1. (a)

(i) $\frac{x}{\arccos(x)} \cdot \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{x}{x^2+e^x} \cdot (2x+e^x)$

$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^n}{n!}$

$\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^{n-1}}{(n-1)!}\right)x$

(b)

condition number for $f(x)$ $= \frac{x}{\sum_{i=0}^{n} \frac{x^i}{i!}} \cdot \sum_{i=0}^{n-1} \frac{x^i}{i!}$

$= \frac{x+x^2+\frac{x^3}{2}+\cdots+\frac{x^n}{(n-1)!}}{1+x+}$

condition number for $\exp(x)$ $= \frac{x}{e^x} e^x = x$

(c) $|\tilde{f}(\tilde{x},\tilde{y}) - f(x,y)|$

$$= \left| f(x+\varepsilon_x\theta, y+\varepsilon_y\theta) \cdot \frac{\varepsilon_x f_x(x+\varepsilon_x\theta, y+\varepsilon_y\theta) + \varepsilon_y f_y(x+\varepsilon_x\theta, y+\varepsilon_y\theta) + \varepsilon_f(\tilde{x},\tilde{y})}{f(x+\varepsilon_x\theta, y+\varepsilon_y\theta)} \right|$$

$$= \left| f(x+\varepsilon_x\theta, y+\varepsilon_y\theta) \cdot \left(C_{x+\varepsilon_x\theta} \cdot \frac{\varepsilon_x}{x+\varepsilon_x\theta} + C_{y+\varepsilon_y\theta} \frac{\varepsilon_y}{y+\varepsilon_y\theta} + \frac{\varepsilon_f(\tilde{x},\tilde{y})}{f(x+\varepsilon_x\theta, y+\varepsilon_y\theta)} \right) \right|$$

$$\leq |f(x+\varepsilon_x\theta, y+\varepsilon_y\theta)| \cdot \left| C_{x+\varepsilon_x\theta} \frac{\varepsilon_x}{x+\varepsilon_x\theta} + C_y \varepsilon_y\theta \frac{\varepsilon_y}{y+\varepsilon_y\theta} \right| + |\varepsilon_f(\tilde{x},\tilde{y})|$$

$$f'(x)h - f'(x)h$$

$$f(x) + f'(x)h + f''(x)\frac{h^2}{2} + \cdots - f(x) + f'(x)h - f''(x)\frac{h^2}{2} + \cdots$$

2. (i) $\frac{f(x+h) - f(x-h)}{2h}$

$$= \frac{1}{2h}\left(\sum_{i=0}^{\infty} f^{(i)}(x)\frac{h^i}{i!} - \sum_{i=0}^{\infty} f^{(i)}(x)\frac{(-h)^i}{i!} \right)$$

$$= \frac{1}{2h}\, 2 \cdot \sum_{i=0}^{\infty} f^{(2i+1)}(x)\frac{h^{(2i+1)}}{(2i+1)!}$$

$$\sum_{i=0}^{\infty} f^{(2i+1)}(x)\frac{h^{2i}}{(2i+1)!}$$

$$\left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| = \left| \sum_{i=0}^{\infty} f^{(2i+1)}(x)\frac{h^{2i}}{(2i+1)!} - f'(x) \right|$$

$$= \left| f'(x) + f^{(3)}\frac{h^2}{3!} + f^{(5)}\frac{h^4}{5!} + \cdots - f'(x) \right|$$

$$= \left| \sum_{i=1}^{\infty} f^{(2i+1)}(x)\frac{h^{(2i)}}{(2i+1)!} \right|$$

(ii) Let $A :=$ approximation error $= \left| \sum_{i=1}^{\infty} f^{(2i+1)}(x)\frac{h^{(2i)}}{(2i+1)!} \right|$

$\varepsilon_{f(\hat{x})} :=$ numerical error of $f(x)$, so $\varepsilon_{f(\hat{x})} = |\hat{f}(\hat{x}) - f(x)|$

$$\left| \frac{f(\hat{x}+h) - f(\hat{x}-h)}{2} - f'(x) \right| \approx \left| \frac{f(x+h) - f(x-h) \pm 2\varepsilon_{f(\hat{x})}}{2h} - f'(x) \right|$$

$$= \left| \frac{\varepsilon_{f(\hat{x})}}{h} + A \right|$$

(iii) Let $\bar{E} := \left| \frac{\varepsilon f(\hat{x})}{h} + \sum_{i=1}^{\infty} f^{(2i+1)}(x) \frac{h^{(2i)}}{(2i+1)!} \right|$

$$\frac{d\bar{E}}{dh} = -\varepsilon f(\hat{x}) (h^{-2}) + \sum_{i=1}^{\infty} f^{(2i+1)}(x) \cdot 2i \cdot \frac{h^{(2i-1)}}{(2i+1)!}$$

To minimize $\bar{E}$, let $\frac{d\bar{E}}{dh} = 0$

$$-\frac{\varepsilon f(\hat{x})}{h^2} + \sum_{i=1}^{\infty} f^{(2i+1)}(x) \cdot 2i \frac{h^{(2i-1)}}{(2i+1)!} = 0$$

for large $\varepsilon f(\hat{x})$, h should be large to reduce $\bar{E}$

for small $\varepsilon f(\hat{x})$, h should be small to reduce $\bar{E}$.

(iv)

$$\bar{E} = \frac{\varepsilon f(\hat{x})}{h} + \sum_{i=1}^{\infty} f^{(2i+1)}(x) \frac{h^{(2i)}}{(2i+1)!}$$

$f^{0}(x) = \sin(x)$

$f^{1}(x) = \cos(x)$

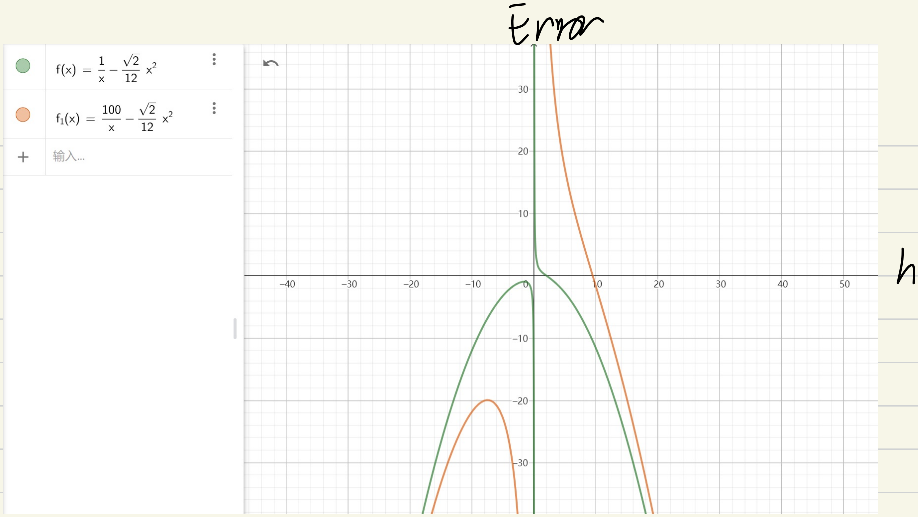
$f^{2}(x) = -\sin(x)$

$f^{3}(x) = -\cos x$

when $f(x) = \sin(x)$, $x = \frac{\pi}{4}$

$$= \frac{\varepsilon f(\hat{x})}{h} - \cos(x) \cdot \frac{h^2}{3!} + \cos(x) \frac{h^4}{5!} + \cdots$$

$$\approx \frac{\varepsilon f(\hat{x})}{h} - \frac{\sqrt{2}}{2} \frac{h^2}{6}$$



Since ε is the numerical error in f and x, it is always positive.

When the absolute value of h is increasing, the greater the ε, the greater the total error.

The observation aligns with my theory.

3. pf : $|x_{k+1} - x^*| \le M|x_k - x^*|^2$ where x^* is a simple root of $f(x)$
k is a large natural number.

$$e_{k+1} = x_{k+1} - x^*$$

$$= x_k - \frac{f(x_k)}{g(x_k)} - x^*$$

$$= e_k - \frac{f(x_k)}{\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}}$$

$$= e_k - \frac{(f(x_k))^2}{f(x_k + f(x_k)) - f(x_k)}$$

$$= \frac{e_k(f(x_k + f(x_k)) - f(x_k)) - f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}$$

Since x^* is a simple root $f(x^*) = 0$ and $f'(x^*) \neq 0$

$$f(x_k) = f(x^* + e_k) \approx f(x^*) + e_k f'(x^*) + \frac{1}{2} e_k^2 f''(x^*)$$

$$= e_k f'(x^*) + \frac{1}{2} e_k^2 f''(x^*)$$

$$= e_k A \quad \text{for } A = f'(x^*) + \frac{1}{2} e_k f''(x^*)$$

$$f(x_k + f(x_k)) \approx f(x^* + e_k + f(x_k))$$

$$= f(x^*) + (e_k + f(x_k)) f'(x^*)$$

$$+\frac{1}{2}\left(e_k + f(x_k)\right)^2 f''(x^*)$$

$$= 0 + e_k f'(x^*) + f(x_k) f'(x^*)$$

$$+\frac{1}{2}\left(e_k + f(x_k)\right)^2 f''(x^*)$$

$$\approx e_k f'(x^*) + e_k A f'(x^*) + \frac{1}{2} e_k^2 (1+A)^2 f''(x^*)$$

$$e_{k+1} \approx \frac{e_k\left(f(x_k + f(x_k)) - f(x_k)\right) - f(x_k)^2}{f(x_k + f(x_k)) - f(x_k)}$$

$$(f(x_k))^2 \approx \left(e_k f'(x^*) + \frac{1}{2} e_k^2 f''(x^*)\right)^2$$

$$= e_k^2 f'(x^*)^2 + e_k f'(x^*) f''(x^*) + O(e_k^4)$$

$$f(x_k + f(x_k)) - f(x_k) \approx e_k f'(x^*) + e_k A f'(x^*) + \frac{1}{2} e_k^2 (1+A)^2 f''(x^*)$$

$$- e_k f'(x^*) - \frac{1}{2} e_k^2 f''(x^*)$$

$$= e_k (f'(x^*))^2 + \frac{1}{2} e_k^2 \left(3 f'(x^*) + (f'(x^*))^2\right) f''(x^*) + O(e_k^3)$$

$$e_k\left(f(x_k + f(x_k)) - f(x_k)\right) \approx e_k^2 (f'(x^*))^2 + \frac{1}{2} e_k^3 \left(3 f'(x^*) + (f'(x^*))^2\right) f''(x^*)$$

$$+ O(e_k^4) - e_k^2 (f'(x^*))^2 - e_k^3 f'(x^*) f''(x^*)$$

$$+ O(e_k^4)$$

$$\approx \frac{1}{2} e_k^3 \left(f'(x^*) + (f'(x^*))^2\right) f''(x^*) + O(e_k^4)$$

$$e_{k+1} \approx \frac{\frac{1}{2} e_k^3 \left(f'(x^*) + (f'(x^*))^2\right) f''(x^*) + O(e_k^4)}{e_k (f'(x^*))^2 + \frac{1}{2} e_k^2 \left(3 f'(x^*) + f'(x^*)^2\right) f''(x^*) + O(e_k^3)}$$

$$\approx \frac{\frac{1}{2} e_k^3 \left(f'(x^*) + f'(x^*)^2\right) f''(x^*) (1 + O(e_k))}{e_k (f'(x^*))^2 (1 + O(e_k))}$$

$$\approx \frac{1}{2} e_k^2 \frac{f''(x^*)}{f'(x^*)} (1 + f'(x^*))$$

Let $M = \frac{1}{2} \frac{f''(x^*)}{f'(x^*)} (1 + f'(x^*))$

so $e_{k+1} \leq M e_k^2$. and Steffensen's method converges quadratically.

4. Please check for the file hw1q4.py for the full code.
The roots between -10 and 10 with tolerance
10^-7 are [-9.4247779, -6.2831853, -3.1415927,
3.1415926, 6.2831853, 9.424778]

5. Please check for the files hw1q5a.py and hw1q5b.py for the full code.

(a) the minimum x values are [-7.8539815598704197, -7.8539815575953424, -1.5707546841026927, -1.5711324913221059, -1.570752999412242, 4.7124103956495915, 4.712587072291541, 4.712389000566194]. And the minimum value is around -1.

(b) x at minimum values: -7.725251836940207, -7.725251836943057, -7.7252517628158595, -7.725251825941354, 7.725251792502424, 7.725251858907085, 7.725251837003621, 7.725251842508407, minimum values are -0.12837455352589913, -0.12837455352589913, -0.1283745535258988, -0.12837455352589913, -0.12837455352589902, -0.1283745535258991, -0.12837455352589913, -0.12837455352589913

```python
import math

def f(x):
    if x != 0:
        return math.sin(x) / x
    else:
        return 1

def df(x):
    if x != 0:
        return (math.cos(x) / x) - (math.sin(x) / (x**2))
    else:
        return 0



def newton():
    roots = []
    tol = 10 ** (-7)
    for num in range(-10, 11):
        x_hat = num
        counter = 0
        while x_hat <= 10 and x_hat >= -10:
            fx = f(x_hat)
            dfx = df(x_hat)
            if (abs(dfx) < tol):
                if fx != 0:
                    print(f"drop guess: {x_hat}")
                    break # if f'(x) == 0 and f(x) != 0, then initial guess x_hat is not a good guess, so we discard this inital guess and pass to the next
            if abs(f(x_hat)) < tol:
                if x_hat not in roots and x_hat + tol not in roots and x_hat - tol not in roots:
                    roots.append(x_hat)
                    print(f"found used {counter} rounds, x_hat = {x_hat}")
                break
            else:
                x_hat -= fx / dfx
                x_hat = round(x_hat, 7)
            counter += 1

    return roots

print(newton())
```

```python
import math

# f(x) = sin(x)
import math

def f(x):
    return math.sin(x)

def df(x):
    return math.cos(x)

def secant_method_update(xk, xk_, df):
    return xk - df(xk) * (xk - xk_) / (df(xk) - df(xk_))

def mini():
    tol = 10 ** (-8)
    roots = []
    prevx = -10
    for num in range(-9, 10):
        x_hat = num  # Save current value of x_hat
        while x_hat >= -10 and x_hat <= 10:
            x_hat_ = secant_method_update(x_hat, prevx, df)
            if abs(df(x_hat_)) > tol:
                prevx = x_hat
                x_hat = x_hat_
            else:
                if (df(x_hat + 0.5) >= 0 and df(x_hat - 0.5) <= 0):
                    roots.append(x_hat)
                    minim = f(x_hat)
                break
        prevx = num
    return roots, minim

print(mini())
```

```python
import math

# f(x) = -sin(x) / x

def f(x):
    return -math.sin(x) / x

def df(x):
    return -(x * math.cos(x) - math.sin(x)) / x ** 2

def secant_method_update(xk, xk_, df):
    return xk - df(xk)*(xk - xk_) / (df(xk) - df(xk_))

def mini():
    tol = 10 ** (-8)
    prevx = -10
    roots = []
    min = []
    for num in range(-9, 11):
        x_hat = num
        while abs(x_hat) >= -10 and abs(x_hat) <= 10:
            if (x_hat == 0) or (prevx == 0): break

            if abs(df(x_hat)) < tol:
                if df(x_hat - 0.5) < 0 and df(x_hat + 0.5) > 0:
                    roots.append(x_hat)
                    min.append(f(x_hat))
                break
            else:
                x_hat_ = secant_method_update(x_hat, prevx, df)
                if abs(x_hat_) > 0 and abs(x_hat_) < tol: break
                prevx, x_hat = x_hat, x_hat_
                print("iteration", num , "prevx = ", prevx, "x_hat_ = ", x_hat_)

        prevx = num
    return roots, min

print(mini())
```

6. $x_{k+1} = x_k - \frac{\psi(x_k)}{\psi'(x_k)}$

$= x_k - \frac{f(x_k)/f'(x_k)}{(f(x_k)/f'(x_k))'}$

$= x_k - \frac{f(x_k)/f'(x_k)}{1-(f'(x_k))^{-2} f''(x_k) f(x_k)}$

$= x_k - \frac{f(x_k) \cdot f'(x_k)}{(f'(x_k))^2 - f''(x_k) \cdot f(x_k)}$

$[f(x) \cdot (f'(x))^{-1}]'$

$= f'(x) \cdot [f'(x)]^{-1}$

$+ (-(f'(x))^{-2} \cdot f''(x) \cdot f(x))$

$= 1 - (f'(x))^{-2} f''(x) \cdot f(x)$

$\square$