Spin, Electric Charge, and Color Charge on the FCC Lattice:
A Lattice Realization of the SU(2)–U(1)–SU(3) Gauge Hierarchy

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Abstract

We present a unified lattice framework in which spin, electric charge, and color charge emerge from the topology of the face–centered cubic (FCC) lattice. The coexistence of triangular and square minimal loops in the FCC skeleton provides the minimal structure supporting both SO(3) parity and U(1) phase. A quaternionic SU(2) field at each site encodes spin orientation and local phase; its internal U(1) projection produces quantized electric charge in units of $e/6$. On the same lattice, combinatorial classes of plaquette phase quadruples under the dihedral group D_4 define color charge as SU(3) weight–type invariants. The $\mathbb{Z}_{12}$ torsion of the FCC 2–skeleton fixes the common quantization scale $\pi/6$, ensuring coherence among spin, charge, and color. Baryon and meson color neutrality follow naturally from the closure rules of octahedral cells.

1. Geometric Background

FCC Lattice and 2–Skeleton. Let $G = (V, E)$ be the nearest–neighbor graph of the face–centered cubic lattice. We attach triangular and square minimal loops as 2–cells, forming the 2–skeleton

$$C_2 = \mathbb{Z}^F, \qquad C_1 = \mathbb{Z}^E, \qquad \partial_2 : C_2 \to C_1.$$

For each edge $e \in E$ assign a phase $\phi_e \in \mathbb{R}/2\pi\mathbb{Z}$. If $\Phi(\partial_2 f) = 0$ for all minimal faces f, then $\Phi \in Z^1(X; \mathbb{R}/2\pi\mathbb{Z})$.

Lattice Torsion. From the Smith normal form of the boundary matrix ∂_2, one obtains a local torsion element of order 12 in

$$A := C_1/\operatorname{im} \partial_2,$$

implying

$$\phi_e \in \frac{2\pi}{12}\mathbb{Z} \equiv \frac{\pi}{6}\mathbb{Z} \pmod{2\pi}.$$

Hence both electric and color charges will share the same quantization scale $\pi/6$.

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2. Spin: Quaternionic SU(2) Field

At each site $i \in V$, define

$$q_i = e^{i\phi_i \hat{n}_i \cdot \vec{\sigma}} = \cos\frac{\phi_i}{2} + i\sin\frac{\phi_i}{2}\,\hat{n}_i \cdot \vec{\sigma} \in SU(2),$$

with local axis $\hat{n}_i$ and phase ϕ_i.

2.1 Link and Wilson Loops

$$U_{ij} = q_i q_j^{-1} \in SU(2), \qquad W(\ell) = \prod_{(ij)\in\ell} U_{ij}.$$

For minimal loops, $W(\ell) \in \{\pm 1\} = Z(SU(2))$, recording the even/odd winding of spin-1/2. Mapping to $SO(3)$ identifies ± 1, giving a *projectively flat* configuration.

2.2 Octahedral Bianchi Constraint

For every octahedral cell $\mathcal{O}$,

$$\prod_{p \subset \partial\mathcal{O}} W(p) = +1,$$

which forbids monopole–like defects and allows only even central flux.

3. Electric Charge: U(1) Phase Projection

3.1 t'Hooft-Type Abelian Projection

Given a local axis m_i, define

$$u_{ij} = \frac{\mathrm{Tr}\left(\frac{1+m_i\cdot\sigma}{2}\,U_{ij}\right)}{\left|\mathrm{Tr}\left(\frac{1+m_i\cdot\sigma}{2}\,U_{ij}\right)\right|} = e^{ia_{ij}}, \qquad a_{ij} \in \mathbb{R}/2\pi\mathbb{Z}.$$

The loop sum satisfies $\sum_{(ij)\in\ell} a_{ij} = 2\pi n$.

3.2 Charge Quantization

Due to $\mathbb{Z}_{12}$ torsion, $a_{ij} \in (\pi/6)\mathbb{Z}$. Hence each tetrahedral ("quark cell") closure quantizes electric charge in units of $e/6$:

$$Q_i = \frac{e}{6}s_i, \qquad s_i = \mathrm{sign}\left(\mathrm{Tr}\left[q_i(\hat{n}_p \cdot \vec{\sigma})q_i^{-1}(\hat{m}_p \cdot \vec{\sigma})\right]\right),$$

$$Q(\mathcal{T}) = \frac{e}{6}\sum_{i\in\mathcal{T}} s_i \in \{-\tfrac{2}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}, \tfrac{2}{3}\}e.$$

4. Color Charge: Dihedral Class and SU(3) Embedding

4.1 Plaquette Phase Quadruples

For a square plaquette, let

$$(a, b, c, d) \in ((\pi/6)\mathbb{Z}/2\pi\mathbb{Z})^4, \qquad a + b + c + d \equiv 0 \pmod{2\pi}.$$

Write integer form $k_i = (6/\pi)a_i \in \mathbb{Z}_{12}$ with $k_1 + k_2 + k_3 + k_4 \equiv 0 \pmod{12}$.

4.2 Cyclic Words and D_4 Action

Define the cyclic word space

$$\mathcal{W} = \big\{[k_1, k_2, k_3, k_4] \mid k_i \in \mathbb{Z}_{12}, \ \textstyle\sum k_i \equiv 0\big\}/(\text{cyclic shift}).$$

The square dihedral group D_4 acts on $\mathcal{W}$ by rotation and reflection.

    Quark–Plaquette Assumption: Only plaquettes with all four k_i distinct are admissible (degenerate sets are colorless).

    Theorem. For distinct k_i, the number of orbits under D_4 is 3. By Burnside's lemma, since nontrivial elements fix no configurations, $|\mathcal{W}/D_4| = \frac{1}{8} \times 24 = 3$.

4.3 Color Charge Function

Define

$$\kappa: \ \mathcal{W}_{\mathrm{adm}}/D_4 \longrightarrow \{\pm r, \pm g, \pm b, 0\},$$

assigning the three orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ to $\{r, g, b\}$ by convention. Orientation reversal $\iota: [k_1, k_2, k_3, k_4] \mapsto [k_1, k_4, k_3, k_2]$ acts as $\kappa(\iota[w]) = -\kappa([w])$ (anticolor). If k_i are not distinct, $\kappa = 0$.

4.4 Closure Sector and Admissible Sets

Within the modular-12 closure sector $(k_1 + k_2 + k_3 + k_4 \equiv 0)$, the representative integer quadruples $(k_i \in \{-5, \ldots, 6\})$ number 42. Imposing 3D octahedral consistency requires $0 \in K = \{0, x, y, z\}$, leaving 14 equivalence classes after symmetries are factored out.

4.5 SU(3) Static Embedding

Associate $\{r, g, b\}$ with the fundamental weights $\{\omega_1, \omega_2, \omega_3\}$:

$$\omega_1 + \omega_2 + \omega_3 = 0, \qquad \alpha_1 = \omega_1 - \omega_2, \ \alpha_2 = \omega_2 - \omega_3, \ \alpha_3 = \alpha_1 + \alpha_2.$$

Orientation reversal corresponds to $(t_3, t_8) \mapsto -(t_3, t_8)$ or $\omega \mapsto -\omega$. When the 14 admissible (a, b, c) triples are projected onto the Cartan plane (T_3, T_8), their coordinates lie precisely on integer lattice points in the (ω_1, ω_2) basis.

5. Binding Rules and 3D Consistency

5.1 Meson

A plaquette and its reverse orientation combine as

$$\omega + (-\omega) = 0,$$

implying color neutrality.

5.2 Baryon (Octahedral Closure)

For three mutually orthogonal plaquettes (in XY, YZ, ZX planes) sharing edges to close an octahedron, the closure condition is

$$K = \{0, x, y, z\}, \qquad x + y + z \equiv 0 \pmod{12}.$$

Arranged as
$$XY : [0, x, y, z], \quad YZ : [0, y, z, x], \quad ZX : [0, z, x, y],$$

all eight triangular faces close simultaneously. When the three plaquette colors differ,

$$\omega_1 + \omega_2 + \omega_3 = 0,$$

forming a color singlet baryon.

6. Global Topological Sectors

On a 3-torus with noncontractible loops C_x, C_y, C_z,

$$W_x, W_y, W_z \in \{\pm 1\}, \qquad \mathcal{H}_{\text{global}} \cong H^2(T^3, \mathbb{Z}_2) \cong \mathbb{Z}_2^3.$$

These global Z_2 choices fix the background spin parity sector.

7. Hierarchical Summary

Level	Geometry	Gauge Group	Observable
Spin	triangular/square loops	$SU(2)$ (center Z_2)	$W(\ell) = \pm 1$
Charge	internal phase (Hopf fiber)	$U(1)$	$a_{ij} \in (\pi/6)\mathbb{Z}, e/6$
Color	plaquette cyclic orbit	$SU(3)$ (Cartan T^2)	$\kappa \in \{\pm r, \pm g, \pm b, 0\}$
Consistency	octahedral closure	Z_2 Bianchi	even flux, 14 reps

8. Logical Cohesion

1. Common quantum unit: $\pi/6$ for both charge and color.

2. Common antisymmetry: spin parity (Z_2) $\leftrightarrow$ color reversal (order 2).

3. Common closure rule: $\sum k_i \equiv 0 \pmod{12}$ governs charge loops and color plaquettes alike.

4. Observables are not traces but combinatorial orbit invariants—topological markers of degenerate sectors.

9. Physical Interpretation

- Fractional charges are pinned by lattice $\mathbb{Z}_{12}$ torsion, stabilized by the Z_2 Bianchi constraint.

- Color degrees of freedom survive as combinatorial invariants κ within degenerate energy layers.

- Meson and baryon color neutrality arise geometrically from plaquette and octahedral closure.

- Global Z_2^3 sectors may act as parity backgrounds controlling defect condensation.

10. Conclusion

The FCC lattice provides a natural discrete arena where the SU(2) spin layer, its internal U(1) phase, and the combinatorial SU(3) color structure share a single quantization scale and closure rule. Spin is *projectively flat*, charge arises from the internal U(1) projection, and color emerges from the D_4 cyclic orbits of $\mathbb{Z}_{12}$ plaquette phases. Baryon and meson neutrality follow from the same modular–12 closure. Thus, fractional charge and color confinement appear as geometric consequences of FCC lattice topology.