Exercise for Lecture 11 and 12

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Exercise 11-1

a. The baseline category group is Independent.
b. The prediction equation for $log(\hat{\pi}_R/\hat{\pi}_D)$ is obtained by:

$$log(\hat{\pi}_R/\hat{\pi}_D) = log(\hat{\pi}_R/\hat{\pi}_I) - log(\hat{\pi}_D/\hat{\pi}_I) = (1.0 + 0.3x) - (3.3 - 0.2x) = -2.3 + 0.5x$$

   Since $exp(0.5) = 1.65$, the estimated odds of preferring Republicans over Democrats increase by 65% for every $10,000 increase in annual income.
c. The prediction equation for $\hat{\pi}_I$ is (see slide 13)

$$\hat{\pi}_I = \frac{1}{1 + \exp(3.3 - 0.2x) + \exp(1.0 + 0.3x)}.$$

d. The prediction equations for $\hat{\pi}_R$ and $\hat{\pi}_D$ are as shown below (see slide 13):

$$\hat{\pi}_R = \frac{\exp(1.0 + 0.3x)}{1 + \exp(1.0 + 0.3x) + \exp(3.3 - 0.2x)},$$

$$\hat{\pi}_D = \frac{\exp(3.3 - 0.2x)}{1 + \exp(1.0 + 0.3x) + \exp(3.3 - 0.2x)}.$$

   Given that the denominators are the same for $\hat{\pi}_R$ and $\hat{\pi}_D$, the range of our interest can be obtained by solving the following inequality:

$$\begin{aligned} \exp(1.0 + 0.3x) &> \exp(3.3 - 0.2x) \\ 1.0 + 0.3x &> 3.3 - 0.2x \\ 0.5x &> 2.3 \\ x &> 2.3/0.5 = 4.6. \end{aligned}$$

   Therefore, the when annual income is larger than $46,000 we have that $\hat{\pi}_R > \hat{\pi}_D$, that is, the estimated probability of preferring a Republican for President is at least as large as the estimated probability of preferring a Democrat for President.

Exercise 11-2

a. (i) Keeping x_2 and x_3 fixed, for any given response category j ($j = 1, \ldots, 4$), the estimated odds that job satisfaction is lower than higher are multiplied by $\exp(-0.54) = 0.58$ when x_1 increases 1 unit:

$$\left.\frac{\hat{P}(Y \le y)}{\hat{P}(Y > y)}\right|_{x_1} \times \underbrace{\exp(-0.54)}_{0.58} = \left.\frac{\hat{P}(Y \le y)}{\hat{P}(Y > y)}\right|_{x_1+1}.$$

   Thus, the odds *decrease* as x_1 increases since $0.58 < 1$. This implies that $\hat{P}(Y \le y)$ *decreases* as x_1 increases, or equivalently, it implies that $\hat{P}(Y > y)$ *increases* as x_1 increases.
   In short: Job satisfaction tends to increase as x_1 increases.

(ii) Keeping x_1 and x_3 fixed, for any given response category j ($j = 1, \ldots, 4$), the estimated odds that job satisfaction is lower than higher are multiplied by $\exp(0.60) = 1.82$ when x_2 increases 1 unit:

$$\left.\frac{\hat{P}(Y \leq y)}{\hat{P}(Y > y)}\right|_{x_2} \times \underbrace{\exp(0.60)}_{1.82} = \left.\frac{\hat{P}(Y \leq y)}{\hat{P}(Y > y)}\right|_{x_2+1}.$$

Thus, the odds *increase* as x_2 increases since $1.82 > 1$. This implies that $\hat{P}(Y \leq y)$ *increases* as x_2 increases, or equivalently, it implies that $\hat{P}(Y > y)$ *decreases* as x_2 increases.
In short: Job satisfaction tends to decrease as x_2 increases.

(iii) Keeping x_1 and x_2 fixed, for any given response category j ($j = 1, \ldots, 4$), the estimated odds that job satisfaction is lower than higher are multiplied by $\exp(1.19) = 3.29$ when x_3 increases 1 unit:

$$\left.\frac{\hat{P}(Y \leq y)}{\hat{P}(Y > y)}\right|_{x_3} \times \underbrace{\exp(1.19)}_{3.29} = \left.\frac{\hat{P}(Y \leq y)}{\hat{P}(Y > y)}\right|_{x_3+1}.$$

Thus, the odds *increase* as x_3 increases since $3.29 > 1$. This implies that $\hat{P}(Y \leq y)$ *increases* as x_3 increases, or equivalently, it implies that $\hat{P}(Y > y)$ *decreases* as x_3 increases.
In short: Job satisfaction tends to decrease as x_3 increases.

b. From the previous answer, we conclude that job satisfaction tends to increase as x_1 increases and as x_2 and x_3 decrease. Therefore, the most favorable combination of predictor values to achieve the highest predicted job satisfaction probability is $x_1 = 4$ and $x_2 = x_3 = 1$.

Exercise 11-3

a.

```r
library(VGAM)

# Load data:
happy <- read.csv("Datasets/happy.csv", header = TRUE)

# Fit the nominal logistic regression model:
happy.fit <- vglm(happiness ~ income,
                  family = multinomial(refLevel = "very"),
                  data   = happy)

# Results:
coef(happy.fit, matrix = TRUE)
```

```
##             log(mu[,1]/mu[,3]) log(mu[,2]/mu[,3])
## (Intercept)         -2.5551795        -0.35128561
## income              -0.2275057        -0.09615234
```

```r
summary(happy.fit)
```

```
## Call:
## vglm(formula = happiness ~ income, family = multinomial(refLevel = "very"),
##     data = happy)
##
## Coefficients:
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept):1 -2.55518    0.72559  -3.522 0.000429 ***
## (Intercept):2 -0.35129    0.26837  -1.309 0.190554
## income:1      -0.22751    0.34119  -0.667 0.504903
```

```
## income:2      -0.09615    0.12202  -0.788 0.430694
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Names of linear predictors: log(mu[,1]/mu[,3]), log(mu[,2]/mu[,3])
##
## Residual deviance: 926.2112 on 1198 degrees of freedom
##
## Log-likelihood: -463.1056 on 1198 degrees of freedom
##
## Number of Fisher scoring iterations: 6
##
## Warning: Hauck-Donner effect detected in the following estimate(s):
## '(Intercept):1'
##
##
## Reference group is level  3  of the response
```

b.

$$\hat{\pi}_{\text{not}} = \frac{\exp(-2.5552 - 0.2275x)}{1 + \exp(-2.5552 - 0.2275x) + \exp(-0.3513 - 0.0962x)}$$

$$\hat{\pi}_{\text{pretty}} = \frac{\exp(-0.3513 - 0.0962x)}{1 + \exp(-2.5552 - 0.2275x) + \exp(-0.3513 - 0.0962x)}$$

$$\hat{\pi}_{\text{very}} = \frac{1}{1 + \exp(-2.5552 - 0.2275x) + \exp(-0.3513 - 0.0962x)}$$

c. The odds of being 'not' happy over being 'very' happy are multiplied by $\exp(-0.2275) = 0.80$ as 'income' increases by 1 unit. That is, when income increases, the probability of being unhappy decreases since $0.80 < 1$.

d.

```r
null.fit <- vglm(happiness ~ 1,
                 family = multinomial(refLevel = "very"),
                 data   = happy)
lrtest(happy.fit, null.fit)
```

```
## Likelihood ratio test
##
## Model 1: happiness ~ income
## Model 2: happiness ~ 1
##    #Df  LogLik Df  Chisq Pr(>Chisq)
## 1 1198 -463.11
## 2 1200 -463.58  2 0.9439     0.6238
```

The test result is $\chi^2(2) = 0.94$, $p = .62$. We conclude that the two models do not differ significantly at 5% significance level, and thus income is not significantly related to happiness.

e. We can compute this either manually,

$$\hat{\pi}_{\text{very}} = \frac{1}{1 + \exp(-2.5552 - 0.2275(2)) + \exp(-0.3513 - 0.0962(2))} = .61$$

or using the `predict()` function,

```r
predict(happy.fit,
        newdata = data.frame(income = 2),
        type    = "response") # see the outcome under 'very'
```

```
##         not    pretty      very
## 1 0.0302356 0.3562457 0.6135187
```

Exercise 11-4

a.

```r
happy.ordfit <- vglm(happiness ~ income,
                     family = cumulative(parallel = TRUE),
                     data   = happy)
coef(happy.ordfit, matrix = TRUE)
```

```
##             logitlink(P[Y<=1]) logitlink(P[Y<=2])
## (Intercept)         -3.2466452         -0.2377552
## income              -0.1117422         -0.1117422
```

The estimated model equations are

$$\text{logit}(P(Y \leq 1)) = -3.247 - 0.112x$$
$$\text{logit}(P(Y \leq 2)) = -0.238 - 0.112x.$$

b. First of all, the outcome variable 'happiness' is categorical with three levels, hence $(3-1) = 2$ model equations are estimated. Furthermore, the classic ordinal logistic regression model assumes that there is a constant x effect across both cumulative logits (in the R code above, this pertains to the `parallel = TRUE` command). This is the reason why there are two intercepts (one per equation), but one common estimated x effect (-0.112) for both equations.

c. For any j (1 or 2), the odds of being in the 'unhappy' direction (i.e., $Y \leq j$) rather than in the 'happy' direction (i.e., $Y > j$) are multiplied by $\exp(-0.112) = 0.89$ when 'income' increases by 1 unit. That is, when income increases, the probability of being unhappy decreases since $0.89 < 1$.

d.

```r
null.ordfit <- vglm(happiness ~ 1,
                    family = cumulative(parallel = TRUE),
                    data   = happy)
lrtest(happy.ordfit, null.ordfit)
```

```
## Likelihood ratio test
##
## Model 1: happiness ~ income
## Model 2: happiness ~ 1
##   #Df  LogLik Df  Chisq Pr(>Chisq)
## 1 1199 -463.13
## 2 1200 -463.58  1 0.8876     0.3461
```

The test result is $\chi^2(1) = 0.89$, $p = .35$. We conclude that the two models do not differ significantly at 5% significance level, and thus income is not significantly related to happiness.

e. We can compute this either manually,

$$\begin{aligned} 1 - P(Y \leq 2)|_{x=2} &= 1 - \frac{\exp[-0.238 - 0.112(2)]}{1 + \exp[-0.238 - 0.112(2)]} \\ &= 1 - .39 \\ &= .61 \end{aligned}$$

or using the `predict()` function,

```
predict(happy.ordfit,
        newdata = data.frame(income = 2),
        type    = "response") # see the outcome under 'very'
```

```
##          not    pretty      very
## 1 0.03017419 0.3565176 0.6133082
```