Available online at www.sciencedirect.com
ScienceDirect
Economics Letters 98 (2008) 129 – 135
economics letters
www.elsevier.com/locate/econbase

Bounded memory, overparameterized forecast rules, and instability ☆

Christophre Georges *

Department of Economics, Hamilton College, Clinton, NY 13323, USA

Received 15 September 2006; received in revised form 30 January 2007; accepted 4 April 2007
Available online 27 August 2007

Abstract

We consider an environment in which traders with finite memory update their forecast rules at random intervals by OLS. In this context, overparameterization of the forecast rules can destabilize the learning dynamics. This instability tends to be attenuated by greater memory and less frequent updating.

Keywords: Learning; Expectations; Agent-based modeling

JEL classification: D83; D84; E44

1. Introduction

We report on the behavior of artificial agents who base their trading decisions on forecast rules which they fit to recent data by OLS. The rules are overparameterized relative to the fundamentals of the market. Further, memory is limited, and agents face random wait times between updates.

Simulations indicate that if traders use minimum state variable forecasting rules, price dynamics conform to a noisy version of the fundamental rational expectations equilibrium as predicted by Honkapohja and Mitra (2003). However, when traders fit overparameterized forecast rules to the data, the learning dynamics can become unstable. This instability tends to increase in the updating rate and

☆ I thank Seppo Honkapohja and an anonymous referee for helpful comments. Computational resources were provided by NSF grants CHE-0116435 and CHE-0457275.
* Tel.: +1 315 859 4472; fax: +1 315 859 4477.
E-mail address: cgeorges@hamilton.edu.
URL: http://academics.hamilton.edu/economics/cgeorges/.

0165-1765/$ - see front matter
doi:10.1016/j.econlet.2007.04.023

decrease in the memory of the traders. For nonlinear misspecification, the instability can persist even with relatively large memories. Thus, the correspondence between the learning equilibrium and the fundamental rational expectations equilibrium is fragile.

2. Background

The stationary rational expectations equilibrium of the model below is strongly expectationally stable in the sense of Evans and Honkapohja (2001). Consequently, we would expect to see convergence to that REE locally under uninterrupted least squares learning by a representative agent with unbounded memory who employs either a minimum state variable (MSV) forecast rule or a linearly overparameterized forecast rule. Honkapohja and Mitra (2003) consider least squares learning by a representative agent with finite memory in a context similar to the one studied here. They assume an MSV forecast rule and are able to show that, while bounded memory causes learning to be incomplete, forecasts are nonetheless asymptotically unbiased. There is however no similar analytical result for overparameterized forecast rules in the case of bounded memory.

The traders considered below have bounded memory, overparameterized forecast rules and heterogeneous expectations. The behavior of the model is studied by means of simulation. This exercise falls under the rubric of agent-based computational economics. See the volume by Tesfatsion and Judd (2006) (and particularly the chapter by LeBaron (2006)) for relevant surveys of this literature. Georges (2006) provides a complementary analysis for an artificial currency market with learning via an evolutionary algorithm.

3. A simple market environment

There are two assets, a stock that pays a stochastic dividend d_t in each period t, and a bond with a fixed rate of return r in each period. Dividends are given by $d_t = \bar{d} + \varepsilon_t$, where $\bar{d}$ is constant and the ε_t are iid with zero mean and finite variance. Denote the price of the stock in period t, P_t.

In each period t, each trader i constructs a forecast $F_t^i\,[P_{t+1} + d_{t+1}]$ of the price plus dividend of the stock in the following period. We will assume the market clearing price in period t is the price that equalizes the forecasted returns on the two assets for an average trader. I.e., the price of the stock in period t satisfies

$$P_t = \frac{\bar{F}_t[P_{t+1} + d_{t+1}]}{1 + r} \tag{1}$$

where $\bar{F}_t\,[P_{t+1} + d_{t+1}]$ is the forecast of a representative agent (to be defined below).[1]

4. Rational expectations benchmark

There is a unique stationary rational expectations equilibrium in this simple model: $P_t = P^* \;\; \forall\, t$, where $P^* \equiv \frac{\bar{d}}{r}$.[2] Thus, given the dividend process assumed above, any volatility in price represents a deviation from the stationary REE.

[1] This assumption provides us a very simple equilibrium condition to study the effect of forecast rule specification and updating and it allows easy comparison with benchmark models in the literature. Georges (2005) examines a similar model under an explicit trading protocol.

[2] There are also rational bubble and sunspot equilibria, but the stationary REE is unique.

It will be useful to define $x_t \equiv P_t + d_t$ and $\tilde{x}_t \equiv x_t - x^*$, where $x^* \equiv P^* + \bar{d} = \frac{1+r}{r} \cdot \bar{d}$, and note that the stationary REE can be expressed as $x_t = x^* + \varepsilon_t \ \forall t$, or equivalently $\tilde{x}_t = \varepsilon_t \ \forall t$.

5. Forecast rules

We suppose that, due to uncertainty about the structure of the dividend process and/or the other agents' beliefs, traders form their forecasts of future prices inductively. Specifically, all agents are technical traders who use forecast rules with a common functional form

$$F_t^i[\tilde{x}_{t+1}] = a_{0t}^i + a_{1t}^i \tilde{x}_t + a_{2t}^i \cdot \tilde{x}_{t-1} + a_{3t}^i \cdot \tilde{x}_{t-1}^2 + a_{4t}^i \cdot \tilde{x}_{t-2} + a_{5t}^i \cdot \tilde{x}_{t-2}^2 + a_{6t}^i \cdot \tilde{x}_{t-1} \cdot \tilde{x}_{t-2} \tag{2}$$

where $a_{0t}^i, \ldots, a_{6t}^i$ are scalars that can vary across agents i and time t.[3,4] Functional form (2) is arbitrary but simple and follows the spirit of Grandmont's (1998) "uncertainty principle" as well as Gigerenzer and Selten's (2001) "fast and frugal heuristics."

We will take the average forecast $\bar{F}$ used in equilibrium condition (1) to be the forecast (2) using the algebraic averages ($\bar{a}_{0t}$, $\bar{a}_{1t}$,...) of the traders' rule parameters. Note that the stationary REE forecast rule is given by $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6) = (0,0,0,0,0,0,0)$.

6. Forecast rule updating

At the start of each period t, each agent updates her forecast rule parameters with common probability pupdate. If agent i updates her rule in t, she chooses the rule that minimizes the sum of squared forecast errors over the preceeding M periods (where M stands for memory). I.e., the new rule parameters a_{jt}^i minimize

$$\sum_{k=1}^{M} (x_{t-k} - F[x_{t-k}])^2. \tag{3}$$

Thus, agents learn using a finite memory least squares learning algorithm. Note that, while not all agents will update their rules in a given period t, each agent who is selected to update her rule at t will select the same new rule.

7. Simulations

First consider the case in which all agents use minimum state variable (McCallum, 1983) forecast rules $F_t^i[\tilde{x}_{t+1}] = a_0^i$. This is the general rule (2) with $a_1 = a_2 = \ldots = a_6 = 0$ and is the most parsimonious form consistent with an REE of the model. If for all agents $a_0^i = 0$, then the market will be at the stationary REE.

[3] The results do not change significantly if forecasts are conditioned on x or on deviations $(x - x^*)$ from the stationary REE expected value as in (2).

[4] Assuming that x_t is not in traders' information sets at time t, traders form iterated forecasts of x_{t+1} by first forecasting the current period's value x_t using the observed values from the preceeding three periods.

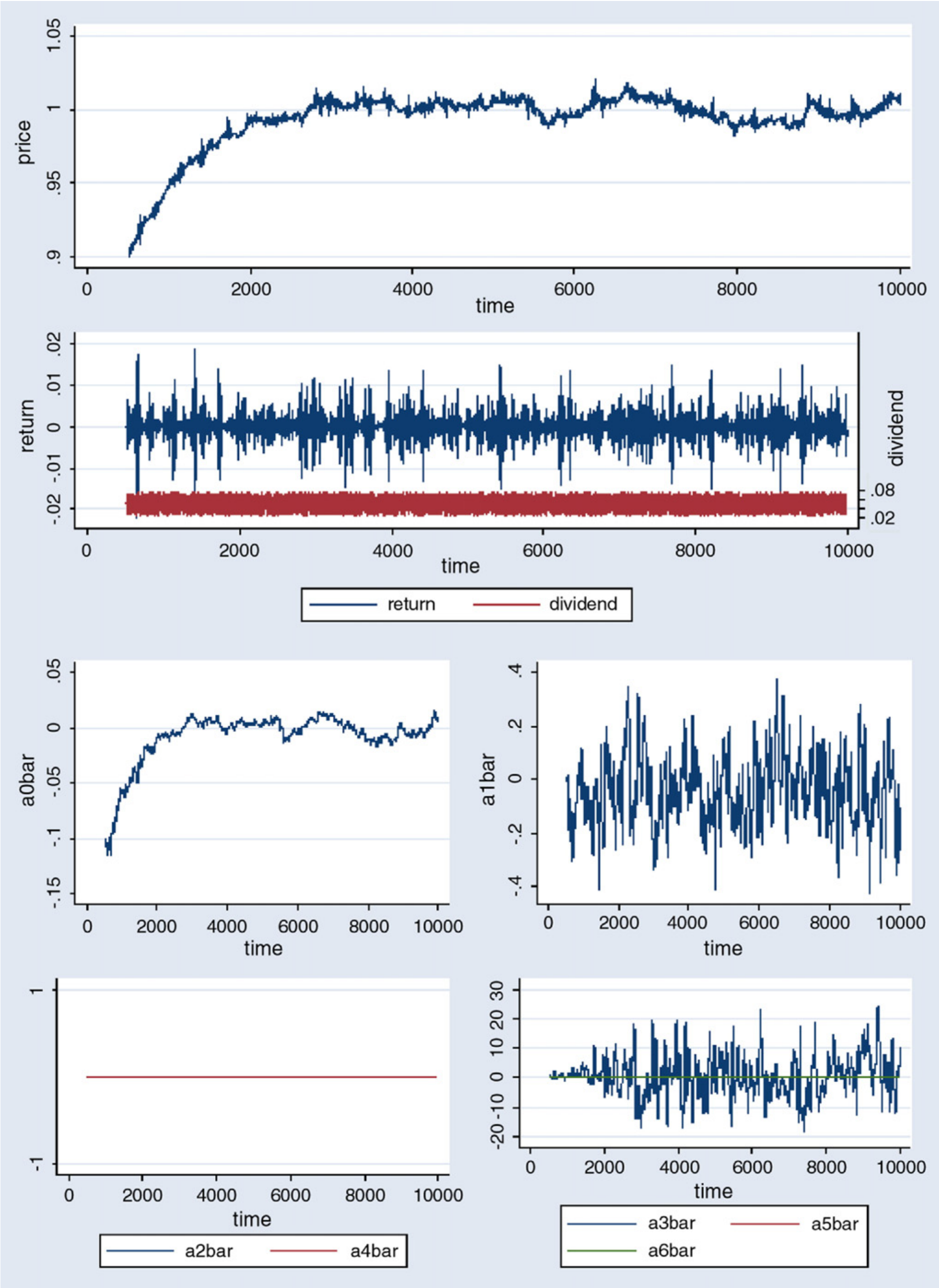

Fig. 1. Run with memory $M=50$ and pupdate$=0.1$ Forecast rules are of nonlinear AR(2) form: $F_t^i[\tilde{x}_{t+1}]=a_{0t}^i+a_{1t}^i\tilde{x}_t+a_{3t}^i\cdot\tilde{x}_{t-1}^2$, which is (2) subject to $a_2^i=a_4^i=a_5^i=a_6^i=0$. $\overline{d}=0.05$ and $r=0.05$, and so $P^*=1$, and $x^*=1.05$. Dividend shocks are uniformly distributed on $(-0.025,0.025)$ and occur with probability 0.8 in any period. Shown here are price, returns (log price differences), and average rule parameters $(\bar{a}_0,\ldots,\bar{a}_6)$ for 10,000 rounds of a representative run. The corresponding stationary REE rule is (0,0,0,0,0,0,0). In this run, price is initially set to 0.9 and agents' initial rule parameters are $(-0.1,0,0,0,0,0,0)$.

Simulations confirm the prediction of Honkapohja and Mitra (2003). There are persistent dynamics in the agents' forecast rule parameters a_0^i, and thus in the price level P_t, due to the finite memory and ongoing dividend shocks. However, the a_0^i dynamics become centered on 0 and the price level dynamics become centered on P^*. Thus, forecasts appear to be asymptotically unbiased, and price dynamics converge to a noisy version of the stationary REE.

As we move to overparameterized forecast rules, we see two phenomena emerge. First, there is still a general attraction to the stationary REE as in MSV case, in the sense that we often see the forecast rule parameter values drawn to fluctuate around (0,0,0,0,0,0,0) and the price level to fluctuate around P^*. However, the degree of excess price volatility is heightened as is the intensity of volatility clustering and excess kurtosis in returns. The latter are signature features of empirical financial market data (e.g., Lux and Ausloos (2002)). An illustration is provided in Fig. 1.

Second, we now also see the occasional emergence of explosive bubbles and crashes — i.e., states in which the learning dynamics near the stationary REE become unstable.[5,6]

Tables 1a–1c indicate the frequency of the formation of explosive bubbles and crashes for different specifications of the forecast rule.

We see three broad patterns here. The incidence of explosive bubbles and crashes is generally increasing in the degree of overparameterization, decreasing in memory M, and increasing in the rate of updating pupdate (i.e., in the rate of learning).

Increasing the degree of overparameterization and reducing M increases the variance of the OLS parameter estimates and thus promotes the incidence of extreme rules. Increasing pupdate increases the number of agents who adopt new rules in any period. When a new rule is extreme, it can lead to large price movements which in turn feedback onto the selection of rules in subsequent periods.[7,8]

8. Conclusion

We consider the evolution of price in an artificial financial market in which traders with bounded memory engage in intermittent least squares learning. Overparameterization of the forecast rules is shown to generate instability in the learning dynamics. The degree of instability is decreasing in memory and increasing in the rate of updating and the degree of overparameterization. Thus, it is shown that, with finite memory, the correspondence between least squares learning equilibria and stationary rational expectations equilibria established by Honkapohja and Mitra may not be robust to overparameterization of forecast rules.

In related work, we have found similar results but with lower incidence of instability for less directed learning by an evolutionary algorithm (Georges and Wallace, 2004) and somewhat higher incidence of

[5] In the current model, these bubbles and crashes do not self correct and the price level blows up. We may follow Grandmont (1998) in anticipating that agents would alter their expectation generating mechanisms under such circumstances, but do not pursue this here.

[6] Overparameterization of forecast rules is crucial to the emergence of instability. If all agents use the MSV rule $F_t^i[\tilde{x}_{t+1}]=a_t^i$, then for any memory length M, unbounded bubbles are impossible.

[7] A similar property is displayed by the Santa Fe Artificial Stock Market (Arthur et al., 1997) in which long memories and slow exploration rates were found to facilitate convergence to an REE. Similarly, in Brock and Hommes (1997), there is a critical level of the intensity of choice above which the REE become locally unstable.

[8] Volatility and the incidence of bubbles are also decreasing in the rate of return r on the bond. However, the patterns in Table 1a,b,c are robust, even for unrealistically large rates (e.g., $r=1.5$). A similar relationship between volatility and r is reported by Bullard et al. (2005), who find that arbitrarily high excess volatility can be generated in an "exuberance equilibrium" for discount factors arbitrarily close to one.

Table 1a
Number of runs out of 100 in which an explosive bubble forms within 10,000 trading periods for 42 combinations of memory M and update rate pupdate

		Pupdate							
		0.01	0.1	0.2	0.3	0.4	0.5	…	1.0
M	10	0	0	1	22	56	52		50
	15	0	0	0	4	24	40		55
	20	0	0	0	2	13	29		45
	25	0	0	0	0	5	14		35
	30	0	0	0	0	1	4		23
	…								
	50	0	0	0	0	0	0		1

For each combination of parameter values, the 100 runs differ by random seed. Forecast rules are AR(2): $F_t^i\,[\tilde{x}_{t+1}]=a_{0t}^i+a_{1t}^i\tilde{x}_t+a_{2t}^i\cdot\tilde{x}_{t-1}$. Initial forecast rule parameter values are set to zero.

Table 1b
Number of runs out of 100 in which an explosive bubble forms within 10,000 trading periods

		Pupdate							
		0.01	0.1	0.2	0.3	0.4	0.5	…	1.0
M	10	1	96	99	98	98	97		98
	30	0	42	79	90	94	98		100
	60	0	10	15	24	33	41		70
	90	0	4	4	4	6	7		20
	120	0	1	1	1	2	2		40
	150	0	0	0	0	1	1		1

Same as Table 1a except forecast rules are nonlinear AR(2): $F_t^i[\tilde{x}_{t+1}]=a_{0t}^i+a_{1t}^i\tilde{x}_t+a_{3t}^i\cdot\tilde{x}_{t-1}^2$.

Table 1c
Number of runs out of 100 in which an explosive bubble forms within 10,000 trading periods

		Pupdate							
		0.01	0.1	0.2	0.3	0.4	0.5	…	1.0
M	20	24	99	99	99	100	98		99
	100	0	78	92	98	99	98		100
	200	0	14	22	24	30	36		50
	300	0	1	1	3	5	6		13
	400	0	0	1	1	2	2		3
	500	0	1	1	1	1	1		2

Same as Table 1a–b except forecast rules are of form (2) — all seven parameters active.

instability for the model with an explicit trading protocol (Georges, 2005). We have also considered more sophisticated traders who engage in specification testing before selecting which regressors to include in the forecast rule at each time of updating. While it may be computationally implausible for agents to perform systematic on-line specification testing (Aragones et al., 2004), any such testing will limit the degree of overparameterization that is implemented in forecasts. However, due to limited memory, this will not eliminate overparameterization and the resultant excess volatility.[9]

References

Aragones, Enriqueta, Gilboa, Itzhak, Postlewaite, Andrew, Schmeidler, David, 2004. Fact Free Learning, Manuscript. Yale University.

Arthur, W. Brian, Holland, John H., LeBaron, Blake, Palmer, Richard, Taylor, Paul, 1997. Asset pricing under endogenous expectations in an artificial stock market. In: Arthur, W. Brian, Durlauf, Steven N., Lane, David A. (Eds.), The Economy as Evolving Complex System II. Addison-Wesley, Reading, MA, pp. 15–44.

Brock, William A., Hommes, Cars H., 1997. A rational route to randomness. Econometrica 65 (5), 1059–1095.

Bullard, James, Evans, George W., Honkapohja, Seppo, 2005. Near-Rational Exuberance. Federal Reserve Bank of St. Louis. Manuscript.

Cho, In-Koo, Kasa, Kenneth, 2006. Learning and Model Validation. Manuscript, U. Illinois.

Evans, George W., Honkapohja, Seppo, 2001. Learning and Expectations in Macroeconomics. Princeton U. Press.

Georges, Christophre, 2005. Staggered Updating in an Artificial Financial Market. Hamilton College. Manuscript.

Georges, Christophre, 2006. Learning with misspecification in an artificial currency market. Journal of Economic Behavior and Organization 60, 70–84.

Georges, Christophre, Wallace, John C., 2004. Learning Dynamics and Non-linear Misspecification in an Artificial Financial Market. Hamilton College. Manuscript.

Gigerenzer, Gerd, Selten, Reinhard (Eds.), 2001. Bounded Rationality: The Adaptive Toolbox. MIT Press.

Grandmont, Jean-Michel, 1998. Expectations formation and stability of large socioeconomic systems. Econometrica 66 (4), 741–781.

Honkapohja, Seppo, Mitra, Kaushik, 2003. Learning with bounded memory in stochastic models. Journal of Economic Dynamics and Control 27 (8), 1437–1457.

LeBaron, Blake, 2006. In: Tesfatsion, Leigh, Judd, Kenneth (Eds.), Agent-based Computational Finance. Handbook of Computational Economics, vol. 2. Elsevier, pp. 1187–1233.

Lux, Thomas, Ausloos, Marcel, 2002. Market fluctuations i: scaling, multi-scaling and their possible origins. In: Bunde, A., Schellnhuber, H.-J. (Eds.), Theories of Disasters: Scaling Laws Governing Weather, Body and Stock Market Dynamics. Springer, Berlin.

McCallum, Bennett T., 1983. On non-uniqueness in rational expectations models: an attempt at perspective. Journal of Monetary Economics 11, 139–168.

Tesfatsion, Leigh, Judd, Kenneth (Eds.), 2006. Handbook of Computational Economics, 2: Agent Based Computational Economics. Elsevier.

[9] On model validation and learning, see also Cho and Kasa (2006).