Particle, Kinetic, and Hydrodynamic Models of Swarming

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Abstract. We review the state-of-the-art in the modelling of the aggregation and collective behavior of interacting agents of similar size and body type, typically called swarming. Starting with individual-based models based on "particle"-like assumptions, we connect to hydrodynamic/macroscopic descriptions of collective motion via kinetic theory. We emphasize the role of the kinetic viewpoint in the modelling, in the derivation of continuum models and in the understanding of the complex behavior of the system.

Keywords. swarming, kinetic theory, particle models, hydrodynamic descriptions, mean fields, pattern formation

1 Introduction

Everyone at some point in his life has been surprised and astonished by the observation of beautiful swinging movements of certain animals such as birds (starlings, geese,...), fishes (tuna, capelin,...), insects (locusts, ants, bees, termites,...) or certain mammals (wildebeasts, sheep,...). These coherent and synchronized structures are apparently produced without the active role of a leader in the grouping, phenomena denominated self-organization [CDFSTB03, PE99, BCCCCGLOPPVZ09] and it has been reported even for some microorganisms such as myxobacteria [KW98].

Most of the basic models proposed in the literature are based on discrete models [VCBCS95, GC04, CKFL05, YEECBKMS09] incorporating certain effects that we might call the "first principles" of swarming. These first principles are based on modelling the "sociological behavior" of animals with very simple rules such as the social tendency to produce grouping (attraction/aggregation), the inherent minimal space they need to move without problems and feel comfortably inside the group (repulsion/collisional avoidance) and the mimetic adaptation or synchronization to a group (orientation/alignment). They model animals as

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simple particles following certain microscopic rules determined by their position and velocity inside the group and by the local density of animals. These rules incorporate the "sociological" or "behavioral" component in the modelling of the animals movement. Even if these minimal models contain very basic rules, the patterns observed in their simulation and their complex asymptotic behavior is already very challenging from the mathematical viewpoint.

The source of this tendency to aggregation can also be related to other factors rather than sociological as survival fitness of grouping against predators, collaborative effort in food finding, etc. Moreover, we can incorporate other interaction mechanisms between animals as produced by certain chemicals, pheromone trails for ants, the interest of the group to stay close to their roost, physics of swimming/flying, etc. Although the minimal models based on "first principles" are quite rich in complexity, it is interesting to incorporate more effects to render them more realistic, see [BCCCCGLOPPVZ09, HH08, BEBSVPSS09] for instance.

Along this work, we take as reference models the one proposed in [MEBS03, LRC00, DCBC06] for self-propelled particles with attraction and repulsion effects and the simple model of alignment in [CS07, CS07-2]. On the one hand, the authors in [DCBC06] classify the different "zoology" of patterns: translational invariant flocks, rotating single and double mills, rings and clumps; for different parameter values. On the other hand, in the simpler alignment models [CS07, CS07-2], we get generically a flocking behavior. We will elaborate starting from these basic bricks to conclude with continuum models capable of simulating the collective behavior in case of analyzing systems with a large number of agents N. Control of large agent systems are important not only for the somehow bucolic example of the animal behavior but also for pure control engineering for robots and devices with the aim of unmanned vehicle operation and their coordination, see [CHDB07, CS07, PGE09] and the references therein.

When the number of agents is large as in migration of fish [YBTBG08] or in myxobacteria [PE99, KW98], the use of continuum models for the evolution of a density of individuals becomes essential. Some continuum models were derived phenomenologically [TT95, TB04, TBL06, BCM07] including attraction-repulsion mechanisms through a mean force and spatial diffusion to deal with the anti-crowding tendency. Other continuum models are based on hydrodynamic descriptions [CDMBC07, CDP09] derived by means of studying the fluctuations or the mean-field particle limits. The essence of the kinetic modelling is that it does connect the microscopic world, expressed in terms of particle models, to the macroscopic one, written in terms of continuum mechanics systems. A very recent trend of research has been launched in this direction in the last few years, see for instance [DM08, DM08-2, HT08, CDP09, CFRT09] for different kinetic models in swarming and [HL09, CDP09, CCR, FHT09] for the particle to hydrodynamics passage. The analysis, asymptotic behavior, numerical simulation, pattern formation and their stability in many of these models still remain a unexplored research territory.

This book chapter is organized as follows: first we concentrate on particle models, then we consider the next level of modelling going to the kinetic equations of multiagent systems in section 3 and finally we discuss some hydrodynamic descriptions of these systems. The objective of this book chapter is to make a review of the state-of-the-art in this interesting new topic of research with an emphasis in the kinetic model viewpoint.

2 Particle Models

In the following we detail the description of particle models: first those based on the combination of self-propelling, friction and attraction-repulsion phenomena, and then those based on the Cucker–Smale description.

2.1 Regions of Influence

In collective motion of groups of animals, three fundamental regions of influence are distinguished, see [Ao82, HW92]. The first region is characterized by the tendency of moving apart from another individual in near proximity, typically in order to avoid physical collision or being of mutual obstacle. In the immediate further proximity, this region of *avoidance or*

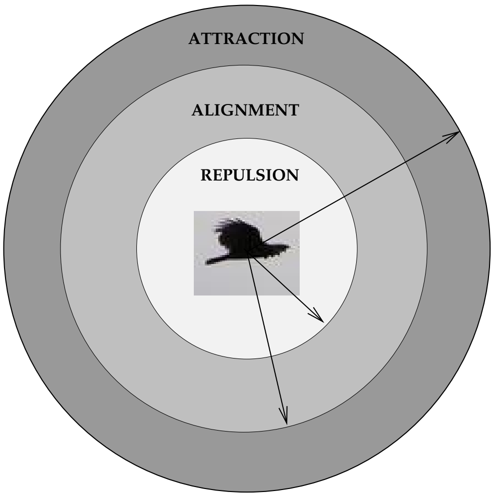

Figure 1: Three-zone model

repulsion is then substituted by a region of *orientation or alignment*, where the individual tries to identify the possible direction of the group and to align with it. When the individual finds itself too far apart from the group, it will try to reach the others which are located in the far distance region of *attraction*, see Fig. 1 for a scheme of this model. The strength of these different effects depends on the distance, and on the number of other individuals located instantaneously in the different zones; hence the direction of the individual will be changed according to the weighted superposition of these effects.

In this review, our starting point consists of some particle models, also called Individual-Based models, incorporating attraction, repulsion, and orientation mechanisms. However, these are certainly simplified models which do not take into account several other important aspects: cone of visibility, closed-neighbor interaction, noise, roosting, aerodynamics, etc. It is worth mentioning that these three region models have been improved by adding many of these different effects and analysed for different type of animals and particular species; see for instance the works of theoretical biologists, applied mathematicians and physicists in [CKJRF02, MEBS03, KH03, HK05, KZH06, VPG04, VPG05, BTTYB09, LRC00, LLE08, EVL07], the works for fish [Bi07, BEBSVPSS09, YBTBG08, HH08] with the aim of studying migration patterns for the capelin around Iceland, and the recent study [BCCCCGLOPPVZ09, HCH] for birds, focusing on starlings aggregation patterns in Rome.

In the first part of this chapter we address particle, kinetic, and hydrodynamic models for describing mathematically the basic three-zones model. Despite their simplicity, these minimal models show how the combination of these simple rules can produce striking phenomena, such as pattern formations: flocks and single or double mills, as indeed in nature we can observe, see Fig. 2.

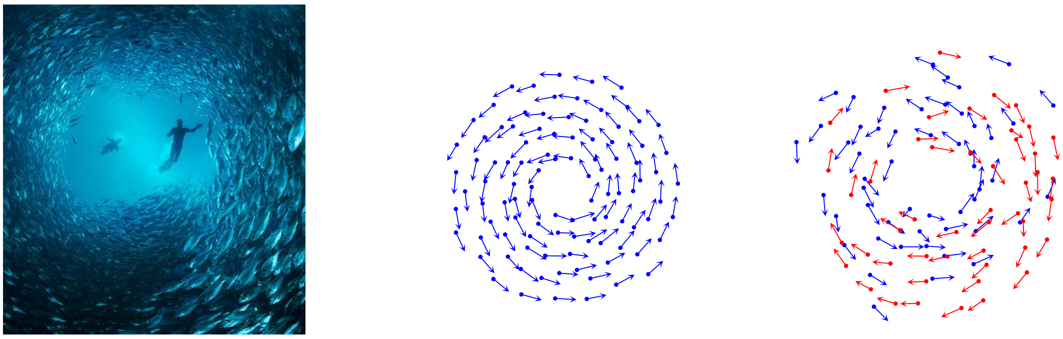

Figure 2: Mills in nature and in models!

In the second part of this chapter, we discuss several improvements in the modeling including other important aspects, like the fact that animals are not influenced simultaneously by all the other individuals, either because they have a special cone of visibility or because, in a very short time, they can simply track the motion of a relatively small number of other individuals; for example it is estimated statistically that certain birds, for instance starlings, decide their direction according to mutual topological rather than metrical distance, and in particular to the behavior of the proximal 6-7 birds only [BCCCCGLOPPVZ09]. This number is perhaps sufficiently small to allow the individual bird to take quickly a decision, and sufficiently large to avoid that the group easily split. Another effect is due to non-deterministic behaviors, simply related to the impossibility of an individual to apply the "ideal rule" in a very short time, these leads to small random errors in some of the decisions taken.

2.2 Self-Propelling, Friction, and Attraction–Repulsion Model

The first model we review, as proposed in [DCBC06], is given as follows:

Self-Propelling, Friction, and Attraction–Repulsion Particle Model:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = (\alpha - \beta|v_i|^2)v_i - \dfrac{1}{N}\displaystyle\sum_{j\neq i} \nabla U(|x_i - x_j|) \ , \end{cases} \tag{2.1}$$

for $i = 1, \ldots, N$, where α, β are nonnegative parameters, $U : \mathbb{R}^d \to \mathbb{R}$ is a given potential modelling the short-range repulsion and long-range attraction, and N is the number of individuals.

The term associated to α models the *self-propulsion* of individuals, whereas the term corresponding to β represents a *friction* following the Rayleigh's law. The combination and the balance of these two terms result in the tendency of the system (if we ignore the effect due to the repulsion and attraction term) to reach the asymptotic speed $|v| = \sqrt{\alpha/\beta}$, however not influencing the orientation of the velocities. A typical choice for U is the Morse potential which is radial and given by

$$U(x) = k(|x|), \quad k(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R} \ , \tag{2.2}$$

where C_A, C_R and ℓ_A, ℓ_R are the strengths and the typical lengths of attraction and repulsion, respectively. The most relevant situations for biological applications are determined

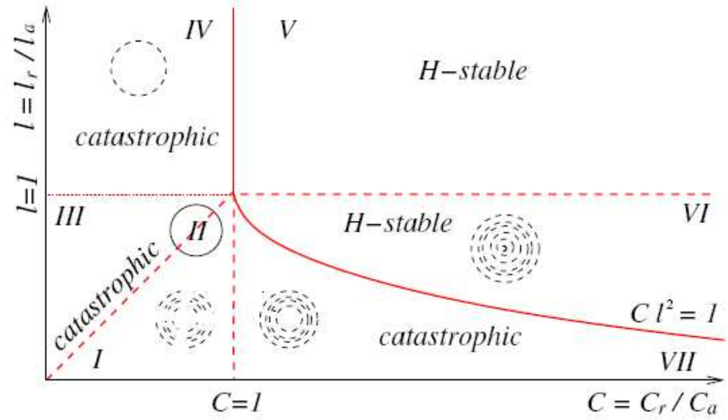

Figure 3: Classification of different regions corresponding to the model in [DCBC06], figure reprinted with the permission of authors.

for $C := C_R/C_A > 1$ and $\ell = \ell_R/\ell_A < 1$, which correspond to long-range attraction and short-range repulsion.

These quantities rule the combination of both the effects, resulting in different regimes [DCBC06]. In particular, there are two fundamental observable situations: the first of stability, for $C\ell^d > 1$, where individuals form crystalline-like patters. Here, for N sufficiently large, particles find an optimal spacing and maintain a fixed relative distance from each other; in the second regime, for $C\ell^d < 1$, so-called *catastrophic*, individuals tend to a rotational motion of constant speed, if initially well separated, $|v| = \sqrt{\alpha/\beta}$, and single or double mills are observed, see Fig. 2. The area of these rotating patterns stabilizes as N goes to infinity, see [CDP09] and the discussion therein. The regions determining these regimes are readily classified in the graphics of Fig. 3 in two dimensions, in terms of the fundamental parameters $C = C_R/C_A$ and $\ell = \ell_R/\ell_A$. Let us remark that the potential is not scaled by $1/N$ in [DCBC06], this fact does not affect the type of patterns obtained in each regime but their behavior for large N is different, more comments in [CDP09]. We have presented the model with the scaled interaction potential to be able to get a meaningful limit as $N \to \infty$ while keeping the total mass of the system finite.

2.3 The Cucker–Smale Model

The model proposed by Cucker–Smale in [CS07, CS07-2] takes into account only an alignment mechanism of the individuals by adjusting/averaging their relative velocities with all the others. The strength of this averaging process depends on the mutual distance, and closer individuals have more influence than the far distance ones. For a system of N individuals this model is described by the following dynamical system:

The Cucker–Smale Particle Model of Flocking:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} H(|x_i - x_j|)(v_j - v_i) \ , \end{cases} \tag{2.3}$$

for $i = 1, \ldots, N$, where the *communication rate* H is given by

$$H(x) = a(|x|), \quad a(r) = \frac{K}{(\varsigma^2 + |r|^2)^\gamma} \ ,$$

for positive parameters K, ς, and $\gamma \geq 0$.

For $x, v \in \mathbb{R}^{d\times N}$, denote

$$\Gamma(x) = \frac{1}{2}\sum_{i\neq j} |x_i - x_j|^2 \ , \tag{2.4}$$

and

$$\Lambda(v) = \frac{1}{2}\sum_{i\neq j} |v_i - v_j|^2 \ . \tag{2.5}$$

Then, different regimes are achieved in this model too, depending on the parameters K, ς, γ.

Theorem 1 ([CS07, CS07-2]) *Assume that one of the following conditions holds.*

(i) $\gamma < 1/2$*;*

(ii) $\gamma \geq 1/2$ *and, for a suitable constant* $\nu \geq 1/3$*,*

$$\left[\left(\frac{1}{2\gamma}\right)^{\frac{1}{2\gamma-1}} - \left(\frac{1}{2\gamma}\right)^{\frac{2\gamma}{2\gamma-1}}\right]\left(\frac{(\nu K)^2}{8N^2\Lambda(0)}\right)^{\frac{1}{2\gamma-1}} > 2\Gamma(0) + \varsigma^2 \ ,$$

then there exists a constant B_0 such that $\Gamma(x(t)) \leq B_0$ for all $t \in \mathbb{R}$, while $\Lambda(v(t))$ converges towards zero as $t \to \infty$, and the vectors $x_i - x_j$ tend to a limit vector $\widehat{x}_{ij}$, for all $i, j \leq N$.

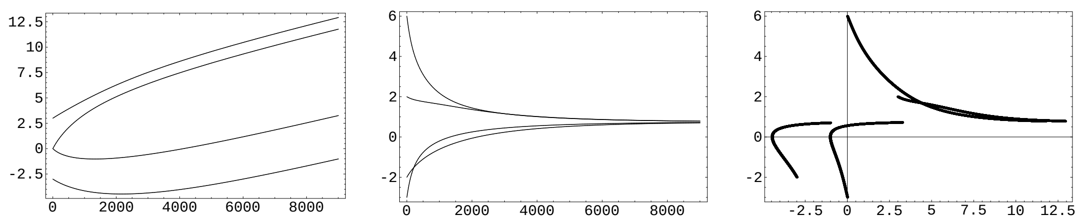

Figure 4: We illustrate in these figures the dynamics of four particles in 1D, i.e., ($d = 1$), as modeled by (2.3) for $\gamma \leq 1/2$, with initial positions and velocities $(0, 6)$, $(3, 2)$, $(-3, -2)$, and $(0, -3)$. Left: dynamics of positions, Center: dynamics of velocities, Right: dynamics in phase-space.

Let us remark that the theorem was improved in [HL09, CFRT09] in the case $\gamma = 1/2$ for any initial data. Let us also note that the regime $\gamma \leq 1/2$ neither depends on N, d nor on the initial configuration of the system; in this case, called *unconditional flocking*, the behavior of the population is perfectly specified, all the individuals tend to move with the same mean velocity and to form a group with fixed mutual distances, but not necessarily in a crystalline-type pattern and with an spatial profile that depends on the initial condition. In the regime $\gamma > 1/2$, flocking can be expected under sufficient *density conditions*, i.e., $\Gamma(0)$ is small enough and K is large enough, and the initial relative velocities are not too large, i.e., $\Lambda(0)$ is small.

Despite the theoretical and fundamental nature of this result, surprising and remarkable applications of the Cucker–Smale principle have recently been found in spacecraft flight control [PGE09] in the context of the ESA-mission DARWIN. Darwin will be a flotilla of four or five free-flying spacecrafts that will search for Earth-like planets around other stars and analyze their atmospheres for the chemical signature of life. The fundamental problem is to ensure that, with a minimal amount of fuel expenditure, the spacecraft fleet keep remaining in flight (flock), without loosing mutual radio contact, and eventually scattering.

3 Kinetic Models

Unlike the control of a finite number of agents, the numerical simulation of a rather large population of interacting agents can constitute a serious difficulty which stems from the accurate solution of a possibly very large system of ODEs. Borrowing the strategy from the kinetic theory of gases, we may want instead to consider a density distribution of agents, depending on spatial position, velocity, and time evolution, which interact with stochastic influence (corresponding to classical collisional rules in kinetic theory of gases) – in this case the influence is spatially "smeared" since two individuals do interact also when they are far apart. Hence, instead of simulating the behavior of each individual agent, we would like to describe the collective behavior encoded by the density distribution whose evolution is governed by one sole mesoscopic partial differential equation, see [CIP94] for classical references.

Let $f(x, v, t)$ denote the density of individuals in the position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ at time $t \geq 0$, $d \geq 1$. The kinetic model for the evolution of $f = f(x, v, t)$ can be easily derived by standard methods of kinetic theory, considering that the change in time of $f(x, v, t)$ depends both on transport (individuals moving freely if they do not interact with others), and interactions with other individuals. Discarding other effects, this change in density depends on a balance between the gain and loss of individuals with velocity v due to binary interactions.

3.1 Formal Derivation from Boltzmann-Type Equations

Let us assume that two individuals with positions and velocities (x, v) and (y, w) modify their velocities after the interaction, according to

$$\begin{aligned} v^* &= \mathscr{C}(x, v; y, w) \\ w^* &= \mathscr{C}(y, w; x, v) , \end{aligned}$$

where $\mathscr{C}$ is a suitable *interaction rule*. This leads to the following integro-differential equation of Boltzmann type,

$$\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f \right) (x, v, t) = Q(f, f)(x, v, t) , \tag{3.6}$$

where

$$Q(f,f)(x,v) = \varepsilon \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\frac{1}{J(x,v;y,w)} f(x,v_*)f(y,w_*) - f(x,v)f(y,w) \right) dw\, dy \ . \tag{3.7}$$

In (3.7) (v_*,w_*) are the *pre-interaction velocities* that generate the couple (v,w) after the interaction. In (3.6) $J(x,v;y,w)$ is the Jacobian of the transformation of (v,w) into (v^*,w^*) via $\mathscr{C}$. The role of the *interaction operator* Q is to provide a stochastic description of the interactions happening between each pair of individuals. Hence, the mesoscopic behavior is described by means of an averaging process. We are in particular interested in the interaction rules suggested by the particle models (2.1) and (2.3), i.e., respectively

(C1) $\mathscr{C}(x,v;y,w) = v + \eta\left[(\alpha-\beta|v|^2)v - \nabla U(|x-y|)\right]$;

(C2) $\mathscr{C}(x,v;y,w) = [1-\eta\, a(|x-y|)]\, v + \eta\, a(|x-y|)w$.

Other interaction rules will be considered as well later where, for instance, noise will also be included. Let us remark that we are assuming that the previous change of variables is well-defined, i.e., that $\mathscr{C}(x,v;y,w)$ is invertible under suitable assumptions. The presence of the Jacobian in the interaction operator (3.7) can be avoided by considering the weak formulation. By a weak solution of the initial value problem for (3.6), corresponding to the initial density $f_0(x,v)$, we shall mean any density satisfying the weak form of (3.6)-(3.7) given by

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^{2d}} \varphi(x,v)f(x,v,t)\, dv\, dx &+ \int_{\mathbb{R}^{2d}} (v\cdot\nabla_x\varphi(x,v))f(x,v,t)\, dv\, dx \\ &= \varepsilon\int_{\mathbb{R}^{4d}} (\varphi(x,v^*)-\varphi(x,v))f(x,v,t)f(y,w,t)\, dv\, dx\, dw\, dy\end{aligned} \tag{3.8}$$

for $t>0$ and all smooth functions φ with compact support, and such that

$$\lim_{t\to 0^+}\int_{\mathbb{R}^{2d}} \varphi(x,v)f(x,v,t)\, dx\, dv = \int_{\mathbb{R}^{2d}} \varphi(x,v)f_0(x,v)\, dx\, dv \ . \tag{3.9}$$

The form (3.8) is easier to handle, and it is the starting point to explore the evolution of macroscopic quantities (i.e., of moments for $\varphi(v) = 1, v, v^2$).

The phenomenon of pattern formation is mainly related to the large-time behavior of the solution of (3.6). An accurate description can be still furnished as well by resorting to simplified models for large time. This idea has been first used in dissipative kinetic theory by McNamara and Young [MY93] to recover from the Boltzmann equation in a suitable asymptotic procedure, simplified models of nonlinear frictions for the evolution of the gas density [BCP97, Tos04]. Similar asymptotic procedures have been subsequently used to recover Fokker–Planck type equations for wealth distribution [CT07], or opinion formation [Tos06].

By expanding $\varphi(x,v^*)$ in Taylor's series of v^*-v up to the second order the weak form of the interaction integral takes the form

$$\begin{aligned}\int_{\mathbb{R}^{4d}} &(\varphi(x,v^*)-\varphi(x,v))f(x,v,t)f(y,w,t)\, dx\, dv\, dy\, dw \\ &= \int_{\mathbb{R}^{4d}} (\nabla_v\varphi(x,v)\cdot(v^*-v))\, f(x,v,t)\, f(y,w,t)\, dx\, dv\, dy\, dw \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{4d}} \left[\sum_{i,j=1}^{d} \frac{\partial^2\varphi(x,\tilde{v})}{\partial v_i^2}(v_j^*-v_j)^2\right] f(x,v)f(y,w)\, dx\, dv\, dy\, dw \ ,\end{aligned} \tag{3.10}$$

with $\tilde{v} = \theta v + (1-\theta)v^*$, $0 \le \theta \le 1$. Now we resume the specific interaction rules (C1) and (C2) and we rewrite the first and second order terms of the Taylor's series as follows:

for $v^* - v = \eta\left[(\alpha - \beta|v|^2)v - \nabla U(|x-y|)\right]$

$$
\begin{aligned}
&\int_{\mathbb{R}^{4d}} (\varphi(x,v^*) - \varphi(x,v)) f(x,v,t) f(y,w,t)\, dx\, dv\, dy\, dw \qquad (3.11)\\
&\quad = \eta \int_{\mathbb{R}^{4d}} \left(\nabla_v \varphi(x,v) \cdot \left[(\alpha - \beta|v|^2)v - \nabla U(|x-y|)\right]\right) f(x,v,t)\, f(y,w,t)\, dx\, dv\, dy\, dw \\
&\qquad + \frac{\eta^2}{2} \int_{\mathbb{R}^{4d}} \left[\sum_{i,j=1}^{d} \frac{\partial^2 \varphi(x,\tilde{v})}{\partial v_i^2} \left[(\alpha - \beta|v|^2)v - \nabla U(|x-y|)\right]_j^2\right] f(x,v) f(y,w)\, dx\, dv\, dy\, dw\ ,
\end{aligned}
$$

for $v^* - v = \eta\, a(|x-y|)(w-v)$

$$
\begin{aligned}
&\int_{\mathbb{R}^{4d}} (\varphi(x,v^*) - \varphi(x,v)) f(x,v,t) f(y,w,t)\, dx\, dv\, dy\, dw \qquad (3.12)\\
&\quad = \eta \int_{\mathbb{R}^{4d}} \left(\nabla_v \varphi(x,v) \cdot (w-v)\right) a(|x-y|)\, f(x,v,t)\, f(y,w,t)\, dx\, dv\, dy\, dw \\
&\qquad + \frac{\eta^2}{2} \int_{\mathbb{R}^{4d}} \left[\sum_{i,j=1}^{d} \frac{\partial^2 \varphi(x,\tilde{v})}{\partial v_i^2} (w_j - v_j)^2\right] a(|x-y|)^2 f(x,v) f(y,w)\, dx\, dv\, dy\, dw\ .
\end{aligned}
$$

For $\eta \ll 1$, i.e., for the strength of the interaction being very small, such that $\varepsilon\eta = \lambda$ constant and $\varepsilon\eta^2 \ll 1$, a regime that is expected at large time, we may approximate the interaction operator with the sole first order term, i.e.,

$$
\begin{aligned}
&\frac{\partial}{\partial t} \int_{\mathbb{R}^{2d}} \varphi(x,v) f(x,v,t)\, dv\, dx + \int_{\mathbb{R}^{2d}} (v \cdot \nabla_x \varphi(x,v)) f(x,v,t)\, dv\, dx \\
&\approx \begin{cases} \lambda \displaystyle\int_{\mathbb{R}^{4d}} \left(\nabla_v \varphi(x,v) \cdot \left[(\alpha - \beta|v|^2)v - \nabla U(|x-y|)\right]\right) f(x,v,t)\, f(y,w,t)\, dx\, dv\, dy\, dw & \text{for } (C1) \\ \lambda \displaystyle\int_{\mathbb{R}^{4d}} (\nabla_v \varphi(x,v) \cdot (w-v)) a(|x-y|)\, f(x,v,t)\, f(y,w,t)\, dx\, dv\, dy\, dw & \text{for } (C2) \end{cases}
\end{aligned}
$$

or, in the strong form, we derive two corresponding nonlinear friction-type equations

Self-Propelling, Friction, and Attraction–Repulsion Kinetic Model:

$$
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \lambda\left[(\nabla_x U * \rho) \cdot \nabla_v f - \nabla_v \cdot ((\alpha - \beta|v|^2) v f)\right]\ . \qquad (3.13)
$$

where

$$
\rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\, dv\ . \qquad (3.14)
$$

and $*$ is the x-convolution,

and

The Cucker–Smale Kinetic Model of Flocking:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \lambda \nabla_v \cdot [\xi(f) f] \tag{3.15}$$

where

$$\xi(f)(x,v,t) = [(H(x)\nabla_v W(v)) * f] = \int_{\mathbb{R}^{2d}} \frac{K(v-w)}{(\varsigma^2 + |x-y|^2)^\beta} f(y,w,t)\, dy\, dw \ , \tag{3.16}$$

and $W(v) = \frac{1}{2}|v|^2$, $H(x) = a(|x|)$, and $*$ is the (x,v)-convolution.

3.2 Formal Derivation via Mean-Field Limit

The equations derived above via *grazing collision limit* from the general Boltzmann-type equations (3.6) can also be computed via *mean-field limit*. Here we present a simple formal description of this procedure for the Cucker–Smale model, see also [HL09, CCR]. Similar computations can be performed for (2.1) and more general models [BH77, Dou79, Neu77, CCR].

Define the empirical distribution density (or the atomic probability measure) associated to a solution $(x(t), v(t))$ of (2.3) and given by

$$f^N(x,v,t) = \frac{1}{N}\sum_{i=1}^N \delta(x - x_i(t))\delta(v - v_i(t)) \ ,$$

where δ is the Dirac delta and $\mathcal{P}(\mathbb{R}^k)$ denotes the space of probability measures on $\mathbb{R}^k$.

Let us assume that the particles remain in a fixed compact domain $(x_i(t), v_i(t)) \in \bar{\Omega} \subset \mathbb{R}^d \times \mathbb{R}^d$ for all N in the time interval $t \in [0,T]$. It is easy to check that this assumption for the Cucker–Smale model (2.3) is fulfilled if for instance the initial configuration is obtained as an approximation of an initial compactly supported probability measure f_0. Since for each t the measure $f^N(t) := f^N(\cdot,\cdot,t)$ is a probability measure in $\mathcal{P}(\mathbb{R}^{2d})$ together with the uniform support in N, then Prohorov's theorem implies that the sequence is weakly-$*$-relatively compact.

Hence, there exists a subsequence $(f^{N_k})_k$ and $f : \mathbb{R} \to \mathcal{P}(\mathbb{R}^{2d})$ such that

$$f^{N_k} \to f \quad (k \to \infty) \qquad \text{with } w^* - \text{convergence in } \mathcal{P}(\mathbb{R}^{2d}) \ ,$$

pointwise in time. We shall give a formal derivation of the evolution equation satisfied by the limit measure. Let us consider a test function $\varphi \in C_0^1(\mathbb{R}^{2d})$ and we compute

$$
\begin{aligned}
\frac{d}{dt}\langle f^N(t),\varphi\rangle &= \frac{1}{N}\sum_{i=1}^N \frac{d}{dt}\varphi(x_i(t),v_i(t)) \\
&= \frac{1}{N}\sum_{i=1}^N \nabla_x\varphi(x_i(t),v_i(t))\cdot v_i(t) + \frac{1}{N^2}\sum_{i,j=1}^N H(x_i-x_j)\left[\nabla_v\varphi(x_i(t),v_i(t))\cdot(v_j(t)-v_i(t))\right] \\
&= \langle f^N(t),\nabla_x\varphi\cdot v\rangle - \left(\frac{1}{N^2}\sum_{i,j=1}^N H(x_i-x_j)\left[\nabla_v\varphi(x_i(t),v_i(t))\cdot v_i(t)\right]\right) \\
&\qquad + \left(\frac{1}{N}\sum_{i=1}^N\left[\frac{1}{N}\sum_{j=1}^N H(x_i-x_j)\,v_j(t)\right]\cdot\nabla_v\varphi(x_i(t),v_i(t))\right) \\
&= \left\langle f^N(t),\nabla_x\varphi\cdot v\right\rangle - \left\langle f^N(t),\left[\frac{1}{N}\sum_{j=1}^N H(x-x_j)\right]\nabla_v\varphi(x,v)\cdot v\right\rangle \\
&\qquad + \left\langle f^N(t),\left(\frac{1}{N}\sum_{j=1}^N H(x-x_j)v_j(t)\right)\cdot\nabla_v\varphi(x,v)\right\rangle .
\end{aligned}
$$

We can easily compute

$$
\frac{1}{N}\sum_{j=1}^N H(x-x_j) = \frac{1}{N}\sum_{j=1}^N\langle H(x-y),\delta(y-x_j)\rangle_x = H*\rho_{f^N}(x,t) ,
$$

where

$$
\rho_{f^N}(x,t) = \int_{\mathbb{R}^d} f^N(x,v,t)\,dv = \left\langle 1,\frac{1}{N}\sum_{j=1}^N\delta(y-x_j)\delta(v-v_j)\right\rangle_v ;
$$

Similarly one deduces

$$
\frac{1}{N}\sum_{j=1}^N H(x-x_j)\,v_j(t) = H*m_{f^N}(t,x) ,
$$

where

$$
m_{f^N}(x,t) = \int_{\mathbb{R}^d} vf^N(x,v,t)\,dv = \left\langle v,\frac{1}{N}\sum_{j=1}^N\delta(y-x_j)\delta(v-v_j)\right\rangle_v .
$$

Collecting these formal computations we obtain

$$
\frac{d}{dt}\langle f^N(t),\varphi\rangle = \left\langle f^N(t),\nabla_x\varphi\cdot v + H*m_{f^N}\cdot\nabla_v\varphi - H*\rho_{f^N}\nabla_v\varphi\cdot v\right\rangle .
$$

After integration by part, in both x and v, we obtain

$$
\left\langle \frac{\partial f^N}{\partial t} + v\cdot\nabla_x f^N - \nabla_v\cdot\left[\xi(f^N)f^N\right],\varphi\right\rangle = 0 ,
$$

or, in the strong form,

$$
\frac{\partial f^N}{\partial t} + v\cdot\nabla_x f^N = \nabla_v\cdot\left[\xi(f^N)f^N\right] ,
$$

where ξ is defined in (3.16). For the limit of $k \to \infty$ of the subsequence f^{N_k} this leads formally to

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot [\xi(f) f] \ .$$

which is exactly (3.15) for $\eta = 1$. The mean-field limit $N \to \infty$ introduced above can be proved rigorously by using the techniques in [HL09, CCR], and also for the model (2.1) [BH77, Dou79, Neu77, CCR].

3.3 Rigorous Derivation of the Mean-Field Limit

Here, we would like to address the main results achieved so far concerning the solutions of (3.13) and (3.15). To this end, we need to introduce further notations and preliminaries. On the probability space $\mathcal{P}(\mathbb{R}^k)$ we define the so-called *Monge-Kantorovich-Rubinstein* distance,

$$W_1(\mu,\nu) = \sup \left\{ \left| \int_{\mathbb{R}^k} \varphi(x) d\mu(x) - \int_{\mathbb{R}^k} \varphi(x) d\nu(x) \right|, \varphi \in \mathrm{Lip}(\mathbb{R}^k), \mathrm{Lip}(\varphi) \leq 1 \right\} \ , \qquad (3.17)$$

where $\mathrm{Lip}(\mathbb{R}^k)$ denotes the set of Lipschitz functions on $\mathbb{R}^k$ and $\mathrm{Lip}(\varphi)$ the Lipschitz constant of φ. The main results in [CCR] show the well-posedness of the kinetic models (3.13) and (3.15) in the set of probability measures. Let us emphasize that the main technical issue is to overcome the lack of global Lipschitz character of the fields in phase space not allowing for direct application of known results [BH77, Dou79, Neu77]. In fact, we are able to overcome their lack of Lispchitzianity at infinity but not at the origin for the attraction-repulsion potential U. Therefore, in order to ensure the validity of the following results, assume further that we substitute in (2.1) and in (3.13) the Morse potential U with the smoother version

$$U(x) = k_2(|x|), \quad k_2(r) = -C_A e^{-r^2/\ell_A^2} + C_R e^{-r^2/\ell_R^2} \ . \qquad (3.18)$$

This new potential does not change significantly the qualitative behavior of the particle model as described in Fig. 2.2. The precise theorem in [CCR] states:

Theorem 2 *Assume that $f_0 \in \mathcal{P}(\mathbb{R}^{2d})$ is a compactly supported initial datum. Then there exists a unique measure valued solution $f \in C([0,+\infty);\mathcal{P}(\mathbb{R}^{2d}))$ to equation (3.13), respectively to (3.15), (i.e., it is a solution in the sense of the distributions), and there is a function $R = R(T)$ such that for all $T > 0$,*

$$\operatorname{supp} f(t) \subset B_{R(T)} \subset \mathbb{R}^{2d}, \quad \textit{for all } t \in [0,T] \ .$$

Moreover, the solution depends continuously with respect to the initial data in the following sense. Assume that $f_0, g_0 \in \mathcal{P}(\mathbb{R}^{2d})$ are two compactly supported initial data, and consider the respective measure valued solutions f, g to (3.13), respectively to (3.15). Then, there exists a strictly increasing function $r : [0,\infty) \to \mathbb{R}_0^+$ with $r(0) = 0$ depending only on the size of the supports of f_0 and g_0 such that

$$W_1(f(t), g(t)) \leq r(t) W_1(f_0, g_0), \quad t \geq 0 \ .$$

In Theorem 2 the main difference between the self-propulsion model (3.13) and the kinetic Cucker–Smale model (3.15) is on the growth on the supports in space and velocity in time, i.e., the function $R(T)$, as we will see below.

The previous theorem does not give any information about the asymptotic behavior of (3.13) or (3.15), but it does imply as an important consequence the convergence of the particle method via mean-field limit.

Corollary 3 *Given $f_0 \in P(\mathbb{R}^{2d})$ compactly supported and a sequence of f_0^N empirical measures associated with the initial data for the particle system (2.1), respectively to (2.3), (with $x_i(0)$ and $v_i(0)$ possibly varying with N), in such a way that*

$$\lim_{N\to\infty} W_1(f_0^N, f_0) = 0 \,.$$

Consider f_t^N the empirical measure associated with the solutions of the particle system (2.1), respectively of the particle system (2.3), with initial conditions $x_i(0)$, $v_i(0)$. Then,

$$\lim_{N\to\infty} W_1(f_t^N, f_t) = 0 \,,$$

for all $t \geq 0$, where $f = f(x, v, t)$ is the unique measure solution to (3.13), respectively to (3.15), with initial data f_0.

Concerning the asymptotic behavior, there are no results so far which establish the convergence to stable patterns for solutions of (3.13). In particular a classification of the asymptotic behavior of this kinetic equation as for the corresponding particle model (2.1) is lacking. We propose in Sect. 6 numerical experiments which highlight some of the features of this kinetic model. Moreover, when considering macroscopic quantities (moments) and passing to hydrodynamic equations, it has been possible to show, under special assumptions, that single and double mills (Fig. 2) are stationary solutions for the kinetic model as well [CDP09], although it is widely open, as mentioned above, how to establish and predict their formation and to analyze their stability.

Differently from the situation encountered in the kinetic model (3.13), again for the Cucker–Smale model we are able to provide a result of non-universal unconditional flocking which generalizes Theorem 1 for measure valued solution of (3.15), see [CFRT09, HT08, HL09]. In order to present this result we need a bit of notation: given a measure $\mu \in \mathcal{P}(\mathbb{R}^{2d})$, we define its translate μ^h with vector $h \in \mathbb{R}^d$ by:

$$\int_{\mathbb{R}^d} \zeta(x, v)\, \mathrm{d}\mu^h(x, v) = \int_{\mathbb{R}^{2d}} \zeta(x - h, v)\, \mathrm{d}\mu(x, v) \,,$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^{2d})$. We will also denote by μ_x the marginal in the position variable, that is,

$$\int_{\mathbb{R}^{2d}} \zeta(x)\, \mathrm{d}\mu(x, v) = \int_{\mathbb{R}^d} \zeta(x)\, \mathrm{d}\mu_x(x) \,,$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$. With this we can write the main conclusion about the asymptotic behavior, i.e., the convergence relative to the center of mass variables to a fixed density characterized by the initial data and its unique solution.

Theorem 4 *Assume $\gamma \leq 1/2$. Given $f_0 \in \mathcal{P}(\mathbb{R}^{2d})$ compactly supported, there exists $L_\infty(f_0) \in \mathcal{P}(\mathbb{R}^d)$ such that the unique measure-valued solution f to (3.13) satisfies*

$$\lim_{t\to\infty} W_1(f_x^{mt}(t), L_\infty(f_0)) = 0, \qquad \text{where } m = \int_{\mathbb{R}^{2d}} v\, df_0(x, v) \,.$$

Let us mention that it is also proved in [CFRT09] that the support in velocity of the solutions always shrinks in time toward its mean velocity m and it does it exponentially fast. In space, the support of the solutions will not grow in time by fixing our axis in the co-moving frame of the center of mass. In other words, the support in space of the solutions at all times is inside a ball of the form $B(x_c^o + mt, R_x)$ with x_c^o the initial center of mass with R_x fixed.

It has not been proved so far what happens in the kinetic model for the regime $\gamma > 1/2$, and which density conditions are required in order to obtain again flocking, i.e., convergence to a profile traveling with mean velocity. In such a regime we may expect *density phase transitions* in order to obtain flocking in the kinetic model as well, i.e., no flocking is achievable for initial low mean-density distributions and flocking is always achievable when the initial distribution has high mean-density.

4 Hydrodynamic Models

Kinetic models are time-consuming when solving in more than four dimensions (two for position and two for velocity). Therefore, we wish to reduce the dimensionality of the problem by taking a macroscopic limit, in such a way that numerical simulations become affordable.

4.1 Flocking, Single- and Double-Mills

Let us obtain continuum-like equations by computing the evolution of macroscopic quantities starting from (3.13) and (3.15), as usually done in kinetic theory. These macroscopic quantities are the velocity moments of $f(x, v, t)$. The mean velocity field $u(x, t)$ and the temperature $T(x, t)$ are defined by

$$\rho\, u = \int_{\mathbb{R}^d} v f(x, v, t)\, dv \qquad \text{and} \qquad d\, \rho\, T = \int_{\mathbb{R}^d} |v - u|^2 f\, dv\ ,$$

respectively. Integrating (3.13) or (3.15) in v we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0\ .$$

Proceeding by integrating (3.13) or (3.15) against vdv and using integration by parts, we find the momentum equation which involves the second moment of the distribution function. To close the moment system we assume that fluctuations are negligible, i.e., that the temperature $T(x, t) = 0$, and that the velocity distribution is monokinetic: $f(x, v, t) = \rho(x, t)\, \delta(v - u(x, t))$. In this way, we obtain for (3.13) the hydrodynamic systems

Self-Propelling, Friction, and Attraction–Repulsion Hydrodynamic Model:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0\ , \\[2mm] \rho\, \dfrac{\partial u_i}{\partial t} + \operatorname{div}(\rho u u_i) = \rho\, (\alpha - \beta |u|^2) u_i - \rho \left(\dfrac{\partial U}{\partial x_i} * \rho \right)\ . \end{cases} \tag{4.19}$$

and

The Cucker–Smale Hydrodynamic Model of Flocking:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0\ , \\[2mm] \rho\, \dfrac{\partial u_i}{\partial t} + \operatorname{div}(\rho u u_i) = \displaystyle\int_{\mathbb{R}^d} H(x - y)\, \rho(x, t)\, \rho(y, t)\, [u_i(y, t) - u_i(x, t)]\ dy\ . \end{cases} \tag{4.20}$$

The system of equations (4.19) was already proposed in [CDMBC07] based on computations of the empirical measure associated to N particles. Here, the same description is recovered from the monokinetic ansatz applied to the kinetic equations (3.13) or (3.15). In [CDMBC07] the authors discussed the validity of this approximation based on numerical comparisons of the N-particle system and the hydrodynamic system (4.19). They concluded that the hydrodynamic system is a good approximation close to the steady state pattern situations and performed a linear stability analysis around the simple infinite-extent flocking solution $\rho = \rho_0$ and $|u| = \sqrt{\alpha/\beta}$. Also, the system of equations (4.20) was somehow obtained in [HT08], although there they include the evolution of the temperature and they did not perform any momentum closure.

Let us look for steady single-milling and flocking patterns as particular type of monokinetic solutions or hydrodynamic solutions of (3.13) or (3.15). Let us first concentrate on flocking solutions to (3.13). By imposing that the velocity field is constantly u_0 satisfying $\beta|u_0|^2 = \alpha$ in (3.13), we obtain that the flocking solutions of (4.19) with density $\rho(x,t) = \tilde{\rho}(x - tu_0)$ are characterized by

$$U * \tilde{\rho} = C, \quad \tilde{\rho} \neq 0 \,,$$

where C is a constant of integration, as verified numerically in [LRC00]. Solutions of this simple looking equation can be very complicated even in one dimension for regular and singular interaction potentials [FR09, Rao09]. For the system (4.20), it is trivially observed that all densities of the form $\rho(x,t) = \tilde{\rho}(x - tu_0)$ with u_0 constant are solutions.

Now, let us search for single-mill stationary solutions of (4.19) in two dimensions by setting u in a rotatory state,

$$u = \pm\sqrt{\frac{\alpha}{\beta}}\,\frac{x^\perp}{|x|}\,,$$

where $x = (x_1, x_2)$, $x^\perp = (-x_2, x_1)$, and look for $\rho = \rho(|x|)$ radial, then we see that $\nabla_x \cdot u = 0$, $u \cdot \nabla_x \rho = 0$, which implies the continuity equation, and furthermore,

$$(u \cdot \nabla_x)\, u = -\frac{\alpha}{\beta}\,\frac{x}{|x|^2}\,.$$

Thus, (4.19) implies

$$U * \rho = D + \frac{\alpha}{\beta} \ln|\mathbf{x}|\,, \quad \text{whenever} \quad \rho \neq 0, \tag{4.21}$$

where D is a constant of integration, which gives a linear integral equation that can be solved for ρ. Multiple solutions with support filling an interval $[R_0, R_1]$ with $0 < R_0 < R_1$ were found numerically in [LRC00] and matched to single mill patterns in [CDMBC07]. Such solutions represent circular swarms in which all particles move with the same linear speed $\sqrt{\alpha/\beta}$. Given the conditions on the regularity of the potential U allowing for the rigorous existence of annularly supported solutions to (4.21) and studying their stability seems a challenging problem. Double mills however cannot be simply explained with this hydrodynamic approach due to the use of a single macroscopic velocity. They correspond to combination of single-mill solutions seen as distributional solutions of the kinetic model (3.13), see [CDP09] for more details.

4.2 Fluid Dynamic Description of Flocking via Povzner–Boltzmann Equation

In [LP90], Lachowicz and Pulvirenti established an interesting connection between solutions of the Euler equations for compressible fluids, and the solutions of an equation describing the dynamics of a system of particles undergoing elastic collisions at a stochastic distance. More precisely, consider density, velocity and temperature fields $\rho(x,t)$, $u(x,t)$ and $T(x,t)$ which constitute a (smooth) solution of the Euler equations (up to some time t_0 before the appearance of the first singularity), and construct a local Maxwellian function M whose mean density, velocity and temperature are given by ρ, u and T, respectively [CIP94]

$$M(x,v,t) = \frac{\rho(x,t)}{(2\pi T(x,t))^{3/2}} \exp\left(-\frac{(v-u(x,t))^2}{2T(x,t)}\right) .$$

Consider also a system of N particles located at the points $x_1, x_2, \ldots, x_N$ on a domain of $\mathbb{R}^3$, which move freely unless a pair of them undergo an elastic collision, expressed by the formula

$$v_i' = v_i - \frac{1}{2}((v_i - v_j)\cdot n_{ij})n_{ij}\,, \quad v_j' = v_j + \frac{1}{2}((v_i - v_j)\cdot n_{ij})n_{ij}\ . \tag{4.22}$$

where the unit vector n_{ij} is given by

$$n_{ij} = \frac{x_i - x_j}{|x_i - x_j|}\ . \tag{4.23}$$

As usual, v_i' and v_j' denote the outgoing velocities, where the ingoing velocities are given by v_i and v_j, provided that $(v_i - v_j)\cdot n_{ij} < 0$. Each binary collision takes place according to a stochastic law. The collision times for each pair i and j of particles are independent Poisson processes with intensity given by $\varphi(x_i,x_j,v_i,v_j)|(v_i - v_j)\cdot n_{ij}|$, and φ is given by

$$\begin{aligned}\varphi(x_i,x_j,v_i,v_j) &= \frac{1}{N\,\epsilon}\,\frac{1}{\delta^3}\chi(|x_i - x_j| \le \delta)\,\chi(|v_i - v_j| \le \theta)\\ &= \frac{1}{N\,\epsilon}\,B_\delta(|x_i - x_j|)\,\chi(|v_i - v_j| \le \theta)\ .\end{aligned} \tag{4.24}$$

In (4.24) $\chi(I)$ is the characteristic function of the subset I.

The evolution of the system of particles is described by the N-particle distribution function $f^N(x_1, v_1 \ldots, x_N, v_N, t)$ which gives the probability density for finding the N particles in the points $x_1, \cdots, x_N$ with velocities $v_1, \cdots, v_N$ at the time $t \ge 0$. Let the s-particle distribution functions be defined by the marginals

$$f^{N,s}(x_1, v_1 \ldots, x_s, v_s, t) = \int f^N(x_1, v_1 \ldots, x_N, v_N, t)\, dx_{s+1}\, dv_{s+1} \cdots dx_N\, dv_N\ .$$

Then, under some additional hypotheses on the regularity of the solutions to the Euler system in the time interval $[0, t_0]$, it is proven in [LP90] that, for all $\sigma > 0$ there exist $\epsilon_0(\sigma)$, $\delta_0(\sigma,\epsilon)$, $\theta_0(\sigma,\epsilon,\delta)$ and $N_0(\sigma,\epsilon,\delta,\theta)$ such that if $\epsilon \le \epsilon_0$, $\delta \le \delta_0$, $\theta \ge \theta_0$ $N \ge N_0$

$$\sup_{t\in[0,t_0]} \|M - f^{N,1}\| < \sigma\ ,$$

where $f^{N,1}$ is the 1-particle marginal corresponding to the N-particle distribution function $f^N(x_1, v_1 \ldots, x_N, v_N, t)$ with initial conditions

$$f^{N,s}(x_1, v_1 \ldots, x_s, v_s, t = 0) = \prod_{j=1}^{s} M(0; x_j, v_j)\ .$$

Substituting particles with birds, and changing consequently the interaction intensity φ in (4.24), introduces a reasonable model to study the time-space evolution of a population of birds, and, at the same time it establishes an interesting connection with the fluid dynamic picture, in presence of a large population. On the other hand, since the flocking phenomena is heavily dependent from dissipation, the elastic picture provided in [LP90] is not suitable to take into account this effect.

The idea to generalize the binary interactions of the stochastic system of particles to take into account dissipation phenomena has been recently developed in [FHT09], to end up with a consistent dissipative correction of the Euler system.

An intermediate step in the analysis of [LP90] shows that, as the number of particles tend to infinity, the 1-particle marginal $f^{N,1} = f$ satisfies the (elastic) Boltzmann–Povzner equation [Pov62]. As already described in Sect. 3.1, Povzner–like equations are based on collision integrals of the form (3.7).

In presence of an intensity function (4.24) where

$$\varphi(x_i, x_j, v_i, v_j) = \frac{1}{N\,\epsilon}\, B_\delta(|x_i - x_j|)\ ,$$

the elastic Povzner collision operator reads [Pov62]

$$Q_P(f,f)(x,v) = \frac{1}{\epsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B_\delta(|x-y|)\,\left(f(x,v')f(y,w') - f(x,v)f(y,w)\right)\,dw\,dy\ , \qquad (4.25)$$

where the pair v', w' is the post-interaction pair obeying to the elastic law of type (4.22).

The variant of the interaction rules (C1) and (C2) introduced in Sect. 3.1 which leads to a dissipation phenomenon consistent with the elastic picture is a binary interaction in which birds dissipate their relative velocity only along their relative direction. This agrees with

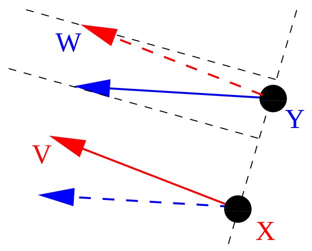

Figure 5: Example of interaction rule as in the Cucker–Smale-Povzner model where the particle with position x and velocity v averages its velocity with a particle in position y and velocity w according to (4.26).

the realistic assumption that agents which are approaching tend to diminish their relative velocity along their relative position, and the same happens in the opposite situation where they are going away.

The consequent microscopic dynamics of two birds (x, v) and (y, w) is fully governed by the interaction coefficient $0 < e(|x - y|) < 1$ which relates the components of the agents velocities before and after an interaction. The change in velocity now reads

$$v^* = v - \frac{1}{2}(1+e)((v-w)\cdot n)n\,, \quad w^* = w + \frac{1}{2}(1+e)((v-w)\cdot n)n\ , \qquad (4.26)$$

where the unit vector n is given by (4.23). This interaction is momentum preserving. Note that, on the contrary to what happens in the Cucker–Smale dynamics, the post interaction velocities collapse into a standard Povzner–type (conservative) interaction as $e = 1$ [Pov62]. In this case the pair (v^*, w^*) becomes the energy preserving pair (v', w')

$$v' = v - \frac{1}{2}((v - w) \cdot n)n\,, \quad w' = w + \frac{1}{2}((v - w) \cdot n)n\;.$$

For dissipative interactions e decreases with increasing degree of dissipation. In agreement with Cucker–Smale model, we assume

$$e(|x - y|) = 1 - \eta\, a(|x - y|)\;, \tag{4.27}$$

where $a(r)$ is the communication rate given in (2.3). The constant η has to be chosen so that $0 < e(|x - y|) < 1$.

If the interactions are such that the dissipation is low, so that $\eta << 1$, an asymptotic analysis similar to that of Sect. 3.1 allows to decompose the collision integral in two parts

$$Q(x, v, t) = Q_P(f, f)(x, v, t) + \eta\, I(f, f)(x, v, t)\;,$$

where Q_P is the classical elastic Povzner collision operator (4.25), and I is a dissipative nonlinear friction operator which is based on elastic interactions between birds

$$I(f, f)(x, v)\, dv \;= \frac{\sigma^2}{\epsilon}\, \mathrm{div}_v \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} n\, b(|x - y|)(q \cdot n) f(x, v') f(y, w')\, dw\, dy\;,$$

where

$$b(|x - y|) = B_\delta(|x - y)|)\, a(|x - y|)\;.$$

Finally, for nearly elastic interactions, with a restitution coefficient satisfying (4.27), the dissipative Povzner equation can be modelled at the leading order as

$$\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right)(x, v, t) = Q_P(f, f)(x, v, t) + \eta\, I(f, f)(x, v, t)\;. \tag{4.28}$$

The nonlinear friction operator I is mass preserving

$$\int_{\mathbb{R}^3} I(f, f)(x, v)\, dv = 0 \qquad \text{and} \qquad \mathcal{A}(\rho, u)(x, t) = \int_{\mathbb{R}^3} v I(f, f)(x, v)\, dv$$

satisfies

$$\mathcal{A}(\rho, u) = \frac{\sigma^2}{\epsilon} \int_{\mathbb{R}^3} n\, b(|x - y|)\, (u(x, t) - u(y, t)) \cdot n\, \rho(x, t)\rho(y, t) dy\;. \tag{4.29}$$

Note that the mean velocity is not a local collision invariant, while by symmetry

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v I(f, f)(x, v)\, dv\, dx = 0\;.$$

Finally, when the density $f(x, v, t)$ is isotropic in the velocity variable, $f = f(x, |v|, t)$

$$\begin{aligned}\mathcal{B}(\rho, u, T)(x, t) &= \int_{\mathbb{R}^3} |v|^2 I(f, f)(x, v)\, dv \qquad (4.30)\\ &= \frac{\sigma^2}{\epsilon}\int_{\mathbb{R}^3} b(|x - y|)\rho(x, t)\rho(y, t)\left[n \cdot u(x, t)\, n \cdot u(y, t) - \left(\frac{1}{3}|u(y, t)|^2 + T(y, t)\right)\right] dy\;.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 I(f,f)(x,v)\,dv\,dx = -\sigma^2 \int_{\mathbb{R}^3} b(|x-y|)((v-w)\cdot n)^2 f(x,v')f(y,w')dv\,dw\,dx\,dy < 0\ .$$

Hence, the nonlinear friction operator is globally dissipative.

Let us make use of the splitting method, very popular in the numerical approach to the Boltzmann equation. To solve the complete dissipative Boltzmann–Povzner equation (4.28), at each time step we consider sequentially the elastic Boltzmann–Povzner equation

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = Q_P(f,f)(x,v,t)\ , \tag{4.31}$$

and the dissipative correction

$$\frac{\partial f}{\partial t} = \frac{\eta}{\epsilon} I(f,f)(x,v,t)\ . \tag{4.32}$$

By the result of Lachowicz and Pulvirenti [LP90] we know that the solution to (4.31) is well approximated by a Maxwellian function M whose moments satisfy the Euler system. Substituting this Maxwellian function in (4.32), the right–hand side can be evaluated exactly using (4.29) and (4.30). Hence, when $\eta/\epsilon \to \lambda$, we obtain the following Euler equations for density $\rho(x,t)$, bulk velocity $u(x,t)$ and temperature $T(x,t)$

Euler Equations with Right-Hand Side Correction for Flocking :

For $A(\rho,u)$ defined by (4.29) and $B(\rho,u,T)$ defined by (4.30)

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) &= 0\ , && (4.33)\\ \frac{\partial}{\partial t}(\rho u_i) + \mathrm{div}(\rho u u_i + \rho T e_i) &= \lambda \mathcal{A}_i(\rho,u)(x,t)\ , && (4.34)\\ \frac{\partial}{\partial t}\left(\rho(\frac{2}{3}T + \frac{1}{2}u^2)\right) + \mathrm{div}\left(\rho u(\frac{1}{2}u^2 + \frac{5}{2}T)\right) &= \lambda \mathcal{B}(\rho,u,T)(x,t)\ . && (4.35) \end{aligned}$$

e_i is the component of the unit vector e in the i-th direction.

5 Variations on the Theme

The models (2.1) and (2.3), although they are able to catch the essence of certain behaviors which are really observed in nature, are clearly far from being realistic [BCCCCGLOPPVZ09]. For instance they assume that the rules of interactions are perfectly applied. In practice, we should assume delays as well as imperfect interaction rule applications, i.e., rules applied with a stochastic random noise. Moreover, in nature very often in a group of individuals there is the emergence of leaders who are responsible of conducting the others. The models presented, although not perfectly symmetrical[1], they do not privilege any individual of the group specifically. In the following we describe the role of emerging leadership as well as noise terms in interaction rules.

[1] The Cucker–Smale model is not completely symmetrical; in fact, birds which are belonging to the physical boundary of the group are much less influenced by others than those which are located in the middle of the group, simply because the interaction strength depends on the mutual distance.

5.1 Leadership, Geometrical Constraints, and Cone of Influence

We address the emergence of instantaneous leaders in the model (2.3) proposed by Cucker–Smale. In the particle model, i.e., in the case of a finite number of agents, it is possible to impose a special order structure to the group. The group can be described as an oriented graph where the directions of the arcs represent the dependencies of individuals. If a node of the graph has no outward connecting arcs, then that node represents a leader within the group. This structure and the corresponding behavior of the Cucker–Smale model under these assumptions have been investigated by Shen [She08]. From a more abstract point of view, this is equivalent to determining for each i^{th}-individual in position $x_i(t)$ and with velocity $v_i(t)$ a set of dependence $\Sigma_i \subset \{1, \dots, N\}$, such that

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j \in \Sigma_i}^{N} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases} \tag{5.36}$$

However, such a rigid structure and definition of a leader, i.e., the i^{th}-individual is a leader if $\Sigma_i = \emptyset$, can be well-understood for a finite number of particles, but it is definitively less clear how can one describe an absolute leader within a continuous distribution, as in a kinetic description. Moreover, in nature we are more used to the concept of *instantaneous leader* than the presence of a leader immutable in time. For instance, in a swarm of birds, it

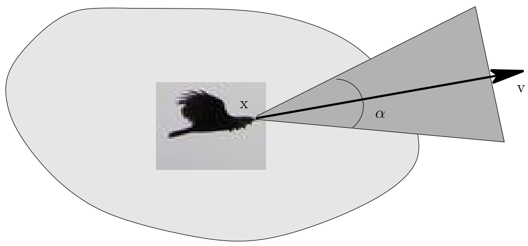

Figure 6: Cone of vision for a bird.

is observed that the leader of the group may change every few seconds. Moreover, in these situations, such as in a swarm of birds flying in formation or in a fish school, it is quite well-understood that the vision geometrical constraints are fundamental in order to determine the structure of a moving group [BCCCCGLOPPVZ09]. Therefore here we would like to propose a modification of the Cucker and Smale model which promotes the emergence of leaders on the basis of the sole vision constraint and can be easily generalized to continuous distributions via mean-field limit:

The Cucker–Smale Particle Model with Leader Emergence:
For $\alpha \in [-1, 1]$ we define the dynamics

$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases} \tag{5.37}$$

where

$$\Sigma_i(t) := \left\{ 1 \le \ell \le N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \ge \alpha \right\}$$

In this situation a bird in position x_i averages its velocity v_i with a rate of communication $a(|x_i - x_j|)$ only with those birds whose positions x_j belong to the *visual cone* $\mathcal{C}_i := \left\{ x \in \mathbb{R}^d : \frac{(x-x_i)\cdot v_i}{|x-x_i||v_i|} \ge \alpha \right\}$. A scheme of this situation is shown in Fig. 6. It is clear that those birds which are located at the boundary of the group and have velocity pointing outward have potential of having an empty visual cone $\mathcal{C}_i = \emptyset$ and hence to become *instantaneous leaders*. In fact, their leadership might be overtaken by the emergence of another leader which suddenly, at a successive time, bursts into their visual cone. However, we can expect that the dynamics provided by (5.36) will promote the emergence of a finite number of triangle-shaped groups with a leader in front, see Fig. 7. The shape of the triangle region containing the group of *followers*, in particular, the angle formed at the vertex where the leader is located is necessarily related to the parameter α determining the visual dependence cone. By mean-field limit, exactly as we proceeded in Sect. 3.2, we easily derive the follow-

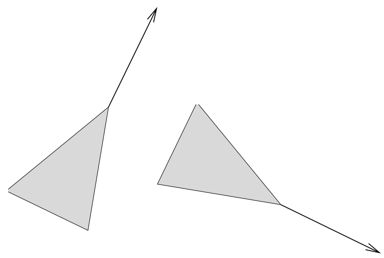

Figure 7: Expected triangle-like groups produced by leaders.

ing kinetic version of the proposed model:

The Cucker–Smale Kinetic Model with Leader Emergence:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \lambda \nabla_v \cdot [\xi(f;\alpha) f] \tag{5.38}$$

where

$$\xi(f;\alpha)(x,v,t) = \int_{\frac{(y-x)\cdot v}{|y-x||v|} \ge \alpha} \frac{K}{(\varsigma^2 + |x-y|^2)^\beta} \left(\int_{\mathbb{R}^d} (v-w) f(y,w,t)\, dw \right) dy \ . \tag{5.39}$$

5.2 Adding White Noise to the Models

It is a natural question to ask what happens if we add simple noise to the particle systems (2.1) and (2.3). The first issue is what are the corresponding kinetic systems equivalent to (3.13) and (3.15) and if the mean-field limit works for these stochastic particle systems. Let us take as an example the system (2.1) with noise written as

Self-Propelling, Friction, and Attraction–Repulsion with Noise Particle Model:

$$\begin{aligned}
&\dot{x}_i = v_i, \\
&dv_i = \left[(\alpha - \beta\,|v_i|^2)v_i - \frac{1}{N}\nabla_{x_i}\sum_{j\neq i} U(|x_i - x_j|) \right] dt + \sqrt{2\sigma}\, d\Gamma_i(t)\ ,
\end{aligned} \tag{5.40}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. Here, $\alpha \in \mathbb{R}$ is the effective friction constant coming from $\alpha = \alpha_1 - \alpha_0$ with $\alpha_0, \alpha_1 > 0$, and α_0 is the linear Stokes friction component and α_1 is the self-propulsion generated by the organisms.

Using Ito's formula to obtain a Fokker–Planck equation for the N-particle distribution and following a BBGKY procedure, it is easy to derive formally the following kinetic Fokker–Planck equation:

Self-Propelling, friction, and Attraction–Repulsion with Noise Kinetic Model:

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \mathrm{div}_v\big[(\alpha - \beta|v|^2)v\, f\big] - \mathrm{div}_v\left[(\nabla_x U * \rho)f\right] = \sigma\Delta_v f\ . \tag{5.41}$$

Analogously, for the Cucker–Smale model with white noise, one will conclude the equation

The Cucker–Smale with Noise Kinetic Model:

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = \lambda\nabla_v\cdot\left[\xi(f)f\right] + \sigma\Delta_v f\,, \tag{5.42}$$

with $\xi(f)$ defined in (3.16). Let us mention that the stochastic particle system related to Cucker–Smale with noise has been recently analysed in [HLL09]. In fact, these Vlasov-Fokker–Planck-like equations can be obtained from the stochastic particle models by coupling methods as it will be proved in [BCC]. It is an open problem to discuss any property about their asymptotic behavior and existence of any stationary solution.

Finally, let us mention that again they serve as bridges between the particle descriptions and the macroscopic descriptions. As shown in [CDP09], one can obtain the macroscopic equations

$$\partial_t\rho = \nabla_x\cdot\big((\nabla_x U * \rho)\rho\big) + \Delta_x\rho \tag{5.43}$$

and

$$\partial_t\rho = \nabla_x\cdot\big((\nabla_x U * \rho)\rho\big)\ , \tag{5.44}$$

as approximations to (5.41) in different scaling regimes. Both (5.43) and (5.44) were proposed in [TB04, TBL06] as continuum models for swarming, and thus, recovered through the presented kinetic theory. They are related to models in granular media equations, see [CMV03, CMV06].

5.3 Adding Nonlinear Dependent Noise to Cucker–Smale

In a recent paper [YEECBKMS09] it is argued that coherence in collective swarm motion is facilitated in presence of a certain degree of randomness (in the sense of imperfect applications of the interaction rules) which has to be weaker at some position around which mean velocity of particles is larger. We would like to describe this special phenomenon, again taking as a reference the model (2.3). Let us now assume that the dynamics is imperfect in the sense that it is perturbed by random noise:

The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt \, , \\ dv_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2\frac{\sigma}{N}\sum_{j=1}^{m} a(|x_j - x_i|)}\, d\Gamma_i(t) \, . \end{cases} \tag{5.45}$$

Here, a denotes again the communication rate function defined as in (2.3) and $\Gamma_i(t)$ $(1 \le i \le N)$ are N independent Wiener processes with values in $\mathbb{R}^d$, and $\sigma \ge 0$ is a constant denoting the coefficient of noise strength.

Notice that the strength of noise for i-th particle is

$$\frac{\sigma}{N}\sum_{j=1}^{N} a(|x_j - x_i|)$$

which is proportional to the summation of distance potentials of i-th particle with all particles.

In order to address a mesoscopic description, this time we resume the approach via Boltzmann equations. Let us assume that the post-interaction velocities (v^*, w^*) of two individuals which have positions and velocities (x, v) and (y, w) before interaction are determined by the rule

$$\begin{aligned} v^* &= (1 - \eta\, a(|x-y|))v + \eta\, a(|x-y|)w + \sqrt{2\sigma\eta\, a(|x-y|)}\,\theta_v \, , \\ w^* &= \eta\, a(|x-y|)v + (1 - \eta\, a(|x-y|))w + \sqrt{2\sigma\eta\, a(|x-y|)}\,\theta_w \, , \end{aligned}$$

where $\eta > 0$ and $\sigma \ge 0$ are constants which will enter into the equation exactly in the same way as in Sect. 3.1, and

$$\begin{aligned} \theta_v &= (\theta_{v,1}, \theta_{v,2}, \cdots, \theta_{v,n}) \in \mathbb{R}^d \, , \\ \theta_w &= (\theta_{w,1}, \theta_{w,2}, \cdots, \theta_{w,n}) \in \mathbb{R}^d \, . \end{aligned}$$

$\theta_{v,i}$ and $\theta_{w,i}$, $(1 \le i \le d)$ are identically distributed independent random variables of zero mean and unit variance. For now, it is also supposed that $\sup_x a(|x|)$ is finite and

$$\eta \sup_x a(|x|) < \frac{1}{2} \, . \tag{5.46}$$

Notice that this assumption can be removed in the later grazing limit since a will be scaled up to a small parameter $\epsilon > 0$. As in Sect. 3.1, the evolution of the density can be described at a kinetic level by the following integro-differential equation of Boltzmann type:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) \ , \tag{5.47}$$

with

$$Q(f,f) = \delta \mathbb{E}\left[\iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{1}{J(|x-y|)} f(x,v_*) f(y,w_*) - f(x,v) f(y,w) \right) dy\, dw \right] \ ,$$

where (v_*, w_*) mean the pre-collisional velocities of particles that generate the pair velocities (v,w) after interaction, and

$$J(|x-y|) = (1 - 2\eta\, a(|x-y|))^d$$

is the Jacobian of the transformation of (v,w) into (v^*, w^*). We resume now the weak formulation of the stochastic problem, and we consider any smooth function $\varphi(x,v)$ with compact support; for a weak solution f it holds that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \varphi(x,v) f(x,v,t)\, dx\, dv = & \int_{\mathbb{R}^{2d}} v \cdot \nabla_x \varphi(x,v) f(x,v,t)\, dx\, dv \\ & + \delta\, \mathbb{E}\left[\int_{\mathbb{R}^{4d}} (\varphi(x,v^*) - \varphi(x,v)) f(x,v,t) f(y,w,t)\, dx\, dy\, dv\, dw \right] \end{aligned} \tag{5.48}$$

for any $t > 0$ and

$$\lim_{t \to 0+} \int_{\mathbb{R}^{2d}} \varphi(x,v) f(x,v,t) dx dv = \int_{\mathbb{R}^{2d}} \varphi(x,v) f_0(x,v) dx dv \ .$$

To carry out the grazing limit, we scale a as $a = \epsilon a_0$, where $\epsilon > 0$ is a small parameter. Suppose that $f = f(x,v,t)$ satisfies (5.47), where f actually takes the form of $f^{\delta,\epsilon}$ which depends on parameters δ and ϵ but the superscripts are omitted for brevity. Let us begin with the weak form (5.48) and we consider the Taylor's expansion

$$\begin{aligned} \varphi(x,v^*) - \varphi(x,v) = & \nabla_v \varphi(x,v) \cdot (v^* - v) + \frac{1}{2} \sum_{|\beta|=2} \partial_v^\beta \varphi(x,v) (v^* - v)^\beta \\ & + \frac{1}{6} \sum_{|\beta|=3} \partial_v^\beta \varphi(x,\tilde{v}) (v^* - v)^\beta \ , \end{aligned}$$

where $\tilde{v}$ is a vector between v^* and v. Recall also that

$$v^* - v = \epsilon \eta a_0(|x-y|)(w - v) + \sqrt{2\sigma\epsilon\eta a_0(|x-y|)}\, \theta_v \ .$$

Then, formally one has

$$\mathbb{E}[\varphi(x,v^*) - \varphi(x,v)] = \epsilon \nabla_v \varphi(x,v) \cdot \eta a_0(|x-y|)(w-v) + \epsilon \Delta_v \varphi(x,v) \cdot \sigma \eta a_0(|x-y|) + O(\epsilon^2) \ .$$

Thus, it holds

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \varphi(x,v)\, f(x,v,t)\, dx\, dv = & \int_{\mathbb{R}^{2d}} v \cdot \nabla_x \varphi(x,v) f(x,v,t)\, dx\, dv \\ & + \delta\epsilon \int_{\mathbb{R}^{4t}} \nabla_v \varphi(x,v) \cdot (w - v) \eta a_0(|x-y|) f(x,v,t) f(y,w,t)\, dx\, dy\, dv\, dw \\ & + \delta\epsilon \int_{\mathbb{R}^{4d}} \sigma \Delta_v \varphi(x,v) \eta a_0(|x-y|) f(x,v,t) f(y,w,t)\, dx\, dy\, dv\, dw \\ & + \delta\epsilon O(\epsilon). \end{aligned}$$

Taking the so-called *grazing limit*, so that $\epsilon \to 0$, $\delta\epsilon \to 1$, then the limit function, still denoted by $f(x, v, t)$, satisfies

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2d}} \varphi(x,v)\, f(x,v,t)\,dx\,dv &= \int_{\mathbb{R}^{2d}} v\cdot\nabla_x\varphi(x,v) f(x,v,t)\,dx\,dv \\ &+ \int_{\mathbb{R}^{4d}} \nabla_v\varphi(x,v)\cdot(w-v)\eta a_0(|x-y|) f(x,v,t) f(y,w,t)\,dx\,dy\,dv\,dw \\ &+ \int_{\mathbb{R}^{4d}} \sigma\Delta_v\varphi(x,v)\eta a_0(|x-y|) f(x,v,t) f(y,w,t)\,dx\,dy\,dv\,dw\ .\end{aligned}$$

This implies that f satisfies formally the equation

The Cucker–Smale Kinetic Model with Nonlinear Dependent Noise:

$$\partial_t f + v\cdot\nabla_x f = \eta\nabla_v\cdot(\xi(f)f) + \eta\sigma(a_0 * \rho)\,\Delta_v f\ .$$

Let us define as usual

$$m(x,t) = \int_{\mathbb{R}^d} v f(x,v,t)\,dv\ .$$

Let us now consider the parameter $\gamma > d$ as introduced in (2.3). Hence $a_0(|x|)$ is now a even summable function which describes weak long-range interactions. Moreover, for simplicity, assume $\eta = 1$, and recast the equation as follows:

$$\partial_t f + v\cdot\nabla_x f + (a_0 * m)\cdot\nabla_v f = (a_0 * \rho)\nabla_v\cdot(\nabla_v f + vf)\ , \tag{5.49}$$

Note that, except for the presence of the convolution $a_0 * \rho$, the right-hand-side is given by a Fokker–Planck operator term $\nabla_v\cdot(\nabla_v f + vf)$. One simple steady state solution of (5.49) is provided by the *global Maxwellian function*

$$\mathbf{M}(v) = \frac{1}{(2\pi)^{d/2}}\exp\left(-|v|^2/2\right)\ . \tag{5.50}$$

In [DFT09, Theorem 1] it is shown that this steady pattern is locally stable. In fact, if we assume an initial datum of the type

$$f_0(x,v) = f(x,v,0) = \mathbf{M}(v) + \sqrt{\mathbf{M}(v)}F_0(x,v)\ ,$$

for a smooth and small function F_0, then, under such smoothness and smallness assumptions, it is possible to show that the evolution of f is also of the type

$$f(x,v,t) = \mathbf{M}(v) + \sqrt{\mathbf{M}(v)}F(x,v,t)\ ,$$

and $F(x,v,t)\to 0 \quad (t\to\infty)$, with a polynomial rate.

6 Numerical Experiments

In this section we illustrate a few numerical experiments which show the dynamics and asymptotic properties of the particle and kinetic models discussed in the previous sections.

Before going into these experiments, a few words in relation to the numerical methods used. The general system of ODEs governing particle models is solved by a standard classical third order Runge-Kutta schemes. As for the kinetic models, several strategies are followed for the discretization of the time-space-velocity domain. A robust choice is the use of a homogeneous grid in both space and velocity with suitable finite differences schemes based on non-oscillatory ENO schemes and a Runge-Kutta discretization in order to advance in time. Another possibility is the use of splitting methods, which allows to decompose the problem into transport phases in different variables, solve each part separately and recombine everything to approximate a solution for the whole problem. With splitting methods the phase space (x, v) is decomposed into either dimensions (Dimensional Splitting), therefore the transport equation is decomposed into the transport along the x-dimension and the transport along the v-dimension; at this point, the DS can be coupled to semi-Lagrangian methods for the solution of each transport block. Although direct finite differences solvers for the transport equation are well-established and robust, they have the drawback of being constrained by the CFL condition. The splitting methods have the advantage of allowing larger time-steps, but their drawback is that they require to solve the characteristics, possibly complicated. We refer to [CV05] for more details on this kind of numerical schemes for kinetic equations.

6.1 Mills, Double Mills, Crystalline Structure Formation, and Flocking

The problem (2.1) depends on six parameters: the self-propulsion coefficient α, the friction coefficient β, the attraction strength C_A, the attraction typical length l_A, the repulsion strength C_R and the repulsion typical length l_R. Different choices for these parameters lead to different configurations of the asymptotics of the problem: when $\alpha > 0$ and $\beta > 0$, the meaningful magnitudes are represented by the ratios $C = \frac{C_R}{C_A}$ and $l = \frac{l_R}{l_A}$. In Fig. 8 we sketch four different situations: the parameters α and β fix the velocities of the particles to $\|v_i\| = \sqrt{\alpha/\beta}$. If the repulsion strength is larger than the attraction strength and the effects of self-propulsion/friction are weak, then we obtain a single milling, as in the (a) sketch of Fig. 8; if, instead, the self-propulsion/friction effects are stronger, we can obtain a flocking with a crystalline structure, as in the (c) sketch of Fig. 8. Double mills are obtained in a short-range repulsion regime in which the attraction strength is larger than the repulsion strength, as we can observe in the (b) sketch of Fig. 8. Finally, whenever the Cucker–Smale interaction is added, we obtain flocking in which the velocity is fixed by the self-propulsion/friction coefficients, as in the (d) sketch of Fig. 8.

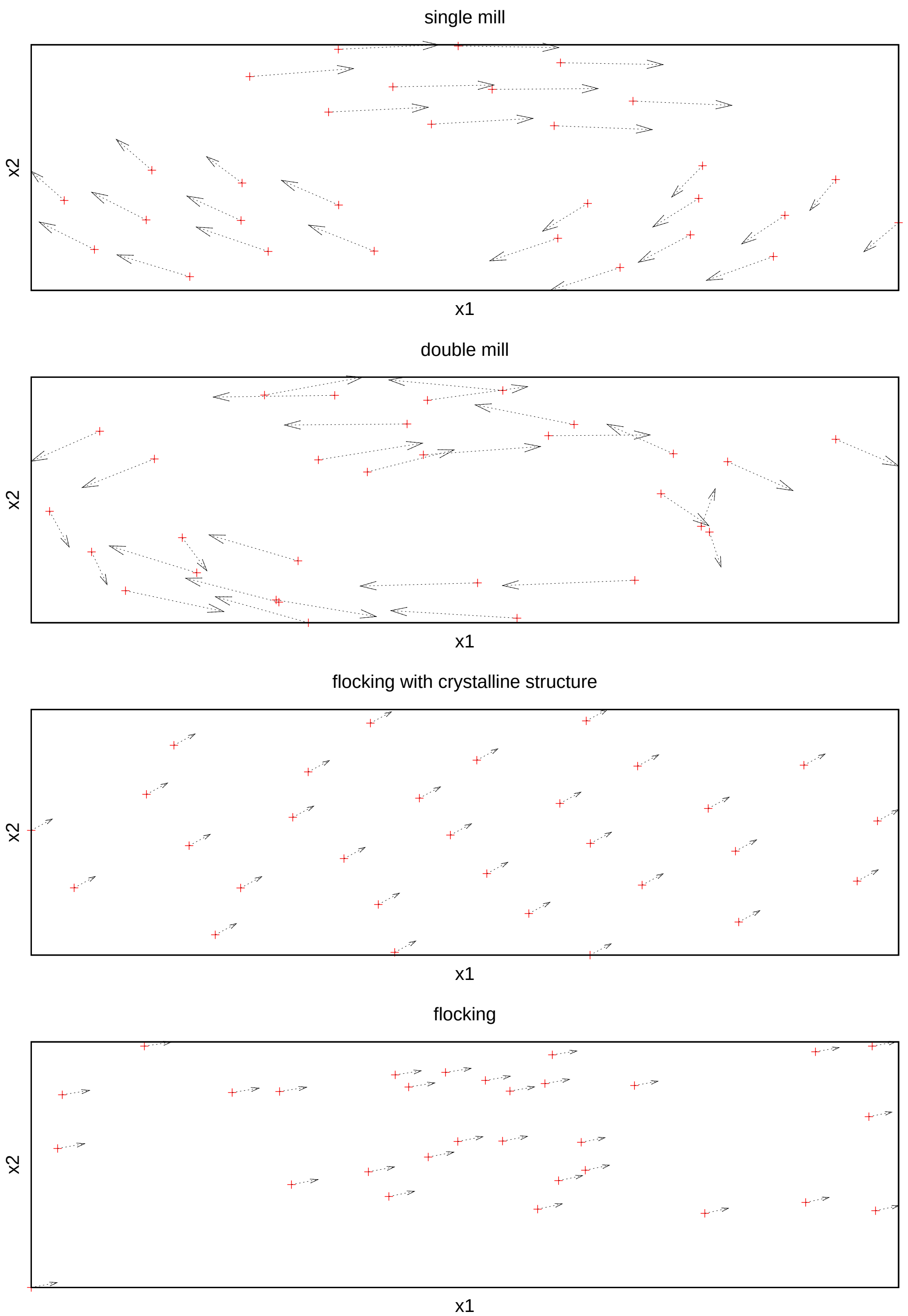

Figure 8: The choice of the parameters in the problem composed by self-propulsion, friction and attraction-repulsion is crucial to obtain completely different asymptotics: (a) single mills, we are in regime $\alpha = 0.07$, $\beta = 0.05$, $C = 2.5 > 1$, $l = 0.02$ and $Cl^2 = 0.01$; (b) double mills, we are in regime $\alpha = 0.15$, $\beta = 0.05$, $C = 0.5 > 1$, $l = 0.2$ and $Cl^2 = 0.02$; (c) flocking with crystalline structure, we are in regime $\alpha = 0.2$, $\beta = 0.1$, $C = 2.5 > 1$, $l = 0.02$ and $Cl^2 = 0.001$; (d) flocking, we are in regime $\alpha = 0.07$, $\beta = 0.05$, $C = 2.5 > 1$, $l = 0.02$ and $Cl^2 = 0.01$, plus we added Cucker–Smale interaction with $\gamma = 0.05$.

6.2 The Cucker–Smale Model both for Particle and Kinetic Regimes

For $\gamma < \frac{1}{2}$ the strength of the interactions makes the whole system converge to a uniform velocity. In the case of the discrete problem, this means that all the particles tend to have the same velocity; in the case of the continuous problem, this means that the distribution function tends to concentrate on a delta function in the velocity space and to be distributed only along the spatial dimension, as we see in Fig. 9. In these simulations we show also that the discrete problem approximates well the continuous problem, as expected theoretically by the mean-field limit.

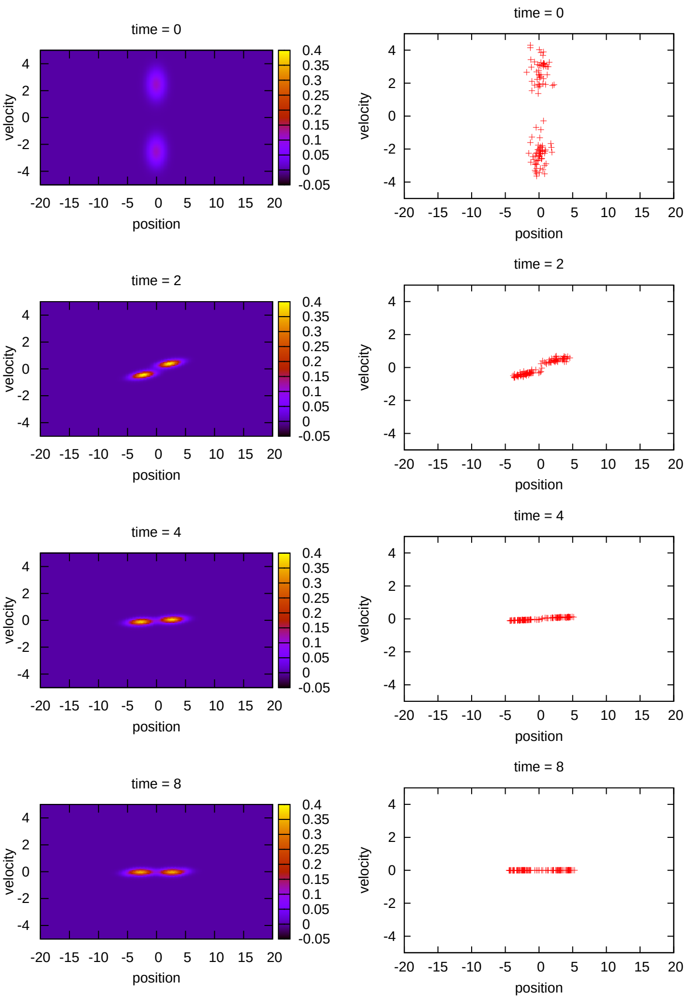

Figure 9: Particle versus Kinetic simulation of the Cucker–Smale model in 1D with $\gamma = 0.05 < \frac{1}{2}$. The kinetic problem is solved through splitting methods.

In case we use a parameter $\gamma > \frac{1}{2}$, then the interaction between the particles may not force them to converge to a uniform velocity, and flocking may depend on the density. In Fig. 10 the system is initialized by distributing two groups around velocities $+2.5$ and -2.5; in the example, we see that there is no convergence to a uniform velocity.

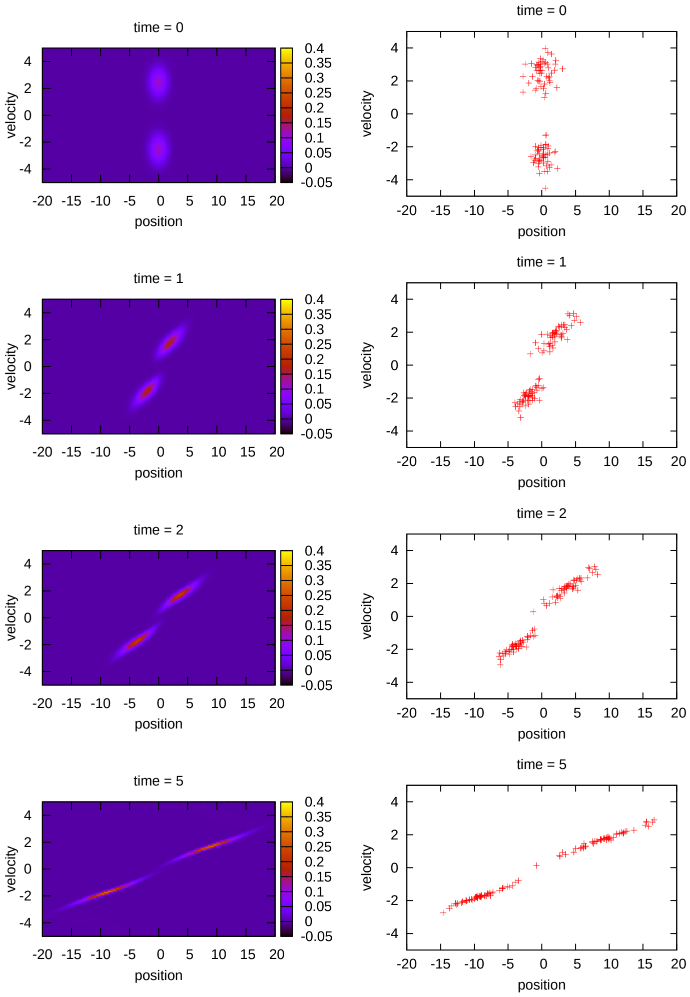

Figure 10: Particle versus Kinetic simulation of the Cucker–Smale model in 1D with $\gamma = 0.95 > \frac{1}{2}$. The kinetic problem is solved through splitting methods.

The convergence rate of the discrete Cucker–Smale system can be influenced by including some non-linear effects; replace the dynamics in (2.3) by

$$\frac{dv_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} H(|x_i - x_j|)(v_j - v_i)\|v_j - v_i\|^{p-2} ,$$

as in [HHK09]. Of course, for $p = 2$ we recover the usual model (2.3). In Fig. 11 we plot

$$\sqrt{\sum_{i=1}^{N} |v_i(t) - \widehat{v}|^2}$$

against time in logarithmic scale. For $p = 2$ the convergence towards the asymptotics is exponential, while for $p > 2$ the convergence has polynomial rate for some of the values studied, and for $p < 2$ the asymptotics is reached in finite time. A detailed study of this problem will be done elsewhere.

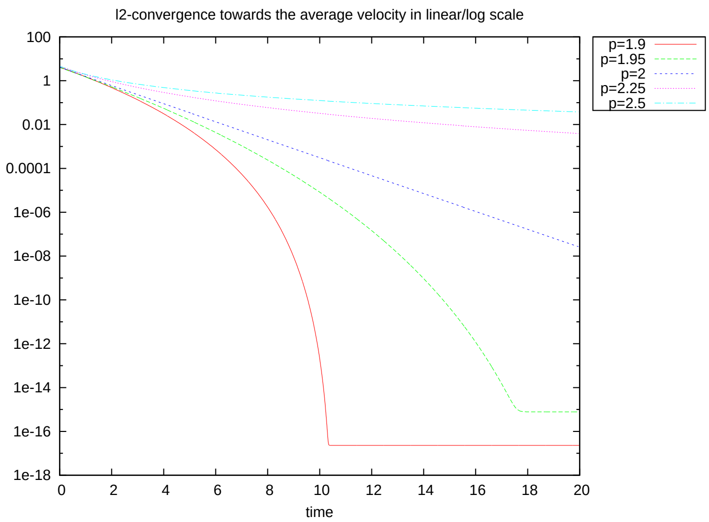

Figure 11: Flocking convergence rate for the nonlinear Cucker–Smale model in 1D.

6.3 Leadership Emergence for Particle Regimes, when Visibility Cone Conditions are Considered

The Cucker–Smale model with $\gamma < \frac{1}{2}$ forces all the particle to the average velocity given by the initial spatial and velocity configuration. When a vision angle α is introduced, then the conditions for the formation of several groups are set; the particles which cannot see the other ones become leaders of a group. The evolution of the system, when different vision angles α are considered, is sketched in Fig. 12; as the angle decreases, the ability of the particles of feeling the presence of the others decreases, therefore more distinct groups with emerging leaders tend to be formed.

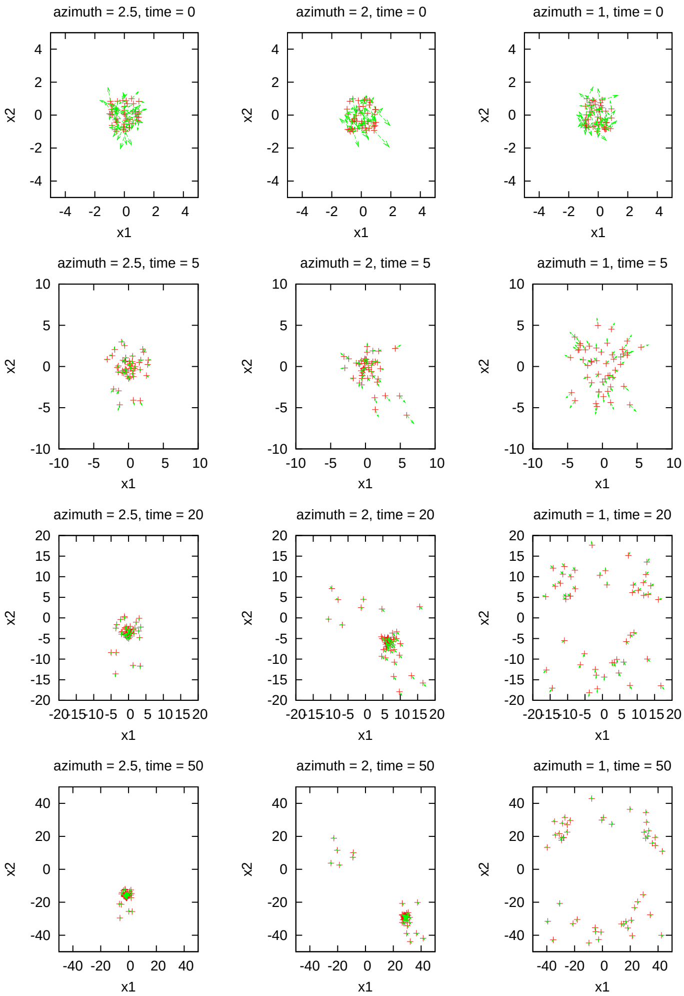

Figure 12: Cucker–Smale model with $\gamma = 0.05$ and different vision angles.

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