$. Assume that f and g in (2) are Lipschitz continuous on $\mathbb{R}^n$. Suppose $\tau > 0$, and a differentiable $Q : \mathbb{R}^n \to \mathbb{R}$ satisfies $Q(0) < \tau^2$ and

$$\Omega^{cc,0}_{Q,\tau^2} \times \mathcal{W} \subseteq \left\{(x,w) \in \mathbb{R}^n \times \mathbb{R}^m : \nabla Q(x) \cdot [f(x) + g(x)w] \leq w^T w\right\}. \tag{3}$$

Consider $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t. If $\|w\|_2^2 < \tau^2 - Q(x_0)$, the solution to (2) with $x(0) = x_0$ satisfies $Q(x(t)) < \tau^2$ for all t, and hence $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all t.

Remark 3.2
Without loss of generality, Q in Theorem 3.1 **(Reach)** can be taken to be zero at $x = 0$. For instance, define $\tilde{Q}(x) := Q(x) - Q(0)$ and $\tilde{\tau}^2 := \tau^2 - Q(0)$. The conditions of Theorem 3.1 hold with $\tilde{Q}$ replacing Q, and the same norm bound (i.e. reachable set) is obtained.

Remark 3.3
Condition (3) can be equivalently expressed as

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{x \in \mathbb{R}^n : \nabla Q(x) \cdot [f(x) + g(x)w] - w^T w \leq 0 \; \forall w \in \mathcal{W}\right\}, \tag{4}$$

and also equivalently

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla Q(x) \cdot [f(x) + g(x)w] - w^T w \leq 0\right\}. \tag{5}$$

Proof
Suppose not, and define $T > 0$ such that

$$Q(x(t)) < \tau^2 \;\; \forall \, t \in [0, T]$$

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and $Q(x(T)) = \tau^2$. Indeed, such a T exists since x and Q are continuous and $Q(x_0) < \tau^2$. Hence, on $[0,T]$, $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$. Since Q is differentiable, x is absolutely continuous, and $w(t) \in \mathcal{W}$ for all t, integrating the dissipation inequality from condition (3) gives

$$Q(x(T)) \leq Q(x_0) + \|w\|^2_{2,T} .$$

Since $\tau^2 = Q(x(T))$, it follows that $\|w\|_2^2 \geq \|w\|^2_{2,T} \geq \tau^2 - Q(x_0)$. This establishes the result by contradiction. □

The special case $\mathcal{W} = \mathbb{R}^m$ is stated as a corollary.

Corollary 3.4
Assume that f and g in (2) are Lipschitz continuous on $\mathbb{R}^n$. Suppose $\tau > 0$, and a differentiable $Q : \mathbb{R}^n \to \mathbb{R}$ satisfies $Q(0) < \tau^2$ and

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathbb{R}^m} \nabla Q(x) \cdot [f(x) + g(x)w] - w^T w \leq 0 \right\} . \tag{6}$$

Consider $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2$. If $\|w\|_2^2 < \tau^2 - Q(x_0)$, the solution to (2) with $x(0) = x_0$ satisfies $Q(x(t)) < \tau^2$ for all t, and hence $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all t.

Theorem 3.1 **(Reach)** can be restated on finite intervals as the following corollary.

Corollary 3.5
Assume the conditions of Theorem 3.1 with $Q(0) = 0$. Then, for all $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$, $T > 0$, and $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|^2_{2,T} < \tau^2 - Q(x_0)$, the solution of (2) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0,T]$. □

The next theorem relaxes the assumption that f and g are globally Lipschitz continuous in exchange for assuming boundedness of $\Omega^{cc,0}_{Q,\tau^2}$.

Theorem 3.6
Suppose f and g in (2) are locally Lipschitz continuous, and hence Lipschitz continuous on any bounded set. Assume all the other conditions of Theorem 3.1 **(Reach)** are satisfied. If in addition

$$\Omega^{cc,0}_{Q,\tau^2} \text{ is bounded,} \tag{7}$$

then the conclusions of Theorem 3.1 (and Corollaries 3.4 - 3.5) remain true.

Proof
By Lemma 9.1, since $\Omega^{cc,0}_{Q,\tau^2}$ is bounded, f and g can be extended to globally Lipschitz continuous functions, $\tilde{f}$ and $\tilde{g}$ such that $f(x) = \tilde{f}(x)$ and $g(x) = \tilde{g}(x)$ for all $x \in \Omega^{cc,0}_{Q,\tau^2}$. The conditions of Theorem 3.1 hold for $\tilde{f}$ and $\tilde{g}$, and hence the conclusions apply to solutions of

$$\dot{x}(t) = \tilde{f}(x(t)) + \tilde{g}(x(t))w(t). \tag{8}$$

Consequently, for all x_0 with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2$ with $\|w\|_2^2 < \tau^2 - Q(x_0)$, the solution of (8) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all t. Since the solution remains in the region where $f = \tilde{f}$ and $g = \tilde{g}$, it must be that the solution to (2) is the same function, and has the properties as claimed. □

3.2. *Reachability Refinement*

The sufficient conditions presented thus far consider any differentiable Q as a barrier function. In the computational approach we pursue, polynomials play a key role, and the choice of Q will be restricted to polynomials of a given degree. This restriction limits the expressiveness of Q, and may introduce "slack" in the differential inequalities (DIEs) (3), (6), meaning that the maximum of the DIE in (5) is 0 for some, but not all values of x. Next we present a simple procedure to partially remove the slack, yielding a function M whose sublevel sets are the same as those of Q. The reachability bound certified by M is generally an improvement of the bound guaranteed by Q.

Theorem 3.7
[Reach Refine:] Suppose $r : \mathbb{R} \to \mathbb{R}$ is piecewise continuous, and $0 < r(\xi) \leq 1$ for all ξ. Assume that f and g in (2) are Lipschitz continuous. Suppose $\tau > 0$, and a differentiable $Q : \mathbb{R}^n \to \mathbb{R}$ satisfies $Q(0) = 0$ and

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla Q(x) \cdot [f(x) + g(x)w] - r(Q(x))w^T w \leq 0 \right\}. \tag{9}$$

Then, for all $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$, $T > 0$, with $Q(x_0) < \tau^2$ and $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t and

$$\|w\|^2_{2,T} < \int_{Q(x_0)}^{\tau^2} r^{-1}(\xi) d\xi, \tag{10}$$

the solution to (2) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0, T]$.

Remark 3.8
The only difference between (5) and (9) is that $w^T w$ is replaced by $r(Q(x))w^T w$. Consequently, if $r(\xi) < 1$ for some ξ, then (9) is a stronger condition than (5) and the new allowable bound on $\|w\|_2$ in (10) is larger than the original bound of $\tau^2 - Q(x_0)$.

Proof
Define

$$M(x) := \int_0^{Q(x)} \frac{1}{r(\xi)} d\xi$$

and $\tau_e^2 := \int_0^{\tau^2} \frac{1}{r(\xi)} d\xi > 0$. Note that $\tau_e \geq \tau$ and M is a differentiable function satisfying $M(0) = 0$ and

$$\nabla M(x) = \frac{1}{r(Q(x))} \nabla Q(x).$$

It follows from the definitions of $M(x)$ and τ_e^2 that $M(x) \leq \tau_e^2$ if and only if $Q(x) \leq \tau^2$. Therefore

$$\Omega^{cc,0}_{Q,\tau^2} = \Omega^{cc,0}_{M,\tau_e^2} \tag{11}$$

and

$$\Omega^{cc,0}_{M,\tau_e^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla M(x) \cdot [f(x) + g(x)w] \leq w^T w \right\}.$$

By Theorem 3.1 **(Reach)**, for all $x_0 \in \Omega^{cc,0}_{M,\tau_e}$ with $M(x_0) < \tau_e^2$, $T > 0$ and $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|^2_{2,T} < \tau_e^2 - M(x_0)$, the solution to (2) satisfies $x(t) \in \Omega^{cc,0}_{M,\tau_e^2}$ for all $t \in [0, T]$. Recalling (from (11)) that $\Omega^{cc,0}_{Q,\tau^2} = \Omega^{cc,0}_{M,\tau_e^2}$ completes the proof. □

In practice, the analyst might select a predefined set $\mathcal{P}$ for which the goal is to show that all states reachable from $x(0) = 0$ and $\|w\|_2^2 < \tau^2$ are contained in $\mathcal{P}$. Augmenting the conditions of Theorems 3.1 **(Reach)** and 3.6 and their respective corollaries with the requirement $\Omega_{Q,\tau^2} \subseteq \mathcal{P}$ is one possible approach.

In particular, we use an adjustable region derived from a given function $p : \mathbb{R}^n \to \mathbb{R}$, called the *shape factor function*, defining $\mathcal{P}_\beta := \Omega_{p,\beta}$. Typically p is simple (quadratic), so that even in high dimensions, its sub-level sets are easily interpreted (in contrast to Q, whose sub-level sets may be difficult to quantify). Much like a cost function in optimal control, the positive-definite function p is chosen by the analyst to reflect the relative importance of the individual state elements.

3.3. $\mathbf{L}_2$ *gains*

In this section, we establish conditions which provide a bound on the $\mathbf{L}_2$ gain of the system under $\mathbf{L}_2$ and pointwise-in-time ($\mathbf{L}_\infty$-like) constraints on w. As in the development of the reachability results, w is interpreted as a disturbance, whose worst-case effect on the output y is being quantified. We obtain bounds on the $\mathbf{L}_2$ norm of y that depend on the assumed bounds on w and x_0. Again, specific computational approaches based on the S-procedure and sum-of-squares, which are outlined in the Appendix, are introduced in Section 5.

Consider the system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))w(t) \\ y(t) &= h(x(t)), \end{aligned} \tag{12}$$

where $t \in \mathbb{R}$, $x(0) = x_0 \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$. We recall the following definition from [1].

Definition 3.9
The system (12) is said to have finite $\mathbf{L}_2$ gain if there exist a finite constant $\rho > 0$ and for every initial condition x_0, a finite constant $\phi(x_0) \geq 0$ such that solutions of (12) satisfy

$$\|y\|_{2,T} \leq \phi(x_0) + \rho\|w\|_{2,T} \tag{13}$$

for all $w \in \mathbf{L}_2$ and for all $T \geq 0$. □

Alternatively, one can define the $\mathbf{L}_2$ gain as follows.

Definition 3.10
The system (12) is said to have finite $\mathbf{L}_2$ gain if there exists a finite constant $\gamma > 0$ such that for every initial condition x_0 there exists a finite constant $\psi(x_0) \geq 0$ with the property

$$\|y\|_{2,T}^2 \leq \psi(x_0) + \gamma^2\|w\|_{2,T}^2, \tag{14}$$

for all $w \in \mathbf{L}_2$ and for all $T \geq 0$. □

The equivalence of Definition 3.9 and Definition 3.10 is shown in the Appendix. Our results use the form in (14).

Theorem 3.11
[$\mathbf{L}_2$ Gain:] Suppose $\mathcal{W} \subseteq \mathbb{R}^m$. Assume that f, g and h in (12) are Lipschitz continuous. Suppose $\gamma > 0$, $R > 0$, and a differentiable $V : \mathbb{R}^n \to \mathbb{R}$ satisfies $V(0) = 0$, and

$$\Omega_{V,R^2}^{cc,0} \setminus 0 \subseteq \{x \in \mathbb{R}^n : V(x) > 0\}, \tag{15}$$

$$\Omega_{V,R^2}^{cc,0} \times \mathcal{W} \subseteq \left\{(x, w) \in \mathbb{R}^n \times \mathbb{R}^m : \nabla V(x) \cdot [f(x) + g(x)w] \leq w^T w - \frac{1}{\gamma^2} h^T(x)h(x)\right\}. \tag{16}$$

Consider $x_0 \in \Omega_{V,R^2}^{cc,0}$ with $V(x_0) < R^2$, $T > 0$ and $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t. If $\|w\|_{2,T}^2 \leq R^2 - V(x_0)$, the solution to (12) with $x_0 = x(0)$ satisfies $x(t) \in \Omega_{V,R^2}^{cc,0}$ for all $t \in [0, T]$ and

$$\|y\|_{2,T}^2 \leq \gamma^2 V(x_0) + \gamma^2 \|w\|_{2,T}^2 . \tag{17}$$

Moreover, any condition on w that ensures $x(t) \in \Omega_{V,R^2}^{cc,0}$ for the solutions will yield the same gain bound.

Proof
The conditions in (16) are stricter than the reachability conditions in (3), hence the norm-bound on w ensures that the trajectories remain in $x(t) \in \Omega_{V,R^2}^{cc,0}$. Hence (16) can be integrated over the solution on $[0, T]$, giving

$$\gamma^2 V(x(T)) + \|y\|_{2,T}^2 \leq \gamma^2 V(x_0) + \gamma^2 \|w\|_{2,T}^2 .$$

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The additional assumption that V is nonnegative on $\Omega^{cc,0}_{V,R^2}$ implies

$$\|y\|_{2,T}^2 \le \gamma^2 V(x_0) + \gamma^2 \|w\|_{2,T}^2 \tag{18}$$

as claimed. Finally, it is clear that the bound is true for any $w \in \mathbf{L}_2$ under the condition that the state trajectories remain in $\Omega^{cc,0}_{V,R^2}$. □

Remark 3.12
Superficially, the $\mathbf{L}_2$ gain supply rate in (16) $(i.e., w^T w - \frac{1}{\gamma^2} h^T(x)h(x))$ can be replaced by a more general supply rate $s(w,x)$. Assuming the same initial condition assumptions, but with a harder-to-verify restriction on w, namely $\int_0^T s(w(t),x(t))\, dt \le R^2 - V(x_0)$, local dissipativity with respect to the supply rate $s(w,x)$ is established, $0 \le V(x(T)) \le V(x(0)) + \int_0^T s(w(t),x(t))\, dt$. The ideas in Section 3.4 can be used obtain $\mathbf{L}_2$ bounds on w instead.

Remark 3.13
Similar to Remark 3.3 condition (16) can be equivalently expressed as

$$\Omega^{cc,0}_{V,R^2} \subseteq \left\{ x \in \mathbb{R}^n : \max_{w \in \mathcal{W}} \nabla V(x) \cdot [f(x) + g(x)w] - w^T w + \frac{1}{\gamma^2} h^T(x)h(x) \le 0 \right\}. \tag{19}$$

Theorem 3.11 **($\mathbf{L}_2$ Gain)** can be restated for $\mathcal{W} = \mathbb{R}^m$ as in Corollary 3.4 but is omitted for brevity.

Corollary 3.14
Suppose f, g and h are locally Lipschitz continuous, the conditions of Theorem 3.11 **($\mathbf{L}_2$ Gain)** hold and, in addition, $\Omega^{cc,0}_{V,R^2}$ is bounded. The conclusion of Theorem 3.11 holds.

Proof
The proof follows by applying Lemma 9.1 to f, g and h. □

3.4. Combining reachability bounds with $\mathbf{L}_2$ gain estimates

As noted at the end of the proof of Theorem 3.11 **($\mathbf{L}_2$ Gain)**, the bound (45) holds for any condition on w and x_0 which ensures that $x(t)$ remains in $\Omega^{cc,0}_{V,R^2}$. In Theorem 3.11, one such condition is $\|w\|_{2,T}^2 < R^2 - V(x_0)$. However, it is advantageous to make a separate reachability analysis (using a new storage function Q) to ascertain bounds on w which keep $x(t) \in \Omega^{cc,0}_{V,R^2}$. In this context, $\Omega^{cc,0}_{V,R^2}$ plays the role of the set $\mathcal{P}$ introduced at the end of Section 3.3. Theorem 3.15 below clarifies this process.

Theorem 3.15
Assume the conditions of Theorem 3.1 **(Reach)** and Theorem 3.11 **($\mathbf{L}_2$ Gain)** hold. If, in addition,

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}, \tag{20}$$

then, for all $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$, $T > 0$, and $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|_{2,T}^2 \le \tau^2 - Q(x_0)$, the solution to (12) satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0,T]$ and $\|y\|_{2,T}^2 \le \gamma^2 V(x_0) + \gamma^2 \|w\|_{2,T}^2$.

Proof
The solution satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0,T]$ by Theorem 3.1. The condition in (18) holds since $w \in \mathbf{L}_2$ and the state trajectories remain in $\Omega^{cc,0}_{V,R^2}$ by (20). □

Obviously, the refinement described in Theorem 3.7 can be used to relax the conditions on w such that $x(t)$ remains in Ω_{Q,τ^2}.

Theorem 3.16
Assume the conditions of Theorems 3.7 **(Reach Refine)** and 3.11 **($\mathbf{L}_2$ Gain)** hold, and $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}$. Then for all x_0 with $M(x_0) < \tau_e^2$, $T > 0$, and all $w \in \mathbf{L}_2$ with $w(t) \in \mathcal{W}$ for all t and $\|w\|^2_{2,T} \leq \tau_e^2 - M(x_0)$, the solution satisfies $x(t) \in \Omega^{cc,0}_{M,\tau_e^2}$ for all $t \in [0,T]$ and $\|y\|^2_{2,T} \leq \gamma^2 V(x_0) + \gamma^2 \|w\|^2_{2,T}$.

Proof
The solution satisfies $x(t) \in \Omega^{cc,0}_{M,\tau_e^2}$ for all $t \in [0,T]$ by Theorem 3.7. The condition in (18) holds since $w \in \mathbf{L}_2$ and the state trajectories remain in $\Omega^{cc,0}_{V,R^2}$ by (20) and (11). □

Remark 3.17
Theorems 3.15 and 3.16 can be applied to locally Lipschitz continuous f, g and h by enforcing that Ω_{V,R^2} is bounded.

4. APPLICATION TO A LOCALLY STABLE SCALAR SYSTEM

4.1. *Reachability calculations*

Since (12) restricts the vector fields $(f(x) + g(x)w)$ to be affine in w, the maximizing w in (5) is $\frac{1}{2}g^T(x)\nabla Q^T(x)$. Plugging this in and setting the maximum to be zero (the limit for (5) to be satisfied) gives a quadratic equality in $\nabla Q(x)$. For scalar systems ($n = m = 1$), it is easy to solve for $\nabla Q(x)$. We demonstrate this method on a simple example, which will also be studied with the SOS methods described in Section 5.

We notate $Q'(x) := \nabla Q(x)$. Consider the system in (12) with $n = m = 1$ and $\mathcal{W} = \mathbb{R}$. At the maximizing w, taking the maximum to be zero, we get $\frac{1}{4}g^2(x)Q'^2(x) + Q'(x)f(x) = 0$. Hence, $Q'(x) = -\frac{4f(x)}{g^2(x)}$ and

$$Q(x) = \int_0^x -\frac{4f(\xi)}{g^2(\xi)}\, d\xi. \tag{21}$$

If $g(x) \neq 0$ for all x, then $Q(x)$ is well defined for all x and

$$Q'(x)\left[f(x) + g(x)w\right] \leq w^2 \tag{22}$$

holds for all $x \in \mathbb{R}$ and for all $w \in \mathbb{R}$. Thus, (6) holds for any $\tau > 0$. For such τ, if f and g are Lipschitz continuous, then the conditions of Theorem 3.1 **(Reach)** are satisfied. If f and g are locally Lipschitz continuous and $\Omega^{cc,0}_{Q,\tau^2}$ is bounded, then the conditions of Theorem 3.6 are satisfied.

For example, consider the system

$$\dot{x} = -x + x^3 + w, \tag{23}$$

which has finite escape times for some inputs with $\|w\|_2 \geq 1$. Equation (21) yields $Q(x) = 2x^2 - x^4$, illustrated in Figure 1.

Since f and g are only locally Lipschitz continuous, we must choose τ such that $\Omega^{cc,0}_{Q,\tau^2}$ is bounded in order to apply Theorem 3.6. Clearly, for $0 < \tau < 1$, $\Omega^{cc,0}_{Q,\tau^2}$ is bounded and the conditions of Theorem 3.6 are satisfied. For example, if $\tau^2 = 0.95$, then $\Omega^{cc,0}_{Q,\tau^2} = [-0.88, 0.88]$, illustrated in Figure 1. Thus, for all $|x_0| < 0.88$ and $w \in \mathbf{L}_2$ with $\|w\|_2 < 0.95 - Q(x_0)$, solutions to (23) satisfy $|x(t)| \leq 0.88$ for all t.

4.2. $\mathbf{L}_2$ *Gain calculations*

Analogously to reachability, the maximizing w in the DIE (19) is $\frac{1}{2}g^T(x)\nabla V^T(x)$ and setting the maximum to zero yields a quadratic inequality in $\nabla V(x)$. Again, we solve this for a scalar system and apply the method on the dynamics in (23). At the maximizing w with zero as the maximum we

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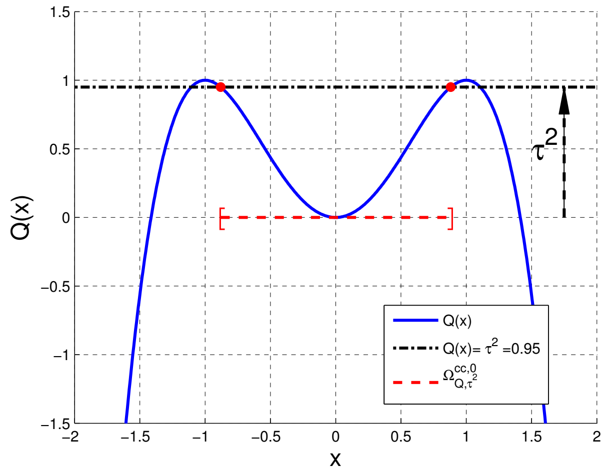

Figure 1. Illustration of Q

get

$$\frac{1}{4}g^2(x)V'^2(x) + V'(x)f(x) + \frac{1}{\gamma^2}h^2(x) = 0. \tag{24}$$

Applying the quadratic formula to (24) yields the following two solutions for $V'(x)$. One choice for $V'(x)$ is

$$V'(x) = \begin{cases} \frac{2\left(-f(x)-\sqrt{f^2(x)-\frac{1}{\gamma^2}g^2(x)h^2(x)}\right)}{g^2(x)} & \text{for } x < 0 \\ \frac{2\left(-f(x)+\sqrt{f^2(x)-\frac{1}{\gamma^2}g^2(x)h^2(x)}\right)}{g^2(x)} & \text{for } x \geq 0. \end{cases}$$

Setting $V(0) = 0$ gives $V(x) = \int_0^x V'(\xi)\, d\xi$. Assume $g(x) \neq 0$ for all $x \in \mathbb{R}$. Note that $V'(x)$ is real for all x such that $f^2(x) - \frac{1}{\gamma^2}g^2(x)h^2(x) \geq 0$. Let R be such that $\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x : V(x) > 0\}$. The inequality

$$V'(x)\left[f(x) + g(x)w\right] \leq w^2 - \frac{1}{\gamma^2}h^2(x) \tag{25}$$

holds for all $x \in \Omega^{cc,0}_{V,R^2}$ and for all $w \in \mathbb{R}$. If f, g, and h are Lipschitz continuous and $V(0) = 0$, then the assumptions of Theorem 3.11 **($\mathbf{L_2}$ Gain)** are satisfied. If f, g and h are locally Lipschitz continuous, $V(0) = 0$, and $\Omega^{cc,0}_{V,R^2}$ is bounded, then the assumptions of Corollary 3.14 are satisfied.

We demonstrate the process on the system in (23). Plugging $f(x)$, $g(x)$ and $h(x)$ into (25) yields

$$V'(x) = \begin{cases} 2\left(x - x^3 - x^2\sqrt{x^4 - 2x^2 + 1 - \frac{1}{\gamma^2}}\right) & \text{for } x < 0, \\ 2\left(x - x^3 + x^2\sqrt{x^4 - 2x^2 + 1 - \frac{1}{\gamma^2}}\right) & \text{for } x \geq 0 \end{cases}$$

and the resultant V is illustrated in Figure 2 with a choice of $\gamma = 2$.

Let $\alpha = \sqrt{\frac{\gamma-1}{\gamma}}$ and note that $V(x)$ is real for all x such that $|x| \leq \alpha$. Thus, for any $R^2 < V(\alpha)$, $\Omega^{cc,0}_{V,R^2}$ is bounded and $\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x : V(x) > 0\}$, satisfying Corollary 3.14. For example, let

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$\gamma = 2$ (V as in Figure 2) and $R^2 = 0.62$, then $\Omega^{cc,0}_{V,R^2} = [-0.68, 0.68]$. Thus, by Corollary 3.14, the solution satisfies $|x(t)| < 0.68$ and

$$\|y\|^2_{2,T} \le 2^2 V(x_0) + 2^2 \|w\|^2_{2,T}$$

for all $|x_0| < 0.68$, $T > 0$, and all $w \in \mathbf{L}_2$ with $\|w\|^2_{2,T} \le 0.62 - V(x_0)$.

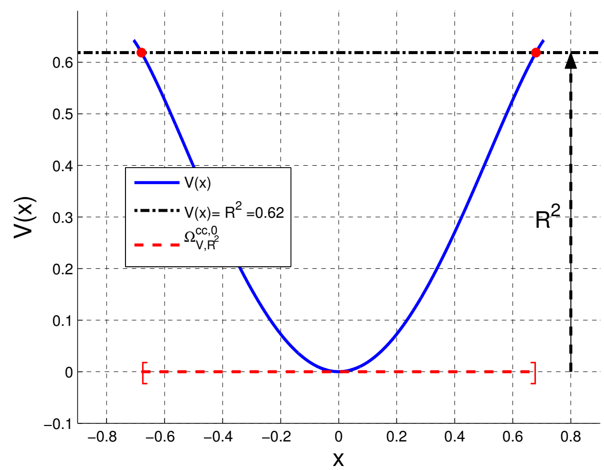

Figure 2. Illustration of V with $\gamma = 2$

We can further improve the bound on the gain by exploiting the reachability argument. From Theorem 3.15, given $\gamma > 0$, we restrict $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}$. In the case of $\gamma = 2$ and $R^2 = 0.62$, we simply equate $\Omega^{cc,0}_{Q,\tau^2} = \Omega^{cc,0}_{V,0.62} = [-0.68, 0.68]$, which results in $\tau^2 = 0.711$. Thus, the bound on $\|w\|_2$ is increased from $\|w\|^2_{2,T} \le 0.62 - V(x_0)$ to

$$\|w\|^2_{2,T} \le 0.711 - Q(x_0),$$

while the bound on the gain remains $\gamma = 2$. The increase is shown in Figure 3. We repeat this procedure for a range of γ values to obtain a curve, shown in Figure 4, of the gain based on the size of the input assuming $x_0 = 0$. Note that the bound on the input approaches 1 as the gain increases, which is expected since (23) has finite escape times for some inputs $\|w\|_{2,T} \ge 1$.

5. PERFORMANCE CHARACTERIZATIONS AS SUM-OF-SQUARES CONDITIONS

The S-procedure, in the Appendix, gives a simple sufficient condition to verify containments of sets described by inequalities. The conditions derived thusfar involve containments of a *particular* connected component of a sublevel set of Q within another set $\mathcal{Z}$. In employing the S-procedure, we will impose more stringent conditions, requiring the entire sublevel set of Q to lie within $\mathcal{Z}$.

5.1. *Reachability, Refinement and* $\mathbf{L}_2$ *Gain Formulations*

In this section we outline the computational methods used to verify the conditions in Section 3 using SOS programming [19, 20, 21], introduced in the Appendix. Assume f, g, and h in (12) are polynomials, and are therefore locally Lipschitz continuous. The Q and V in Theorem 3.6 and Corollary 3.14 will also be restricted to be polynomial.

For reachability, a radially unbounded shape factor function p is assumed given (typically positive definite, quadratic). As mentioned above, the condition $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega_{p,\beta}$ is enforced with the S-procedure, which actually certifies $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$. It is also assumed that $\mathcal{W}$ is defined in terms of a sublevel set of a polynomial function p_W, $\mathcal{W} := \{w : p_W(w) \le 0\}$.

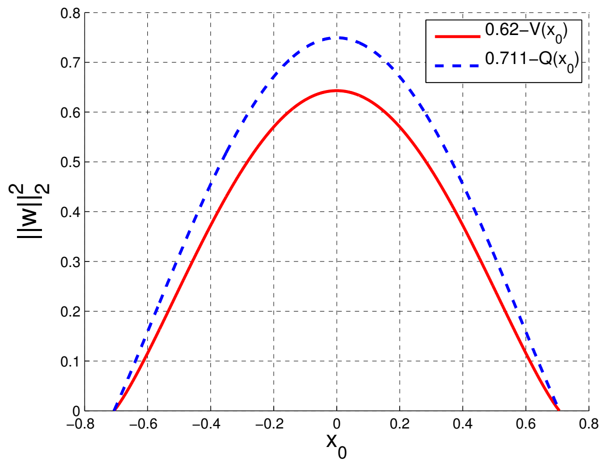

Figure 3. The bound on the input is shown as a function of the initial condition x_0. After applying Theorem 3.15, the input bound is increased for all $x_0 \in [-0.68, 0.68]$

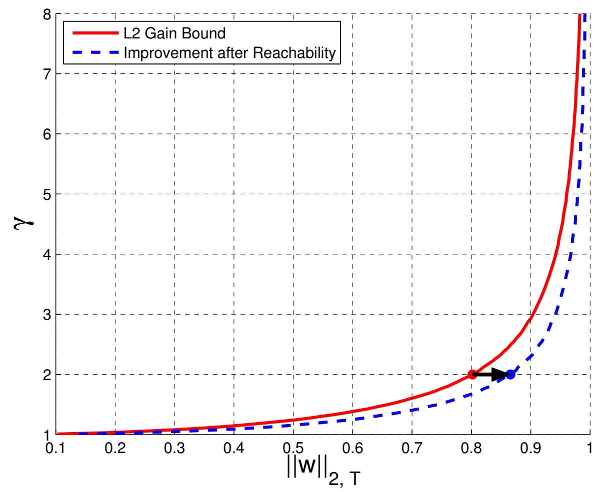

Figure 4. The $\mathbf{L}_2$ gain is improved after applying Theorem 3.15

Two separate applications of the S-procedure are used. The condition $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$ in $\mathbb{R}^n$, and the DIE containment of (5), in $\mathbb{R}^n \times \mathbb{R}^m$. The sets on which these conditions hold dictate the type of SOS polynomial used in the S-procedure. The first condition requires SOS polynomial in Σ_n and the second requires an SOS polynomial in Σ_{n+m}.

Specifically, for given $\tau > 0$, the conditions

$$s_1 \in \Sigma_{n+m}, \quad s_2 \in \Sigma_n, \quad s_p \in \Sigma_{n+m}, \quad Q \in \Sigma_n, \quad Q(0) = 0 \tag{26}$$

$$w^T w - \nabla Q \cdot (f(x) + g(x)w) - s_1(\tau^2 - Q) - s_p p_W \in \Sigma_{n+m}, \tag{27}$$

$$\beta - p - s_2(\tau^2 - Q) \in \Sigma_n \tag{28}$$

guarantee the hypothesis of Theorem 3.6 and $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$.

Some of the "slack" introduced by employing a low-degree polynomial Q in (6) can be removed through a refinement procedure. Given τ and polynomial Q that satisfy Theorem 3.6, $N \in \mathbb{N}$ and numbers $\{r_k\}_{k=1}^N$, we define $\epsilon := \frac{\tau^2}{N}$ and restrict r to be a piecewise constant such that $r(\xi) = r_k$ for all $\epsilon(k-1) \leq \xi \leq \epsilon k$ for $k = 1, \ldots, N$. The refinement is performed by solving N separate,

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uncoupled SOS optimizations, namely for $k = 1, \dots, N$

$$\begin{aligned} \underset{s_{1k}, s_{2k}, s_{pk}}{\text{minimize}} \quad & r_k \\ \text{subject to} \quad & s_{1k} \in \Sigma_{n+m}, \;\; s_{2k} \in \Sigma_{n+m}, \;\; s_{pk} \in \Sigma_{n+m}, \\ & -\left[(\epsilon k - Q)s_{1k} + (Q - \epsilon(k-1))s_{2k} + \nabla Q(f(x) + g(x)w) - r_k w^T w\right] \\ & \qquad - s_{pk} p_W \in \Sigma_{n+m}. \end{aligned}$$

For this r, $\tau_e^2 = \displaystyle\int_0^{\tau^2} \frac{1}{r(\xi)} d\xi = \epsilon \sum_{k=1}^{N} \frac{1}{r_k}$.

For $\mathbf{L}_2$ gain, only one application of the S-procedure is used for the DIE containment of (16) in $\mathbb{R}^n \times \mathbb{R}^m$, which requires the SOS polynomial to be in Σ_{n+m}. If V is restricted to be a polynomial, for given $\gamma > 0$, $R > 0$, and polynomial $l(x) > 0$ for all $x \neq 0$, $l(0) = 0$, the conditions $V(0) = 0$,

$$s_3 \in \Sigma_{n+m}, \;\; s_p \in \Sigma_{n+m}, \tag{29}$$

$$V - l \in \Sigma_n, \tag{30}$$

$$w^T w - \frac{1}{\gamma^2} y^T y - \nabla V \cdot (f(x) + g(x)w) - s_3(R^2 - V) - s_p p_W \in \Sigma_{n+m} \tag{31}$$

satisfy the conditions of Corollary 3.14.

For any such V and R in (29)-(31), $\tau = R$ and $Q = V$ are feasible for (26)-(28).

5.2. *Feasibility Guarantee*

Various standard results from nonlinear system theory show that properties of the linearized dynamical system carry over to local properties of the nonlinear system, for instance, exponential stability of an equilibrium point of an autonomous system. In [22], we explored whether properties of the linearized system implied the corresponding feasibility of the SOS formulation, using quadratic storage functions, for three types of problems: region-of-attraction, reachability and $\mathbf{L}_2$ gain. In [22], the vector field was limited to cubic, and the proof techniques geared toward systems of that class. In this section, we briefly show how the results can be extended to any degree vector field. For brevity, we only consider $\mathbf{L}_2$ gain, although the other results follow as well.

The results here are similar in spirit, though definitely different (and by our admission, of significantly weaker theoretical value) to the results being pursued by [14]. That work is devoted to establishing the optimality of polynomial storage functions, while only resorting to Positivestellensatz-based proofs (generalization of the simple S-procedure).

A main technical lemma will be used in the subsequent claim.

Lemma 5.1
Let $d \geq 2$ be a positive integer. Let $V(x) := x^T Q x$ with $0 \prec Q = Q^T \in \mathbf{R}^{n \times n}$. Let $r(x)$ denote the vector of all monomials of degree 1 through degree $d-1$ and $s(x) = r(x)^T r(x)$. Similarly let $z(x)$ denote the vector of all monomials of degree 2 through degree d. The length of z is denoted n_z. There exists $H \in \mathbf{R}^{n_z \times n_z}$ with $H = H^T \succ 0$ and $s(x)V(x) = z(x)^T H z(x)$.

Proof
Since $Q \succ 0$ there exists $\alpha > 0$ such that $\tilde{Q} := Q - \alpha I \succ 0$. Define the perturbed polynomial $\tilde{V}(x) := x^T \tilde{Q} x$. By assumption, $s(x) = \sum_i r_i(x)^2$ and hence

$$s(x)\tilde{V}(x) = \sum_i r_i(x)^2 x^T \tilde{Q} x = \sum_i (r_i(x)x)^T \tilde{Q} (r_i(x)x). \tag{32}$$

Each term in the sum is SOS with positive definite Gram matrix $\tilde{Q}$. Thus $s(x)\tilde{V}(x)$, being a sum of SOS terms, is itself an SOS polynomial. Since $s(x)\tilde{V}(x)$ contains all monomials of degree 4 through degree $2d$ it has a Gram matrix decomposition of the form $z(x)^T \tilde{H} z(x)$. The existence of a Gram matrix $\tilde{H} \succeq 0$ follows because $s(x)\tilde{V}(x)$ is SOS.

Finally, $s(x)V(x) = s(x)\tilde{V}(x) + \alpha \sum_i r_i(x)^2 x^T x$. $\sum_i r_i(x)^2 x^T x$ is a sum of monomials squared. The sum includes squares of all monomials in $z(x)$ possibly with repeats. Therefore this sum has a Gram matrix decomposition of the form $z(x)^T D z(x)$ where D is diagonal and positive-definite. Thus $s(x)V(x)$ has a Gram matrix decomposition $z^T(x)Hz(x)$ where $H = \tilde{H} + \alpha D \succ 0$. □

Now, write the affine-in-w system in (1) as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) + f_2(x(t)) + g_1(x(t))w(t) \\ y(t) &= Cx(t) + h_2(x(t)) \end{aligned} \tag{33}$$

where f_2, g_1 and h_2 are polynomials. They contain, respectively, terms of degree 2, 1, and 2, and higher. Let $\partial(f_2)$, $\partial(h_2)$ and $\partial(g_1)$ denote the highest degree of monomials within each function. Define

$$d := max\left\{\partial(f_2), \partial(h_2), \partial(g_1) + 1\right\}$$

Suppose the linearization has A Hurwitz, and for some $\gamma > 0$, $\left\| C(sI - A)^{-1}B \right\|_\infty < \gamma$. Then by the bounded-real lemma, there exists $P = P^T \succ 0$ such that

$$\begin{bmatrix} A^T P + PA + \frac{1}{\gamma^2} C^T C & PB \\ B^T P & -I \end{bmatrix} \prec 0.$$

Defining $V(x) := x^T P x$ and $s := \alpha r^T(x) r(x)$ leads to the main SOS constraint as

$$\begin{aligned} &2x^T P \left[Ax + Bw + f_2(x) + g_1(x)w\right] - w^T w \\ &\qquad + \frac{1}{\gamma^2}\left[x^T C^T + h_2^T(x)\right]\left[Cx + h_2(x)\right] + \alpha\left(R^2 - x^T P x\right) r^T(x) r(x). \end{aligned}$$

This is a quadratic form in $[x; w; z(x)]$ as follows. There exists a matrices F and H such that $f_2(x) = Fz(x)$ and $h_2(x) = Hz(x)$. Likewise, there exists a matrix G such that $x^T P g_1(x) w = w^T G z(x)$. By Lemma 5.1, there exists a positive-definite matrix M_V such that $r^T(x)r(x)x^T P x = z^T(x) M_P z(x)$. Finally, there exists a matrix of the form $\begin{bmatrix} I & 0 \\ 0 & \tilde{I} \end{bmatrix}$ such that

$$r^T(x) r(x) = \begin{bmatrix} x \\ z(x) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & \tilde{I} \end{bmatrix} \begin{bmatrix} x \\ z(x) \end{bmatrix}$$

Combining, the expression is

$$\begin{bmatrix} x \\ w \\ z(x) \end{bmatrix}^T \underbrace{\begin{bmatrix} A^T P + PA + \frac{1}{\gamma^2} C^T C + \alpha R^2 I & PB & PF + \frac{1}{\gamma} C^T H \\ B^T P & -I & G \\ F^T P + \frac{1}{\gamma} HC & G^T & \alpha R^2 \tilde{I} + \frac{1}{\gamma^2} H^T H - \alpha M_P \end{bmatrix}}_{K(R,\alpha)} \begin{bmatrix} x \\ w \\ z(x) \end{bmatrix}$$

At $R := 0$, the top/left 2×2 block is negative definite, and $M_P \succ 0$. Hence for α sufficiently large, it follows that $K(0, \alpha) \prec 0$. With such an α chosen, by continuity there exist nonzero R such that $K(R, \alpha) \prec 0$.

The above reasoning is summarized in a theorem.

Theorem 5.2
Assume $x = 0$ is an equilibrium of (1), and express the vector field with linear and nonlinear terms separated, as in (33). If A is Hurwitz, and $\left\| C(sI - A)^{-1}B \right\|_\infty < \gamma$, then there exists $R > 0$, $\epsilon > 0$, quadratic V and polynomial s_3 (with $s_p = 0$) such that equations (29)-(31) are feasible using $l(x) := \epsilon x^T x$.

5.3. Iteration Strategy

Equations (26)-(28) and (29)-(31) constitute a nonconvex optimization problem, namely a linear objective subject to bilinear matrix inequality constraints. Acknowledging the theoretical implications [23], [24], we nevertheless push forward with iterative schemes to generate feasible solutions, and further optimize the cost. We outline an iteration for (29)-(31). An analogous iteration is possible for (26)-(28) by replacing V with Q, R with τ, and s_3 with s_1, s_2.

1. [**Basis Choice:**] Based on the polynomials f, g and h (and their degrees), choose basis functions for the unknowns s_p, s_3 and V. At the time of the writing of this paper, computational restrictions (memory, numerical conditioning, algorithms etc.) place a practical restriction on the overall degree of the DIE polynomial in terms of the number of independent variables $(n+m)$ in the polynomials. This limits the degree of s_p, s_3 and V.
2. [**Initialize:**] If the linearization is stable, choose γ consistent with the $\mathcal{H}_\infty$ norm of the linearization, and find V, R and s_3 from standard $\mathcal{H}_\infty$ LMI calculations.
3. [**R Maximization:**] R, by choice of s_3, such that (29) and (31). Since R and s_3 multiply, this step requires a bisection in R. For each fixed value of R, determining a feasible s_3 is a linear SOS problem.
4. [**V Recenter:**] Hold s_3 fixed, and "recenter" V by finding the analytic center (in R^2, parameters of V, and parameters associated with the kernel representation of the SOS problem) of system of LMI constraints defined by equations (30) and (31).
5. Return to the [**R Maximization**] step, and repeat.

The V recentering step requires additional discussion. While the storage function V that was held fixed in the R maximization step is feasible for the constraints in the V recenter step, it is likely not at the analytic center. Moving it to the center allows R to be reliably increased at the next iteration. A more formal theory for the behavior of this feasibility step is still an open question.

6. EXAMPLES

In the following examples, we utilize the SOS optimization tool SOSOPT and associated nonlinear systems analysis software, available at `http://www.aem.umn.edu/ AerospaceControl/`. Supporting material, documentation, and additional examples can be found at [25].

6.1. Scalar Example from Section 4

We revisit the example in Section 4 and compare the performance of the algebraic approach with the SOS approach with the iteration in Section 5.3. Using quadratic V, Q, the results are compared in Figure 5. The V obtained from the $\mathbf{L}_2$ analysis is used as the shape factor function in the reachability analysis, which improves the bound. The refinement of the reachability results in a further improvement. A lower bound is provided from a power algorithm [26] that attempts to find the worst case input that achieves the induced gain of the nonlinear system.

6.2. Three State Reach Example

Consider the three state system from [27]:

$$\begin{aligned} \dot{x}_1 &= -x_3 + x_2 - x_3 x_2^2 \\ \dot{x}_2 &= -x_2 x_3^2 - x_2 + w \\ \dot{x}_3 &= \frac{1}{2}(x_1 - x_3), \end{aligned} \tag{34}$$

where $x = [x_1\ x_2\ x_3]^T \in \mathbb{R}^3$, $w \in R$. We choose $p(x) := 8x_1^2 - 8x_1x_2 + 4x_2^2 + x_3^2$. Given $\beta > 0$, and a basis for Q, we maximize τ such that the conditions of Theorem 3.1 hold and $\Omega_{Q,\tau^2} \subseteq \Omega_{p,\beta}$, which will further imply that Theorem 3.6 holds since Ω_{Q,τ^2} is bounded.

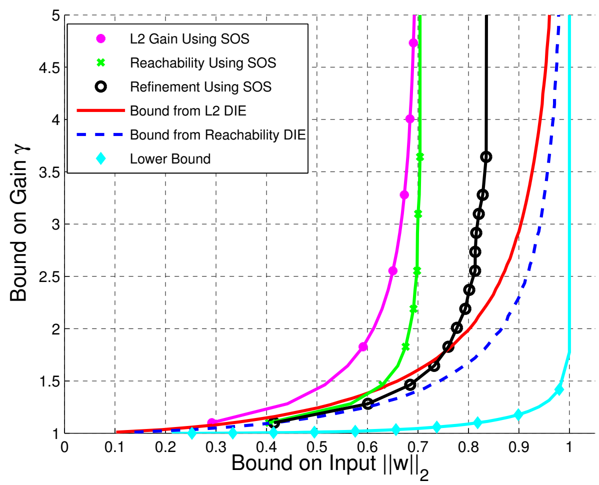

Figure 5. Comparison of the SOS approach with the algebraic approach in Section 4

We perform reachability analysis for a range of β values, followed by the refinement of the reachability bound. First a global analysis with a quartic degree Q is performed (i.e, (6) holds on $\mathbb{R}$ and $Q(x) > 0$ for all $x \neq 0$). Then a local analysis with a quadratic Q, and finally a local analysis with a quartic Q are performed. These results are compared to the system response to the worst-case input, seen in Figure 6. In both of the local analyses we restrict s_1 to be constant and s_2 to be quadratic. The various subplots show a subset of the various results for ease of comparison. The lower bound obtained from a power algorithm [26] tis shown in all three subplots. The upper left shows the induced $\mathbf{L}_2$ gains achieved with the global analysis with quartic V and the local analysis with quadratic V. The local analysis provides better bounds on the reachable set for inputs w of given $\mathbf{L}_2$ norm. This upper left subplot also shows the improvement in both the local and global bounds obtained via the refinement procedure. The upper right subplot compares the local quadratic V results with the local quartic V results. This subplot demonstrates that using higher degree storage functions improves the bounds as expected. This improvement comes at the price of increased computational time. This subplot also shows the improved bounds obtained via the refinement procedure. Finally, the lower left subplot shows the effects of imposing an additional (spatially Euclidean) $\mathbf{L}_\infty$-like constraint on w, $w(t)^T w(t) \leq 2.5^2$ for all t. The two curves show the bounds from local reachability analysis and refinement with the quartic Q. The refinement only provides a small improvement in this case. Under this restriction, the maximum of the input is restricted, thus the upper bound curves are lower than the bound from the power iteration (i.e. the "lower bound") which was computed without the restriction.

6.3. $\mathbf{L}_2$ *Induced Gain Analysis of Generic Transport Longitudinal Model*

This section performs nonlinear analyses for NASA's Generic Transport Model. NASA's Generic Transport Model (GTM) is a remote-controlled 5.5 percent scale commercial aircraft [28, 29]. The main GTM aircraft parameters are provided in Table I. NASA constructed a high fidelity 6 degree-of-freedom Simulink model of the GTM with the aerodynamic coefficients described as look-up tables. This section describes the construction of a polynomial model for the longitudinal dynamics of the GTM. This polynomial model is then used to estimate the $\mathbf{L}_2$ gain for the open-loop longitudinal dynamics as well as a closed-loop model with a simple pitch rate feedback law. Details on the polynomial modeling are provided in [30] and files containing the model can be found at [25].

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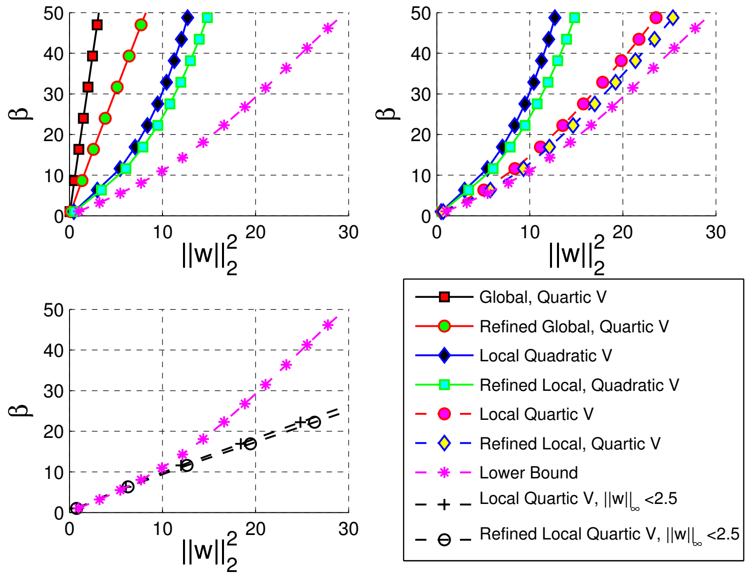

Figure 6. Estimations of global and local reachability estimations are performed on the dynamics in (34) with quadratic and quartic Q and with $\|w\|_\infty \leq 2.5$

Table I. Aircraft and Environment Parameters

Wing Area, S	5.902 ft^2
Mean Aerodynamic Chord, $\bar{c}$	0.9153 ft
Mass, m	1.542 slugs
Pitch Axis Moment of Inertia, I_{yy}	4.254 slugs-ft^2
Air Density, ρ	0.002375 slugs/ft^3
Gravity Constant, g	32.17 ft/s^2

The longitudinal dynamics of the GTM are described by a standard four-state longitudinal model [31]:

$$
\begin{aligned}
\dot{V} &= \frac{1}{m}\left(-D - mg\sin(\theta-\alpha) + T_x\cos\alpha + T_z\sin\alpha\right)\\
\dot{\alpha} &= \frac{1}{mV}\left(-L + mg\cos(\theta-\alpha) - T_x\sin\alpha + T_z\cos\alpha\right) + q\\
\dot{q} &= \frac{(M+T_m)}{I_{yy}}\\
\dot{\theta} &= q
\end{aligned}
$$

where V is the air speed (ft/s), α is the angle of attack (rad), q is the pitch rate (rad/s) and θ is the pitch angle (rad). The control inputs are the elevator deflection δ_{elev} (deg) and engine throttle δ_{th} (percent).

The drag force D (lbs), lift force L (lbs), and aerodynamic pitching moment M (lb-ft) are given by:

$$
\begin{aligned}
D &= \bar{q}SC_D(\alpha, \delta_{elev}, \hat{q})\\
L &= \bar{q}SC_L(\alpha, \delta_{elev}, \hat{q})\\
M &= \bar{q}S\bar{c}C_m(\alpha, \delta_{elev}, \hat{q})
\end{aligned}
$$

where $\bar{q} := \frac{1}{2}\rho V^2$ is the dynamic pressure (lbs/ft^2) and $\hat{q} := \frac{\bar{c}}{2V}q$ is the normalized pitch rate (unitless). C_D, C_L, and C_m are unitless aerodynamic coefficients computed from look-up tables provided by NASA.

The GTM has one engine on the port side and one on the starboard side of the airframe. Equal thrust settings for both engines is assumed. The thrust from a single engine T (lbs) is a function of the throttle setting δ_{th} (percent). $T(\delta_{th})$ is specified as a ninth-order polynomial in NASA's high fidelity GTM simulation model. T_x (lbs) and T_z (lbs) denote the projection of the total engine thrust along the body x-axis and body-z axis, respectively. T_m (lbs-ft) denotes the pitching moment due to both engines. T_x, T_z and T_m are given by:

$$\begin{aligned} T_x(\delta_{th}) &= n_{ENG}T(\delta_{th})\cos(\epsilon_2)\cos(\epsilon_3) \\ T_z(\delta_{th}) &= n_{ENG}T(\delta_{th})\sin(\epsilon_2)\cos(\epsilon_3) \\ T_m(\delta_{th}) &= r_z T_x(\delta_{th}) - r_x T_z(\delta_{th}) \end{aligned}$$

$n_{ENG} = 2$ is the number of engines. $\epsilon_2 = 0.0375$ rad and $\epsilon_3 = -0.0294$ rad are angles that specify the rotation from engine axes to the airplane body axes. $r_x = 0.4498$ ft and $r_z = 0.2976$ ft specify the moment arm of the thrust.

The following terms of the longitudinal are approximated by low-order polynomials:

1. Trigonometric functions: $\sin(\alpha)$, $\cos(\alpha)$, $\sin(\theta - \alpha)$, $\cos(\theta - \alpha)$
2. Engine model: $T(\delta_{th})$
3. Rational dependence on speed: $\dfrac{1}{V}$
4. Aerodynamic coefficients: C_D, C_L, C_m

Constructing polynomial approximations for the trigonometric functions, engine model, and rational dependence on speed is relatively straight-forward. The trigonometric functions are approximated by Taylor series expansions: $\sin z \approx z - \frac{1}{6}z^3$ and $\cos z \approx 1 - \frac{1}{2}z^2$ for z in units of radians. For the engine model, a least squares technique is used to approximate the ninth order polynomial function $T(\delta_{th})$ by a third order polynomial. The least squares technique is also used to compute a linear fit to $\dfrac{1}{V}$ over the desired range of interest from 100 ft/s to 200 ft/s. Finally, polynomial least squares fits are computed for the aerodynamic coefficient look-up tables provided by NASA. A degree seven polynomial model is obtained after replacing all non-polynomial terms with their polynomial approximations and is provided in [30]. The quality of the polynomial approximation was assessed by comparing the trim conditions and simulation responses of the polynomial model and the original model with look-up tables. The polynomial model takes the form:

$$\dot{x} = f(x, u) \tag{35}$$

where $x := [V(\text{ft/s}),\ \alpha(\text{rad}),\ q(\text{rad/s}),\ \theta(\text{rad})]$, and $u := [\delta_{elev}(\text{deg}),\ \delta_{th}(\%)]$.

The $\mathbf{L}_2 \to \mathbf{L}_2$ gain analysis is performed around a level flight condition with speed $V = 150$ ft/s:

$$\begin{bmatrix} V_t \\ \alpha_t \\ q_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} 150 \text{ ft/s} \\ 0.047 \text{ rad} \\ 0 \text{ rad/s,} \\ 0.047 \text{ rad} \end{bmatrix}, \begin{bmatrix} \delta_{elev,t} \\ \delta_{th,t} \end{bmatrix} = \begin{bmatrix} 14.78\ \% \\ 0.051 \text{ rad} \end{bmatrix} \tag{36}$$

The subscript "t" denotes a trim value. A cubic order polynomial longitudinal model is extracted from the 4-state, degree seven polynomial model, Equation (35), by holding δ_{th} at its trim value and retaining terms up to cubic order. The cubic order model has 4 states $[V,\ \alpha,\ q,\ \theta]^T$ and one input δ_{elev}.

The open loop dynamics of the GTM are slightly underdamped. Inner loop pitch rate feedback is typically used to improve the damping of the aircraft. The following proportional pitch rate feedback is used to improve the damping of the GTM aircraft:

$$\delta_{elev} = K_q q + d_{elev} = 0.0698q + d_{elev} \tag{37}$$

DOI: 10.1002/rnc

where d_{elev} denote an input disturbances at the elevator channel.The remainder of the section describes the estimation of the $\mathbf{L}_2$ gain bounds for both the open-loop and closed-loop longitudinal dynamics of the GTM. The $\mathbf{L}_2 \to \mathbf{L}_2$ gain from elevator disturbances (d_{elev}) to pitch rate (q) is estimated for the GTM dynamics.

Figure 7 indicates how the induced gain of the system varies as the size of the elevator disturbances $||d_{elev}||_2$ increases. The horizontal axis indicates the size of the elevator disturbances, $||d_{elev}||_2$, around the trim input value and the vertical axis shows the estimated bounds of the induced gain from d_{elev} to q. For the linearization, the induced gain for the open-loop system is 23.9. Pitch rate feedback reduces the induced gain of the closed-loop system to 16.6.

For several fixed values of γ, the upper bound of the allowable $\mathbf{L}_2$ input is estimated for both the open and closed loop system by maximizing R over the choice of V and s_3 such that (29)-(31) hold using the iteration strategy described in Section 5.3. This analysis is run on both a quadratic (dashed line, black '$\diamondsuit$', open loop; dashed line, red 'x', closed loop) and a quartic (solid line, black '$\diamondsuit$', open loop; solid line, blue 'x', closed loop) choice for V.

On the closed-loop system, the reachability analysis is performed by setting V as the shape factor function and maximizing τ over choice of Q, s_1 and s_2 such that (26)-(28) hold, also restricting the degree of Q to be quadratic (dashed line, red 'o') and quartic (solid line, blue 'o'). Finally, the refinement is performed on the quadratic (dashed line, red '+'), and quartic (solid line, blue '+') results from the reachability. It is clear from the figure that the bound on the allowable input is improved from the reachability analysis, and improved further by the refinement procedure. As expected, the upper bounds on the gain using quartic V and Q are improvements over their quadratic counterparts.

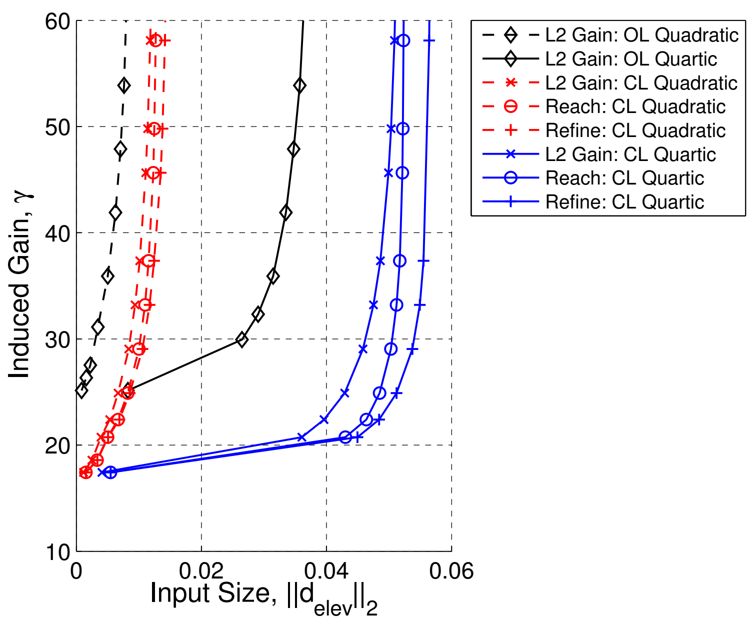

Figure 7. Estimation of induced $\mathbf{L}_2 \to \mathbf{L}_2$ gain bounds from d_{elev} to q for closed-loop GTM model.

7. REACHABILITY AND GAIN ESTIMATES FOR UNCERTAIN SYSTEMS

The previous sections developed conditions and computational algorithms to estimate bounds on the reachable set and $\mathbf{L}_2$ gain of a known nonlinear system. However, the model is an approximation of the true system dynamics and hence it is useful to quantify the effect of uncertainty. This section

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extends the reachability and $\mathbf{L}_2$ gain conditions in Theorems 3.1 and 3.11 to systems with dynamic uncertainty. The uncertainty is modeled in the standard linear fractional transformation framework.

Consider the dynamics of a multivariable, nonlinear system, G

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g_1(x(t))w_1(t) + g_2(x(t))w_2(t), \\ z_1(t) & = & h_1(x(t)), \\ z_2(t) & = & h_2(x(t)), \end{array} \tag{38}$$

where $x(t) \in \mathbb{R}^n$, $z_1(t) \in \mathbb{R}^{p_1}$, $z_2(t) \in \mathbb{R}^{p_2}$, $w_1(t) \in \mathbb{R}^{m_1}$, and $w_2(t) \in \mathbb{R}^{m_2}$. For notational simplicity, define $\tilde{f} : \mathbb{R}^{n \times m_1 \times m_2} \to \mathbb{R}^n$ as $\tilde{f}(x, w_1, w_2) := f(x) + g_1(x)w_1 + g_2(x)w_2$. Let $\Delta : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ be a causal operator that forms a well-posed feedback interconnection with G through

$$w_2(t) = (\Delta(z_2))\,(t) \tag{39}$$

as shown in Figure 8. By well-posedness of the interconnection, we mean that for any $w_1 \in \mathbf{L}_{2e}$, and any initial condition x_0, there exists unique $w_2 \in \mathbf{L}_{2e}$ and absolutely continuous functions x, z_1 and z_2 satsifying equations (38) and (39). This is an assumption about the interaction of G and Δ. It is true, for instance, if Δ is governed by nonlinear ODEs of the form

$$\begin{aligned} \dot{\xi}(t) &= a(\xi(t)) + b(\xi(t))z_2(t) \\ w_2(t) &= c(\xi(t)) + d(\xi(t))z_2(t) \end{aligned}$$

for Lipschitz continuous functions a, b, c and d.

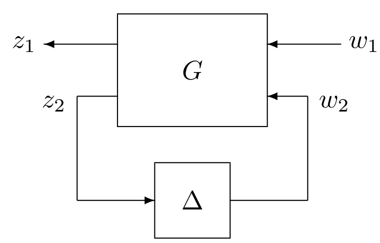

Figure 8. G-Δ interconnection.

If Δ is dissipative with respect to a collection $\{r_i\}_{i=1}^N$ of supply rates, then for all i,

$$\int_0^T r_i(z_2(t), w_2(t))dt \geq 0 \tag{40}$$

for all $z_2 \in \mathbf{L}_{2,e}$ with $w_2 = \Delta(z_2)$, and all $T > 0$.

The following proposition analyzes the interconnection, establishing $\mathbf{L}_2$ gain bounds from w_1 to z_1 valid for all operators Δ that are dissipative with respect to the supply rates $r_1, \ldots, r_N$.

Proposition 7.1

Suppose $\mathcal{W}_1 \subseteq \mathbb{R}^{m_1}$. Assume that f, g_1, g_2, h_1, and h_2 in (38) are Lipschitz continuous. Let $r_1, \ldots, r_N : \mathbb{R}^{p_2 \times m_2} \to \mathbb{R}$. Suppose there exist constant $\tau > 0, R > 0$, positive semidefinite functions $\lambda_1, \ldots, \lambda_N : \mathbb{R}^n \to \mathbb{R}$, and $\beta_1, \ldots, \beta_N : \mathbb{R}^n \to \mathbb{R}$, differentiable functions $Q : \mathbb{R}^n \to \mathbb{R}$

and $V : \mathbb{R}^n \to \mathbb{R}$ that satisfy $Q(0) = V(0) = 0$,

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2} \tag{41}$$

$$\Omega^{cc,0}_{V,R^2} \setminus 0 \subseteq \{x \in \mathbb{R}^n : V(x) > 0\} \tag{42}$$

$$\Omega^{cc,0}_{Q,\tau^2} \subseteq \left\{ x \in \mathbb{R}^n : \nabla Q(x) \cdot \tilde{f}(x, w_1, w_2) \right. \\ \left. \leq w_1^T w_1 - \sum_{i=1}^{N} \lambda_i(x) r_i(h_2(x), w_2),\ \forall (w_1, w_2) \in \mathcal{W}_1 \times \mathbb{R}^{m_2} \right\} \tag{43}$$

$$\Omega^{cc,0}_{V,R^2} \subseteq \left\{ x \in \mathbb{R}^n : \nabla V(x) \cdot \tilde{f}(x, w_1, w_2) \right. \\ \left. \leq w_1^T w_1 - \frac{1}{\gamma^2} h_1^T(x) h_1(x) - \sum_{i=1}^{N} \beta_i(x) r_i(h_2(x), w_2),\ \forall (w_1, w_2) \in \mathcal{W}_1 \times \mathbb{R}^{m_2} \right\}. \tag{44}$$

Consider $x_0 \in \Omega^{cc,0}_{Q,\tau^2}$ with $Q(x_0) < \tau^2$, $T > 0$, and $w_1 \in \mathbf{L}_2$ with $w_1(t) \in \mathcal{W}_1$ for all t. If $\|w_1\|^2_{2,T} < \tau^2 - Q(x_0)$, then the solution to (38)-(39) with $x(0) = x_0$ satisfies $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \in [0, T]$, and

$$\|z_1\|^2_{2,T} \leq \gamma^2 V(x_0) + \gamma^2 \|w_1\|^2_{2,T} \tag{45}$$

for all Δ dissipative with respect to the supply rates $r_1, \ldots, r_N$.

This theorem essentially an extension of Theorem 3.15 for robust performance. We use reachability analysis using Q to ascertain bounds on w which keep $x(t) \in \Omega^{cc,0}_{V,R^2}$. In this context, $\Omega^{cc,0}_{V,R^2}$ plays the role of the shape factor function introduced at the end of Section 3.3.

Proof
First, address the invariance of $\Omega^{cc,0}_{Q,\tau^2}$ under the conditions imposed on x_0 and w_1. Suppose it is not invariant, hence there is a finite T such that $Q(x(T)) = \tau^2$ and $Q(x(t)) < \tau^2$ for all $t < T$. Hence $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$ for all $t \leq T$. Integrate $\dot{Q}$ from 0 to T, using that $x(t) \in \Omega^{cc,0}_{Q,\tau^2}$, and $\Omega^{cc,0}_{Q,\tau^2}$ is contained in a region where $\nabla Q \cdot \tilde{f}$ has useful, known properties. This yields

$$\begin{aligned} Q(x(T)) - Q(x_0) &\leq \|w_1\|^2_{2,T} - \int_0^T \lambda_i r_i dt \\ &\leq \|w_1\|^2_{2,T}. \end{aligned}$$

But $Q(x(T)) = \tau^2$, therefore $\|w_1\|^2_{2,T} \geq \tau^2 - Q(x_0)$. Hence $\Omega^{cc,0}_{Q,\tau^2}$ is invariant under the conditions imposed on x_0 and w_1. Recall $\Omega^{cc,0}_{Q,\tau^2} \subseteq \Omega^{cc,0}_{V,R^2}$, so the solutions remain in $\Omega^{cc,0}_{V,R^2}$ as well. Integrating $\dot{V}$ gives

$$\begin{aligned} V(x(T)) - V(x_0) &\leq \|w_1\|^2_{2,T} - \tfrac{1}{\gamma^2} \|z_1\|^2_{2,T} - \int_0^T \beta_i r_i dt \\ &\leq \|w_1\|^2_{2,T} - \tfrac{1}{\gamma^2} \|z_1\|^2_{2,T}. \end{aligned}$$

Since $V \geq 0$ on $\Omega^{cc,0}_{V,R^2}$, $V(x(T)) \geq 0$, and therefore $\|z_1\|^2_{2,T} \leq \gamma^2 V(x_0) + \gamma^2 \|w_1\|^2_{2,T}$ as claimed. □

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9. APPENDIX

9.1. Lipschitz Extensions

Locally Lipschitz continuous functions can be extended to globally Lipschitz continuous functions as follows [32, 33].

Lemma 9.1
Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous on $\mathcal{B} \subseteq \mathbb{R}^n$ with $\mathcal{B} \neq \emptyset$ and Lipschitz constant L. For each $x \in \mathbb{R}^n$, define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$

$$\tilde{f}(x) := \min_{y \in \mathcal{B}} \ f(y) + L \left\| x - y \right\| . \tag{46}$$

Then $\tilde{f}(x) = f(x)$ for all $x \in \mathcal{B}$. Moreover, $\tilde{f}$ is globally Lipschitz continuous (with Lipschitz constant L).

Proof
For $x \in \mathcal{B}$, clearly $\tilde{f}(x) \leq f(x)$. Using that f is Lipschitz continuous on $\mathcal{B}$ gives $f(x) \leq f(y) + L \left\| x - y \right\|$ for all $y \in \mathcal{B}$. Minimizing the right-hand side over $y \in \mathcal{B}$ preserves the inequality, hence $f(x) \leq \tilde{f}(x)$, which implies $\tilde{f}(x) = f(x)$.

For any $x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^n$, and all $z \in \mathbb{R}^n$, the triangle inequality gives

$$f(z) + L \left\| x_1 - z \right\| \leq f(z) + L \left\| x_2 - z \right\| + L \left\| x_1 - x_2 \right\| .$$

Minimize both sides of this expression over $z \in \mathcal{B}$ to get

$$\tilde{f}(x_1) \leq \tilde{f}(x_2) + L \left\| x_1 - x_2 \right\| . \tag{47}$$

Reversing the role of x_1 and x_2 gives

$$\tilde{f}(x_2) \leq \tilde{f}(x_1) + L \left\| x_2 - x_1 \right\| . \tag{48}$$

Combining (47) and (48) gives $\left| \tilde{f}(x_1) - \tilde{f}(x_2) \right| \leq L \left\| x_1 - x_2 \right\|$ as desired. □

9.2. Sum-of-squares programming and S-procedure

Definition 9.2
A **Monomial** m_α in n variables is a function defined as $m_\alpha(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{Z}_+^n$. The degree of a monomial is defined, $\deg m_\alpha := \sum_{i=1}^n \alpha_i$.

Definition 9.3
A **Polynomial** f in n variables is a finite linear combination of monomials, with $c_\alpha \in \mathbb{R}$:

$$f := \sum_\alpha c_\alpha m_\alpha = \sum_\alpha c_\alpha x^\alpha$$

Define $\mathcal{R}_n$ to be the set of all polynomials in n variables. The degree of f is defined as $\deg f := \max_\alpha \deg m_\alpha$ (provided the associated c_α is non-zero).

We use the notation Σ_n to be the set of sum of squares (SOS) polynomials in n variables.

$$\Sigma_n := \left\{ p \in \mathcal{R}_n \,|\, p = \sum_{i=1}^t f_i^2 \quad , \ f_i \in \mathcal{R}_n, \ i = 1, \ldots, t \right\}$$

Obviously if $p \in \Sigma_n$, then $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$. The particular variables are not noted, and usually there is an obvious $n-$dimensional variable present in the discussion. Similarly, the notation Σ_{n+m} also appears, meaning SOS polynomials in $n + m$ real variables, where, again, the particular variables are hopefully clear from the context of the discussion.

The following lemma is a generalization of the well known $\mathcal{S}$-procedure [13], and is a special case of the Positivstellensatz Theorem [34, Theorem 4.2.2].

Lemma 9.4 (Generalized $\mathcal{S}$-procedure)
Given $\{p_i\}_{i=0}^m \in \mathcal{R}_n$. If there exist $\{s_k\}_{i=1}^m \in \Sigma_n$ such that $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, then

$$\bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, , p_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n \,|\, p_0(x) \geq 0\}.$$

Proof
Since $p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n$, so $p_0 \geq \sum_{i=1}^m s_i p_i \ \forall x$. For any $\bar{x} \in \bigcap_{i=1}^m \{x \in \mathbb{R}^n \,|\, p_i(x) \geq 0\}$, since $s_i(\bar{x}) \geq 0$, so $\sum_{i=1}^m s_i p_i \geq 0$, hence $p_0(\bar{x}) \geq 0$. □

9.3. *Proof that 3.9 and Definition 3.10 are equivalent*

⇒ Assume there exists a positive constant $\rho > 0$ and a function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that for all x_0, $\phi(x_0) \geq 0$ is a nonnegative constant and (13) holds for all x_0, all $w \in \mathbf{L}_2$, and all $T \geq 0$. Squaring both sides of (13) yields

$$\begin{aligned} \|y\|_{2,T}^2 &\leq (\phi(x_0) + \rho\|w\|_{2,T})^2 \\ &= \phi^2(x_0) + \rho^2\|w\|_{2,T}^2 + 2\rho\phi(x_0)\|w\|_{2,T}. \end{aligned}$$

The bound $2\rho\phi(x_0)\|w\|_{2,T} \leq \phi^2(x_0) + \rho^2\|w\|_{2,T}^2$ can be used to upper bound the right hand side of (49) yielding:

$$\|y\|_{2,T}^2 \leq 2\phi^2(x_0) + 2\rho^2\|w\|_{2,T}^2.$$

Hence, (14) holds for all $w \in \mathbf{L}_2$ and for all $T \geq 0$ with $\gamma = \sqrt{2}\rho$ and $\psi(x_0) = 2\phi^2(x_0)$.

⇐ Assume there exists a positive constant $\gamma > 0$ and a function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that for all x_0, $\psi(x_0) \geq 0$ is a nonnegative constant and (14) holds for all x_0, all $w \in \mathbf{L}_2$, and all $T \geq 0$. Since $\psi(x_0)$ and γ are nonnegative,

$$\begin{aligned} \|y\|_{2,T}^2 &\leq \psi(x_0) + \gamma^2\|w\|_{2,T}^2 \\ &\leq \psi(x_0) + \gamma^2\|w\|_{2,T}^2 + 2\gamma\sqrt{\psi(x_0)}\|w\|_{2,T} \\ &= \left(\sqrt{\psi(x_0)} + \gamma\|w\|_{2,T}\right)^2 \end{aligned}$$

Taking square roots of both sides of the inequality recovers (13) with $\rho = \gamma$ and $\phi(x_0) = \sqrt{\psi(x_0)}$.
□