機械学習 Machine Learning

確率分布 Probability Distributions

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離散変数 Discrete variables
ベルヌーイ分布 Bernoulli distribution

$$x \in \{0, 1\}. \quad 0 \leqslant \mu \leqslant 1,$$

$$p(x = 1|\mu) = \mu$$
$$p(x = 0|\mu) = 1 - \mu$$

例：コイン投げ（表 = 1，裏 = 0）
e.g., coin flipping (heads = 1, tails = 0)

x 上の確率分布 The probability distribution over x
（ベルヌーイ分布 Bernoulli distribution）

$$\mathrm{Bern}(x|\mu) = \mu^x (1 - \mu)^{1-x}$$

$$\mathbb{E}[x] = \mu$$ 平均 Mean

$$\mathrm{var}[x] = \mu(1 - \mu)$$ 分散 Variance

離散変数 Discrete variables
ベルヌーイ分布 Bernoulli distribution

$\mathcal{D} = \{x_1, \ldots, x_N\}$ x の観測値 Observed values of x

尤度関数 Likelihood function

$$p(\mathcal{D}|\mu) = \prod_{n=1}^{N} p(x_n|\mu) = \prod_{n=1}^{N} \mu^{x_n}(1-\mu)^{1-x_n}$$

対数尤度関数 Log likelihood function

$$\ln p(\mathcal{D}|\mu) = \sum_{n=1}^{N} \ln p(x_n|\mu) = \sum_{n=1}^{N} \{x_n \ln \mu + (1-x_n)\ln(1-\mu)\}$$

$x=1$の回数

The number of observations of $x=1$

最尤推定量 Maximum likelihood estimator

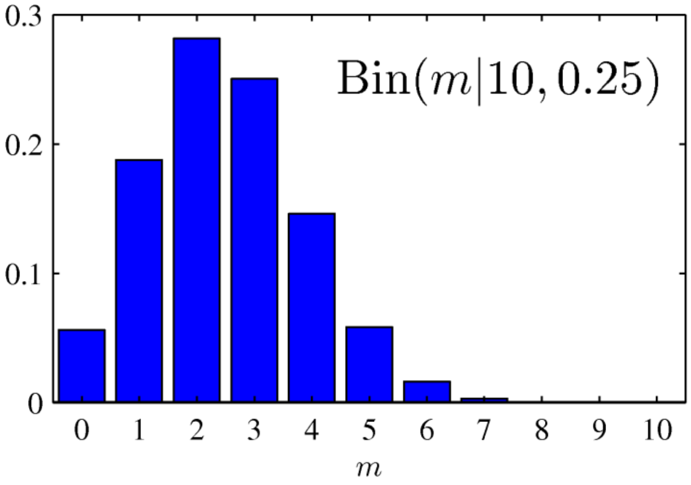
$$\mu_{\mathrm{ML}} = \frac{1}{N}\sum_{n=1}^{N} x_n = \frac{m}{N}$$

離散変数 Discrete variables
二項分布 Binomial distribution

m の分布 The distribution of m
(二項分布 Binomial distribution)

$$\mathrm{Bin}(m|N,\mu) = \binom{N}{m}\mu^m(1-\mu)^{N-m}$$

$$\binom{N}{m} \equiv \frac{N!}{(N-m)!m!}$$



$$\mathbb{E}[m] \equiv \sum_{m=0}^{N} m\mathrm{Bin}(m|N,\mu) = N\mu$$

$$\mathrm{var}[m] \equiv \sum_{m=0}^{N} (m-\mathbb{E}[m])^2\,\mathrm{Bin}(m|N,\mu) = N\mu(1-\mu)$$

離散変数 Discrete variables
多項分布 Multinomial distribution

1対*K*法 1-of-*K* scheme

K: 互いに排他的で，取ることが可能な状態の数
The number of possible mutually exclusive states

例 Example: $K = 6 \qquad x_3 = 1$

$$\mathbf{x} = (0, 0, 1, 0, 0, 0)^{\mathrm{T}} \qquad \sum_{k=1}^{K} x_k = 1$$

離散変数 Discrete variables
多項分布 Multinomial distribution

μ_k： $x_k = 1$となる確率
The probability of $x_k = 1$

$\mu_k \geqslant 0 \qquad \sum_k \mu_k = 1$

$\mathbf{x}$ の分布 The distribution of $\mathbf{x}$

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{K} \mu_k^{x_k} \qquad \boldsymbol{\mu} = (\mu_1, \ldots, \mu_K)^{\mathrm{T}}$$

$$\mathbb{E}[\mathbf{x}|\boldsymbol{\mu}] = \sum_{\mathbf{x}} p(\mathbf{x}|\boldsymbol{\mu})\mathbf{x} = (\mu_1, \ldots, \mu_K)^{\mathrm{T}} = \boldsymbol{\mu}$$

$$\sum_{\mathbf{x}} p(\mathbf{x}|\boldsymbol{\mu}) = \sum_{k=1}^{K} \mu_k = 1$$

離散変数 Discrete variables
多項分布 Multinomial distribution

$\mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ $\mathbf{x}$ の観測値 Observed values of $\mathbf{x}$

尤度関数 Likelihood function

$$p(\mathcal{D}|\boldsymbol{\mu}) = \prod_{n=1}^{N} \prod_{k=1}^{K} \mu_k^{x_{nk}} = \prod_{k=1}^{K} \mu_k^{(\sum_n x_{nk})} = \prod_{k=1}^{K} \mu_k^{m_k}$$

次の式を最大化 Maximize the following equation
(ラグランジュ乗数 λ を使用 using a Lagrange multiplier λ)

$$\underbrace{\sum_{k=1}^{K} m_k \ln \mu_k}_{\ln p(\mathcal{D}|\boldsymbol{\mu})} + \lambda \underbrace{\left(\sum_{k=1}^{K} \mu_k - 1 \right)}_{\sum_k \mu_k = 1} \Rightarrow \mu_k = -m_k/\lambda$$

$\sum_k \mu_k = 1$

$\lambda = -N$

$$\mu_k^{\mathrm{ML}} = \frac{m_k}{N}$$

離散変数 Discrete variables
多項分布 Multinomial distribution

$m_1, \ldots, m_K$ の分布 The distribution of $m_1, \ldots, m_K$
（多項分布 Multinomial distribution）

$$\mathrm{Mult}(m_1, m_2, \ldots, m_K | \boldsymbol{\mu}, N) = \binom{N}{m_1 m_2 \ldots m_K} \prod_{k=1}^{K} \mu_k^{m_k}$$

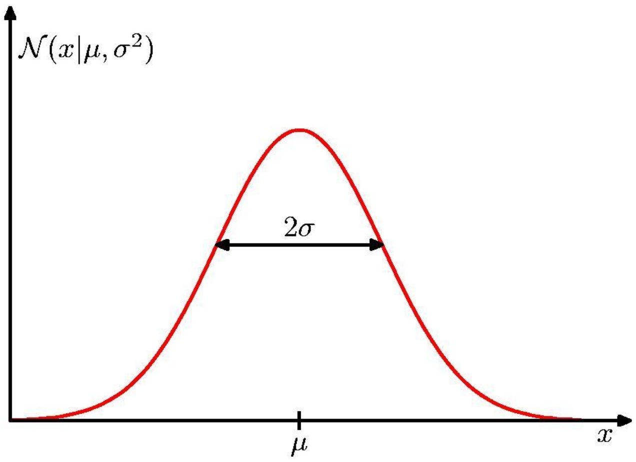
$$\binom{N}{m_1 m_2 \ldots m_K} = \frac{N!}{m_1! m_2! \ldots m_K!} \qquad \sum_{k=1}^{K} m_k = N$$

$$\begin{aligned} \mathbb{E}[m_k] &= N\mu_k \\ \mathrm{var}[m_k] &= N\mu_k(1-\mu_k) \\ \mathrm{cov}[m_j m_k] &= -N\mu_j\mu_k \end{aligned}$$

多変量ガウス分布 The multivariate Gaussian

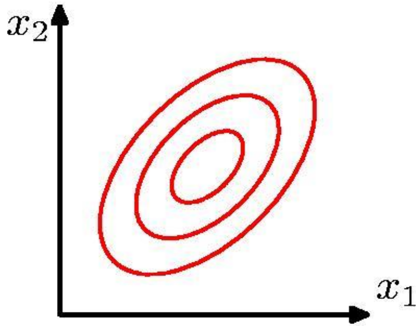
単変量ガウス分布
Univariate Gaussian distribution

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$



多変量ガウス分布
Multivariate Gaussian distribution

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

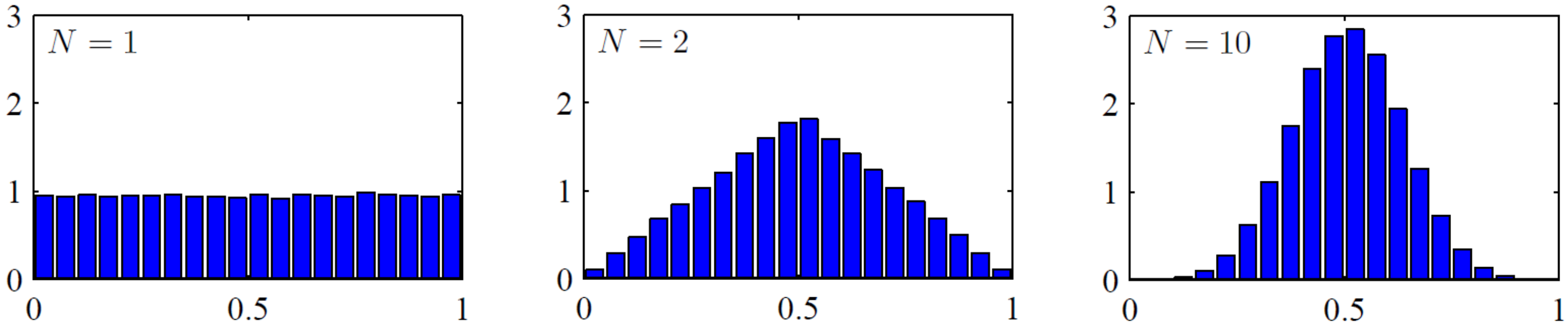


多変量ガウス分布 The multivariate Gaussian

中心極限定理 Central limit theorem Walker (1969)

いくつかの確率変数の和は，足し合わされる変数の数が増えるに従って，
徐々にガウス分布に従うようになる.
The sum of a set of random variables has a distribution that becomes increasingly Gaussian as the number of terms in the sum increases.

例：N 個の一様に分布する量の平均
E.g., the mean of N uniformly distributed numbers $(x_1 + \cdots + x_N)/N$



多変量ガウス分布 The multivariate Gaussian
ガウス分布の幾何 Geometry of the Gaussian

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{u}_j = I_{ij}$$

$$I_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

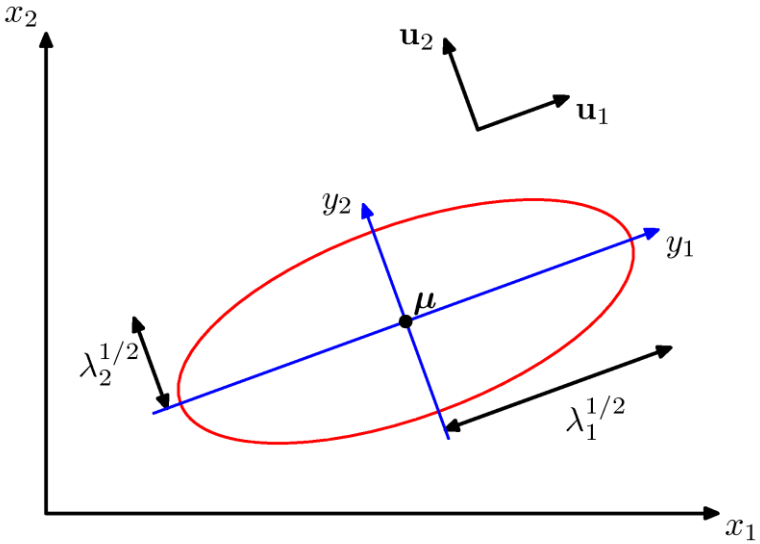
Δ：マハラノビス距離 Mahalanobis distance

$$\Delta^2 = (\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$$

$$\boldsymbol{\Sigma}\mathbf{u}_i = \lambda_i\mathbf{u}_i \qquad \boldsymbol{\Sigma} = \sum_{i=1}^{D}\lambda_i\mathbf{u}_i\mathbf{u}_i^{\mathrm{T}}$$

$$\boldsymbol{\Sigma}^{-1} = \sum_{i=1}^{D}\frac{1}{\lambda_i}\mathbf{u}_i\mathbf{u}_i^{\mathrm{T}}$$

$$\Delta^2 = \sum_{i=1}^{D}\frac{y_i^2}{\lambda_i} \qquad y_i = \mathbf{u}_i^{\mathrm{T}}(\mathbf{x}-\boldsymbol{\mu})$$



多変量ガウス分布 The multivariate Gaussian
ガウス分布の幾何 Geometry of the Gaussian

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

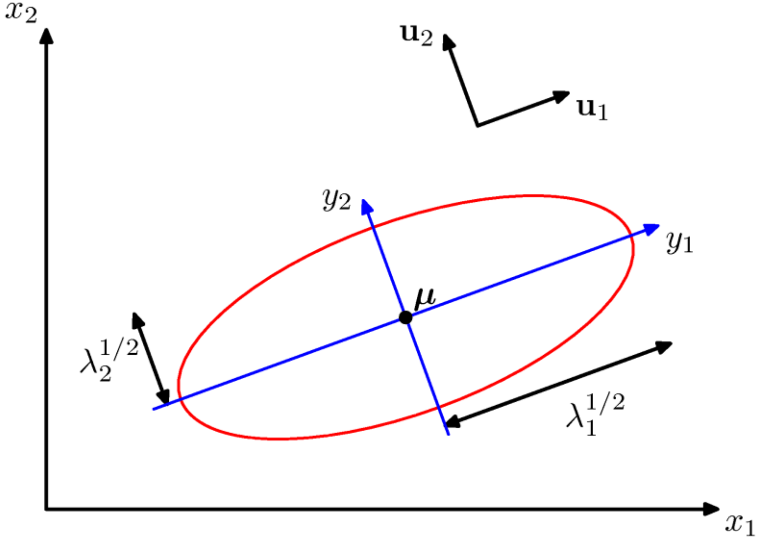
$$p(\mathbf{y}) = p(\mathbf{x})|\mathbf{J}| = \prod_{j=1}^{D} \frac{1}{(2\pi\lambda_j)^{1/2}} \exp\left\{-\frac{y_j^2}{2\lambda_j}\right\} \qquad \mathbf{y} = (y_1, \ldots, y_D)^{\mathrm{T}}$$

$$J_{ij} = \frac{\partial x_i}{\partial y_j} = U_{ji} \qquad \mathbf{y} = \mathbf{U}(\mathbf{x}-\boldsymbol{\mu})$$

$$|\mathbf{J}|^2 = \left|\mathbf{U}^{\mathrm{T}}\right|^2 = \left|\mathbf{U}^{\mathrm{T}}\right| |\mathbf{U}| = \left|\mathbf{U}^{\mathrm{T}}\mathbf{U}\right| = |\mathbf{I}| = 1$$

$$|\mathbf{J}| = 1$$

$$|\mathbf{\Sigma}|^{1/2} = \prod_{j=1}^{D} \lambda_j^{1/2}$$



多変量ガウス分布 The multivariate Gaussian
モーメント Moments

1次モーメント First-order moment

$$\mathbb{E}[\mathbf{x}] = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \mathbf{x}\,\mathrm{d}\mathbf{x}$$
$$= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{z}\right\} (\mathbf{z}+\boldsymbol{\mu})\,\mathrm{d}\mathbf{z}$$
$$= \boldsymbol{\mu}$$

2次モーメント Second-order moment

$$\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \mathbf{x}\mathbf{x}^{\mathrm{T}}\,\mathrm{d}\mathbf{x}$$
$$= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \int \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{z}\right\} (\mathbf{z}+\boldsymbol{\mu})(\mathbf{z}+\boldsymbol{\mu})^{\mathrm{T}}\,\mathrm{d}\mathbf{z}$$

多変量ガウス分布 The multivariate Gaussian
モーメント Moments

2次モーメント Second-order moment

$$\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \int \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{z}\right\} (\mathbf{z}+\boldsymbol{\mu})(\mathbf{z}+\boldsymbol{\mu})^{\mathrm{T}}\,\mathrm{d}\mathbf{z}$$

$$y_j = \mathbf{u}_j^{\mathrm{T}}\mathbf{z}$$

$$\downarrow$$

$$\mathbf{z} = \sum_{j=1}^{D} y_j \mathbf{u}_j$$

$$\frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \int \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{z}\right\} \mathbf{z}\mathbf{z}^{\mathrm{T}}\,\mathrm{d}\mathbf{z}$$

$$= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \sum_{i=1}^{D}\sum_{j=1}^{D} \mathbf{u}_i\mathbf{u}_j^{\mathrm{T}} \int \exp\left\{-\sum_{k=1}^{D}\frac{y_k^2}{2\lambda_k}\right\} y_i y_j\,\mathrm{d}\mathbf{y}$$

$$= \sum_{i=1}^{D} \mathbf{u}_i\mathbf{u}_i^{\mathrm{T}}\lambda_i = \mathbf{\Sigma}$$

$$\mathbf{\Sigma}^{-1} = \sum_{i=1}^{D}\frac{1}{\lambda_i}\mathbf{u}_i\mathbf{u}_i^{\mathrm{T}}$$

$$|\mathbf{\Sigma}|^{1/2} = \prod_{j=1}^{D}\lambda_j^{1/2}$$

$$\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \mathbf{\Sigma}$$

多変量ガウス分布 The multivariate Gaussian
モーメント Moments

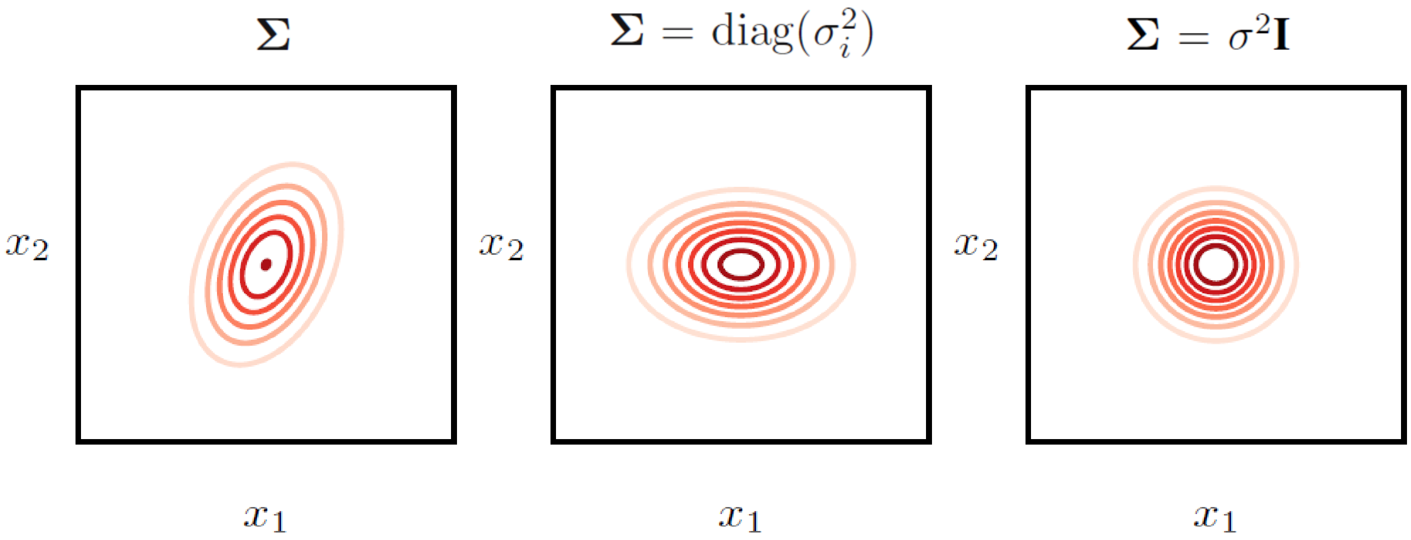
共分散 Covariance

$$\mathrm{cov}[\mathbf{x}] = \mathbb{E}\left[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^{\mathrm{T}}\right]$$

$$\mathbb{E}[\mathbf{x}] = \boldsymbol{\mu} \qquad \mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}} + \boldsymbol{\Sigma}$$

$$\mathrm{cov}[\mathbf{x}] = \boldsymbol{\Sigma}$$

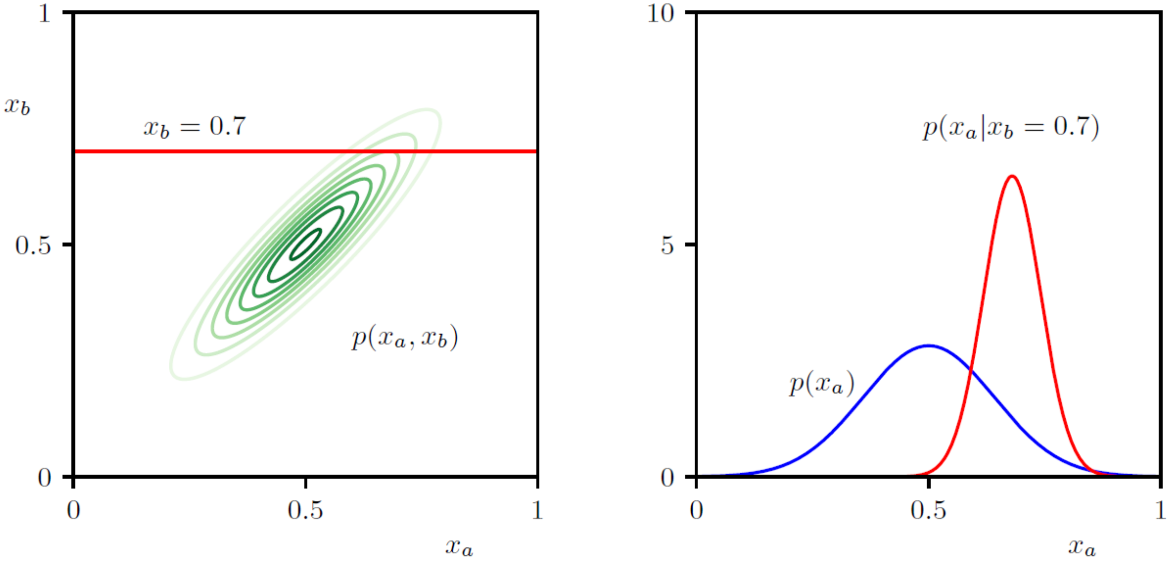
多変量ガウス分布 The multivariate Gaussian
制限 Limitations



自由パラメータの数 The number of free parameters

$D(D+1)/2 + D$ $\quad$ $2D$ $\quad$ $D+1$

多変量ガウス分布 The multivariate Gaussian
条件付き分布 Conditional distribution



$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$p(\mathbf{x}_a|\mathbf{x}_b)$?

多変量ガウス分布 The multivariate Gaussian
条件付き分布 Conditional distribution

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$$\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}^{-1} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}$$

$$\begin{aligned} -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \\ -\frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^{\mathrm{T}}\boldsymbol{\Lambda}_{aa}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^{\mathrm{T}}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b) \\ -\frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^{\mathrm{T}}\boldsymbol{\Lambda}_{ba}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}(\mathbf{x}_b-\boldsymbol{\mu}_b) \end{aligned}$$

多変量ガウス分布 The multivariate Gaussian
条件付き分布 Conditional distribution

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \qquad -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = -\frac{1}{2}\mathbf{x}^{\mathrm{T}}\underline{\boldsymbol{\Sigma}^{-1}}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\underline{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}} + \mathrm{const}$$

$$p(\mathbf{x}_a|\mathbf{x}_b) = \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b})$$

$$\begin{aligned}
-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \\
& -\frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^{\mathrm{T}}\boldsymbol{\Lambda}_{aa}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^{\mathrm{T}}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b) \\
& -\frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^{\mathrm{T}}\boldsymbol{\Lambda}_{ba}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}(\mathbf{x}_b-\boldsymbol{\mu}_b)
\end{aligned}$$

$$= -\frac{1}{2}\mathbf{x}_a^{\mathrm{T}}\underbrace{\boldsymbol{\Lambda}_{aa}}_{\boldsymbol{\Sigma}_{a|b}^{-1}}\mathbf{x}_a + \mathbf{x}_a^{\mathrm{T}}\underbrace{\{\boldsymbol{\Lambda}_{aa}\boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b)\}}_{\boldsymbol{\Sigma}_{a|b}^{-1}\boldsymbol{\mu}_{a|b}} + \mathrm{const}$$

多変量ガウス分布 The multivariate Gaussian
条件付き分布 Conditional distribution

$$\Lambda_{aa} \quad \Vert \quad \boldsymbol{\Sigma}_{a|b}^{-1}$$

$$\{\Lambda_{aa}\boldsymbol{\mu}_a - \Lambda_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)\} \quad \Vert \quad \boldsymbol{\Sigma}_{a|b}^{-1}\boldsymbol{\mu}_{a|b}$$

⇩

$$\boldsymbol{\Sigma}_{a|b} = \Lambda_{aa}^{-1}$$

$$\begin{aligned}\boldsymbol{\mu}_{a|b} &= \boldsymbol{\Sigma}_{a|b}\{\Lambda_{aa}\boldsymbol{\mu}_a - \Lambda_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)\} \\ &= \boldsymbol{\mu}_a - \Lambda_{aa}^{-1}\Lambda_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)\end{aligned}$$

多変量ガウス分布 The multivariate Gaussian
条件付き分布 Conditional distribution

$$\boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{aa}^{-1}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)$$
$$\boldsymbol{\Sigma}_{a|b} = \boldsymbol{\Lambda}_{aa}^{-1}$$

$\Rightarrow$

$$\begin{aligned}\boldsymbol{\mu}_{a|b} &= \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}(\mathbf{x}_b - \boldsymbol{\mu}_b) \\ \boldsymbol{\Sigma}_{a|b} &= \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba}\end{aligned}$$

$\uparrow$

$$\begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}$$

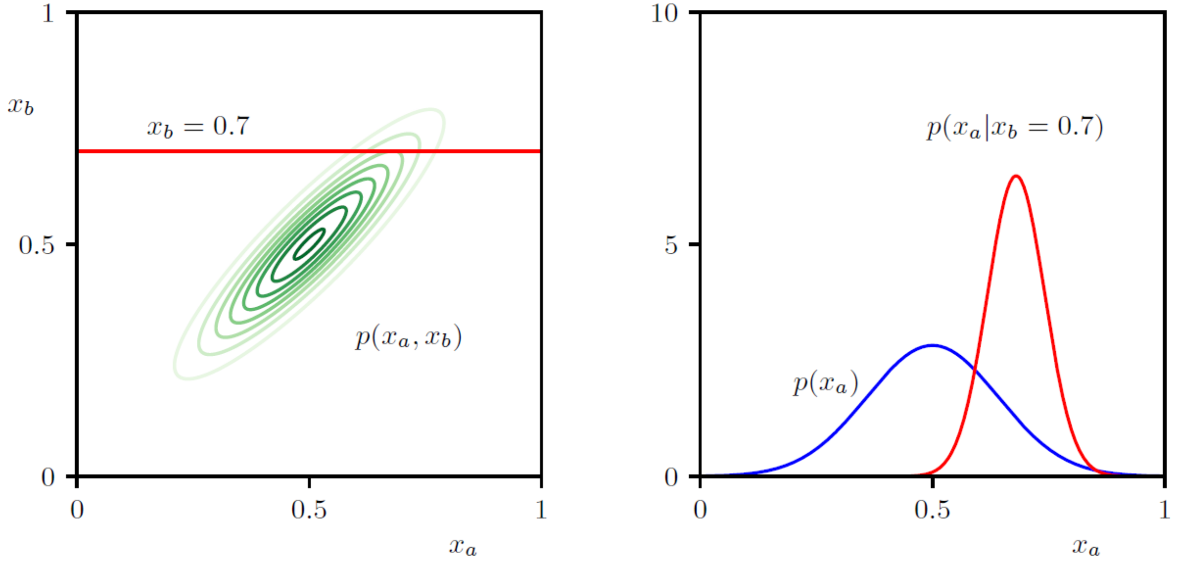
$$\begin{aligned}\boldsymbol{\Lambda}_{aa} &= (\boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba})^{-1} \\ \boldsymbol{\Lambda}_{ab} &= -(\boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba})^{-1}\boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\end{aligned}$$

$\uparrow$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{M} & -\mathbf{M}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{M} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{M}\mathbf{B}\mathbf{D}^{-1} \end{pmatrix}$$

$$\mathbf{M} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}$$

多変量ガウス分布 The multivariate Gaussian
周辺分布 Marginal distribution



$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$p(\mathbf{x}_a)$?

多変量ガウス分布 The multivariate Gaussian
周辺分布 Marginal distribution

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b)\,\mathrm{d}\mathbf{x}_b$$

$$-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) =$$

$$-\frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^{\mathrm{T}}\boldsymbol{\Lambda}_{aa}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^{\mathrm{T}}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b)$$
$$-\frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^{\mathrm{T}}\boldsymbol{\Lambda}_{ba}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}(\mathbf{x}_b-\boldsymbol{\mu}_b)$$

$\mathbf{x}_b$ を含む項 Terms that involve $\mathbf{x}_b$

$$-\frac{1}{2}\mathbf{x}_b^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}\mathbf{x}_b + \mathbf{x}_b^{T}\mathbf{m} \qquad \mathbf{m} = \boldsymbol{\Lambda}_{bb}\boldsymbol{\mu}_b - \boldsymbol{\Lambda}_{ba}(\mathbf{x}_a-\boldsymbol{\mu}_a)$$

$$= \underline{-\frac{1}{2}(\mathbf{x}_b - \boldsymbol{\Lambda}_{bb}^{-1}\mathbf{m})^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}(\mathbf{x}_b - \boldsymbol{\Lambda}_{bb}^{-1}\mathbf{m})} + \frac{1}{2}\mathbf{m}^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}^{-1}\mathbf{m}$$

多変量ガウス分布 The multivariate Gaussian
周辺分布 Marginal distribution

$$\int \exp\left\{-\frac{1}{2}(\mathbf{x}_b - \boldsymbol{\Lambda}_{bb}^{-1}\mathbf{m})^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}(\mathbf{x}_b - \boldsymbol{\Lambda}_{bb}^{-1}\mathbf{m})\right\} \mathrm{d}\mathbf{x}_b \Rightarrow \mathrm{const}$$

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b)\,\mathrm{d}\mathbf{x}_b$$

$\mathbf{x}_a$ に依存する項 Terms that depend on $\mathbf{x}_a$

$$\frac{1}{2}\left[\boldsymbol{\Lambda}_{bb}\boldsymbol{\mu}_b - \boldsymbol{\Lambda}_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)\right]^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}^{-1}\left[\boldsymbol{\Lambda}_{bb}\boldsymbol{\mu}_b - \boldsymbol{\Lambda}_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)\right] \leftarrow \frac{1}{2}\mathbf{m}^{\mathrm{T}}\boldsymbol{\Lambda}_{bb}^{-1}\mathbf{m}$$

$$-\frac{1}{2}\mathbf{x}_a^{\mathrm{T}}\boldsymbol{\Lambda}_{aa}\mathbf{x}_a + \mathbf{x}_a^{\mathrm{T}}(\boldsymbol{\Lambda}_{aa}\boldsymbol{\mu}_a + \boldsymbol{\Lambda}_{ab}\boldsymbol{\mu}_b) + \mathrm{const}$$

$$= -\frac{1}{2}\mathbf{x}_a^{\mathrm{T}}\underbrace{(\boldsymbol{\Lambda}_{aa} - \boldsymbol{\Lambda}_{ab}\boldsymbol{\Lambda}_{bb}^{-1}\boldsymbol{\Lambda}_{ba})}_{=\,\boldsymbol{\Sigma}_a^{-1}}\mathbf{x}_a + \mathbf{x}_a^{\mathrm{T}}\underbrace{(\boldsymbol{\Lambda}_{aa} - \boldsymbol{\Lambda}_{ab}\boldsymbol{\Lambda}_{bb}^{-1}\boldsymbol{\Lambda}_{ba})\boldsymbol{\mu}_a}_{=\,\boldsymbol{\Sigma}_a^{-1}\boldsymbol{\mu}_a} + \mathrm{const}$$

多変量ガウス分布 The multivariate Gaussian
周辺分布 Marginal distribution

$$\boldsymbol{\Sigma}_a = (\boldsymbol{\Lambda}_{aa} - \boldsymbol{\Lambda}_{ab}\boldsymbol{\Lambda}_{bb}^{-1}\boldsymbol{\Lambda}_{ba})^{-1} = \boldsymbol{\Sigma}_{aa}$$

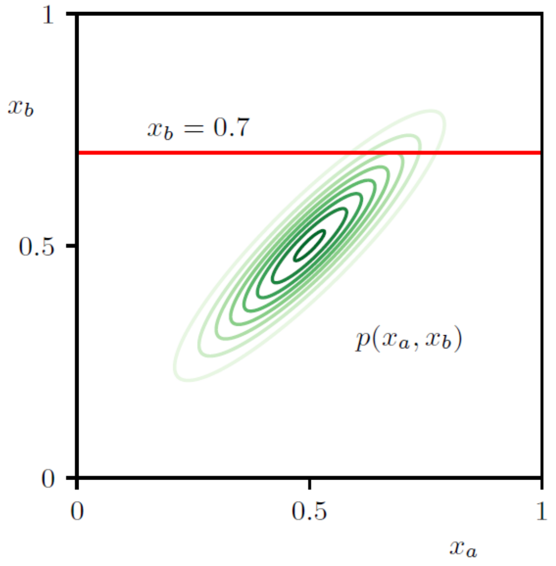
$$\boldsymbol{\Sigma}_a(\boldsymbol{\Lambda}_{aa} - \boldsymbol{\Lambda}_{ab}\boldsymbol{\Lambda}_{bb}^{-1}\boldsymbol{\Lambda}_{ba})\boldsymbol{\mu}_a = \boldsymbol{\mu}_a$$

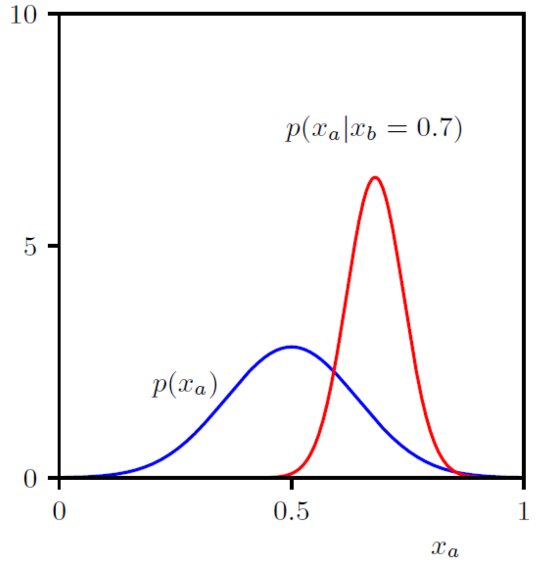
$$\begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} \mathbb{E}[\mathbf{x}_a] &= \boldsymbol{\mu}_a \\ \mathrm{cov}[\mathbf{x}_a] &= \boldsymbol{\Sigma}_{aa} \end{aligned}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{M} & -\mathbf{M}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{M} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{M}\mathbf{B}\mathbf{D}^{-1} \end{pmatrix}$$

$$\mathbf{M} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}$$





$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$$\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}^{-1}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}$$

$$p(\mathbf{x}_a|\mathbf{x}_b) = \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b})$$

$$\boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{aa}^{-1}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)$$

$$\boldsymbol{\Sigma}_{a|b} = \boldsymbol{\Lambda}_{aa}^{-1}$$

$$p(\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa})$$

多変量ガウス分布 The multivariate Gaussian
ベイズの定理 Bayes' theorem

$$
\begin{aligned}
p(\mathbf{x}) &= \mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}\right) \\
p(\mathbf{y}|\mathbf{x}) &= \mathcal{N}\left(\mathbf{y}|\mathbf{A}\mathbf{x}+\mathbf{b}, \mathbf{L}^{-1}\right)
\end{aligned}
$$

$p(\mathbf{y})$?

$p(\mathbf{x}|\mathbf{y})$?

$$
\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}
$$

$$
\begin{aligned}
\ln p(\mathbf{z}) &= \ln p(\mathbf{x}) + \ln p(\mathbf{y}|\mathbf{x}) \\
&= -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu}) \\
&\quad -\frac{1}{2}(\mathbf{y}-\mathbf{A}\mathbf{x}-\mathbf{b})^{\mathrm{T}}\mathbf{L}(\mathbf{y}-\mathbf{A}\mathbf{x}-\mathbf{b}) + \text{const}
\end{aligned}
$$

多変量ガウス分布 The multivariate Gaussian ベイズの定理 Bayes' theorem

$\ln p(\mathbf{z})$ の2次の項 The second-order terms in $\ln p(\mathbf{z})$

$$-\frac{1}{2}\mathbf{x}^{\mathrm{T}}(\boldsymbol{\Lambda}+\mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A})\mathbf{x}-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{L}\mathbf{y}+\frac{1}{2}\mathbf{y}^{\mathrm{T}}\mathbf{L}\mathbf{A}\mathbf{x}+\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{y}$$

$$= -\frac{1}{2}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}^{\mathrm{T}} \underbrace{\begin{pmatrix}\boldsymbol{\Lambda}+\mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A} & -\mathbf{A}^{\mathrm{T}}\mathbf{L}\\ -\mathbf{L}\mathbf{A} & \mathbf{L}\end{pmatrix}}_{\parallel \atop \mathbf{R}} \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = -\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{R}\mathbf{z}$$

$$\mathrm{cov}[\mathbf{z}] = \mathbf{R}^{-1} = \begin{pmatrix}\boldsymbol{\Lambda}^{-1} & \boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}}\\ \mathbf{A}\boldsymbol{\Lambda}^{-1} & \mathbf{L}^{-1}+\mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}}\end{pmatrix}$$

多変量ガウス分布 The multivariate Gaussian
ベイズの定理 Bayes' theorem

$\ln p(\mathbf{z})$ の線形の項 The linear terms in $\ln p(\mathbf{z})$

$$\mathbf{x}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{\mu} - \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{b} + \mathbf{y}^{\mathrm{T}}\mathbf{L}\mathbf{b} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^{\mathrm{T}} \underbrace{\begin{pmatrix} \boldsymbol{\Lambda}\boldsymbol{\mu} - \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{b} \\ \mathbf{L}\mathbf{b} \end{pmatrix}}_{\substack{\Vert \\ \mathbf{R}\,\mathbb{E}[\mathbf{z}]}}$$

$$\begin{aligned}
\mathbb{E}[\mathbf{z}] &= \mathbf{R}^{-1} \begin{pmatrix} \boldsymbol{\Lambda}\boldsymbol{\mu} - \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{b} \\ \mathbf{L}\mathbf{b} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\Lambda}^{-1} & \boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}} \\ \mathbf{A}\boldsymbol{\Lambda}^{-1} & \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Lambda}\boldsymbol{\mu} - \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{b} \\ \mathbf{L}\mathbf{b} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \end{pmatrix}
\end{aligned}$$

多変量ガウス分布 The multivariate Gaussian
ベイズの定理 Bayes' theorem

$$\mathbb{E}[\mathbf{z}] = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \end{pmatrix}$$

$$\mathrm{cov}[\mathbf{z}] = \begin{pmatrix} \boldsymbol{\Lambda}^{-1} & \boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}} \\ \mathbf{A}\boldsymbol{\Lambda}^{-1} & \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}} \end{pmatrix}$$

$\Rightarrow$

$$\begin{aligned} \mathbb{E}[\mathbf{y}] &= \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \\ \mathrm{cov}[\mathbf{y}] &= \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}} \end{aligned}$$

$$\boxed{\begin{aligned} \mathbb{E}[\mathbf{x}_a] &= \boldsymbol{\mu}_a \\ \mathrm{cov}[\mathbf{x}_a] &= \boldsymbol{\Sigma}_{aa} \end{aligned}}$$

周辺分布
Marginal distribution

$$\begin{aligned} p(\mathbf{x}) &= \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}) \\ p(\mathbf{y}|\mathbf{x}) &= \mathcal{N}(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1}) \end{aligned}$$

$$\boxed{p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{L}^{-1} + \mathbf{A}\boldsymbol{\Lambda}^{-1}\mathbf{A}^{\mathrm{T}})}$$

多変量ガウス分布 The multivariate Gaussian
ベイズの定理 Bayes' theorem

$$\mathbb{E}[\mathbf{z}] = \begin{pmatrix} \boldsymbol{\mu} \\ \mathbf{A}\boldsymbol{\mu} + \mathbf{b} \end{pmatrix}$$

$$\operatorname{cov}[\mathbf{z}]^{-1} = \begin{pmatrix} \boldsymbol{\Lambda} + \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A} & -\mathbf{A}^{\mathrm{T}}\mathbf{L} \\ -\mathbf{L}\mathbf{A} & \mathbf{L} \end{pmatrix}$$

$\Rightarrow$

$$\begin{aligned} \mathbb{E}[\mathbf{x}|\mathbf{y}] &= (\boldsymbol{\Lambda} + \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A})^{-1}\left\{\mathbf{A}^{\mathrm{T}}\mathbf{L}(\mathbf{y}-\mathbf{b}) + \boldsymbol{\Lambda}\boldsymbol{\mu}\right\} \\ \operatorname{cov}[\mathbf{x}|\mathbf{y}] &= (\boldsymbol{\Lambda} + \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A})^{-1} \end{aligned}$$

$$\boxed{\begin{aligned} \boldsymbol{\mu}_{a|b} &= \boldsymbol{\Sigma}_{a|b}\left\{\boldsymbol{\Lambda}_{aa}\boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)\right\} \\ \boldsymbol{\Sigma}_{a|b} &= \boldsymbol{\Lambda}_{aa}^{-1} \end{aligned}}$$

条件付き分布
Conditional distribution

$$\begin{aligned} p(\mathbf{x}) &= \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}^{-1}) \\ p(\mathbf{y}|\mathbf{x}) &= \mathcal{N}(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{L}^{-1}) \end{aligned}$$

$$\boxed{p(\mathbf{x}|\mathbf{y}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\Sigma}\{\mathbf{A}^{\mathrm{T}}\mathbf{L}(\mathbf{y}-\mathbf{b}) + \boldsymbol{\Lambda}\boldsymbol{\mu}\}, \boldsymbol{\Sigma})} \qquad \boldsymbol{\Sigma} = (\boldsymbol{\Lambda} + \mathbf{A}^{\mathrm{T}}\mathbf{L}\mathbf{A})^{-1}$$

多変量ガウス分布 The multivariate Gaussian
最尤推定 Maximum likelihood

データ集合 A data set: $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_N)^{\mathrm{T}}$

対数尤度関数 Log likelihood function

$$\ln p(\mathbf{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = -\frac{ND}{2}\ln(2\pi) - \frac{N}{2}\ln|\boldsymbol{\Sigma}| - \frac{1}{2}\sum_{n=1}^{N}(\mathbf{x}_n - \boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu})$$

十分統計量 Sufficient statistics

$$\sum_{n=1}^{N}\mathbf{x}_n \qquad \sum_{n=1}^{N}\mathbf{x}_n\mathbf{x}_n^{\mathrm{T}}$$

多変量ガウス分布 The multivariate Gaussian
最尤推定 Maximum likelihood

対数尤度の最大化 Maximization of the log likelihood function

$$\frac{\partial}{\partial \boldsymbol{\mu}} \ln p(\mathbf{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{n=1}^{N} \boldsymbol{\Sigma}^{-1}(\mathbf{x}_n - \boldsymbol{\mu}) = 0$$

$$\Rightarrow \quad \boldsymbol{\mu}_{\mathrm{ML}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}_n.$$

同様にして Similarly

$$\boldsymbol{\Sigma}_{\mathrm{ML}} = \frac{1}{N} \sum_{n=1}^{N} (\mathbf{x}_n - \boldsymbol{\mu}_{\mathrm{ML}})(\mathbf{x}_n - \boldsymbol{\mu}_{\mathrm{ML}})^{\mathrm{T}}.$$

多変量ガウス分布 The multivariate Gaussian
最尤推定 Maximum likelihood

真の分布の下での最尤推定解の期待値
Expectations of the maximum likelihood solutions under the true distribution

$$\mathbb{E}[\boldsymbol{\mu}_{\mathrm{ML}}] = \boldsymbol{\mu}$$

$$\mathbb{E}[\boldsymbol{\Sigma}_{\mathrm{ML}}] = \frac{N-1}{N}\boldsymbol{\Sigma}$$

不偏推定量 Unbiased estimator

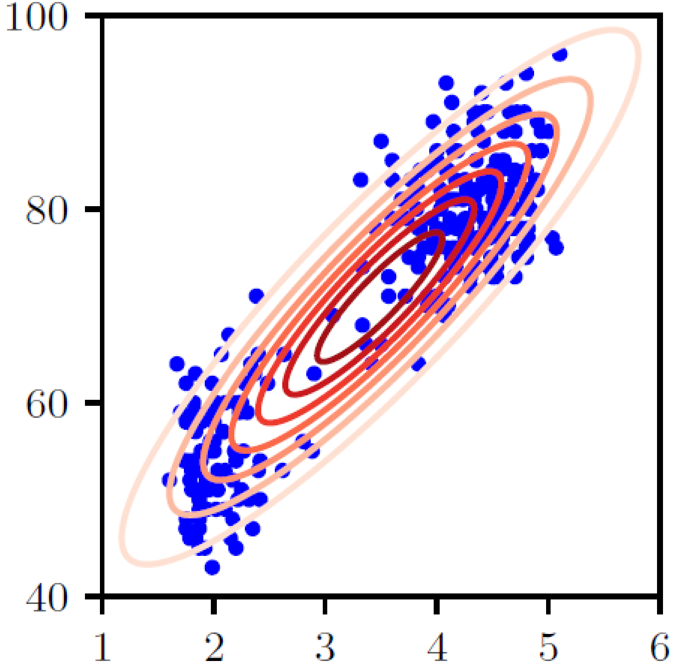
$$\widetilde{\boldsymbol{\Sigma}} = \frac{1}{N-1}\sum_{n=1}^{N}(\mathbf{x}_n - \boldsymbol{\mu}_{\mathrm{ML}})(\mathbf{x}_n - \boldsymbol{\mu}_{\mathrm{ML}})^{\mathrm{T}}.$$

$$\mathbb{E}[\,\widetilde{\boldsymbol{\Sigma}}\,] = \boldsymbol{\Sigma}$$

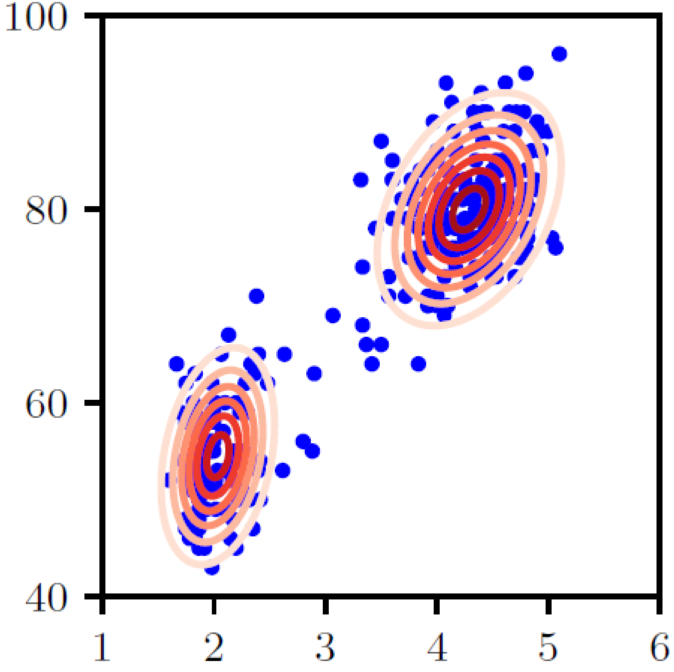
多変量ガウス分布 The multivariate Gaussian
逐次推定 Sequential estimation

$$
\begin{aligned}
\boldsymbol{\mu}_{\mathrm{ML}}^{(N)} &= \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}_n \\
&= \frac{1}{N}\mathbf{x}_N + \frac{1}{N}\sum_{n=1}^{N-1}\mathbf{x}_n \\
&= \frac{1}{N}\mathbf{x}_N + \frac{N-1}{N}\boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)} \\
&= \underbrace{\boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)}}_{\text{古い推定量 Old estimate}} + \underbrace{\frac{1}{N}}_{\text{修正の重み Correction weight}}\underbrace{(\mathbf{x}_N - \boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)})}_{\mathbf{x}_N\text{ による修正 Correction given }\mathbf{x}_N}
\end{aligned}
$$

多変量ガウス分布 The multivariate Gaussian
混合ガウス分布 Mixture of Gaussians



単一のガウス分布
Single Gaussian



2つのガウス分布の線形結合
Mixture of two Gaussians

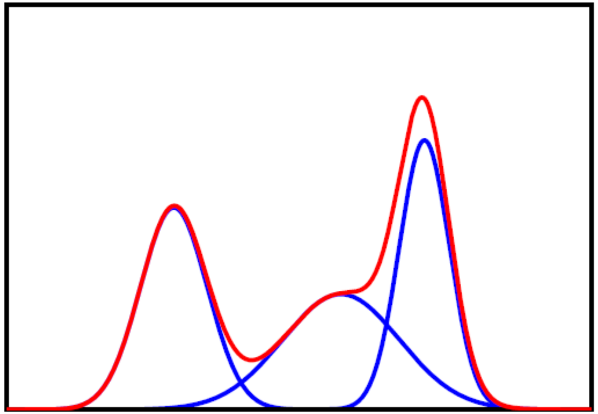
多変量ガウス分布 The multivariate Gaussian
混合ガウス分布 Mixture of Gaussians

混合ガウス分布
Mixture of Gaussians

$$p(\mathbf{x}) = \sum_{k=1}^{K} \pi_k \underline{\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}$$

混合要素 Mixture component

混合係数 Mixing coefficient

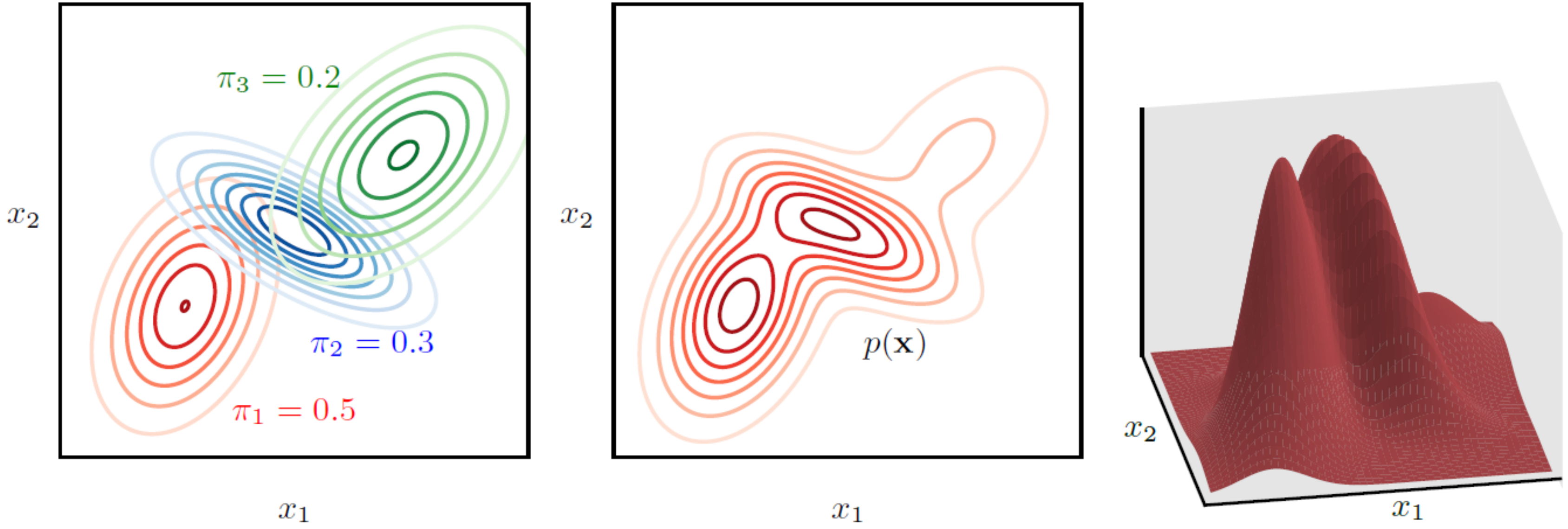


$$\forall k : \pi_k \geqslant 0 \qquad \sum_{k=1}^{K} \pi_k = 1$$

$$\Rightarrow \quad 0 \leqslant \pi_k \leqslant 1$$

$$p(\mathbf{x}) = \sum_{k=1}^{K} p(k) p(\mathbf{x}|k)$$

多変量ガウス分布 The multivariate Gaussian
混合ガウス分布 Mixture of Gaussians



多変量ガウス分布 The multivariate Gaussian
混合ガウス分布 Mixture of Gaussians

混合ガウス分布 Mixture of Gaussians

$$p(\mathbf{x}) = \sum_{k=1}^{K} p(k)p(\mathbf{x}|k) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

負担率 Responsibility:

$$\begin{aligned}
\gamma_k(\mathbf{x}) &\equiv p(k|\mathbf{x}) \\
&= \frac{p(k)p(\mathbf{x}|k)}{\sum_l p(l)p(\mathbf{x}|l)} \\
&= \frac{\pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_l \pi_l \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_l, \boldsymbol{\Sigma}_l)}
\end{aligned}$$

多変量ガウス分布 The multivariate Gaussian
混合ガウス分布 Mixture of Gaussians

パラメータ Parameters

$$\boldsymbol{\pi} \equiv \{\pi_1, \ldots, \pi_K\} \quad \boldsymbol{\mu} \equiv \{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_K\} \quad \boldsymbol{\Sigma} \equiv \{\boldsymbol{\Sigma}_1, \ldots \boldsymbol{\Sigma}_K\}$$

対数尤度関数 Log likelihood function

$$\ln p(\mathbf{X}|\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \sum_{n=1}^{N} \ln \left\{ \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \right\}$$

閉形式の解析解は得られない.
No closed-form analytical solution.

指数型分布族 The exponential family

指数分布族は次式で定義される分布の集合である.
The exponential family of distributions is defined to be the set of distributions of the form

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$$

ただし
where

$$g(\boldsymbol{\eta})\int h(\mathbf{x})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}\,\mathrm{d}\mathbf{x} = 1$$

$\boldsymbol{\eta}$ ：自然パラメータ Natural parameter

指数型分布族 The exponential family

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$$

ベルヌーイ分布 Bernoulli distribution

$$\begin{aligned}
p(x|\mu) &= \mathrm{Bern}(x|\mu) = \mu^{x}(1-\mu)^{1-x} \\
&= \exp\left\{x\ln\mu + (1-x)\ln(1-\mu)\right\} \\
&= (1-\mu)\exp\left\{\underbrace{\ln\left(\frac{\mu}{1-\mu}\right)}_{\to\ \eta} x\right\}
\end{aligned}$$

$$p(x|\eta) = \sigma(-\eta)\exp(\eta x)$$

$$\begin{aligned}
u(x) &= x \\
h(x) &= 1 \\
g(\eta) &= 1 - \sigma(\eta) = \sigma(-\eta).
\end{aligned}$$

$\Rightarrow$ ロジスティックシグモイド関数
Logistic sigmoid function

$$\mu = \sigma(\eta) = \frac{1}{1+\exp(-\eta)}.$$

指数型分布族 The exponential family

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$$

多項分布 Multinomial distribution

$$p(\mathbf{x}|\boldsymbol{\mu}) = \prod_{k=1}^{M} \mu_k^{x_k} = \exp\left\{\sum_{k=1}^{M} x_k \ln \mu_k\right\} \qquad \mathbf{x} = (x_1, \ldots, x_M)^{\mathrm{T}}$$

$$\downarrow$$

$$p(\mathbf{x}|\boldsymbol{\eta}) = \exp(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{x}) \qquad \boldsymbol{\eta} = (\eta_1, \ldots, \eta_M)^{\mathrm{T}}$$

$$\begin{aligned} \eta_k &= \ln \mu_k \\ \mathbf{u}(\mathbf{x}) &= \mathbf{x} \\ h(\mathbf{x}) &= 1 \\ g(\boldsymbol{\eta}) &= 1. \end{aligned}$$

制約 Constraint

$$\boxed{\sum_{k=1}^{M} \mu_k = 1.}$$

指数型分布族 The exponential family

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$$

μ_M を消去 Eliminate μ_M

$$\mu_M = 1 - \sum_{k=1}^{M-1} \mu_k$$

$$\begin{aligned}
&\exp\left\{\sum_{k=1}^{M} x_k \ln \mu_k\right\} \\
&= \exp\left\{\sum_{k=1}^{M-1} x_k \ln \mu_k + \left(1 - \sum_{k=1}^{M-1} x_k\right) \ln\left(1 - \sum_{k=1}^{M-1} \mu_k\right)\right\} \\
&= \exp\left\{\sum_{k=1}^{M-1} x_k \underbrace{\ln\left(\frac{\mu_k}{1 - \sum_{j=1}^{M-1} \mu_j}\right)}_{\to\ \eta_k} + \ln\left(1 - \sum_{k=1}^{M-1} \mu_k\right)\right\}
\end{aligned}$$

$$\Rightarrow \quad \mu_k = \frac{\exp(\eta_k)}{1 + \sum_{j=1}^{M-1} \exp(\eta_j)}.$$

指数型分布族 The exponential family $p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$

多項分布 Multinomial distribution

$$p(\mathbf{x}|\boldsymbol{\eta}) = \left(1 + \sum_{k=1}^{M-1} \exp(\eta_k)\right)^{-1} \exp(\boldsymbol{\eta}^{\mathrm{T}}\mathbf{x})$$

$$\begin{aligned} \boldsymbol{\eta} &= (\eta_1, \ldots, \eta_{M-1}, 0)^{\mathrm{T}} \\ \mathbf{u}(\mathbf{x}) &= \mathbf{x} \\ h(\mathbf{x}) &= 1 \\ g(\boldsymbol{\eta}) &= \left(1 + \sum_{k=1}^{M-1} \exp(\eta_k)\right)^{-1}. \end{aligned}$$

指数型分布族 The exponential family

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$$

ガウス分布 Gaussian distribution

$$\begin{aligned} p(x|\mu,\sigma^2) &= \frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} \\ &= \frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{1}{2\sigma^2}x^2+\frac{\mu}{\sigma^2}x-\frac{1}{2\sigma^2}\mu^2\right\} \end{aligned}$$

$$\downarrow$$

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$$

$$\boldsymbol{\eta} = \begin{pmatrix} \mu/\sigma^2 \\ -1/2\sigma^2 \end{pmatrix} \qquad h(\mathbf{x}) = (2\pi)^{-1/2}$$

$$\mathbf{u}(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix} \qquad g(\boldsymbol{\eta}) = (-2\eta_2)^{1/2}\exp\left(\frac{\eta_1^2}{4\eta_2}\right).$$

指数型分布族 The exponential family $p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}$

制限のある指数型分布族
A restricted set of the exponential family

$$p(\mathbf{x}|\boldsymbol{\lambda}_k, s) = \frac{1}{s}h\left(\frac{1}{s}\mathbf{x}\right)g(\boldsymbol{\lambda}_k)\exp\left\{\frac{1}{s}\boldsymbol{\lambda}_k^{\mathrm{T}}\mathbf{x}\right\}$$

$s > 0$ ：スケールパラメータ Scale parameter

指数型分布族 The exponential family
十分推定量 Sufficient statistics

$$g(\boldsymbol{\eta})\int h(\mathbf{x})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}\,\mathrm{d}\mathbf{x}=1$$

$g(\boldsymbol{\eta})$ の定義より
From the definition of $g(\boldsymbol{\eta})$

$$\Rightarrow\quad \nabla g(\boldsymbol{\eta})\underbrace{\int h(\mathbf{x})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}\,\mathrm{d}\mathbf{x}}_{=\,1/g(\boldsymbol{\eta})}+g(\boldsymbol{\eta})\underbrace{\int h(\mathbf{x})\exp\left\{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})\right\}\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}}_{=\,\mathbb{E}[\mathbf{u}(\mathbf{x})]}=0$$

$$\Rightarrow\quad -\nabla\ln g(\boldsymbol{\eta})=\mathbb{E}[\mathbf{u}(\mathbf{x})]$$

指数型分布族 The exponential family
十分推定量 Sufficient statistics

データ集合 A data set: $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$

尤度関数 Likelihood function

$$p(\mathbf{X}|\boldsymbol{\eta}) = \left(\prod_{n=1}^{N} h(\mathbf{x}_n)\right) g(\boldsymbol{\eta})^N \exp\left\{\boldsymbol{\eta}^{\mathrm{T}} \sum_{n=1}^{N} \mathbf{u}(\mathbf{x}_n)\right\}.$$

最尤推定量が満たすべき条件
Condition to be satisfied by the maximum likelihood estimator

$$-\nabla \ln g(\boldsymbol{\eta}_{\mathrm{ML}}) = \underline{\frac{1}{N} \sum_{n=1}^{N} \mathbf{u}(\mathbf{x}_n)}$$

十分統計量
Sufficient statistics

提出課題 I
Assignment I

提出期限：**10月15日（火曜日）23:59:00 [日本標準時]**
Submission deadline: **October 15 (Tuesday) 23:59:00 [Japan Standard Time]**

提出課題は「一般」チャネルの「ファイル」にアップロードされます.
同チャネルに出現する通知のリンク先から解答を送信（提出）してください.
Assignments will be uploaded to "File" in the "General" channel. Send (submit) your answers via the link that will appear in the same channel.

- 全6回の課題への解答をもとに成績評価を行います.
  Your grades will be based on your answers to all six assignments.
- 解答送信後の解答再送信はできません.
  Once submitted, answers cannot be resubmitted.
- 提出期限は一切延長しません.
  The submission deadline will never be extended.