Shapley values

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Thanks, Wikipedia

Incentive!

Now all of you know enough basic ML to start contributing material to the class.

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For eg: prepare 2-3 slides on something interesting (1hr prep time?):
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Outline

1. *Shapley values (2/3 class)*
2. *Variable and datapoint importance (1/3 class)*

Cooperative game theory

Theory introduced: 1951, Nobel prize: 2012

Problem setup

N players cooperate to produce something of "value".
Value function $v : 2^N \to \mathbb{R}, v(\varnothing) = 0$.
$v([N])$ is the (dollar, say) value actually obtained.
$v(S)$ is the (hypothetical, assumed known) value obtained when subset S work together

How should $v([N])$ be split up amongst the N players?
i.e. how much money $\phi_i(v)$ should player i get?

How about everybody gets 0 dollars?

How about $\phi_1(v) = v(N), \phi_j(v) = 0$ for $j > 1$?

How about $\phi_i(v) = v(N)/N$?

Desiderata ("axioms")

"Efficiency"

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

"Symmetry"

If i and j are two actors who are equivalent in the sense that

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

for every subset S of N which contains neither i nor j, then $\varphi_i(v) = \varphi_j(v)$.

This property is also called *equal treatment of equals*.

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"Linearity"

$$\varphi_i(v + w) = \varphi_i(v) + \varphi_i(w)$$

for every i in N. Also, for any real number a,

$$\varphi_i(av) = a\varphi_i(v)$$

for every i in N.

"Null player" (freeloader)

Player i is "null" if $\forall S$ with $i \notin S, v(S \cup i) = v(S)$

$\varphi_i(v)$ of a null player i in a game v is zero.

The Shapley value

There is a unique function satisfying all four axioms.

$$\boxed{\varphi_i(v) = \frac{1}{n} \sum_{S \subseteq N \setminus \{i\}} \binom{n-1}{|S|}^{-1} \left(v(S \cup \{i\}) - v(S)\right)}$$

$$\varphi_i(v) = \frac{1}{\text{number of players}} \sum_{\text{coalitions excluding } i} \frac{\text{marginal contribution of } i \text{ to coalition}}{\text{number of coalitions excluding } i \text{ of this size}}$$

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An alternative equivalent formula for the Shapley value is:

$$\varphi_i(v) = \frac{1}{n!} \sum_R \left[v(P_i^R \cup \{i\}) - v(P_i^R) \right]$$

where the sum ranges over all $n!$ orders R of the players

and P_i^R is the set of players in N which precede i in the order R.

Eg: the "business game"

Owner provides initial capital, workspace, vision, etc.

Each worker provides additional profit of p.

The value function for this coalitional game is

$$v(S) = \begin{cases} mp, & \text{if } o \in S \\ 0, & \text{otherwise} \end{cases}$$

where m is the cardinality of $S \setminus \{o\}$

Eg: the "glove game"

Player 1 and 2 have left-hand gloves, player 3 has right-hand glove

A coalition has value one if they have a complete pair, else zero

The value function for this coalitional game is

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3\},\{2,3\},\{1,2,3\}\}\,; \\ 0 & \text{otherwise.} \end{cases}$$

More properties

"Negative externality"

If v is a subadditive set function, i.e., $v(S \sqcup T) \leq v(S) + v(T)$,

then for each agent i: $\varphi_i(v) \leq v(\{i\})$.

"Positive externality"

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“Anonymity”

Relabeling the indices of the players leaves their Shapley value unchanged.

“Coalitions”

$$\varphi_C(v) = \sum_{T \subseteq N \setminus C} \frac{(n - |T| - |C|)!\, |T|!}{(n - |C| + 1)!} \sum_{S \subseteq C} (-1)^{|C| - |S|} v(S \cup T) \,.$$

Variable importance

How would you define v(S)?

Datapoint importance

How would you define v(S)?
(regression vs classification)

4 groups?

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