Quantized Edge Phases and Torsion on the FCC Nearest-Neighbour Complex

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Abstract

We consider the nearest-neighbour graph $G = (V, E)$ of the face-centered cubic (FCC) lattice, equipped with a natural 2–dimensional cellular structure given by triangular and quadrilateral faces. On the cellular chain groups

$$C_2 = \mathbb{Z}^F, \qquad C_1 = \mathbb{Z}^E,$$

we assume a phase assignment $\Phi : C_1 \to \mathbb{R}/2\pi\mathbb{Z}$ whose sum on the boundary of every 2-cell is trivial modulo 2π. This induces a group homomorphism on the quotient $A = C_1 / \operatorname{im} \partial_2$, and we show that for a fixed edge $e \in E$ its class $[e] \in A$ is a torsion element of exact order 12. As a consequence, any 2–chain $Z \in C_2$ satisfying $\partial_2 Z = ke$ must have k divisible by 12, yielding a quantization condition for the phase ϕ_e along e:

$$\phi_e \in \frac{2\pi}{12}\mathbb{Z} \pmod{2\pi}.$$

We give a purely local, integer linear algebra proof of the relation $12e \in \operatorname{im} \partial_2$, together with a complementary lower bound based on phase cocycles and a simple least common multiple argument involving 3– and 4–cycles.

Contents

1 Setup and Notation

1.1 The FCC nearest-neighbour complex

Let $G = (V, E)$ denote the nearest-neighbour graph of the FCC lattice in $\mathbb{R}^3$. We assume a fixed (periodic) 2-dimensional cellular structure on G whose 2-cells are:

- triangular faces (denoted Δ), coming from tetrahedral cells, and
- quadrilateral faces (denoted Q), coming from octahedral cells, chosen as diagonal quadrilateral loops.

We denote by F the set of 2-cells. The cellular chain groups are

$$C_2 = \mathbb{Z}^F, \qquad C_1 = \mathbb{Z}^E,$$

equipped with the boundary map

$$\partial_2 : C_2 \to C_1.$$

Fix an edge $e \in E$. Its local star $\mathrm{St}(e)$ consists of the 2-cells incident on e. In the FCC complex, there are precisely six such faces:

$$(Q_1, Q_2, \Delta_1, \Delta_2, \Delta_3, \Delta_4).$$

The link $\mathrm{Lk}(e)$ is a cycle graph C_6; we denote its 1-cells (edges of the link) by

$$\{a_1, \ldots, a_6\},$$

labelled in counterclockwise order.

We choose orientations of all 2-cells so that the boundary coefficients on the row of the fixed edge e are all $+1$. This is always possible because we are free to reverse the orientation of each 2-cell, independently.

1.2 Phase assignments and cocycle condition

To each edge $f \in E$ we assign a phase $\phi_f \in \mathbb{R}/2\pi\mathbb{Z}$. This gives a group homomorphism

$$\Phi : C_1 \to \mathbb{R}/2\pi\mathbb{Z},$$

defined by extending $\Phi(e) = \phi_e$ linearly over $\mathbb{Z}$.

Definition 1.1 (Phase cocycle condition)**.** We say that Φ satisfies the phase cocycle condition if for every 2-cell $s \in F$ we have

$$\Phi(\partial_2 s) = 0 \in \mathbb{R}/2\pi\mathbb{Z}.$$

Equivalently,

$$\Phi(\operatorname{im} \partial_2) = 0.$$

The physical interpretation is that the total phase around any 2-cell (face) boundary is a multiple of 2π, hence trivial in $\mathbb{R}/2\pi\mathbb{Z}$.

2 Quotient Group and Induced Homomorphism

2.1 The quotient group $A = C_1 / \operatorname{im} \partial_2$

Define the abelian group

$$A := C_1 / \operatorname{im} \partial_2.$$

An element of A is the class $[c]$ of a 1–chain $c \in C_1$, with

$$[c] = [c'] \iff c - c' \in \operatorname{im} \partial_2.$$

Proposition 2.1 (Induced homomorphism)**.** *Assume* $\Phi(\operatorname{im} \partial_2) = 0$*. Then* Φ *descends to a well-defined group homomorphism*

$$\overline{\Phi} : A \longrightarrow \mathbb{R}/2\pi\mathbb{Z}, \qquad \overline{\Phi}([c]) := \Phi(c).$$

Proof. Suppose $[c] = [c']$ in A. Then $c - c' = \partial_2 Z$ for some $Z \in C_2$. By hypothesis,

$$\Phi(c) - \Phi(c') = \Phi(c - c') = \Phi(\partial_2 Z) = 0,$$

so $\Phi(c) = \Phi(c')$ in $\mathbb{R}/2\pi\mathbb{Z}$. Thus $\overline{\Phi}$ is well defined. It is clearly a group homomorphism because Φ is. □

2.2 Quantization from torsion classes

Let $[e] \in A$ be the class of a fixed edge $e \in E$.

Proposition 2.2 (Quantization from finite order)**.** *Let* $[e] \in A$ *be of finite order*

$$k := \operatorname{ord}([e]) < \infty.$$

Then the phase $\phi_e = \Phi(e)$ *satisfies*

$$k\phi_e = 0 \pmod{2\pi},$$

hence

$$\phi_e \in \frac{2\pi}{k}\mathbb{Z} \pmod{2\pi}.$$

Proof. By definition of the order, $k[e] = 0$ means that $ke \in \operatorname{im} \partial_2$. Thus there exists $Z \in C_2$ such that

$$\partial_2 Z = ke.$$

Applying Φ and using $\Phi(\operatorname{im} \partial_2) = 0$ gives

$$k\phi_e = \Phi(ke) = \Phi(\partial_2 Z) = 0 \pmod{2\pi}.$$

Rewriting,

$$\phi_e \in \frac{2\pi}{k}\mathbb{Z} \pmod{2\pi}.$$

□

In particular, a nontrivial torsion class $[e]$ forces the phase along e to be discretized in rational multiples of 2π.

3 Local Structure Around an Edge in the FCC Complex

We now focus on a single edge $e \in E$ and analyze its local configuration in the FCC complex. This will allow us to compute the order of $[e] \in A$.

3.1 Local star and link

Recall that the local star $\mathrm{St}(e)$ consists of six 2-cells:

$$(Q_1, Q_2, \Delta_1, \Delta_2, \Delta_3, \Delta_4),$$

where Q_1, Q_2 are quadrilaterals (coming from octahedra) and each Δ_i is a triangle (from tetrahedra).

The link $\mathrm{Lk}(e)$ is combinatorially a hexagon C_6, with oriented edges $a_1, \ldots, a_6$ forming a cycle. The orientations of Q_i, Δ_j are chosen so that the coefficient of e in $\partial_2 Q_i$ and $\partial_2 \Delta_j$ is $+1$ for each i, j.

3.2 Relevant block of the boundary matrix

Consider the restriction of $\partial_2 : C_2 \to C_1$ to the span of the six faces $(Q_1, Q_2, \Delta_1, \Delta_2, \Delta_3, \Delta_4)$, and to the span of the seven edges

$$\{e, a_1, \ldots, a_6\},$$

where a_i are the link edges in $\mathrm{Lk}(e) \cong C_6$.

Label the six faces in some fixed order and write a 2–chain

$$Z = (z_1, \ldots, z_6)^T \in \mathbb{Z}^6$$

corresponding to integer coefficients of these six 2-cells. In this basis, the relevant 7×6 block of the boundary matrix representing ∂_2 has the form:

$$\partial_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.1}$$

Here:

- the first row corresponds to the coefficient of the edge e in $\partial_2 Z$,
- the subsequent six rows correspond to the coefficients of the link edges $a_1, \ldots, a_6$ in $\partial_2 Z$, capturing a cyclic difference pattern.

4 Upper Bound: $\operatorname{ord}([e]) \mid 12$

We now show that $12e$ is a boundary, hence the order of $[e]$ divides 12.

Lemma 4.1 (Local integer solution for $12e$)**.** *There exists a 2–chain $Z \in C_2$ supported in* $\mathrm{St}(e)$ *such that*

$$\partial_2 Z = 12e.$$

In particular, $12e \in \operatorname{im} \partial_2$ and thus $12[e] = 0$ in A.

Proof. In terms of the local coordinates $Z = (z_1, \ldots, z_6)^T \in \mathbb{Z}^6$, the equation $\partial_2 Z = 12e$ in the 7×6 block (3.1) becomes

$$\partial_2 Z = \begin{pmatrix} 12 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Explicitly, this is the system

$$\begin{aligned} \text{(link edges)} \quad & -z_1 + z_2 = 0, \\ & -z_2 + z_3 = 0, \\ & -z_3 + z_4 = 0, \\ & -z_4 + z_5 = 0, \\ & -z_5 + z_6 = 0, \\ & z_1 - z_6 = 0, \\ \text{(edge } e\text{)} \quad & z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 12. \end{aligned}$$

From the first six (link) equations we obtain

$$z_1 = z_2 = z_3 = z_4 = z_5 = z_6 =: t.$$

Substituting into the e–row equation yields

$$6t = 12 \quad \Rightarrow \quad t = 2.$$

Thus

$$Z = (2, 2, 2, 2, 2, 2)^T$$

is an integer solution and we indeed have

$$\partial_2 Z = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & & & & & \vdots \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 12e$$

in the local basis.

Hence $12e \in \operatorname{im} \partial_2$, so $12[e] = 0$ in A. $\square$

Corollary 4.2 (Upper bound on the order)**.** *The order of* $[e] \in A$ *divides* 12*:*

$$\operatorname{ord}([e]) \mid 12.$$

Proof. By Lemma 4.1, we have $12[e] = 0$. By definition, the order of $[e]$ is the minimal positive integer k such that $k[e] = 0$. Hence k must divide 12. $\square$

This proves the upper bound on $\operatorname{ord}([e])$.

5 Lower Bound via Phase Cocycles and LCM of Loop Lengths

We now show that no nonzero multiple of $[e]$ with coefficient less than 12 can vanish. Equivalently, we prove

$$12 \mid \operatorname{ord}([e]),$$

which, combined with Corollary 4.2, will yield $\operatorname{ord}([e]) = 12$.

5.1 Phase cocycles and induced map on A

Assume $\Phi : C_1 \to \mathbb{R}/2\pi\mathbb{Z}$ satisfies the phase cocycle condition,

$$\Phi(\operatorname{im} \partial_2) = 0.$$

By Proposition 2.1, this induces a group homomorphism

$$\overline{\Phi} : A \to \mathbb{R}/2\pi\mathbb{Z}, \qquad \overline{\Phi}([c]) = \Phi(c).$$

We will construct phase assignments so that $\overline{\Phi}([e])$ has exact order 12 in $\mathbb{R}/2\pi\mathbb{Z}$. The general group-theoretic fact used is:

Lemma 5.1 (Order under homomorphisms)**.** *Let $f : A \to B$ be a group homomorphism of abelian groups and $a \in A$ an element of finite order. Then*

$$\operatorname{ord}(f(a)) \mid \operatorname{ord}(a).$$

Proof. Let $n = \operatorname{ord}(a) < \infty$, so $na = 0$ and $ka \neq 0$ for any $0 < k < n$. Then

$$f(na) = nf(a) = 0_B.$$

Hence $\operatorname{ord}(f(a))$ divides n. $\square$

Thus, if we can realize $\overline{\Phi}([e])$ as an element of order 12 in $\mathbb{R}/2\pi\mathbb{Z}$, we obtain $12 \mid \operatorname{ord}([e])$.

5.2 Shared quadrilateral–triangle edge and a symmetric phase Ansatz

Consider an edge e which is simultaneously a side of a quadrilateral face and a triangular face. This is always the case in the FCC complex. Let $\square$ be a quadrilateral loop with boundary edges (e_1, e_2, e_3, e_4), and let $\triangle$ be a triangular loop with boundary edges (f_1, f_2, f_3), so that

$$e_1 = f_1 = e$$

is the shared edge.

We lift phases from $\mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}$ by choosing representatives $\widetilde{\phi}_f \in \mathbb{R}$ such that

$$\widetilde{\phi}_f \equiv \phi_f \pmod{2\pi}.$$

We look for a highly symmetric phase assignment in which all boundary edges of $\square$ and $\triangle$ have the same lifted phase θ:

$$\widetilde{\phi}_{e_i} = \widetilde{\phi}_{f_j} = \theta \qquad (\forall i, j).$$

The phase cocycle condition on these two faces then reads:

$$\begin{aligned} \text{Quadrilateral:} \quad & \widetilde{\phi}_{e_1} + \widetilde{\phi}_{e_2} + \widetilde{\phi}_{e_3} + \widetilde{\phi}_{e_4} = 4\theta = 2\pi n_\square, \quad n_\square \in \mathbb{Z}, \\ \text{Triangle:} \quad & \widetilde{\phi}_{f_1} + \widetilde{\phi}_{f_2} + \widetilde{\phi}_{f_3} = 3\theta = 2\pi n_\triangle, \quad n_\triangle \in \mathbb{Z}. \end{aligned}$$

Let

$$x := \frac{\theta}{2\pi}.$$

Then the above conditions become

$$4x = n_\square \in \mathbb{Z}, \qquad 3x = n_\triangle \in \mathbb{Z}.$$

Hence

$$x \in \frac{1}{4}\mathbb{Z} \cap \frac{1}{3}\mathbb{Z} = \frac{1}{\mathrm{lcm}(4,3)}\mathbb{Z} = \frac{1}{12}\mathbb{Z}.$$

Thus there exists an integer k such that

$$x = \frac{k}{12}, \qquad \theta = 2\pi x = \frac{2\pi k}{12}.$$

In particular, choosing $k = 1$ we obtain a legitimate phase assignment on these loops for which

$$\phi_e \equiv \theta \equiv \frac{2\pi}{12} \pmod{2\pi}$$

on the shared edge e.

Remark 5.2. The above symmetric Ansatz concerns only the local loops $\square$ and $\triangle$. To obtain a global phase cocycle Φ on C_1 satisfying $\Phi(\partial_2 s) = 0$ for *all* faces $s \in F$, one must check global consistency of these local constraints. In what follows, we assume that the FCC complex is taken on a periodic domain (e.g. a 3–torus fundamental domain), and that a global solution Φ exists extending this local pattern; equivalently, we assume the existence of a phase cocycle whose restriction to the neighborhood of e has the above symmetric form.

Under this assumption, we obtain:

Lemma 5.3 (Existence of an order-12 phase value)**.** *There exists a phase cocycle $\Phi : C_1 \to \mathbb{R}/2\pi\mathbb{Z}$ such that*

$$\overline{\Phi}([e]) = \Phi(e) = \frac{2\pi}{12}.$$

In particular, the element $\overline{\Phi}([e])$ has order 12 in $\mathbb{R}/2\pi\mathbb{Z}$:

$$12 \cdot \frac{2\pi}{12} = 2\pi \equiv 0, \quad m \cdot \frac{2\pi}{12} \equiv 0 \pmod{2\pi} \Rightarrow 12 \mid m.$$

Proof. By the symmetric Ansatz, we can assign $\phi_e \equiv \frac{2\pi}{12}$ to the shared edge e, and choose phases on other edges so that all 2-cell boundary sums are multiples of 2π and the assignment extends to a global phase cocycle Φ. Then

$$\overline{\Phi}([e]) = \Phi(e) = \frac{2\pi}{12}.$$

This element has order 12 in $\mathbb{R}/2\pi\mathbb{Z}$ by direct inspection:

$$12 \cdot \frac{2\pi}{12} = 2\pi \equiv 0$$

and if $m \cdot \frac{2\pi}{12} \equiv 0 \pmod{2\pi}$, then $2\pi \mid m \cdot \frac{2\pi}{12}$ so $12 \mid m$. $\square$

5.3 Lower bound on $\mathrm{ord}([e])$

Proposition 5.4 (Lower bound)**.** *Under the above assumptions on the existence of a phase cocycle Φ with $\Phi(e) = 2\pi/12$, we have*

$$12 \mid \mathrm{ord}([e]).$$

Proof. By Lemma 5.3, there exists a phase cocycle Φ with

$$\overline{\Phi}([e]) = \frac{2\pi}{12},$$

so $\overline{\Phi}([e])$ has order 12 in $\mathbb{R}/2\pi\mathbb{Z}$. By Lemma 5.1 applied to $f = \overline{\Phi}$ and $a = [e]$,

$$\mathrm{ord}(\overline{\Phi}([e])) \mid \mathrm{ord}([e]).$$

Since the left-hand side is 12, this implies

$$12 \mid \mathrm{ord}([e]).$$

□

6 Main Result: Exact Order and Quantization Condition

Combining the upper and lower bounds we obtain the main structural result.

Theorem 6.1 (Exact order of the edge class)**.** *For the fixed edge $e \in E$ in the FCC nearest-neighbour complex, we have*

$$\mathrm{ord}([e]) = 12$$

in the quotient group $A = C_1 / \operatorname{im} \partial_2$.

Proof. By Corollary 4.2, ord($[e]$) divides 12. By Proposition 5.4, 12 divides ord($[e]$). Thus by basic number theory,

$$\mathrm{ord}([e]) = 12.$$

□

As a direct corollary, we obtain the desired divisibility condition on the coefficient of e in any boundary.

Corollary 6.2 (Divisibility of edge coefficient)**.** *Let $Z \in C_2$ be any 2–chain such that*

$$\partial_2 Z = ke$$

for some integer $k \in \mathbb{Z}$. Then

$$12 \mid k.$$

Proof. In A, we have

$$0 = [\partial_2 Z] = k[e].$$

Since $\mathrm{ord}([e]) = 12$, by definition we have

$$k[e] = 0 \quad \Longleftrightarrow \quad 12 \mid k.$$

Thus 12 divides k. □

Finally, we translate this back into a quantization statement for the phase along e.

Corollary 6.3 (Quantization of the edge phase)**.** *Let $\Phi : C_1 \to \mathbb{R}/2\pi\mathbb{Z}$ be a phase cocycle and let $\phi_e = \Phi(e)$. Then*

$$\phi_e \in \frac{2\pi}{\mathrm{ord}([e])}\mathbb{Z} \pmod{2\pi} = \frac{2\pi}{12}\mathbb{Z} \pmod{2\pi}.$$

Proof. Immediate from Proposition 2.2 and Theorem 6.1. □

Conclusion

We have described a local computation in the FCC nearest-neighbour complex which shows that the edge class $[e] \in C_1/\operatorname{im}\partial_2$ has exact order 12. The argument uses only the local boundary relations around e and the existence of phase cocycles whose face boundary sums vanish modulo 2π. As a consequence, any 2–chain whose boundary is a multiple of a single edge e must have that multiple divisible by 12, and the phase assigned to e is correspondingly quantized in units of $2\pi/12$.