Counting Distinct Plaquette Phase Configurations under Dihedral Symmetry

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Abstract

We consider a square plaquette whose four directed links carry distinct quantized phases (a, b, c, d). When rotations and reflections of the plaquette are regarded as symmetry operations, two assignments related by such a symmetry are considered equivalent. By applying Burnside's lemma to the dihedral group D_4, we compute the number of inequivalent configurations. The result shows that there exist precisely three distinct configurations, consistent with the classical formula $(n-1)!/2$ for $n = 4$. This simple enumeration highlights the combinatorial structure of discrete gauge-phase arrangements on a symmetric lattice plaquette.

1 Introduction

In lattice models of gauge theory and condensed matter systems, a plaquette often serves as the minimal closed loop through which link phases or fluxes are defined [3]. When link variables are quantized, their relative order around the plaquette becomes physically meaningful only up to the symmetries of the square. Counting such equivalence classes of assignments reveals how many genuinely distinct local configurations exist under lattice symmetry.

This paper derives that count for the simplest case: a plaquette with four distinct link phases (a, b, c, d) and the full dihedral symmetry group D_4 acting on it. Despite the simplicity, the problem directly parallels the classification of discrete flux patterns or phase windings under rotational and mirror symmetries [2].

2 Mathematical Setting

Let

$$X = \{\, (x_1, x_2, x_3, x_4) \mid (x_1, x_2, x_3, x_4) \text{ is a permutation of } (a, b, c, d) \,\},$$

so that $|X| = 4! = 24$.

The dihedral group of the square,

$$D_4 = \{e, r_{90}, r_{180}, r_{270}, s_x, s_y, s_d, s_{d'}\},$$

acts on X by permuting indices according to the square's rotations and reflections [4]. Two configurations are *equivalent* if they lie in the same orbit of this action.

The goal is to determine the number of distinct orbits $|X/D_4|$.

3 Burnside's Lemma

For a finite group G acting on a finite set X,

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g), \tag{1}$$

where $\mathrm{Fix}(g)$ denotes the number of elements of X fixed by the group element g [1].

Here, $|G| = |D_4| = 8$ and $|X| = 24$.

4 Fixed Configurations

Element g	Description of action	$\mathrm{Fix}(g)$
e	Identity	24
r_{90}, r_{270}	4-cycle permutations	0
r_{180}	Swaps opposite links $(1\,3)(2\,4)$	0
$s_x, s_y, s_d, s_{d'}$	Reflections swapping link pairs	0

Since all phases are distinct, no nontrivial symmetry can fix a configuration. Substituting into (1) gives

$$|X/D_4| = \frac{1}{8}(24) = 3.$$

5 Result and Interpretation

Theorem 1. *Let four distinct phases (a, b, c, d) be assigned to the four directed links of a square plaquette, and identify configurations related by the dihedral group D_4. Then the number of inequivalent configurations is*

$$N = 3.$$

This coincides with the classical combinatorial expression for cyclic arrangements of n distinct symbols under reflection:

$$N = \frac{(n-1)!}{2}, \qquad (n = 4).$$

Physically, the three equivalence classes correspond to the three distinct cyclic orderings in which four distinct quantized link phases can circulate around the plaquette, consistent with rotational and mirror symmetry.

6 Discussion

This enumeration illustrates how symmetry drastically reduces the apparent combinatorial complexity of local gauge configurations. Although 24 raw permutations exist, the square's eight symmetry operations identify them into only three types. Such reductions underpin the combinatorial basis of symmetry-protected flux quantization and discrete gauge structures.

In general, for n distinct directed link variables on an n-gon with dihedral symmetry D_n, the number of inequivalent configurations follows the same pattern:

$$N(n) = \frac{(n-1)!}{2},$$

a result that smoothly generalizes the current derivation and mirrors Pólya's enumeration framework [1].

7 Conclusion

Applying Burnside's lemma to the dihedral group D_4, we established that four distinct link phases on a square plaquette produce exactly three inequivalent configurations under full symmetry. This provides a compact combinatorial foundation for classifying local phase arrangements in discrete lattice models and related gauge systems.

References

References

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