Von Neumann's minimax theorem

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Two-player zero sum game

M(r,c) is the payout to each player

	c1	c2	c3
r1	(2,-2)	(1,-1)	(-1,1)
r2	(3,-3)	(0,0)	(-2,2)
r3	(-1,1)	(2,-2)	(1,-1)
r4	(0,0)	(0,0)	(1,-1)

Both players want to maximize *their* payout.

Two-player zero sum game

M(r,c) is the amount row player gets (from column player)

	c1	c2	c3
r1	2	1	-1
r2	3	0	-2
r3	-1	2	1
r4	0	0	1

Row player trying to maximize
Column player trying to minimize

M(r,c) is the payout to each player

	c1	c2	c3
r1	(2,-2)	(1,-1)	(-1,1)
r2	(3,-3)	(0,0)	(-2,2)
r3	(-1,1)	(2,-2)	(1,-1)
r4	(0,0)	(0,0)	(1,-1)

Both players want to maximize *their* payout.

Who plays first? Two options

What if the row player first announces their "move", *then* column player responds?

	c1	c2	c3
r1	2	1	-1
r2	3	0	-2
r3	-1	2	1
r4	0	0	1

Row player trying to maximize
Column player trying to minimize

Results in (r4,c1) or (r4,c2)
(Row) payoff equals 0.

$$\max_{r \in [R]} \min_{c \in [C]} M(r, c) = 0$$

Play order $\longrightarrow$

"Solve order" $\longleftarrow$

To play first, row player must calculate $\min_c M(1,c), \min_c M(2,c), \min_c M(3,c)$ and play the move that maximizes those.

Let's reverse roles: now column player first

	c1	c2	c3
r1	2	1	-1
r2	3	0	-2
r3	-1	2	1
r4	0	0	1

Row player trying to maximize
Column player trying to minimize

Results in (c3,r3) or (c3,r4)
(Row) payoff equals 1.

$$\min_{c\in[C]} \max_{r\in[R]} M(r,c) = 1$$

Play order →

"Solve order" ←

To play first, column player must calculate $\max_r M(r,1)$, $\max_r M(r,2)$, etc
and play the move that minimizes those.

$$\min_{c \in C} \max_{r \in [R]} M(r, c) = 1$$

$$\max_{r \in [R]} \min_{c \in C} M(r, c) = 0$$

Theorem: $\displaystyle \min_{c \in [C]} \max_{r \in [R]} M(r, c) \geq \max_{r \in [R]} \min_{c \in [C]} M(r, c)$

Proof: $\displaystyle \max_{r \in [R]} M(r, c) \geq M(r', c)$ for any r', c

$\displaystyle \min_{c \in [C]} \max_{r \in [R]} M(r, c) \geq \min_{c \in [C]} M(r', c)$ for any r'

A "pure" strategy is a commitment to play a single action.

For "pure strategies", advantageous to go second.

A "mixed" strategy is a distribution over actions.

Row player announces a distribution p over their actions.

Column player announces a distribution q over their actions.

Then, $r \sim p, c \sim q$ and a payoff $M(r, c)$ is realized.

Expected payoff is $\mathbb{E}_{r \sim p, c \sim q}[M(r, c)] = p^T M q = \sum_{r \in [R]} \sum_{c \in [C]} M(r, c) p_r q_c$

Theorem: $\min_{q \in \Delta_C} \max_{p \in \Delta_R} p^T M q = \max_{p \in \Delta_R} \min_{q \in \Delta_C} p^T M q = v^*$

Value of the game

$\exists p \in \Delta_R \, \forall q \in \Delta_C \; p^T M q \geq v^*$

$\exists q \in \Delta_C \, \forall p \in \Delta_R \; p^T M q \leq v^*$

For "mixed strategies", order does not matter!

Zero-sum games [edit]

The minimax theorem was first proven and published in 1928 by John von Neumann,[3] who is quoted as saying "*As far as I can see, there could be no theory of games … without that theorem … I thought there was nothing worth publishing until the Minimax Theorem was proved*".[4]

Formally, von Neumann's minimax theorem states:

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be compact convex sets. If $f : X \times Y \to \mathbb{R}$ is a continuous function that is concave-convex, i.e.

$f(\cdot, y) : X \to \mathbb{R}$ is concave for fixed y, and
$f(x, \cdot) : Y \to \mathbb{R}$ is convex for fixed x.

Then we have that

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Sion's minimax theorem

From Wikipedia, the free encyclopedia

In mathematics, and in particular game theory, **Sion's minimax theorem** is a generalization of John von Neumann's minimax theorem, named after Maurice Sion.

It states:

Let X be a compact convex subset of a linear topological space and Y a convex subset of a linear topological space. If f is a real-valued function on $X \times Y$ with

$f(x, \cdot)$ upper semicontinuous and quasi-concave on Y, $\forall x \in X$, and
$f(\cdot, y)$ lower semicontinuous and quasi-convex on X, $\forall y \in Y$

then,

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$