Assignment 3

Name:__________________; UID:__________________

1 Question 1 20 pt

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. Let $\rho(A)$ denote the largest value of the absolute value of all eigenvalues of A. i) Prove that for any $\epsilon > 0$, there exists an invertible matrix S such that we have

$$\|SAS^{-1}\|_2 - \epsilon \le \rho(A) \le \|A\|_2 \tag{1}$$

ii) Prove that when A can be diagolized by an orthogonal matrix, then $\|A\|_2 = \rho(A)$.

2 Question 2 15 pt

Given a matrix $A \in \mathbb{R}^{m\times n}, m > n$ and a vector $b \in \mathbb{R}^m$, by using the normal equation and the LU decomposition, write code to solve the following minimization problem

$$\min_x \|Ax - b\|_2. \tag{2}$$

Generate a random 100×10 matrix A and a 100 dimensional vector b, then find x using your code that solves the above problem.

3 Question 3 15 pt

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two vector norms. Prove that for any $b \in \mathbb{R}^n$, there exists two constants $\alpha, \beta > 0$ such that

$$\begin{aligned} \|b\|_1 &\le \alpha\|b\|_2, \\ \|b\|_2 &\le \beta\|b\|_1. \end{aligned} \tag{3}$$

4 Question 4 15 pt

For a non-square matrix $A \in \mathbb{R}^{m\times n}$, if $x \in \mathbb{R}^n$ and $x + \delta x$ solve the two least square problems

$$\|Ax - b\|_2, \tag{4}$$

and

$$\|A(x + \delta x) - (b + \delta b)\|_2, \tag{5}$$

where $b, b + \delta b \in \mathbb{R}^m$, prove that

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \|A\|_2 \|(A^T A)^{-1} A\|_2 \frac{\|\delta b\|_2}{\|b\|_2}. \tag{6}$$

5 Question 5 20pt

Given a set of vectors $v_1, ..., v_n$, write code to perform the gram-schmidt orthogonalization. (https://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process)

Randomly generate 100 vectors in $\mathbb{R}^{200}$ to test your code.

6 Question 6 15pt

Suppose A is a square matrix in $R^{n \times n}$. Furthermore, suppose A has full column rank. Prove that the QR factorization of A is unique.