1 Methodology - Coverage Controller

We consider a group of N vehicles, each of them denoted Q_i, $i = 1, \cdots, N$, with dynamics described by

$$\dot{q_i} = p_i, \quad \dot{p_i} = u_i.$$

Let $\Omega \subseteq \mathbb{R}^2$ be a compact domain containing zero, and define $\Omega(t) = \Omega + q_d(t)$, where $q_d(t)$ is the solution for the system

$$\begin{cases} \dot{q_d} & = p_d \\ \dot{p_d} & = f_d(q_d, p_d), \end{cases}$$

we call $q_d(t)$ the marker point of the moving domain $\Omega(t)$.

Define $q_{ij} := q_i - q_j$, and $p_{ij} := p_i - p_j$ and denote by $P_{\partial\Omega(t)}(q_i)$ the closest point of $\partial\Omega(t)$ to q_i (i.e., the projection of q_i on $\partial\Omega(t)$). Also, define $h_i := q_i - P_{\partial\Omega(t)}(q_i)$, and denote by $[[h_i]]$ the signed distance of q_i from $\partial\Omega(t)$.

The proposed control force is given by:

$$u_i = -\sum_{j\neq i}^{N} f_I(\|q_{ij}\|)\frac{q_{ij}}{\|q_{ij}\|} - \frac{1}{N}\sum_{j\neq i}^{N} f_{al}(\|q_{ij}\|)\, p_{ij} - f_h([[h_i]])\frac{h_i}{[[h_i]]} - a(p_i - p_d) \tag{1}$$

The position and velocity of the i th vehicle relative to the marker of the moving domain are given by:

$$\begin{cases} x_i & := q_i - q_d \\ v_i & := p_i - p_d. \end{cases}$$

Note that the inter-vehicle position and velocity in this new framework satisfy:

$$\begin{aligned} x_{ij} &:= x_i - x_j = q_i - q_d - (q_j - q_d) = q_{ij}, \\ v_{ij} &:= v_i - v_j = p_i - p_d - (p_j - p_d) = p_{ij}, \end{aligned}$$

it means the relative positions are invariant to the change of coordinates. Moreover, the vehicle domain distance satisfies $h_i = q_i - P_{\partial\Omega(t)}(q_i) = (q_i - q_d) - P_{\partial\Omega(t)-q_d}(q_i - q_d) = x_i - P_{\partial\Omega(0)}(x_i)$. This allow us to rewrite (1) as

$$u_i = -\sum_{j\neq i}^{N} f_I(\|x_{ij}\|)\frac{x_{ij}}{\|x_{ij}\|} - \frac{1}{N}\sum_{j\neq i}^{N} f_{al}(\|x_{ij}\|)\, v_{ij} - f_h([[h_i]])\frac{h_i}{[[h_i]]} - a v_i \tag{2}$$

Let us consider the potential

$$V_h(x_i) = \int_{-\frac{r_d}{2}}^{\left[\left[x_i - P_{\partial\Omega(0)}(x_i)\right]\right]} f_h(s)\, ds$$

which satisfies

$$\nabla_{x_i} V_h(x_i) = f_h\left(\left[\left[x_i - P_{\partial\Omega(0)}(x_i)\right]\right]\right)\nabla_{x_i}\left(\left[\left[x_i - P_{\partial\Omega(0)}(x_i)\right]\right]\right) = f_h([[h_i]])\frac{h_i}{[[h_i]]}$$

where we have used the identity $\nabla_{x_i}\left(\left[\left[x_i - P_{\partial\Omega(0)}(x_i)\right]\right]\right) = \frac{x_i - P_{\partial\Omega(0)}(x_i)}{\left[\left[x_i - P_{\partial\Omega(0)}(x_i)\right]\right]}$.

Similarly, it can be shown that the inter-vehicle force is the negative gradient of the potential

$$V_I(x_{ij}) = \int_{r_d}^{\|x_{ij}\|} f_I(s)\, ds,$$

to finally get:

$$u_i = \underbrace{-\sum_{j\neq i}^{N} \nabla_{x_i} V_I(x_{ij})}_{\text{Inter Vehicle}} - \underbrace{\frac{1}{N}\sum_{j\neq i}^{N} f_{al}(\|x_{ij}\|)\, v_{ij}}_{\text{Velocity Alignment}} - \overbrace{\underbrace{\nabla_{x_i} V_h(x_i)}_{\text{Domain Vehicle}} - \underbrace{a v_i}_{\text{Speed Alignment}}}^{\text{Navigational feedback}} \tag{3}$$

Consider the candidate for Lyapunov function consisting in kinetic plus (artificial) potential energy:

$$\Phi = \frac{1}{2}\sum_{i=1}^{N}\Big(\dot{x}_i \cdot \dot{x}_i + \sum_{j\neq i}^{N} V_I\left(x_{ij}\right) + V_h\left(x_i\right)\Big).$$

Note that each term in Φ is non-negative, and Φ reaches its absolute minimum value when the vehicles are totally stopped.

The derivative of Φ with respect to time can be calculated as:

$$\begin{aligned}\dot{\Phi} &= \sum_{i=1}^{N}\dot{x}_i \cdot \Big(u_i + \sum_{j\neq i}^{N}\nabla_{x_i} V_I\left(x_{ij}\right) + \nabla_{x_i} V_h\left(x_i\right)\Big) \\ &= \sum_{i=1}^{N}\dot{x}_i \cdot \left(-\frac{1}{N}\sum_{j\neq i}^{N} f_{al}\left(\|x_{ij}\|\right) v_{ij} - a v_i\right)\end{aligned}$$

For the (extra) alignment term, write

$$\sum_{i=1}^{N} v_i \cdot \sum_{j\neq i}^{N} f_{al}\left(\|x_{ij}\|\right)\left(v_i - v_j\right) = \frac{1}{2}\sum_{i=1}^{N} v_i \cdot \sum_{j\neq i}^{N} f_{al}\left(\|x_{ij}\|\right)\left(v_i - v_j\right) + \frac{1}{2}\sum_{j=1}^{N} v_j \cdot \sum_{i\neq j}^{N} f_{al}\left(\|x_{ij}\|\right)\left(v_j - v_i\right),$$

where in the second term in the right-hand-side we simply rename $i \leftrightarrow j$ as indices of summation. From there, use that $\|x_{ij}\| = \|x_{ji}\|$ to get:

$$\sum_{i=1}^{N} v_i \cdot \sum_{j\neq i}^{N} f_{al}\left(\|x_{ij}\|\right)\left(v_i - v_j\right) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} f_{al}\left(\|x_{ij}\|\right)\|v_i - v_j\|^2$$

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With the minus sign in front this gives a negative-definite term. Conclude that $\dot{\Phi}$ is negative semidefinite and equal to zero if and only if $\dot{x}_i = 0$ for all i (i.e., all vehicles are at equilibrium in the relative framework).