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Cucker-Smale Flocking Under General Interaction Topologies ⋆

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Abstract: In the literature, the study of the Cucker-Smale (C-S) model is all restricted to several special interaction topologies. In this paper, we establish the flocking behavior of the C-S model under general interaction topologies, which contain all existing topology structures in relation to the C-S model as special cases. In particular, the topology with multiple leaders is included. The flocking results are guaranteed under some conditions on the system parameters and the initial states only. The critical exponent below which unconditional convergence holds is given, depending only on the interaction topology. It is consistent with the ones obtained with some known special topologies in the literature. The convergence results are shown by employing a new combination of perturbation method and algebraic graph theory.

Keywords: Flocking; Cucker-Smale model; general interaction topologies.

1. INTRODUCTION

The purpose of this paper is to study the flocking behavior of the Cucker-Smale model (Cucker and Smale, 2007a,b) under general interaction topologies. Flocking, as a typical example of collective behavior, is a phenomenon in which a large number of self-propelled agents, using only limited environmental information and simple rules, organize into an ordered motion. Flocking phenomena are ubiquitous in nature. Examples include flocking of birds, schooling of fish, herding of quadrupeds and swarming of bacteria. The description of the flocking behavior has gained increasing interest from various disciplines, e.g., biology (Couzin et al., 2005; Topaz et al., 2006), physics (Vicsek et al., 1995; Vicsek and Zefeiris, 2012), applied mathematics (Bellomo et al., 2012; Degond and Liu, 2012) and control theory (Jadbabaie et al., 2003; Olfati-Saber, 2006). This is due to its wide applications in many applied engineering areas including mobile sensor networks, cooperative robots and formation flying spacecrafts (Beard et al., 2001; Cortés et al., 2004). In applied mathematics, the main goal of flocking study is to model and analyze this fascinating phenomenon. It is well-acknowledged that mathematical models for flocking can always offer deeper insights into various observed complex patterns in the real world if such models can indeed capture the very essence. Among many flocking models, our interest in this paper lies in the flocking model proposed by Cucker and Smale (Cucker and Smale, 2007a,b). This model has significant potential applications. The authors of Perea et al. (2009) suggest to use the Cucker-Smale flocking mechanism to ensure the formation of the various spacecrafts forming the Darwin space mission.

The Cucker-Smale (C-S) model postulates the following behavior: each agent adjusts its velocity by adding to it a weighted average of the differences of its velocity from those of the other agents. That is, for a flock of N agents labeled $\{1,\dots,N\}$ its behavior in discrete-time is specified by

$$\begin{cases} x_i[n+1] = x_i[n] + hv_i[n] \\ v_i[n+1] = v_i[n] + \displaystyle\sum_{j\in\mathcal{N}_i} ha_{ij}(v_j[n]-v_i[n]) \end{cases} \tag{1}$$

where h is the time step, $x_i[n]$ and $v_i[n] \in \mathbb{R}^m$ denote the position and velocity of agent i at time nh respectively and $\mathcal{N}_i \subseteq \{1,\dots,N\}$ denotes the subflock of agents that directly influence agent i. This influence is characterized by the C-S weight function given by

$$a_{ij} = \frac{H}{(1+\|x_i[n]-x_j[n]\|^2)^\beta}. \tag{2}$$

Here $H > 0$ and $\beta \geq 0$ are system parameters, and $\|\cdot\|$ denotes the 2-norm in $\mathbb{R}^m$. For $1 \leq i, j \leq N$, define

$$X_{ij}[n] = \|x_i[n]-x_j[n]\| \text{ and } V_{ij}[n] = \|v_i[n]-v_j[n]\|.$$

We say that the agents under system (1) converge to flocking if the following is satisfied: for all $1 \leq i, j \leq N$,

$$\lim_{n\to\infty} V_{ij}[n] = 0 \text{ and } \sup_{n\geq 0} X_{ij}[n] < \infty. \tag{3}$$

Taking the limit when $h \to 0$, we obtain the corresponding continuous-time system:

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$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(v_j(t) - v_i(t)). \end{cases} \tag{4}$$

It is well-known that the interaction topology plays an important role in the analysis of flocking. Research on the C-S model can be roughly categorized depending on whether the interaction topology is symmetric or asymmetric. For the case of symmetric interaction topology, the all-to-all interaction topology, i.e., the topology is the weighted complete graph, was well-understood (Cucker and Smale, 2007a). Depending on the parameter β, *unconditional* and *conditional flocking* was established by the self-bounding approach for both discrete-time and continuous-time systems. These results were further extended to the more general setting where the interaction topology is a weighted connected graph (Cucker and Smale, 2007b). The proof of convergence motivated a surging interest in the C-S model. Extensions of the C-S model were quickly made in several directions, e.g., adding stochastic noises (Cucker and Mordecki, 2008; Ha et al., 2009) and avoiding collisions (Cucker and Dong, 2010, 2011). The flocking analysis in these cases heavily relied on the conservation of momentum, a property that follows from the symmetry of the underlying graph of interactions. However, when the interaction topology is asymmetric, that is, the topology is a weighted diagraph, to the best of our knowledge, there is no general systematic approach for the C-S model in this case. Only two special classes of weighted digraphs are reported in the literature. The first one is due to Shen (Shen, 2007) where a hierarchical leadership is introduced so that the resulting adjacency matrix can become lower triangular. In the language of graph theory, the hierarchical leadership requires that the corresponding weighted digraph has a single leader and does not contain directed cycles. The hierarchical leadership is further studied in Cucker and Dong (2009). The rooted leadership structure is introduced by Li and Xue (Li and Xue, 2010), which still requires the flock to have a single leader but allows directed cycles between followers. Obviously, the rooted leadership is more general than the hierarchical leadership. The relaxed rooted leadership is further studied in Li et al. (2014). Note that the analysis in Li et al. (2014); Li and Xue (2010) is only for discrete-time case and cannot be applied to continuous-time case.

Based on the above discussions, a natural question one may ask is whether we can establish the flocking results for both the discrete-time and continuous-time C-S model under general interaction topologies, containing the symmetric and asymmetric interaction topologies mentioned above as special cases. In this paper we provide an affirmative answer to this question. Actually, we first consider a more general flocking model for both discrete-time and continuous-time cases. Sufficient conditions are derived to ensure the convergence to flocking, which only depend on the initial states. As an application, the flocking results are provided for the C-S model under general interaction topologies. These results are guaranteed by the parameters and the initial conditions only. The study of general interaction topologies in relation to the C-S model is partially motivated by physical and artificial systems where multiple leaders exist and play a positive role in the flocking behavior (Nagy et al., 2010; Tarcai et al., 2011). We believe that this is relevant, not only theoretically but also for applications such as the one in Perea et al. (2009) in which the spacecrafts can be endowed with general interaction topologies.

Our extension from special topologies to general topologies is not trivial. The main reason for this is that all the existing approaches for the C-S model are no longer applicable to the case with general interaction topology. Note that for all these special topologies mentioned above, the asymptotic velocity for flocking is known a priori since the C-S model under these special topologies always possesses some invariants, either the mean of all the initial velocities or the velocity of the leader. With the help of these invariants, the flocking problem can often be converted to a stability problem of the reduced systems. However, the C-S model under general interaction topologies does not possess any invariants. Hence new approaches are needed to tackle this problem. A novel approach, based on a new combination of perturbation method and algebraic graph theory, is introduced in this paper. Our approach is different from known ones for the C-S model. We borrow the idea from the work (Martin et al., 2014) in which the perturbation method is used for the analysis of other flocking models under the symmetric topology.

Similar to all the existing works on the C-S model, our results also obtain the critical exponent β_c for flocking, below which unconditional convergence to flocking holds. We point out that the critical exponent β_c is tight for some special interaction topologies. In particular, for the all-to-all interaction topology, β_c is given by 1/2, the same as in the original work of Cucker and Smale (Cucker and Smale, 2007a). The critical exponent β_c is consistent with that in Li and Xue (2010) when the rooted leadership is considered.

The rest of this paper is organized as follows. Section 2 presents some preliminaries on algebraic graph theory. In Section 3, we investigate the discrete-time flocking model. Section 4 studies the corresponding continuous-time flocking model. This paper is concluded in Section 5.

Notation: Matrix ordering is meant componentwise, i.e., for $\mathcal{A} = (a_{ij})_{N\times N}$, $\mathcal{A} \geq 0$ stands for $a_{ij} \geq 0$ for all i, j, and similarly for $\mathcal{A} > 0$. For a real number c, denote by $\lfloor c \rfloor$ the largest integer no greater than c.

2. DIRECTED GRAPHS

Directed graph is used to model the interaction topology among agents. A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, \dots, N\}$ of vertices and an arc set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, where an arc is an ordered pair of vertices of $\mathcal{V}$. If $(j, i) \in \mathcal{E}$, then vertex j is called a neighbor of vertex i. The neighbor set of vertex i is denoted by $\mathcal{N}_i$, that is, $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$. For the case that $(i, i) \in \mathcal{E}$, we understand that $\mathcal{G}$ has a self-loop at vertex i. A digraph is said to be simple if it contains no loops and parallel arcs. We say that a nonnegative matrix $\mathcal{A} = (a_{ij})_{N\times N} \geq 0$ is an adjacency matrix of a given digraph $\mathcal{G}$ if $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$. In this case, $\mathcal{G}$ is usually called a weighted digraph, denoted by $\mathcal{G}(\mathcal{A})$. A walk in $\mathcal{G}$ from i_0 to i_k is a sequence $i_0, \dots, i_k$ such that $(i_l, i_{l+1}) \in \mathcal{E}$ for $l = 0, \dots, k-1$. The integer k

(the number of its arcs) is the length of the walk. Note that a walk can include the same vertices. If the vertices of a walk are distinct, then it is called a (directed) path. A cycle is a path such that the origin and terminus are the same. We can see that if there is a walk from i to j, then there is also a path from i to j. A digraph is said to be strongly connected if for any two distinct vertices i and j, there exists a path from i to j. A weaker concept is the existence of a (directed) spanning tree. We say that a digraph has a spanning tree if we can find a vertex (called a root) such that there exists a path from it to any other vertex. The length of a shortest path from vertices i to j is called the distance from i to j and denoted by $d(i,j)$. We use the convention that $d(i,j)=\infty$ if there is no path from i to j. The diameter of a digraph, denoted by d, is the maximum distance from any vertex to another of the digraph, i.e., $d=\max_{i,j\in\mathcal{V}} d(i,j)$. It is clear that a directed graph is of finite diameter if and only if it is strongly connected. We now consider the case that $\mathcal{G}(\mathcal{A})$ has a spanning tree. Denote by $\mathcal{V}_r$ the set of the roots of all possible trees. It is easy to see that the induced subgraph, denoted by $\mathcal{G}[\mathcal{V}_r]$, is strongly connected. Let d_r be the diameter of the induced subgraph $\mathcal{G}[\mathcal{V}_r]$. Denote by $\overline{\mathcal{V}}_r$ the complement of the set $\mathcal{V}_r$, i.e., $\overline{\mathcal{V}}_r=\mathcal{V}\setminus\mathcal{V}_r$,. It thus follows that all the possible arcs between $\mathcal{V}_r$ and $\overline{\mathcal{V}}_r$ are from $\mathcal{V}_r$ to $\overline{\mathcal{V}}_r$. We see that $\mathcal{G}(\mathcal{A})$ is strongly connected when $\overline{\mathcal{V}}_r$ is empty. Define

$$\overline{d}_r=\max_{i\in\mathcal{V}_r}\max_{j\in\overline{\mathcal{V}}_r} d(i,j).$$

Let

$$D=\max\{d_r,\overline{d}_r\}.$$

We understand the constant D as the width of the directed graph $\mathcal{G}(\mathcal{A})$. This quantity D will play a crucial role in establishing our main results.

3. DISCRETE-TIME CONVERGENCE

In this section, we study the discrete-time C-S model (1). To this end, we first discuss the general model for flocking: for $1\le i\le N$,

$$\begin{cases} x_i[n+1]=x_i[n]+hv_i[n] \\ v_i[n+1]=v_i[n]+\displaystyle\sum_{j\in\mathcal{N}_i} ha_{ij}(X_{ij}[n])(v_j[n]-v_i[n]). \end{cases} \tag{5}$$

The interaction topology of the flock is denoted by a simple digraph $\mathcal{G}$. The weight functions $a_{ij}(X_{ij}[n])$ quantify the way the agents influence each other. It is reasonable that the weight functions satisfy the following assumption.

Assumption 1. For each agent i and $j\in\mathcal{N}_i$, the function $a_{ij}(r)$ is positive and non-increasing, i.e., $a_{ij}(r_1)\ge a_{ij}(r_2)$ as $0\le r_1\le r_2$.

We now rewrite the system (5) in a more concise form. Let $\mathcal{A}_n$ be the adjacency matrix of $\mathcal{G}$. Let $\mathcal{L}_n$ be the Laplacian matrix of $\mathcal{A}_n$, that is, $\mathcal{L}_n=\mathcal{D}_n-\mathcal{A}_n$ where the diagonal matrix $\mathcal{D}_n=\mathrm{diag}(d_1[n],\ldots,d_N[n])$ is defined as $d_i[n]=\sum_{j\in\mathcal{N}_i} a_{ij}(X_{ij}[n])$. Consequently, we have $\mathcal{L}_n\mathbf{1}=0$ where $\mathbf{1}$ is the N-dimensional column vector of ones. Then the system (5) can be rewritten as

$$\begin{cases} \boldsymbol{x}[n+1]=\boldsymbol{x}[n]+h\boldsymbol{v}[n] \\ \boldsymbol{v}[n+1]=\mathcal{S}[n]\boldsymbol{v}[n] \end{cases} \tag{6}$$

where $\boldsymbol{x}[n]=(x_1[n],\ldots,x_N[n])^T$, $\boldsymbol{v}[n]=(v_1[n],\ldots,v_N[n])^T$ and $\mathcal{S}[n]=\mathcal{I}-h\mathcal{L}_n$. Precisely, $\mathcal{S}[n]=(s_{ij}[n])_{N\times N}$ with

$$s_{ij}[n]=\begin{cases} ha_{ij}(X_{ij}[n]), & j\in\mathcal{N}_i \\ 1-hd_i[n], & i=j \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{S}[n]\boldsymbol{v}[n]$ means that $\mathcal{S}[n]$ is acting on $\mathbb{R}^{Nm}$. Using Assumption 1, we have for $j\in\mathcal{N}_i$,

$$a_{ij}(X_{ij}[n])\le a_{ij}(0).$$

Assume that the time step h satisfies

$$0<h\le\frac{1}{2\max_{1\le i\le N}\sum_{j\in\mathcal{N}_i} a_{ij}(0)}. \tag{7}$$

Then we have, for $1\le i\le N$ and all $n\ge 0$,

$$s_{ii}=1-hd_i[n]\ge\frac{1}{2} \tag{8}$$

and for $j\in\mathcal{N}_i$ and all $n\ge 0$,

$$0<s_{ij}=ha_{ij}(X_{ij}[n])\le\frac{1}{2}. \tag{9}$$

As a result, the matrix $\mathcal{S}[n]$ is stochastic, i.e., nonnegative with each row summing to 1.

We now assume that the interaction topology $\mathcal{G}$ has a spanning tree. As above, let $\mathcal{V}_r$ denote the set of all possible roots of the flock and similarly for the definitions of the set $\overline{\mathcal{V}}_r$ and the width D. It should be noted that the agents in $\overline{\mathcal{V}}_r$ do not influence the behavior of the agents in $\mathcal{V}_r$. The set $\mathcal{V}_r$ can be regarded as the leader set of the flock. Given a solution $(\boldsymbol{x}[n],\boldsymbol{v}[n])$ of the system (6), define

$$X[n]=\max_{i\in\mathcal{V}_r,j\in\mathcal{V}} X_{ij}[n] \text{ and } V[n]=\max_{i\in\mathcal{V}_r,j\in\mathcal{V}} V_{ij}[n].$$

These two quantities $X[n]$ and $V[n]$ can be thought of as the position and velocity diameters of the flock related to the set $\mathcal{V}_r$ at time n, respectively. We can see that to obtain (3), it is sufficient to show

$$\lim_{n\to\infty} V[n]=0 \quad\text{and}\quad \sup_{n\ge 0} X[n]<\infty.$$

This is because we have the relation: for all $1\le i,j\le N$,

$$X_{ij}[n]\le 2X[n] \quad\text{and}\quad V_{ij}[n]\le 2V[n].$$

Therefore, we can analyze the flocking behavior in terms of $X[n]$ and $V[n]$.

In the following we fix a solution $(\boldsymbol{x}[n],\boldsymbol{v}[n])$ of the system (6) starting from the initial state $(\boldsymbol{x}[0],\boldsymbol{v}[0])$. We are now in a position to state the flocking result for the system (6).

Theorem 2. Consider the system (6) *with the time step h satisfying* (7). *Assume that the interaction topology $\mathcal{G}$ has a spanning tree. Assume also that there exists a constant $\rho>0$ such that*

$$V[0]\le\frac{\rho h^{D-1}L^D}{2D} \tag{10}$$

where

$$L=\min_{(j,i)\in\mathcal{E}} a_{ij}(X_{ij}[0]+\rho). \tag{11}$$

Then the flocking occurs. Furthermore, for all $n\ge 0$,

$$V[n]\le\left(1-(hL)^D\right)^{\lfloor\frac{n}{D}\rfloor}V[0]$$

and

$$X[n]\le X[0]+\frac{\rho}{2}.$$

We have the following result for the discrete-time C-S model (1).

Theorem 3. Consider the discrete-time C-S model (1). *Let the time step h be such that $0 < h \le \frac{1}{2(N-1)H}$. Assume that the interaction topology $\mathcal{G}$ has a spanning tree. Assume also that one of the following three hypotheses holds:*

(i) $\beta < \frac{1}{2D}$,

(ii) $\beta = \frac{1}{2D}$ *and* $V[0] < \frac{h^{D-1}H^D}{2D}$,

(iii) $\beta > \frac{1}{2D}$ *and*

$$V[0] \le \frac{(2\alpha-1)^{2\alpha-1}\left(2X[0](1-\alpha)+K\right)h^{D-1}H^D}{2D\left((2\alpha-1)^2+(2\alpha X[0]+K)^2\right)^{\alpha}} \tag{12}$$

where

$$\alpha = D\beta \quad \text{and} \quad K = \sqrt{4\alpha^2X^2[0]+2\alpha-1}.$$

Then the flocking occurs. Moreover, $V[n]$ converges to zero exponentially as $n \to \infty$.

Proof. We now apply Theorem 2 to show the results. To this end, we first note that $X_{ij}[n] \le 2X[n]$ for $(j,i) \in \mathcal{E}$ and $n \ge 0$. Considering the C-S weight function (2), we see that

$$L \ge \frac{H}{(1+(2X[0]+\rho)^2)^{\beta}}.$$

Therefore, a sufficient condition for (10) to hold is given by

$$V[0] \le \frac{\rho h^{D-1}H^D}{2D(1+(2X[0]+\rho)^2)^{D\beta}} \triangleq F(\rho). \tag{13}$$

(i) Assume $\beta < \frac{1}{2D}$. Then $2D\beta < 1$. As a function of ρ, $F(\rho) \to \infty$ as $\rho \to \infty$. Consequently, for all initial states $(\boldsymbol{x}[0], \boldsymbol{v}[0])$, (13) always holds for appropriately large $\rho > 0$. The convergence results now readily follow from Theorem 2.

(ii) Assume now $\beta = \frac{1}{2D}$. We see that $F(\rho)$ is increasing of ρ. In addition, when $\rho \to \infty$,

$$F(\rho) \to \frac{h^{D-1}H^D}{2D}.$$

In particular, if

$$V[0] < \frac{h^{D-1}H^D}{2D}$$

then (13) can be satisfied with some $\rho > 0$. Now Theorem 2 implies the desired results.

(iii) Assume finally $\beta > \frac{1}{2D}$. By some elementary computations, we obtain that the function $F(\rho)$ attains its maximum at the point ρ^* given by

$$\rho^* = \frac{2X[0](1-\alpha)+K}{2\alpha-1}.$$

Furthermore, $F(\rho^*)$ is equal to the term in the right-hand side of inequality (12). We now apply Theorem 2 to complete the proof. □

4. CONTINUOUS-TIME CONVERGENCE

In this section we consider the following continuous-time dynamical system, for $1 \le i \le N$,

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(X_{ij}(t))(v_j(t)-v_i(t)). \end{cases} \tag{14}$$

Here, for the weight functions a_{ij}, Assumption 1 is still satisfied. Define

$$\bar{a} = \max_{(j,i)\in\mathcal{E}} a_{ij}(0).$$

Since the values of the a_{ii}'s do not influence the system (14), let

$$a_{ii} = N\bar{a} - \sum_{j\in\mathcal{N}_i} a_{ij}(X_{ij}).$$

It is easy to see that, for $1 \le i \le N$,

$$\sum_{j=1}^{N} a_{ij}(X_{ij}) = N\bar{a} \text{ and } a_{ii} \ge \bar{a}. \tag{15}$$

In the following we fix a solution $(\boldsymbol{x}(t), \boldsymbol{v}(t))$ of (14) with initial condition $(\boldsymbol{x}(0), \boldsymbol{v}(0))$.

We are now in a position to state the main result for the system (14).

Theorem 4. Consider the system (14). *Assume that the interaction topology $\mathcal{G}$ has a spanning tree. Assume also that there exists a constant $\rho > 0$ such that*

$$V(0) \le \frac{\rho L^D}{2De^{DN\bar{a}}} \tag{16}$$

where

$$L = \min_{(j,i)\in\mathcal{E}} a_{ij}(X_{ij}(0)+\rho). \tag{17}$$

Then the flocking occurs. Furthermore, for all $t \ge 0$,

$$V(t) \le \left(1 - e^{-DN\bar{a}}L^D\right)^{\lfloor \frac{t}{D} \rfloor} V(0)$$

and

$$X(t) = X(0) + \frac{\rho}{2}.$$

The flocking result for the continuous-time C-S system (4) is presented as follows. Its proof is the same as that of Theorem 3.

Theorem 5. Consider the continuous-time C-S system (4). *Assume that the interaction topology $\mathcal{G}$ has a spanning tree. Assume also that one of the following three hypotheses holds:*

(i) $\beta < \frac{1}{2D}$,

(ii) $\beta = \frac{1}{2D}$ *and* $V(0) < \frac{H^D}{2De^{DNH}}$,

(iii) $\beta > \frac{1}{2D}$ *and*

$$V(0) \le \frac{(2\alpha-1)^{2\alpha-1}\left(2X(0)(1-\alpha)+K\right)H^D}{2De^{DNH}\left((2\alpha-1)^2+(2\alpha X(0)+K)^2\right)^{\alpha}}$$

where

$$\alpha = D\beta \quad \text{and} \quad K = \sqrt{4\alpha^2X^2(0)+2\alpha-1}.$$

Then the flocking occurs. Moreover, $V(t)$ converges to zero exponentially as $t \to \infty$.

Remark 6. The rationale of the formula in part (iii) of Theorems 3 and 5 is clear. It shows that convergence to flocking cannot occur if the agents have very different initial velocities and very spread initial positions. In addition, the formula here is different from those obtained in Cucker and Smale (2007a); Li and Xue (2010). This might be due to distinct approaches adopted.

5. CONCLUSION

In this paper we first consider a general flocking model with long-distance interaction topology, for both discrete-time and continuous-time cases. The general flocking results are guaranteed by the initial conditions only. The C-S flocking is then derived by the application of these general results to the C-S weighted function. The critical exponent below which unconditional flocking holds is given, depending only on the interaction topology. It is shown to be tight for some existing special topologies. Our approach adopted here is different from those appearing for the C-S model in the literature. Future works will focus on the adaptation of this novel approach to other important models for flocking.

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