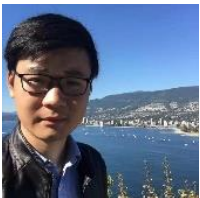


## Lecture 6

# SOFT AND HARD CONSTRAINED TRAJECTORY OPTIMIZATION



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# Outline



1. Introduction



2. Soft-constrained Optimization



3. Hard-constrained Optimization



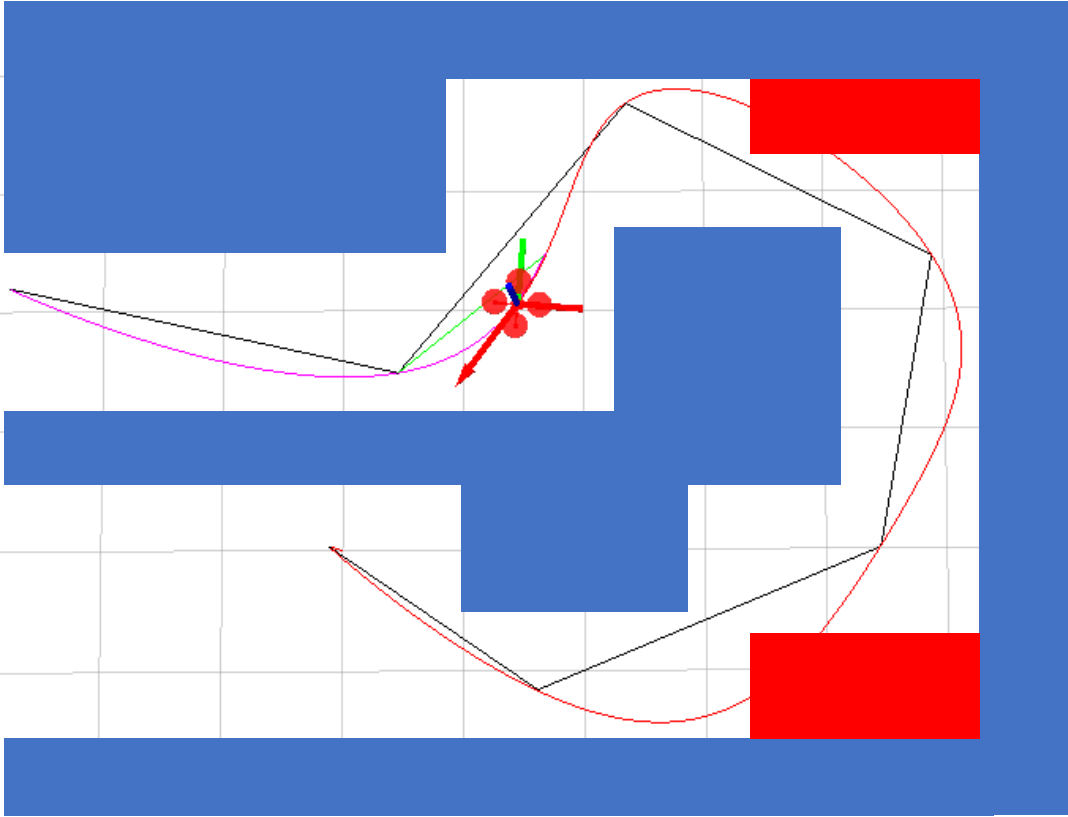
4. Case Study



5. Homework

# Introduction

# Minimum snap trajectory optimization

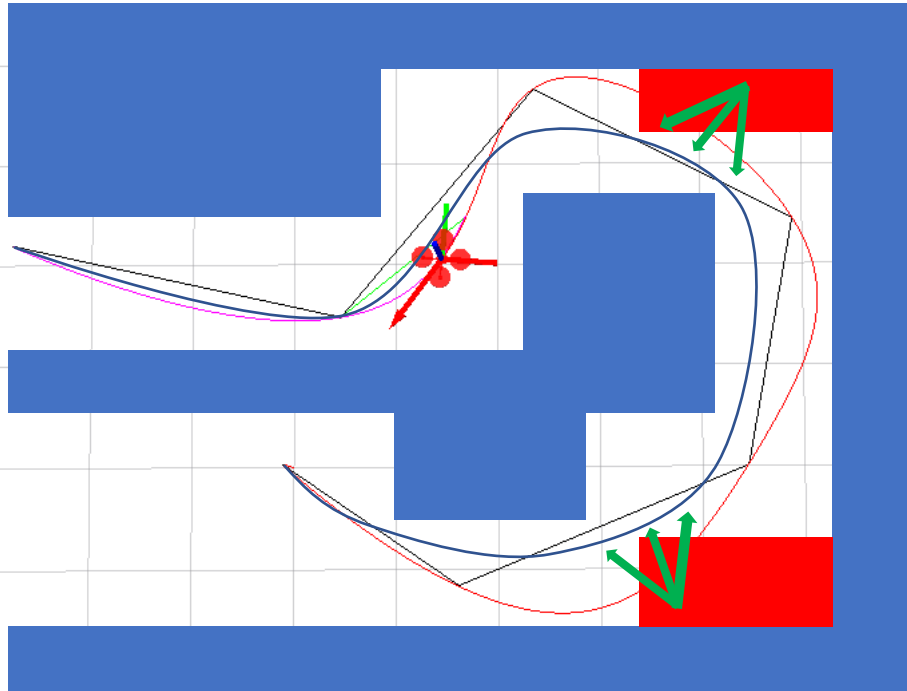


- We only constrain intermediate waypoints of the trajectory should pass.
- Computational cheap and easy to implement.
- No constraints on the trajectory itself.
- The “overshoot” of the trajectory unavoidable.

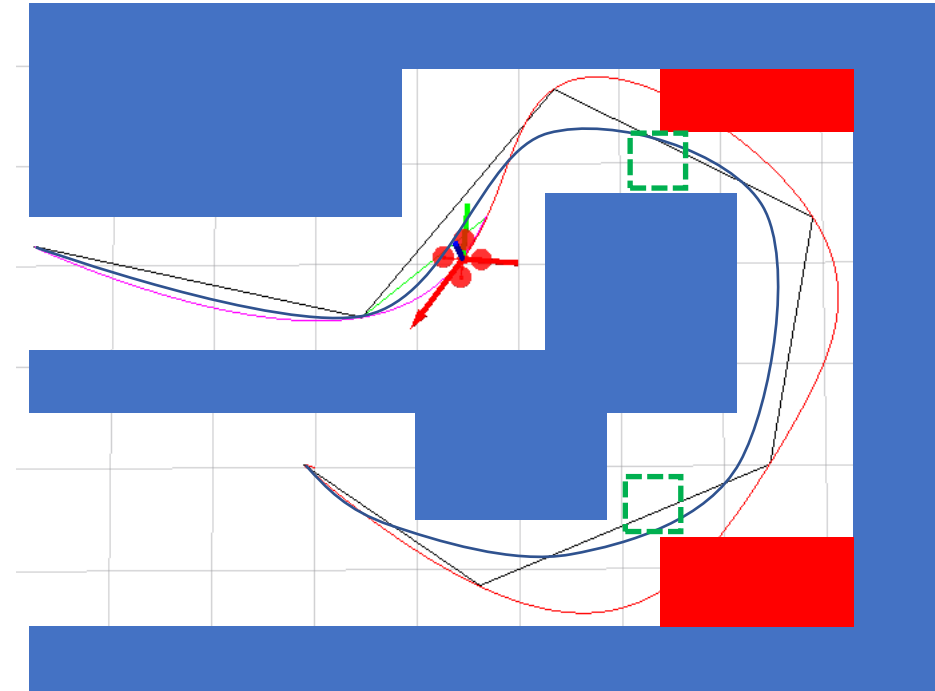
The basic minimum snap framework is good for smooth curve generation, not for collision avoidance.

# Minimum snap with safety constraints

Adding forces



Adding bounds



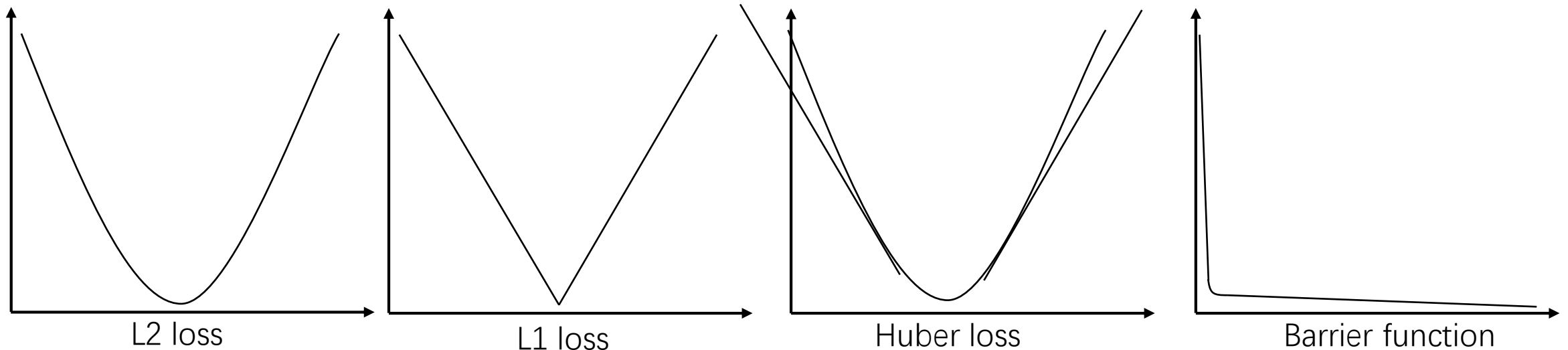


# Hard/ Soft constraints

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \boxed{\begin{array}{ll} g_i(x) = c_i, & i = 1, \dots, n \quad \text{Equality constraints} \\ h_j(x) \geq d_j, & j = 1, \dots, n \quad \text{Inequality constraints} \end{array}} \end{array} \quad \longrightarrow \quad \text{Required to be strictly satisfied.}$$

$$\min \quad f(x) + \boxed{\lambda_1 \cdot g(x) + \lambda_2 \cdot h(x)}$$

- Penalty terms / loss functions.
- Constraints which are preferred but not strictly required.
- Various kinds of loss functions.



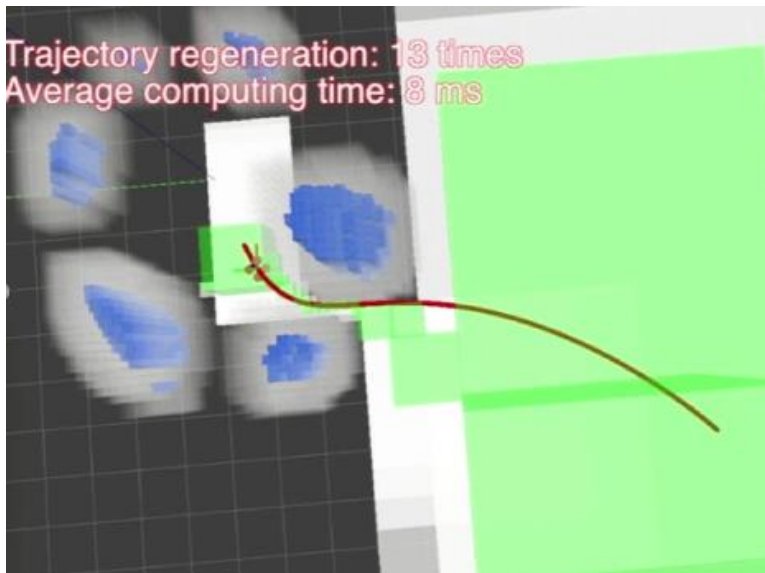
# Hard-constrained Optimization

# Corridor-based Trajectory Optimization





# Corridor-based Smooth Trajectory Generation

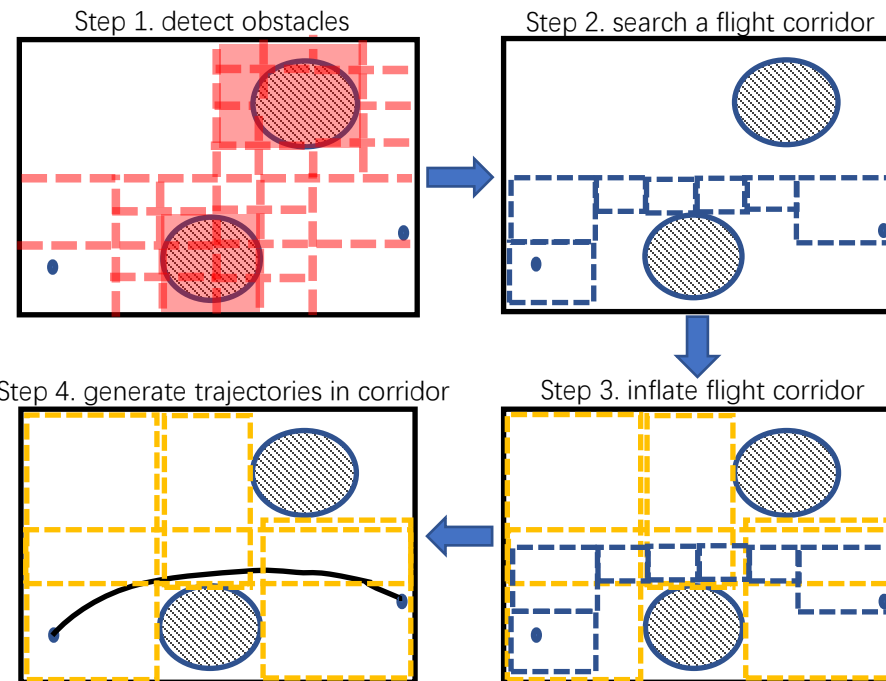


- Differential flatness property

$$\{x, y, z, \dot{x}, \dot{y}, \dot{z}, \phi, \theta, \varphi, p, q, r\} \rightarrow \{x, y, z, \varphi\}$$

- Piecewise polynomial trajectory

$$f_{\mu}(t) = \begin{cases} \sum_{j=0}^N p_{1j}(t - T_0)^j & T_0 \leq t \leq T_1 \\ \sum_{j=0}^N p_{2j}(t - T_1)^j & T_0 \leq t \leq T_1 \\ \vdots & \vdots \\ \sum_{j=0}^N p_{Mj}(t - T_{M-1})^j & T_0 \leq t \leq T_1 \end{cases}$$



- Cost function (minimum jerk)

$$J = \sum_{\mu \in \{x, y, z\}} \int_0^T \left( \frac{d^k f_{\mu}(t)}{dt^k} \right)^2 dt$$

- Boundary constraints
- Continuity constraints
- Safety constraints

$$\begin{aligned} \min \quad & \mathbf{p}^T \mathbf{H} \mathbf{p} \\ \text{s.t.} \quad & \mathbf{A}_{eq} \mathbf{p} = \mathbf{b}_{eq} \\ & \mathbf{A}_{lq} \mathbf{p} \leq \mathbf{b}_{lq} \end{aligned}$$

Quadratic Program



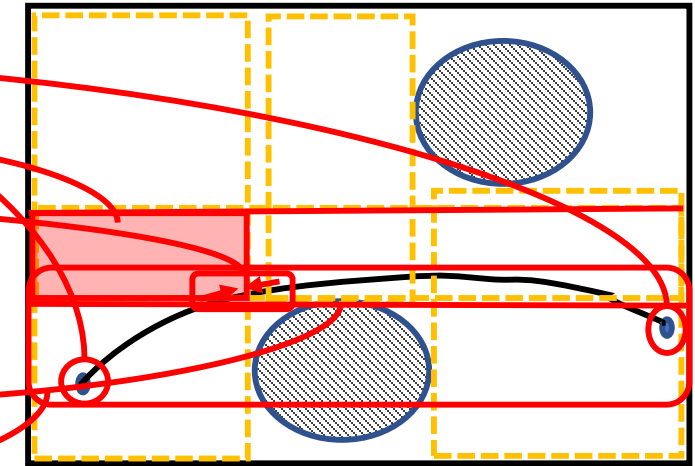
# Problem formulation

- **Instant** linear constraints:

- Start, goal state constraint ( $\mathbf{A}\mathbf{p} = \mathbf{b}$ )
- Transition point constraint ( $\mathbf{A}\mathbf{p} = \mathbf{b}, \mathbf{A}\mathbf{p} \leq \mathbf{b}$ )
- Continuity constraint ( $\mathbf{A}\mathbf{p}_i = \mathbf{A}\mathbf{p}_{i+1}$ )

- **Interval** linear constraints:

- Boundary constraint ( $\mathbf{A}(t)\mathbf{p} \leq \mathbf{b}, \forall t \in [t_l, t_r]$ )
- Dynamic constraint ( $\mathbf{A}(t)\mathbf{p} \leq \mathbf{b}, \forall t \in [t_l, t_r]$ )
  - Velocity constraints
  - Acceleration constraints

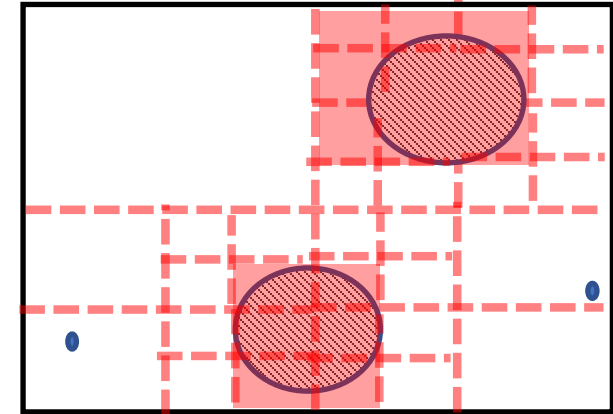
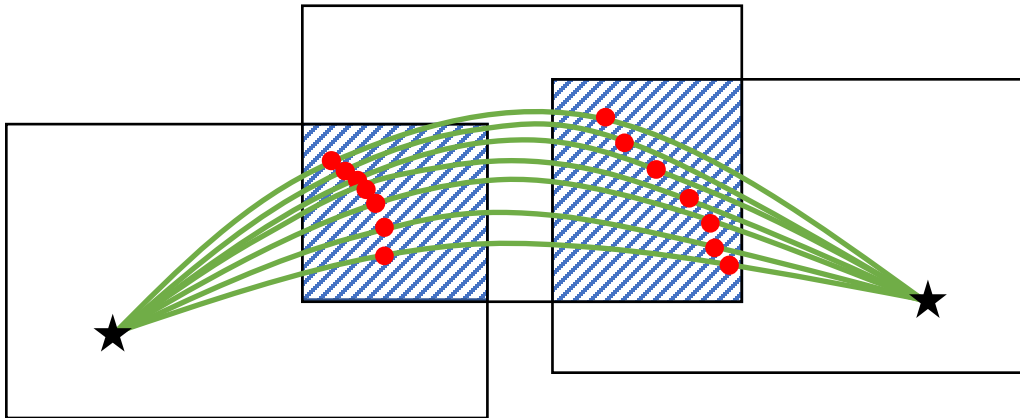




# Advantages

## Many advantages:

- Efficiency: path search in the reduced graph, convex optimization in the corridor are efficient.
- High quality: corridor provides large optimization freedom.



$$\begin{aligned} \min \quad & \mathbf{p}^T \mathbf{H} \mathbf{p} \\ \text{s.t.} \quad & \mathbf{A}_{eq} \mathbf{p} = \mathbf{b}_{eq} \\ & \mathbf{A}_{lq} \mathbf{p} \leq \mathbf{b}_{lq} \end{aligned}$$

Quadratic Program

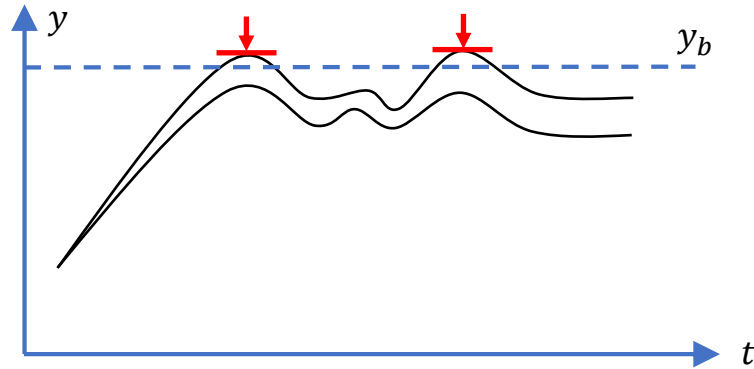


# Disadvantages

## Problem:

All constraints are enforced on piecewise joint points only, how to guarantee they are active along all the trajectory ?

- Iteratively check extremum and add extra constraints.



- Iterative solving is time consuming.
- If strictly no feasible solution meets all constraints. We have to run 10 iterations to determine the status of the solution ?

- Checking extremum is yet another polynomial root finding problem.
  - Up to quartic function, it's easy.
  - Higher order polynomials need numerical solutions.

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$



# Polynomial roots finding

Matlab function “roots”

For a general polynomial function:  $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ ,  
It has the **companion matrix**:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

The characteristic polynomial  $\det(xI - A)$  of  $A$  is the polynomial  $p(x)$ .

## roots

Polynomial roots

### Syntax

```
r = roots(p)
```

### Description

`r = roots(p)` returns the roots of the polynomial represented by `p` as a column vector. Input `p` is a vector containing `n+1` polynomial coefficients, starting with the coefficient of  $x^n$ . A coefficient of 0 indicates an intermediate power that is not present in the equation. For example, `p = [3 2 -2]` represents the polynomial  $3x^2 + 2x - 2$ .

The `roots` function solves polynomial equations of the form  $p_1x^n + \dots + p_nx + p_{n+1} = 0$ . Polynomial equations contain a single variable with nonnegative exponents.

### Algorithms

The `roots` function considers `p` to be a vector with `n+1` elements representing the `n`th degree characteristic polynomial of an `n`-by-`n` matrix, `A`. The roots of the polynomial are calculated by computing the eigenvalues of the companion matrix, `A`.

```
A = diag(ones(n-1,1),-1);  
A(1,:) = -p(2:n+1)./p(1);  
r = eig(A)
```

The results produced are the exact eigenvalues of a matrix within roundoff error of the companion matrix, `A`. However, this does not mean that they are the exact roots of a polynomial whose coefficients are within roundoff error of those in `p`.

# Bezier Curve Optimization



# Trajectory basis changing

- Use **Bernstein** polynomial basis.
- Change to basis of the trajectory from **monomial** polynomial to **Bernstein** polynomial

$$P_j(t) = p_j^0 + p_j^1 t + p_j^2 t^2 + \dots + p_j^n t^n$$



$$B_j(t) = c_j^0 b_n^0(t) + c_j^1 b_n^1(t) + \dots + c_j^n b_n^n(t) = \sum_{i=0}^n c_j^i b_n^i(t)$$

$$b_n^i(t) = \binom{n}{i} \cdot t^i \cdot (1-t)^{n-i}$$

Bézier curve

- Bézier curve is just a special polynomial, it can be mapped to monomial polynomial by:  $p = M \cdot c$ . And all previous derivations still hold.

For 6 order:  $M =$

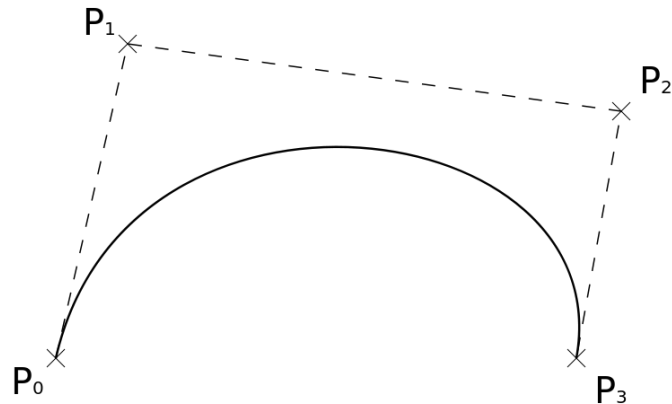
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 15 & -30 & 15 & 0 & 0 & 0 & 0 \\ -20 & 60 & -60 & 20 & 0 & 0 & 0 \\ 15 & -60 & 90 & -60 & 15 & 0 & 0 \\ -6 & 30 & -60 & 60 & -30 & 6 & 0 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix}$$



# Trajectory basis changing

## Properties:

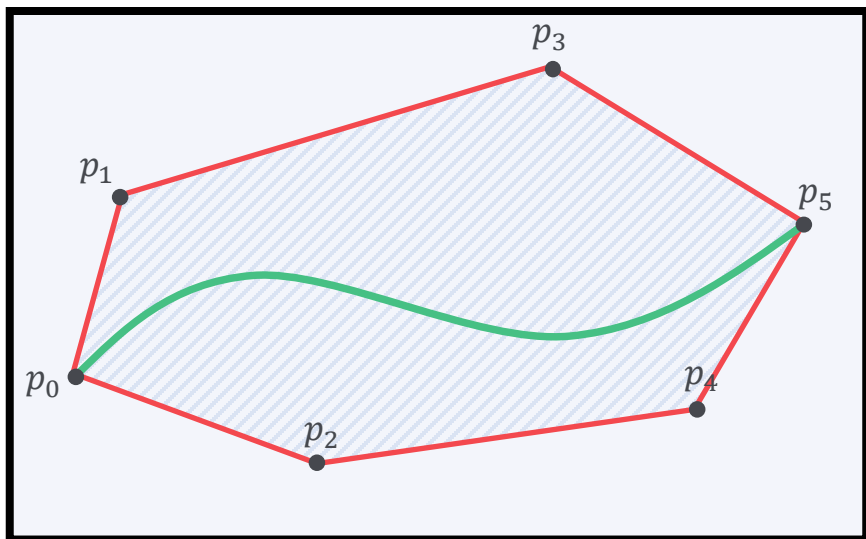
- **Endpoint interpolation.** The Bezier curve always starts at the first control point, ends at the last control point, and never pass any other control points.
- **Convex hull.** The Bezier curve  $B(t)$  consists of a set of control points  $c_i$  are entirely confined within the convex hull defined by all these control points.
- **Hodograph.** The derivative curve  $B'(t)$  of a Bezier curve  $B(t)$  is called as hodograph, and it is also a Bezier curve with control points defined by  $n \cdot (c_{i+1} - c_i)$ , where  $n$  is the degree.
- **Fixed time interval.** A Bezier curve is always defined on  $[0,1]$







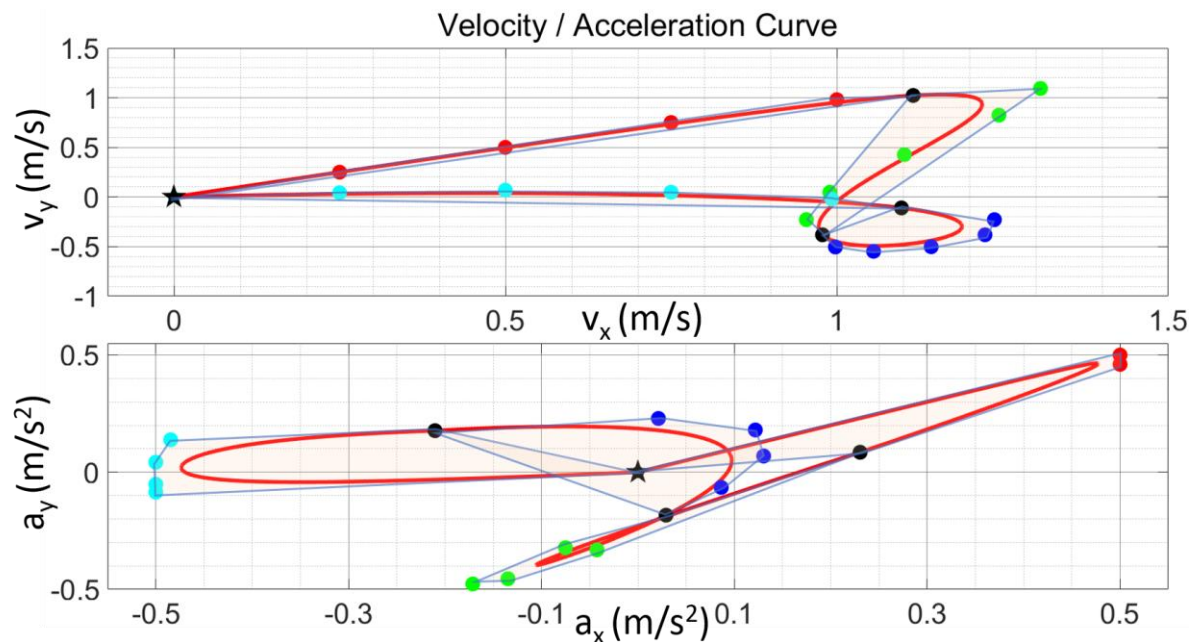
# Convex hull property



- The flight corridor consists of convex polygons.
- Each cube corresponds to a piece of Bezier curve.
- Control points of this curve are enforced inside the polygon.
- The trajectory is entirely inside the convex hull of all points.



The trajectory is entirely inside the flight corridor.



# Trajectory Generation Formulation

Define higher order ( $l^{th}$ ) control points:

$$a_{\mu j}^{0,i} = c_{\mu j}^i, a_{\mu j}^{l,i} = \frac{n!}{(n-l)!} \cdot (a_{\mu j}^{l-1,i+1} - a_{\mu j}^{l-1,i}), \quad l \geq 1$$

- Boundary Constraints:

$$a_{\mu j}^{l,0} \cdot s_j^{(1-l)} = d_{\mu j}^{(l)}$$

- Continuity Constraints:

$$a_{\mu j}^{\phi,n} \cdot s_j^{(1-\phi)} = a_{\mu,j+1}^{\phi,0} \cdot s_{j+1}^{(1-\phi)}, \quad a_{\mu j}^{0,i} = c_{\mu j}^i.$$

- Safety Constraints:

$$\beta_{\mu j}^- \leq c_{\mu j}^i \leq \beta_{\mu j}^+, \quad \mu \in \{x, y, z\}, \quad i = 0, 1, 2, \dots, n,$$

- Dynamical Feasibility Constraints:

$$v_m^- \leq n \cdot (c_{\mu j}^i - c_{\mu j}^{i-1}) \leq v_m^+,$$

$$a_m^- \leq n \cdot (n-1) \cdot (c_{\mu j}^i - 2c_{\mu j}^{i-1} + c_{\mu j}^{i-2})/s_j \leq a_m^+$$

Stack all of these

min

$$\mathbf{c}^T \mathbf{Q}_o \mathbf{c}$$

s.t.

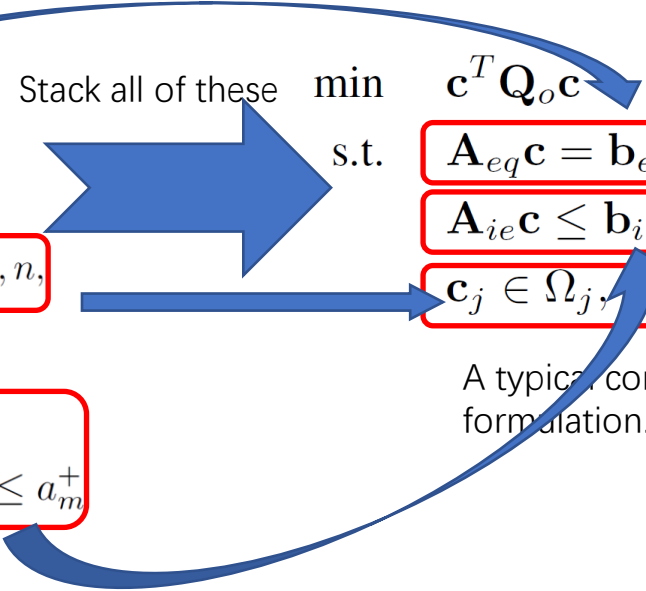
$$\mathbf{A}_{eq} \mathbf{c} = \mathbf{b}_{eq},$$

$$\mathbf{A}_{ie} \mathbf{c} \leq \mathbf{b}_{ie},$$

$$\mathbf{c}_j \in \Omega_j, \quad j = 1, 2, \dots, m,$$

A typical convex QP formulation.

We only solve this program once to determine whether there is a qualified trajectory exists.





## Simulation results

**Blue curve : trajectory  
in execution horizon**

**Red curve :  
current trajectory**

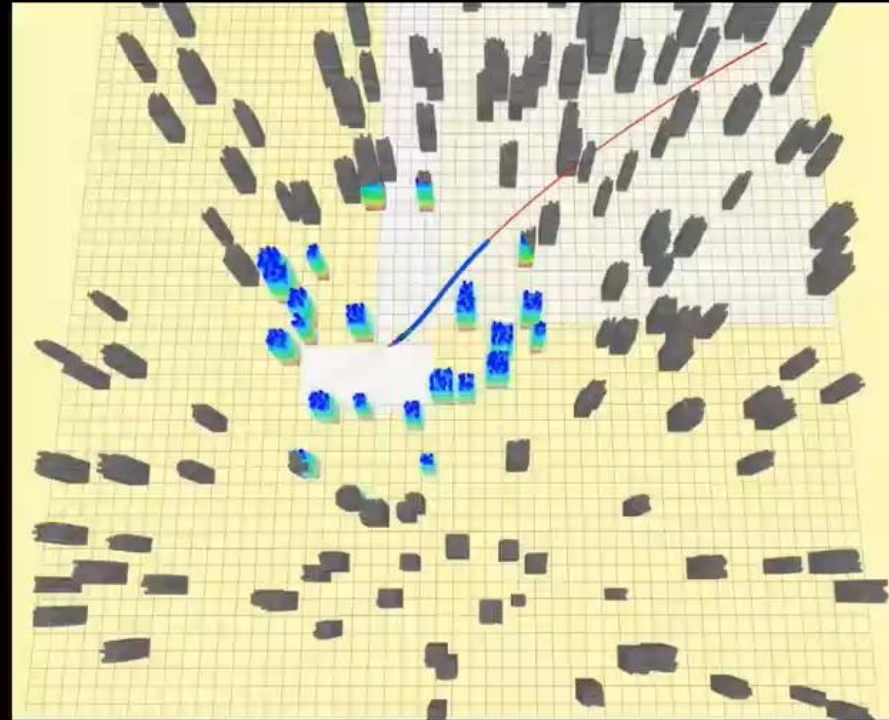
**White cube :  
flight corridor**

**Colorful voxels :  
mapped obstacles**

**Grey voxels :  
un-mapped obstacles**

**Green arrow : velocity**

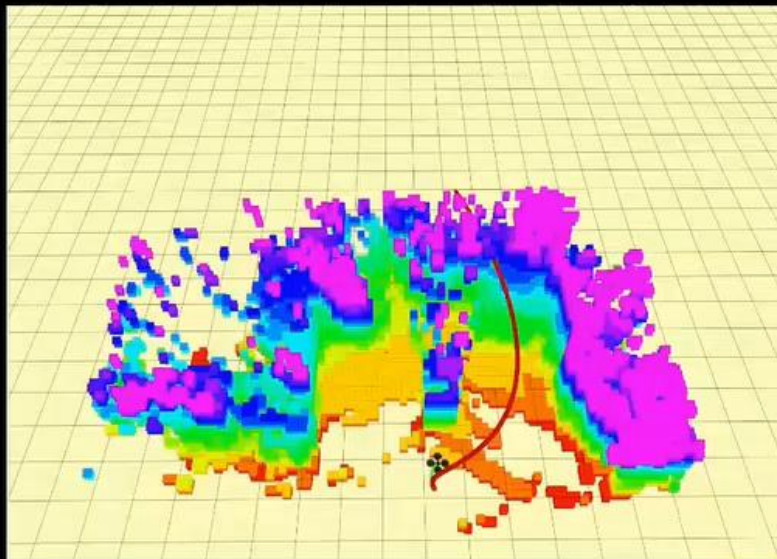
**Yellow arrow : acceleration**



**Flight corridor is projected to  
x-y plane for visualization**



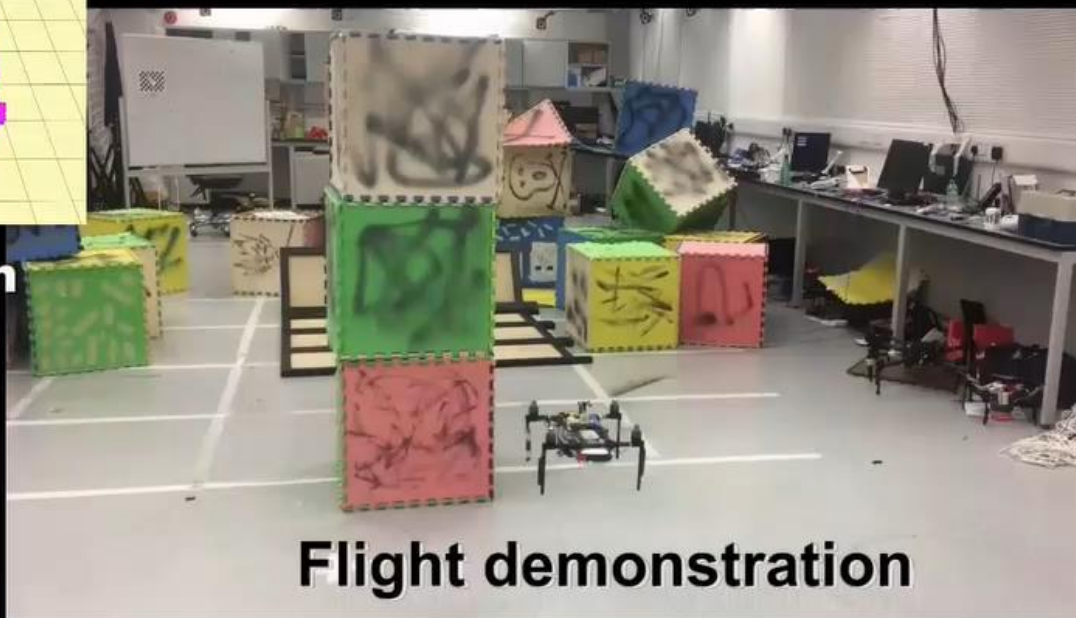
# Experimental results



**Trajectory visualization**

**Speed : 1x**

**Indoor autonomous flight 1**  
**navigating among messy obstacles**  
**2 trajectories (re)generated**  
**Average computing time : 41.2 ms**



**Flight demonstration**

*Online Safe Trajectory Generation For Quadrotors Using Fast Marching Method and Bernstein Basis Polynomial, Fei Gao et al.*

Source code released at: <https://github.com/HKUST-Aerial-Robotics/Btraj>



A complete UAV online motion planning framework

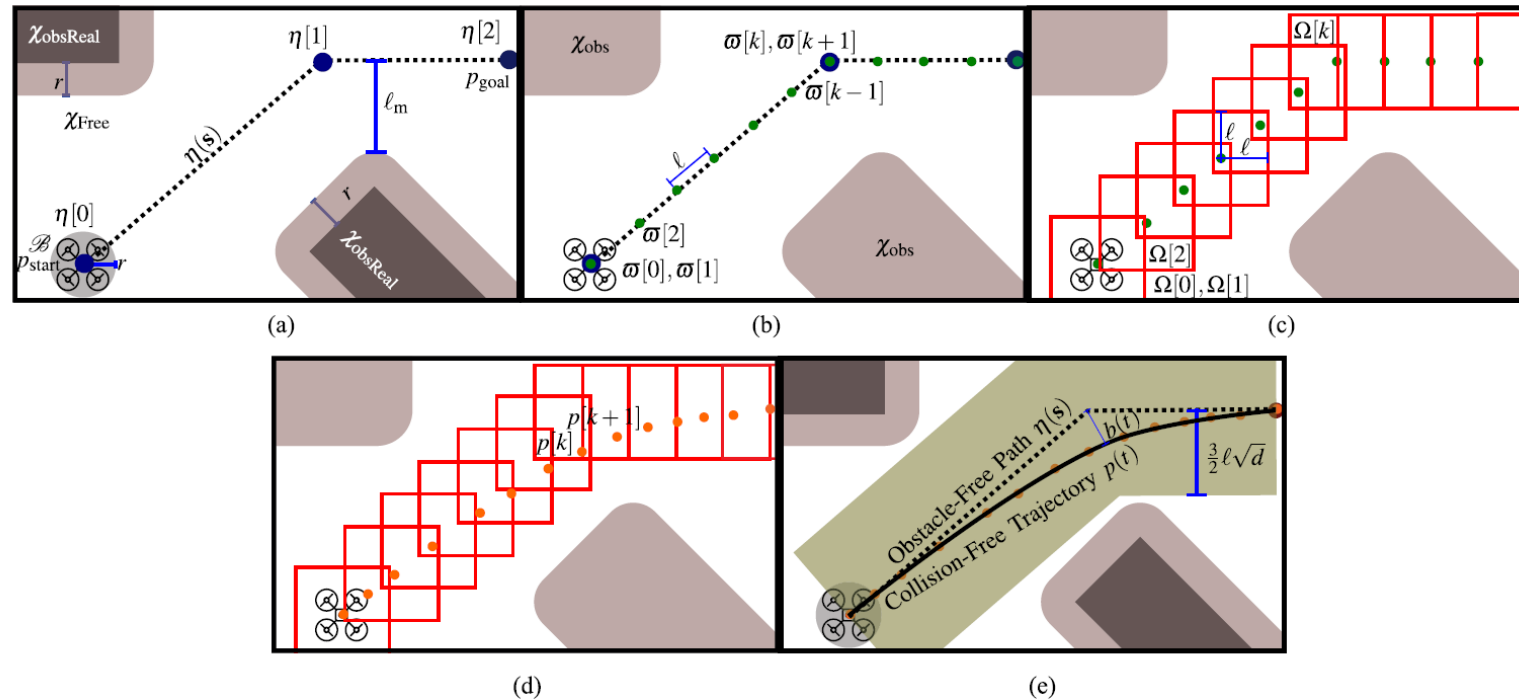
# Other Options





# Dense constraints

- Adding numerous constraints at discrete time ticks.
- Piecewise-constant accelerations at each tick.
- QP program solution.



- Always generates over-conservative trajectories.
- Too many constraints, the computational burden is high.



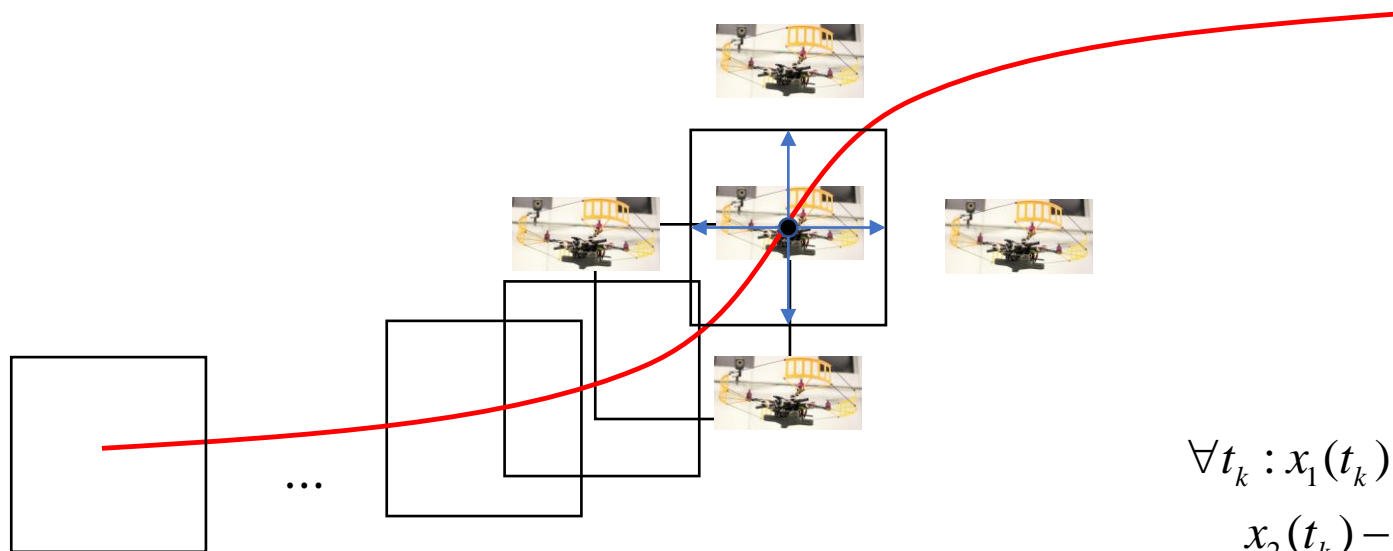
## Dense constraints



*A hybrid method for online trajectory planning of mobile robots in cluttered environments*, L Campos-Macías et. al



# Mixed integer optimization



$$\forall t_k : x_1(t_k) - x_2(t_k) \leq d_x$$

$$\text{or } x_2(t_k) - x_1(t_k) \leq d_x$$

$$\text{or } y_1(t_k) - y_2(t_k) \leq d_y$$

$$\text{or } y_2(t_k) - y_1(t_k) \leq d_y$$

$$\text{or } z_1(t_k) - z_2(t_k) \leq d_z$$

$$\text{or } z_2(t_k) - z_1(t_k) \leq d_z$$

- Mixed-integer QP.
- 'Big M' method.
- Super slow.

$$\forall t_k : x_1(t_k) - x_2(t_k) - c_0 \cdot M \leq d_x$$

$$x_2(t_k) - x_1(t_k) - c_1 \cdot M \leq d_x$$

$$y_1(t_k) - y_2(t_k) - c_2 \cdot M \leq d_y$$

$$y_2(t_k) - y_1(t_k) - c_3 \cdot M \leq d_y$$

$$z_1(t_k) - z_2(t_k) - c_4 \cdot M \leq d_z$$

$$z_2(t_k) - z_1(t_k) - c_5 \cdot M \leq d_z$$

$$\sum_{i=0}^5 c_i = 5, M = 100000$$

$$c_i \in \{0, 1\}$$





## Dense constraints



*Mixed-integer quadratic program trajectory generation for heterogeneous quadrotor teams, D. Mellinger et al.*

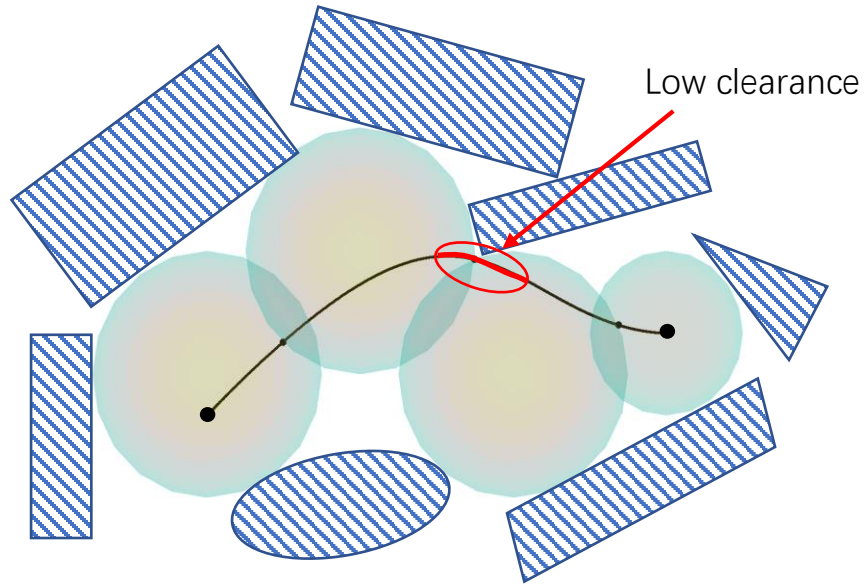
# Soft-constrained Optimization

# Distance-based Trajectory Optimization



# Motivation

Hard-constrained methods



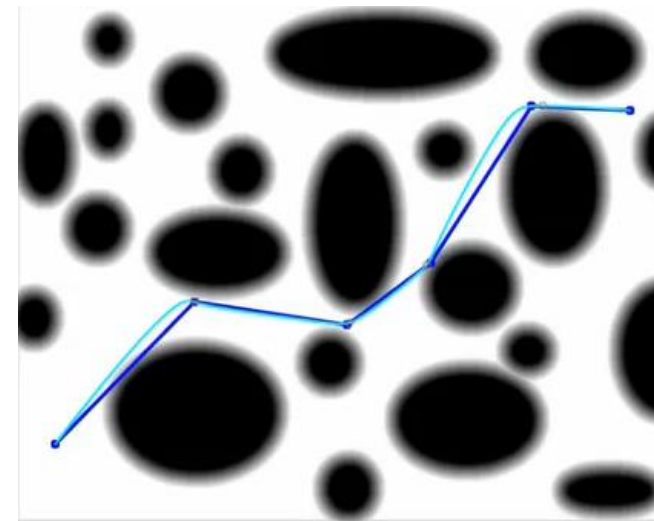
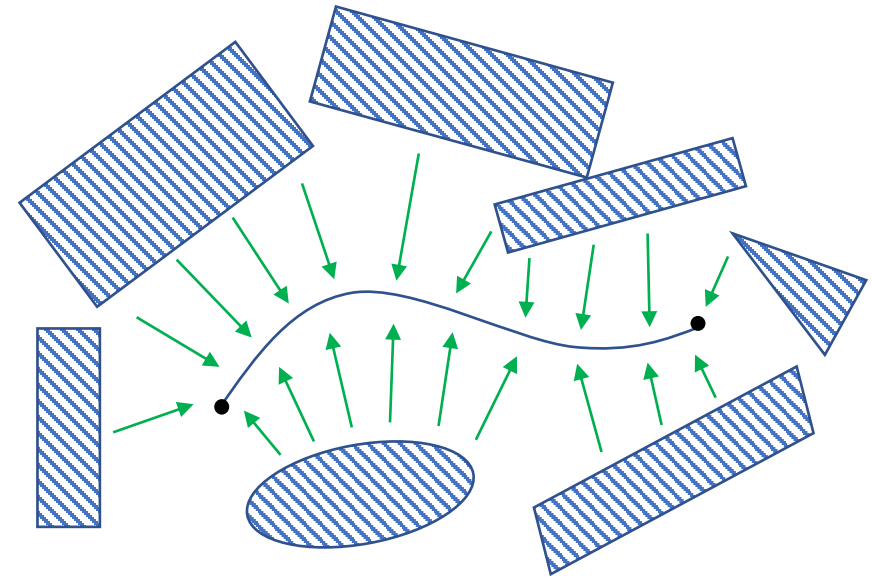
Vision-based drone:

- Limited sensing range and quality
- Noisy depth estimation

Hard-constrained method:

- Treat all free space equally
- Solution space is sensitive to noise

Soft-constrained methods





# Problem formulation

- Differential flatness property

$$\{x, y, z, \dot{x}, \dot{y}, \dot{z}, \phi, \theta, \varphi, p, q, r\} \rightarrow \{x, y, z, \varphi\}$$

- Piecewise polynomial trajectory

$$f_{\mu}(t) = \begin{cases} \sum_{j=0}^N p_{1j}(t - T_0)^j & T_0 \leq t \leq T_1 \\ \sum_{j=0}^N p_{2j}(t - T_1)^j & T_0 \leq t \leq T_1 \\ \vdots & \vdots \\ \sum_{j=0}^N p_{Mj}(t - T_{M-1})^j & T_0 \leq t \leq T_1 \end{cases}$$

- Objective function

$$J = J_s + J_c + J_d$$

$$= \underbrace{\lambda_1 J_1}_{\text{Smoothness cost}} + \underbrace{\lambda_2 J_2}_{\text{Collision cost}} + \underbrace{\lambda_3 J_3}_{\text{Dynamical cost}}$$

- Smoothness cost: minimum snap formulation

$$J_s = \sum_{\mu \in \{x, y, z\}} \int_0^T \left( \frac{d^k f_{\mu}(t)}{dt^k} \right)^2 dt$$

$$= \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \mathbf{C}^T \mathbf{M}^{-T} \mathbf{Q} \mathbf{M}^{-1} \mathbf{C} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix} = \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_{FF} & \mathbf{R}_{FP} \\ \mathbf{R}_{PF} & \mathbf{R}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}$$

- Collision cost: penalize on the distance to nearest obstacle

$$J_c = \int_{T_0}^{T_M} c(p(t)) ds$$

$$= \sum_{k=0}^{T/\delta t} c(p(T_k)) \|v(t)\| \delta t, \quad T_k = T_0 + k \delta t$$

- Dynamical Cost: penalize on the velocity and acceleration where exceeds limits (similar to collision term).



# Objective/Jacobian evaluation

- Smoothness cost: minimum snap formulation

$$J_s = \sum_{\mu \in \{x, y, z\}} \int_0^T \left( \frac{d^k f_\mu(t)}{dt^k} \right)^2 dt$$

$$= \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \mathbf{C}^T \mathbf{M}^{-T} \mathbf{Q} \mathbf{M}^{-1} \mathbf{C} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix} = \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_{FF} & \mathbf{R}_{FP} \\ \mathbf{R}_{PF} & \mathbf{R}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{d}_F \\ \mathbf{d}_P \end{bmatrix}$$

- Collision cost: penalize on the distance to nearest obstacle

$$J_c = \int_{T_0}^{T_M} \underbrace{c(p(t))}_{\text{Distance penalty at a point along the trajectory}} ds$$

$$= \sum_{k=0}^{T/\delta t} c(p(T_k)) \|v(t)\| \delta t, \quad T_k = T_0 + k\delta t$$

- The Jacobian with respect to free derivatives  $\mathbf{d}_{p\mu}$  is:

$$\frac{\alpha J_c}{\alpha \mathbf{d}_{p\mu}} = \sum_{k=0}^{T/\delta t} \left\{ \nabla_\mu c(p(T_k)) \|v\| \mathbf{F} + c(p(T_k)) \frac{v_\mu}{\|v\|} \mathbf{G} \right\} \delta t, \quad \mu \in \{x, y, z\}$$

- The Jacobian with respect to free derivatives  $\mathbf{d}_{p\mu}$  is:

$$\frac{\alpha J_s}{\alpha \mathbf{d}_{p\mu}} = 2\mathbf{d}_F^T \mathbf{R}_{FP} + 2\mathbf{d}_P^T \mathbf{R}_{PP}$$

$L_{dp}$  is the right block of matrix  $\mathbf{M}^{-1} \mathbf{C}$  which corresponds to the free derivatives on the  $\mu$  axis  $\mathbf{d}_{p\mu}$ .

$$\mathbf{F} = \mathbf{T} L_{dp}, \quad \mathbf{G} = \mathbf{T} V_m L_{dp}.$$

$\nabla_\mu c(\cdot)$  is the gradient in  $\mu$  axis of the collision cost.

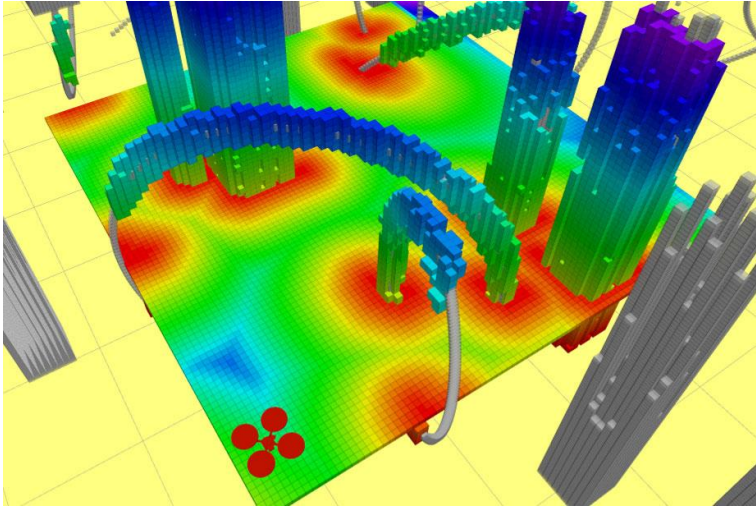
$V_m$  maps the coefficients of the position to the coefficients of the velocity.  $\mathbf{T} = [T_k^0, T_k^1, \dots, T_k^n]$

$$\mathbf{H}_o = \left[ \frac{\partial^2 f_o}{\partial \mathbf{d}_{P_x}^2}, \frac{\partial^2 f_o}{\partial \mathbf{d}_{P_y}^2}, \frac{\partial^2 f_o}{\partial \mathbf{d}_{P_z}^2} \right],$$

$$\frac{\partial^2 f_o}{\partial \mathbf{d}_{P\mu}^2} = \sum_{k=0}^{\tau/\delta t} \left\{ \mathbf{F}^T \nabla_\mu c(p(T_k)) \frac{v_\mu}{\|v\|} \mathbf{G} + \mathbf{F}^T \nabla_\mu^2 c(p(T_k)) \|v\| \mathbf{F} \right. \\ \left. + \mathbf{G}^T \nabla_\mu c(p(T_k)) \frac{v_\mu}{\|v\|} \mathbf{F} + \mathbf{G}^T c(p(T_k)) \frac{v_\mu^2}{\|v\|^3} \mathbf{G} \right\} \delta t, \quad (10)$$

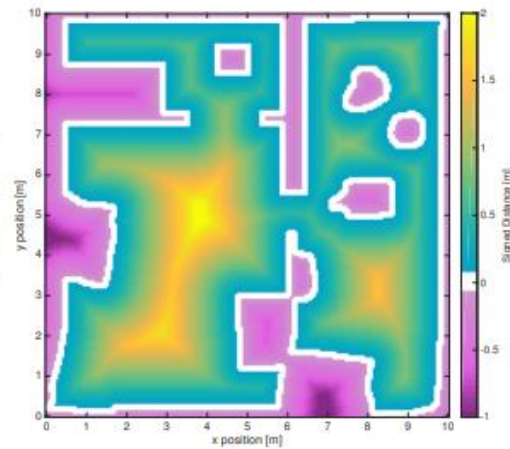
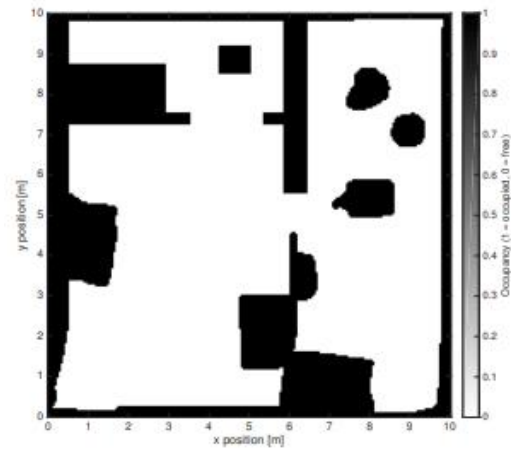
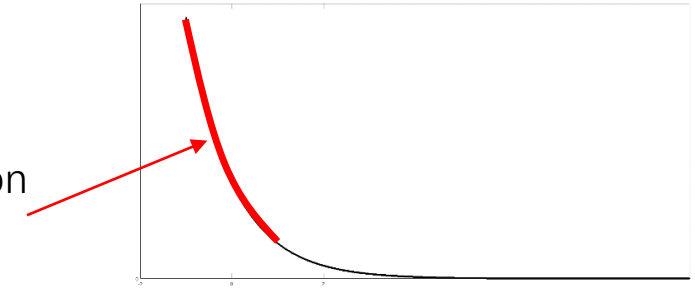


# Euclidean signed distance field (ESDF)

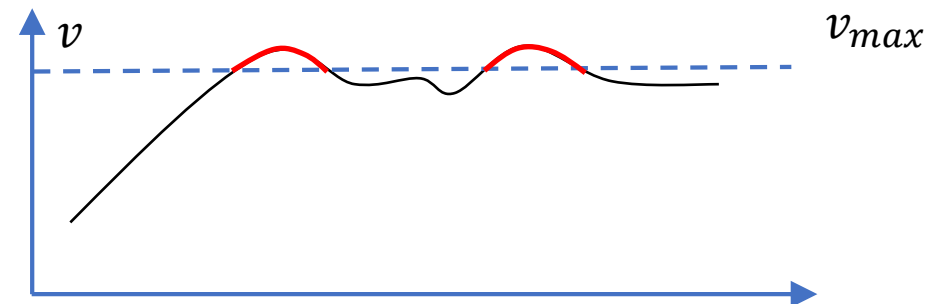


Use **exponential** function as cost function  $c$ , to prevent trajectory from be near to obstacle

Penalty explosion  
near obstacles



- Dynamical Cost





# Numerical optimization

minimize  $f(x)$

- Produce sequence of points  $x^{(k)} \in \text{dom } f, k = 0, 1, \dots$  with

$$f(x^{(k)}) \rightarrow p^*$$

- Can be interpreted as iterative methods for solving optimality condition

$$\nabla f(x^*) = 0$$





## Descent method

$$x^{k+1} = x^k + t^k \Delta x^k \text{ with } f(x^{k+1}) < f(x^k)$$

- Other notations:  $x^+ = x + t\Delta x$ ,  $x := x + t\Delta x$
- $\Delta x$  is the step, or search direction;  $t$  is the step size, or step length

---

General descent method

**given** a starting point  $x \in \text{dom } f$ .

**repeat**

1. Determine a descent direction  $\Delta x$ .
2. Line search. Choose a step size  $t > 0$ .
3. Update.  $x := x + t\Delta x$ .

**until** stopping criterion is satisfied.

---



# Line search

## Line search types

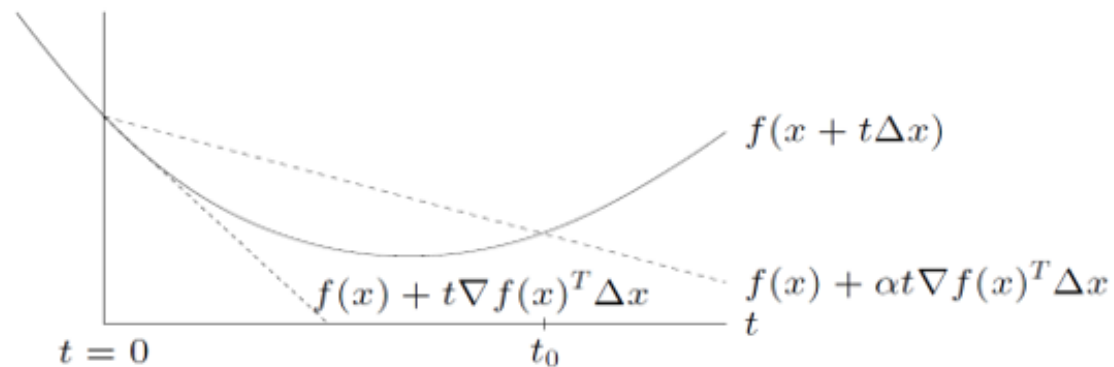
**exact line search:**  $t = \operatorname{argmin}_{t>0} f(x + t\Delta x)$

**backtracking line search** (with parameters  $\alpha \in (0, 1/2), \beta \in (0, 1)$ )

- starting at  $t = 1$ , repeat  $t := \beta t$  until

$$f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$$

- graphical interpretation: backtrack until  $t \leq t_0$





# First-order method

## Gradient descent method

General descent method with  $\Delta x = -\nabla f(x)$

---

**given** a starting point  $x \in \text{dom } f$ .

**Repeat**

1.  $\Delta x := -\nabla f(x)$ .
2. Line search. Choose step size  $t$  via exact or backtracking line search.
3. Update.  $x := x + t\Delta x$ .

**until** stopping criterion is satisfied.

---

- Stopping criterion usually of the form  $\|\nabla f(x)\|_2 \leq \epsilon$



## Second-order method

### Newton method

*Taylor expansion at  $x^{(k)}$ , and second – order similarity  $\Delta x = -\nabla^2 f(x)^{-1} \nabla f(x)$*

---

**given** a starting point  $x \in \text{dom } f$ .

**Repeat**

1.  $\Delta x := -\nabla^2 f(x)^{-1} \nabla f(x)$ .
2. Line search.  $t = \text{argmin}_{t>0} f(x^{(k)} - t \nabla^2 f(x)^{-1} \nabla f(x))$ .
3. Update.  $x := x + t \Delta x$ .

**until** stopping criterion is satisfied.

---

- Stopping criterion usually of the form  $\|\nabla f(x)\|_2 \leq \epsilon$
- $\nabla^2 f(x)$  is the Hessian matrix of the  $f(x)$  at  $x$
- For Gauss-newton:  $\nabla^2 f(x) \approx J_f^T J_f$ ,  $J_f$  is Jacobi matrix,  $\Delta x = -(J_f^T J_f)^{-1} J_f^T$



## Second-order method

### Levenberg-Marquardt method

Improvement of Gauss-newton method:  $\Delta x = -(J_f^T J_f + \lambda I)^{-1} J_f^T$

---

**given** a starting point  $x \in \text{dom } f$ , start  $\lambda_0 > 0$ .

**Repeat**

1.  $\Delta x = -(J_f^T J_f + \lambda I)^{-1} J_f^T$ .
2. Update.  $\lambda$ , the updating is controlled by the gain ratio
3. Update.  $x := x + \Delta x$ .

**until** stopping criterion is satisfied.

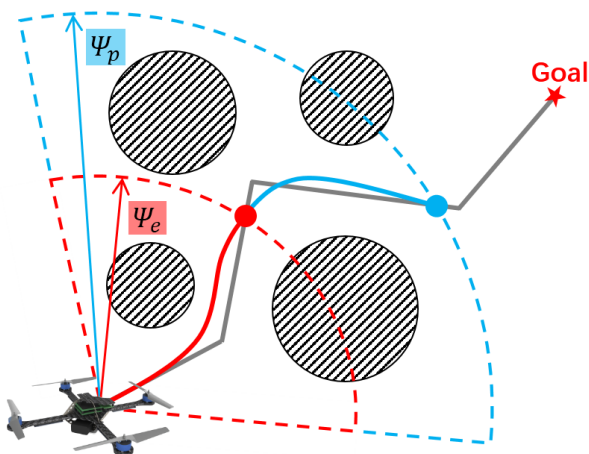
---

- Stopping criterion usually of the form  $\|\nabla f(x)\|_2 \leq \epsilon$
- When  $\lambda \rightarrow 0$ , LM method  $\rightarrow$  Gauss-newton method
- When  $\lambda \rightarrow \infty$ , LM method  $\rightarrow$  Gradient descent method



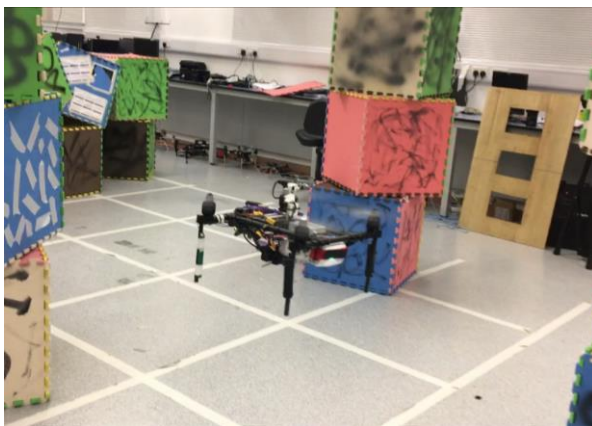
# Planning strategy

- **Receding horizon re-planning**     $\psi_p$  : *planning horizon*     $\psi_e$  : *execution horizon*



- Get a global plan, search at the beginning of the navigation, or initialize as a straight line.
- Find a local path a local target by using the front-end.
- Generate a local trajectory by using the back-end.
- Track the trajectory within the execution horizon

- **Exploration strategy**

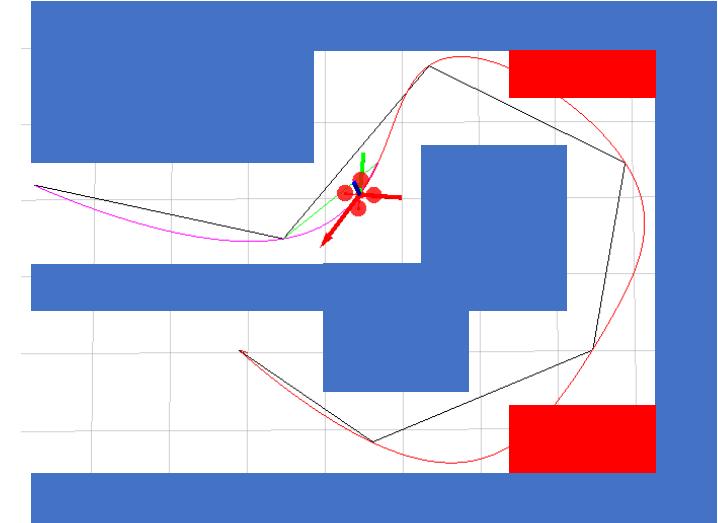


- Activated when outlier in mapping module blocks the way.
- Generate safe but short trajectories in nearby regions.
- Hover and observe.

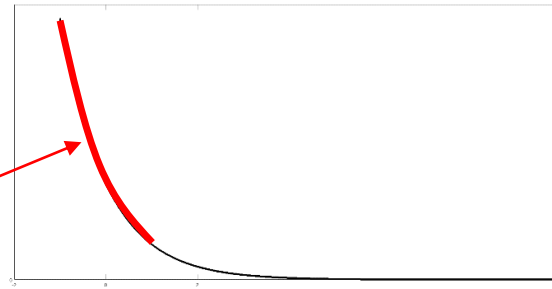


# Planning strategy

- Initialize as minimum snap trajectory
  - Good smoothness.
  - Collision may cost by overshoot. Unsafe initial value.
- Initialize as straight-line trajectory following the path
  - Poor smoothness.
  - Collision free. Safe initial value.
- ✓ • Given collision-free initial trajectory, safety is achieved by using a **exponential penalty function** on collision cost.



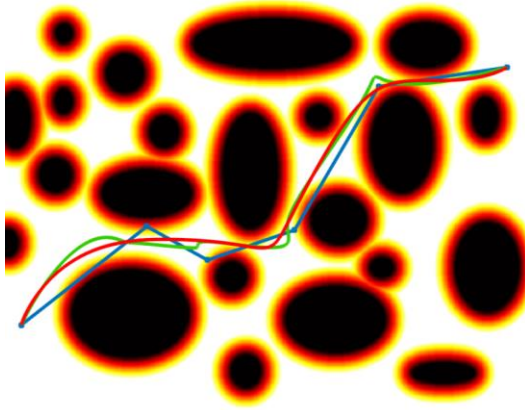
Penalty explosion  
near obstacles



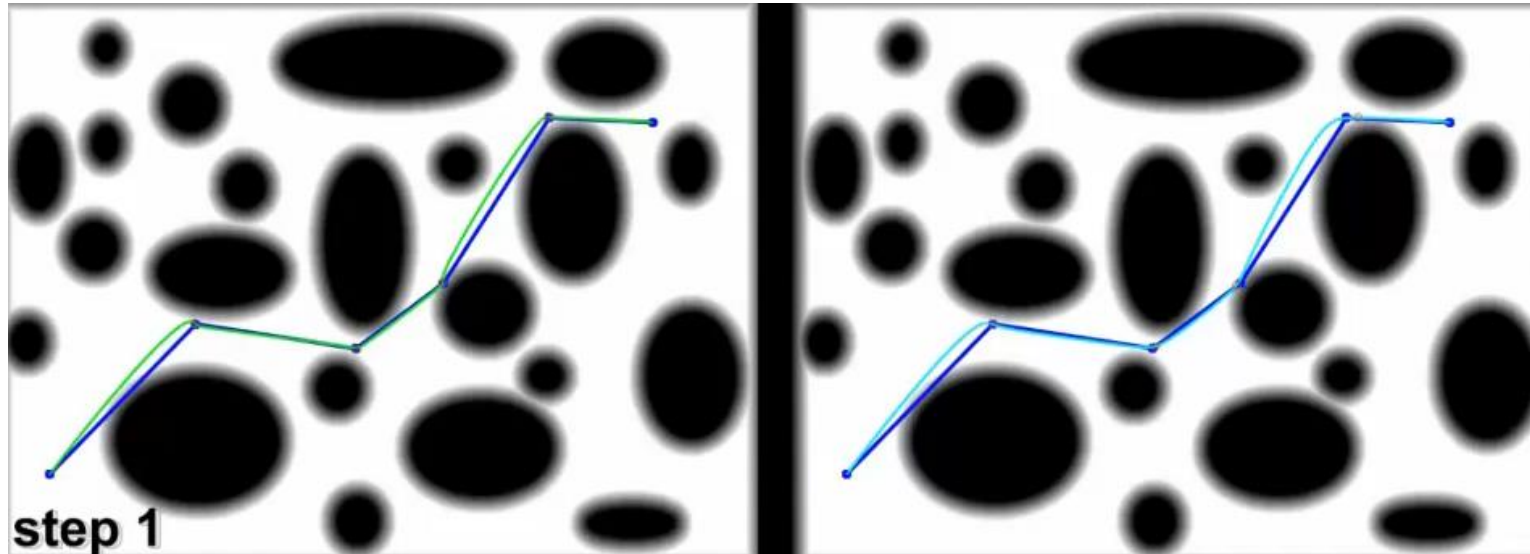


# Optimization strategy

- two-step optimization



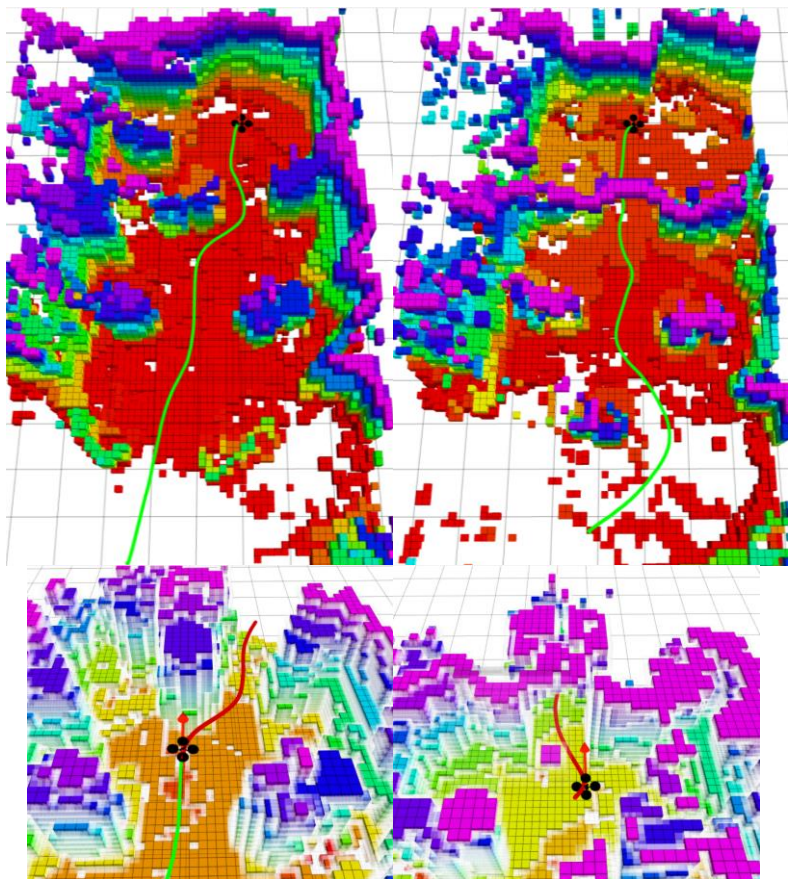
- Optimize the trajectory with collision cost only.
- Re-allocate time and re-parametrize the trajectory.
- Optimize the objective with a smoothness term and dynamical penalty term added.





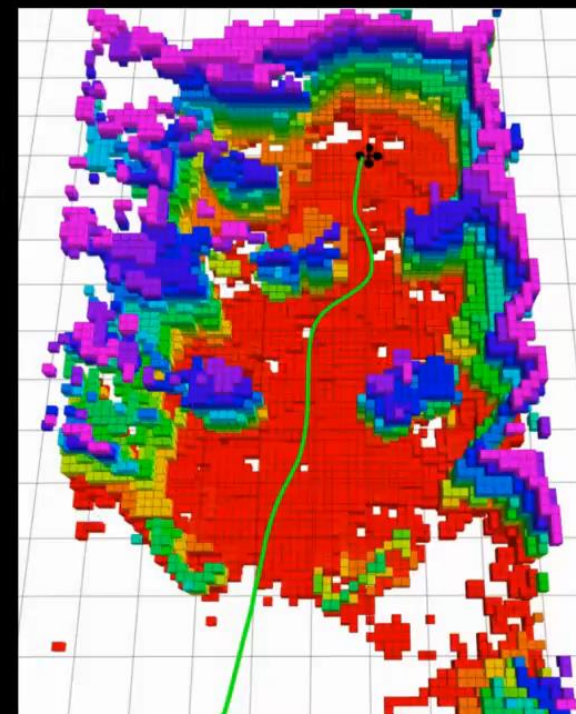


# Results

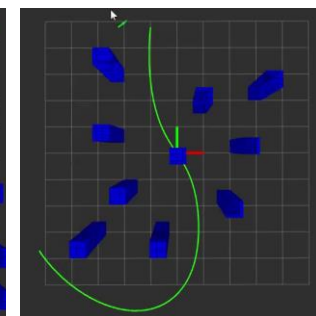
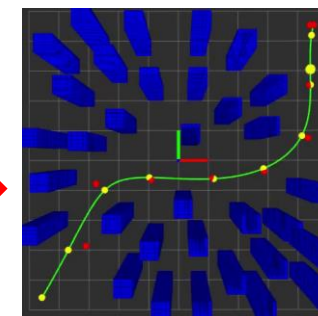


香港科技大學  
THE HONG KONG  
UNIVERSITY OF SCIENCE  
AND TECHNOLOGY

*Color code : height*  
*Green curve:*  
*previous trajectory*  
*Red curve:*  
*current trajectory*



Source code released at:  
[https://github.com/HKUST-Aerial-Robotics/grad\\_traj\\_optimization](https://github.com/HKUST-Aerial-Robotics/grad_traj_optimization)  
A tool for local UAV trajectory optimization



# Case Study

# Fast Planner



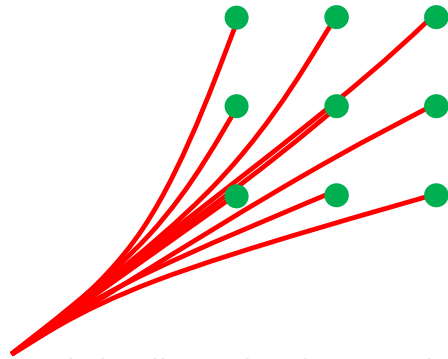
**Kinodynamic path searching**  
**+ B-Spline trajectory optimization**  
**+ time adjustment**



# B-Spline trajectory optimization

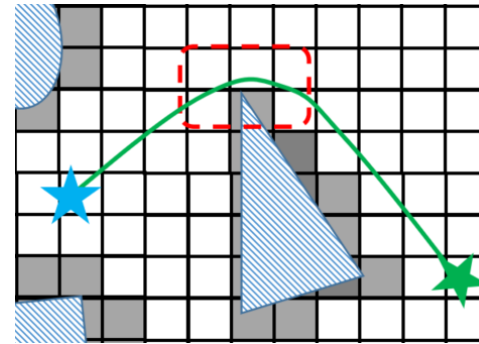
- Limitations of initial path

a) Suboptimality



only search in discretized control space

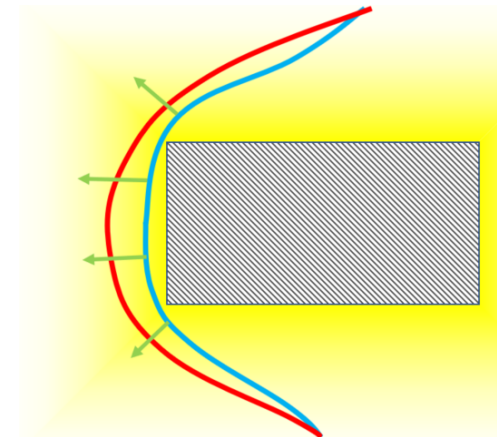
b) Low clearance / high collision risk



some segments may be very close to obstacles

- Gradient-based trajectory optimization

- Improve smoothness
- Use Euclidean distance field (EDF) to improve clearance





## B-Spline trajectory optimization

- Drawbacks of previous gradient-based methods:
  - Expensive computation of collision cost

$$f_{collision} = \int_{T_1}^{T_2} c(p(t)) ds = \int_{T_1}^{T_2} c(p(t)) \|v(t)\| dt = \sum_{k=0}^{(T_2-T_1)/\delta t} c(p(T_k)) \|v(t)\| \delta t$$

small  $\delta t$  for accuracy  $\rightarrow$  large  $(T_2-T_1)/\delta t \rightarrow$  high complexity

- Also so for dynamic feasibility costs

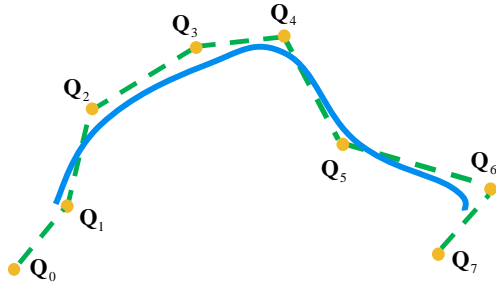
$$f_{vel} = \sum_{\mu \in \{x,y,z\}} \int_{T_1}^{T_2} c_v(v_\mu(t)) ds = \sum_{\mu \in \{x,y,z\}} c_v(v_\mu(t)) \|a_\mu(t)\| dt$$

$$f_{acc} = \sum_{\mu \in \{x,y,z\}} \int_{T_1}^{T_2} c_a(a_\mu(t)) ds = \sum_{\mu \in \{x,y,z\}} c_a(a_\mu(t)) \|j_\mu(t)\| dt$$



# B-Spline trajectory optimization

- Solution: B-Spline trajectory representation
  - Uniform B-Spline



Determined by its degree  $p_b$ , time interval  $\Delta t$ , and a set of  $\{Q_0, Q_1, \dots, Q_N\}$ .

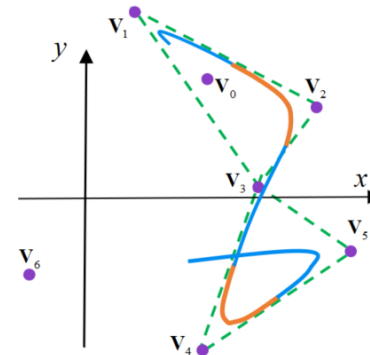
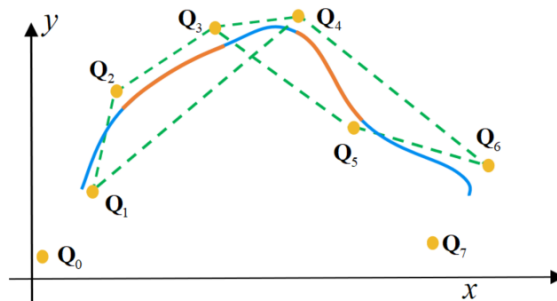
$$t \in [t_i, t_{i+1}) \xrightarrow{\text{normalize}} s(t) = (t - t_i)/\Delta t$$

$$p(s(t)) = \begin{pmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{p_b} \end{pmatrix}^T M_{p_b+1} \begin{pmatrix} Q_{i-p_b} \\ Q_{i-p_b+1} \\ Q_{i-p_b+2} \\ \vdots \\ Q_i \end{pmatrix}$$

- Advantages

- a) Convex hull property: simplify safety & dynamic constraints
- b) Continuity: no need of constraints at segments joints

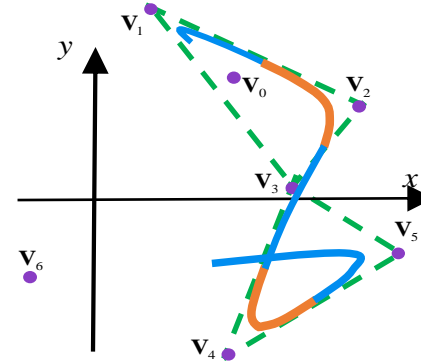
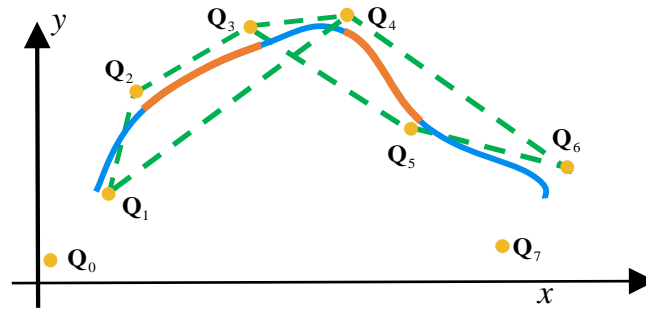
}  $\rightarrow$  Speed up optimization





# B-Spline trajectory optimization

- Convex hull property



Each segment of a B-spline is **bounded** by the corresponding convex hull of control points (left), and so is the derivative (right).

- Simplify safety & dynamic constraints

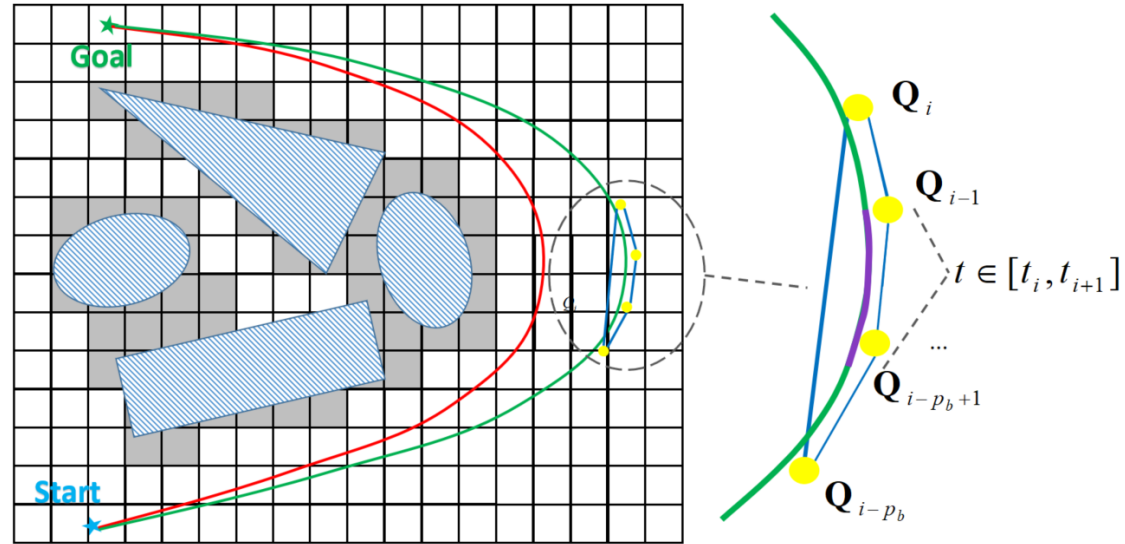
Convex hulls are within feasible space → <sup>convex hull property</sup> The whole trajectory is feasible!





# B-Spline trajectory optimization

- Ensure safety by convex hull



Push control points away  
from obstacles



Convex hulls are  
pushed to safe space



The whole trajectory  
is bounded in safe region



# B-Spline trajectory optimization

- Problem formulation  $f_{total} = \lambda_1 f_s + \lambda_2 f_c + \lambda_3 (f_v + f_a)$

- Smoothness - elastic band cost function

$$f_s = \sum_{i=p_b-1}^{N-p_b+1} \left\| \underbrace{(\mathbf{Q}_{i+1} - \mathbf{Q}_i)}_{F_{i+1,i}} - \underbrace{(\mathbf{Q}_{i-1} - \mathbf{Q}_i)}_{F_{i-1,i}} \right\|^2$$

- Safety - potential function in EDF

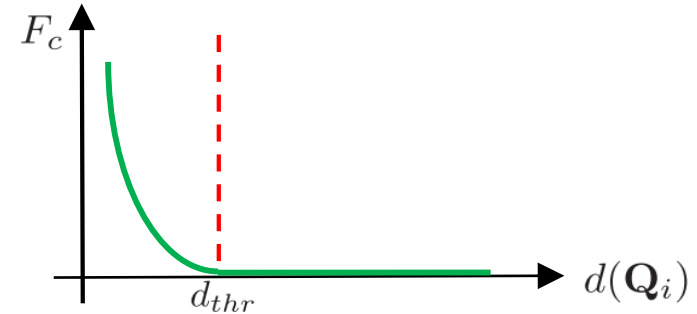
$$f_c = \sum_{i=p_b}^{N-p_b} F_c(d(\mathbf{Q}_i))$$

$$F_c(d(\mathbf{Q}_i)) = \begin{cases} (d(\mathbf{Q}_i) - d_{thr})^2 & d(\mathbf{Q}_i) \leq d_{thr} \\ 0 & d(\mathbf{Q}_i) > d_{thr} \end{cases}$$

- Dynamic feasibility

$$f_v = \sum_{\mu \in \{x,y,z\}} \sum_{i=p_b}^{N-p_b} F_v(V_{i\mu}), \quad f_a = \sum_{\mu \in \{x,y,z\}} \sum_{i=p_b-2}^{N-p_b} F_a(A_{i\mu})$$

$$F_v(v_\mu) = \begin{cases} (v_\mu^2 - v_{\max}^2)^2 & v_\mu^2 > v_{\max}^2 \\ 0 & v_\mu^2 \leq v_{\max}^2 \end{cases}$$



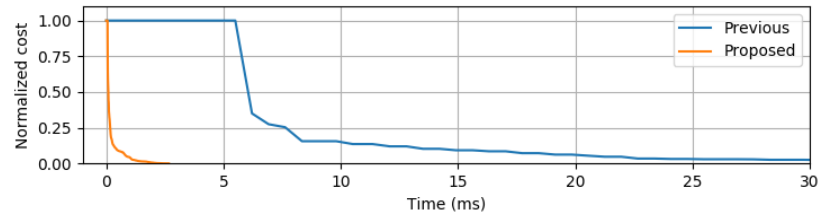
Solve this non-linear optimization by the L-BFGS algorithm



# Results

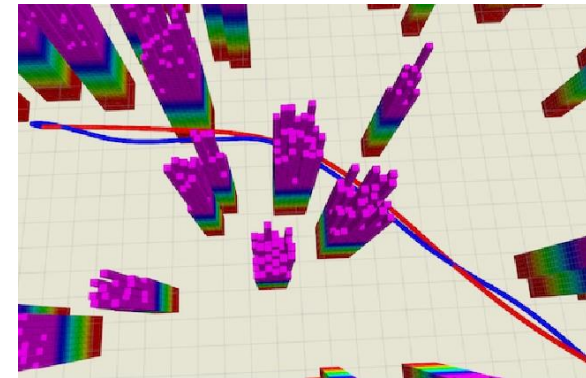
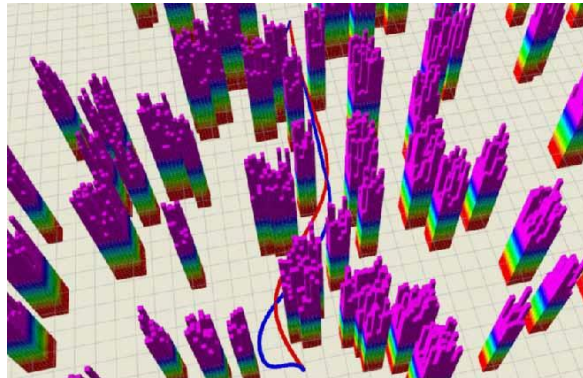
- Comparisons

a) Faster convergence



b) Higher trajectory quality

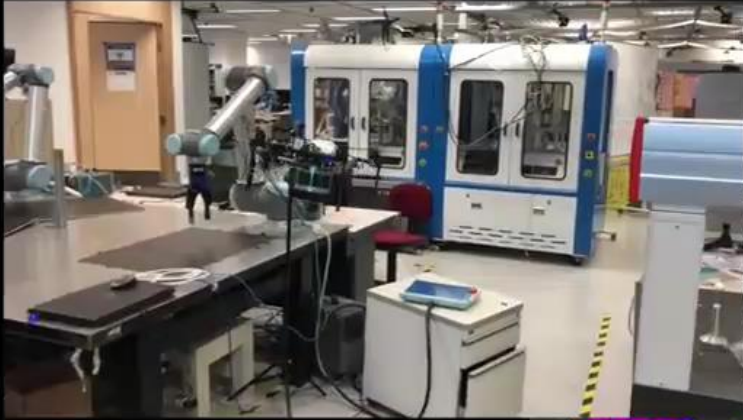
	Integral of Jerk <sup>2</sup> ( $m^2/s^5$ )			Comp. Time(s)
	Mean	Max	Std	
Previous [1]	43.913	181.495	18.394	0.010
Proposed	<b>35.932</b>	<b>131.913</b>	<b>13.118</b>	0.001



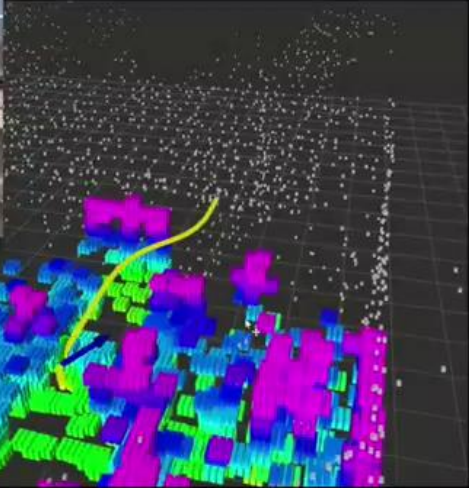
Trajectories generated by the proposed (red) and previous [1] (blue) method. Computation time for proposed method is 10 times shorter!



# Results



Average Speed: 1.27m/s  
Maximum Speed: 1.77m/s

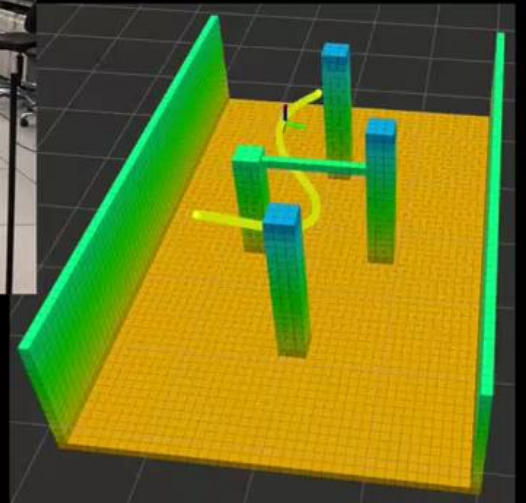


Autonomous  
Flight 1



The goal positions are set  
manually by the stick.

Play Speed: 2X





# Homework

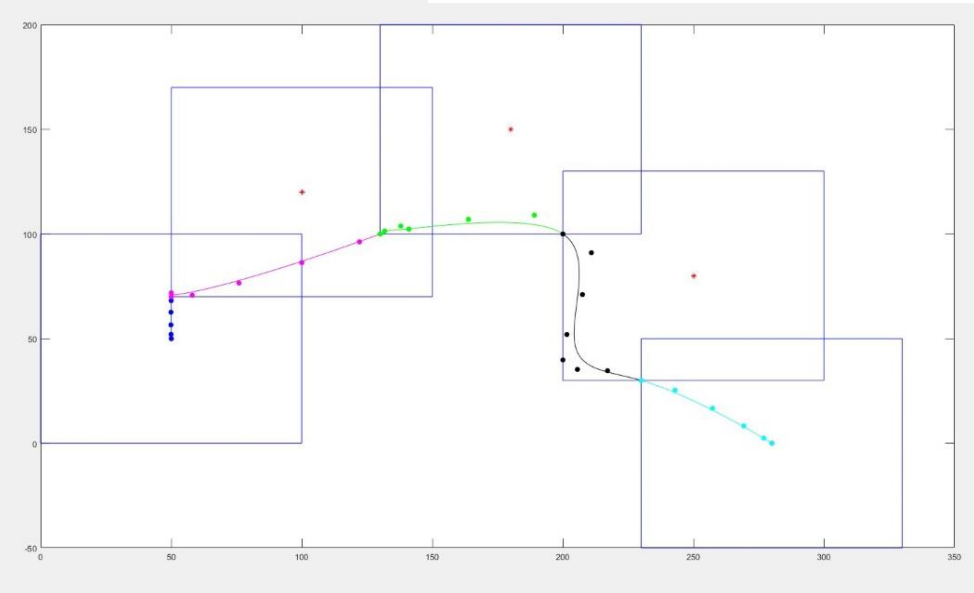
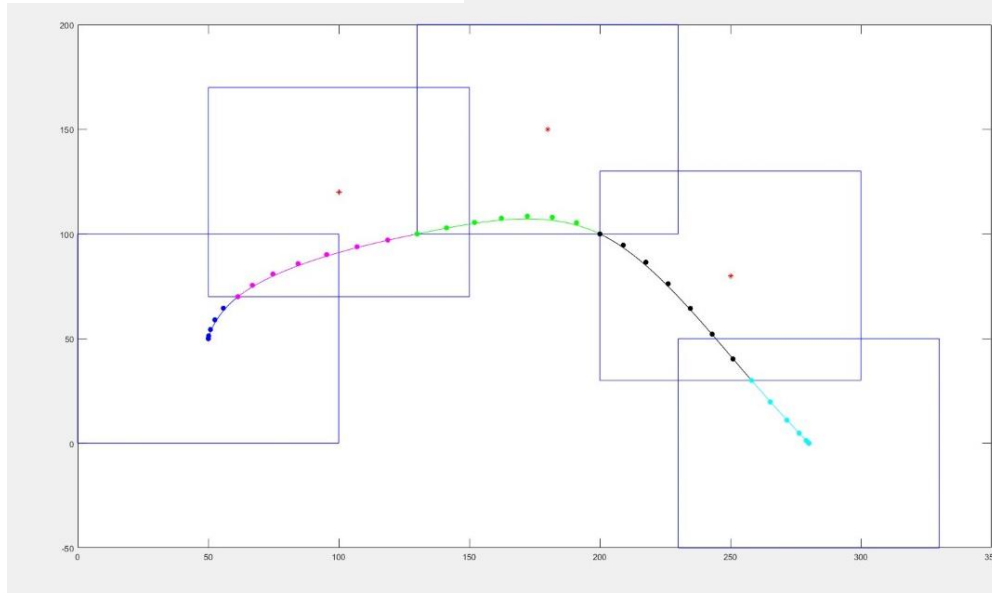
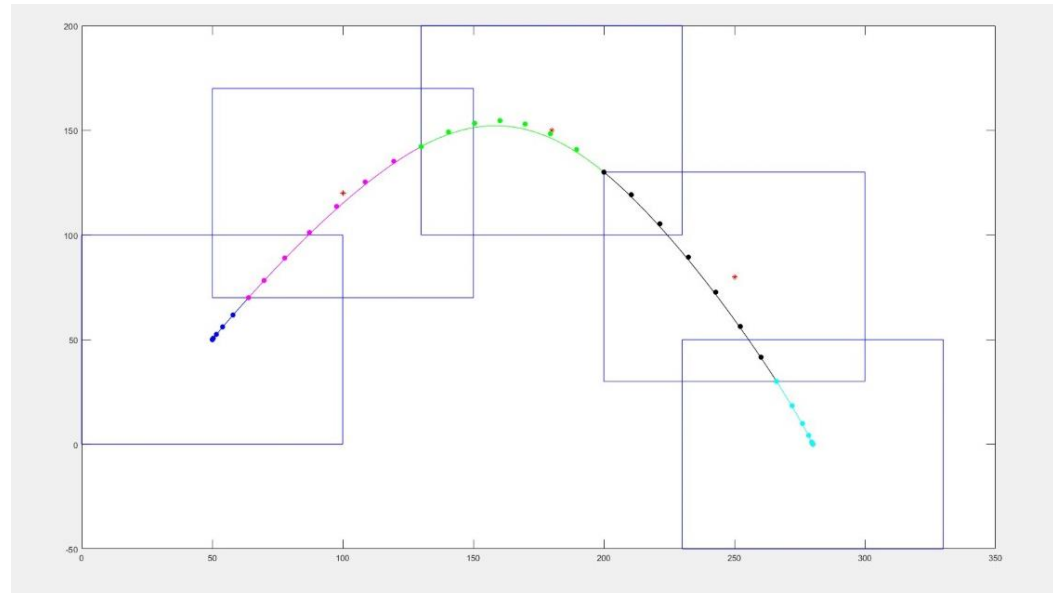


## Homework

- In matlab, write a corridor-constrained piecewise Bezier curve generation.
- The conversion between Bezier to monomial polynomial is given.
- The corridor is pre-defined.
- Only position needs to be constrained.
- TA provides a video tutorial.



# Homework





**Thanks for Listening!**