A3. Emergence of Time: Link · Loop Consistency

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Abstract

We develop a mathematically consistent framework in which time emerges not as a background parameter but as a gauge–invariant functional of local activity in an SU(2) lattice gauge geometry. Two complementary layers are introduced: link–based proper time derived from plaquette–averaged holonomy, and loop–based proper time depending only on the conjugation–invariant holonomy angle of closed contours. The Loop–Equivalence Axiom (ELT) is imposed, requiring that any two simple loops with identical holonomy angle exhibit the same proper–time, independent of shape or length. All definitions, domain conditions, and regularity constraints ensuring full mathematical consistency are stated explicitly. The continuum limit yields a dependence on $F_{\mu\nu}F^{\mu\nu}$, and a structural correspondence with Jacobi–type path–integral lapses is shown. Coordinate time (graph synchronisation) and physical dilation (gauge–invariant activity) become sharply distinguished, forming a coherent emergent–time mechanism.

1 Principle: Time as Emergent Activity

Time is treated as an emergent quantity determined by local activity. For any object (cell, link, loop) with activity–rate $\beta \in (-1, 1)$,

$$\gamma^{-1} = \sqrt{1-\beta^2},$$

and with $\beta = \tanh(\cdots)$ we guarantee $|\beta| < 1$, yielding

$$d\tau = t_p\sqrt{1-\beta^2}, \qquad 0 < d\tau \le t_p.$$

2 Links and Loops: Gauge–Invariant Geometry

Links. An oriented link $e = (i \to i + \hat{\mu})$ carries

$$U_\mu(i) = \mathbf{q}_{i+\hat{\mu}}\mathbf{q}_i^{-1} \in SU(2), \qquad U_\mu(i) \mapsto g_{i+\hat{\mu}}U_\mu(i)g_i^{-1}.$$

Loops. For a closed contour C,

$$W(C) = \prod_{e\in C} U_e, \qquad W(C) \mapsto g_{i_0}W(C)g_{i_0}^{-1}.$$

Define

$$\Theta_C = \arccos\Big(\tfrac{1}{2}\,\mathrm{Tr}\,W(C)\Big) \in [0, \pi].$$

Plaquettes. Minimal loops satisfy

$$U_\Box = W(C_\Box), \qquad \Theta_\Box = \Theta_{C_\Box}.$$

3 Two Levels of Activity: Link vs. Loop

3.1 Link–based Activity

Let $\mathcal{P}(e)$ be the set of plaquettes containing e. Assume

$$\omega_\Box > 0, \qquad \sum_{\Box \ni e} \omega_\Box > 0.$$

Define

$$\Theta_e^{(\mathrm{GI})} = \left(\frac{\sum_{\Box \ni e} \omega_\Box \, \Theta_\Box^2}{\sum_{\Box \ni e} \omega_\Box} \right)^{1/2}, \qquad \Theta_\Box = \arccos(\tfrac{1}{2} \mathrm{Tr}\, U_\Box).$$

Boundary links with $\mathcal{P}(e) = \varnothing$ take $\Theta_e^{(\mathrm{GI})} = 0$.

The activity and proper time are

$$\beta_e = \tanh\Big(\kappa_1 \tfrac{\Theta_e^{(\mathrm{GI})}}{\pi}\Big), \qquad d\tau_e = t_p \sqrt{1 - \beta_e^2}.$$

3.2 Loop–based Activity

For a closed loop C,

$$\beta_C = \tanh\Big(\kappa_{\mathrm{loop}} \tfrac{\Theta_C}{\pi}\Big), \qquad \kappa_{\mathrm{loop}} = \mathrm{const},$$

so the loop proper–time is

$$T_{\mathrm{loop}}(C) = t_p \sqrt{1 - \beta_C^2}.$$

4 Loop–Equivalence Axiom (ELT)

$$\boxed{\Theta_{C_1} = \Theta_{C_2} \implies T_{\mathrm{loop}}(C_1) = T_{\mathrm{loop}}(C_2)}$$

Since T_{loop} depends only on Θ_C and κ_{loop} is loop–independent, the axiom holds.

5 Coordinate Time vs. Proper Time

For a worldline γ,

$$\tau[\gamma] = \sum_{e \in \gamma} t_p \sqrt{1 - \beta_e^2}, \qquad \beta_e = \tanh\Big(\kappa_1 \tfrac{\Theta_e^{(\mathrm{GI})}}{\pi}\Big).$$

This differs from the intrinsic $T_{\mathrm{loop}}(C)$ of loop objects.

6 Cell and Block Effective Proper Time

$$\mathcal{A}_i^2 = c_F \Big(\tfrac{\|\mathcal{F}_i\|}{\Omega_F}\Big)^2 + c_R \Big(\tfrac{\|F_\star(i)\|}{\Omega_R}\Big)^2 + c_\Omega \Big(\tfrac{\|\nabla \mathbf{q}_i\|}{\Omega_\Omega}\Big)^2, \qquad \beta_i = \tanh(\kappa \mathcal{A}_i),$$

$$d\tau_i = t_p \sqrt{1 - \beta_i^2}.$$

Define the pressure weight

$$P_i = p_0 (1 - \alpha m_i) F_\star(i), \qquad F_\star(i) \geq 0, \quad 0 \leq \alpha m_i < 1.$$

For a block B,

$$\Delta T_{\mathrm{eff}}(B) = \frac{\sum_{i \in B} P_i \, d\tau_i}{\sum_{i \in B} P_i}, \qquad \sum_{i \in B} P_i > 0.$$

7 Local YM Coupling

$$\mathcal{L}_{\mathrm{YM}}(i) = \frac{1}{2g_{\mathrm{eff}}^2(i)} \mathrm{Tr}[\mathcal{F}_{\mu\nu}(i)\mathcal{F}^{\mu\nu}(i)],$$

$$\frac{1}{g_{\mathrm{eff}}^2(i)} = \frac{1}{g_0^2} + \tilde{\lambda}(1 - \alpha m_i)\, F_\star(i).$$

8 Branch Continuity

The SU(2) logarithm uses the principal branch $|\theta| \le \pi$, with nearest–branch tracking to avoid 2π jumps.

9 Continuum Limit

$$U_\square = \exp(ia^2 F_{\mu\nu} + O(a^3)), \qquad \Theta_\square \simeq a^2 \|F_{\mu\nu}\|.$$

Thus

$$T_{\mathrm{loop}}(C) \simeq t_p \Big[1 - \tfrac{1}{2} \big(\kappa_{\mathrm{loop}} \tfrac{\Theta_C}{\pi} \big)^2 \Big] ,$$

consistent with a dependence on $F_{\mu\nu}F^{\mu\nu}$.

10 Relation to Path–Integral Lapses

$$S_J[\gamma] = \int d\lambda\, \sqrt{2(E_{\mathrm{tot}} - U_{\mathrm{eff}})}\, \sqrt{\tfrac{ds_{\mathrm{conf}}^2}{d\lambda^2}},$$

$$d\tau_{\mathrm{eff}} = \frac{1}{\sqrt{2}} \frac{\sqrt{ds_{\mathrm{conf}}^2}}{\sqrt{E_{\mathrm{tot}} - U_{\mathrm{eff}}}}.$$

If ds_{conf}^2 and U_{eff} depend only on the angle Θ, then $d\tau_{\mathrm{eff}} = f(\Theta)$, the same class of gauge–invariant scalar lapse functions as

$$d\tau = t_p \sqrt{1 - \beta^2(\Theta)}.$$

11 Summary

Time along links is synchronised by t_p, while physical dilation is determined by gauge–invariant loop/cell activity. Loops with identical holonomy angle Θ share identical proper–time, independent of shape.