Assignment 6

Name:________________; UID:________________

1 Question 1 20 pt

Write code to implement i) the composite trapezium formula and ii) the composite Simpson's formula. Numerically calculate the following integrations

$$1.\ \int_{-1}^{1} \exp(x^3) + \sin(x^2) + x\mathrm{d}x \tag{1}$$

$$2.\ \int_{0}^{2} \cos(x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1)\mathrm{d}x \tag{2}$$

Take different mesh size $x_i - x_{i-1} = 1, 1/2, 1/4, ..., 1/256$. Draw the values for the two methods w.r.t. different $x_i - x_{i-1} = 1, 1/2, 1/4, ..., 1/256$.

2 Question 2 20 pt

Given $f(x), x \in [0,1]$, write code to find a polynomial $p_n(x)$ such that $\|f(x) - p_n(x)\|_2$ (the 2-norm for a function in $[0,1]$) is minimized. Test your code with

$$\begin{aligned} f_1 &= \exp(x), \\ f_2 &= \sin(x). \end{aligned} \tag{3}$$

Test with different n and draw the curve for $f(x) - p_n(x)$.

3 Question 3 20 pt

(a) Prove that the function set $\{f_i(x) = \cos(ix)\}$ are orthogonal to each other on the interval $[0, 2\pi]$.
(b) Prove that if the polynomial set $\{p_i(x)\}$ forms an orthogonal set on the interval $[a,b], a < b$, where p_i is a polynomial of degree i, then $p_i(x), i \geq 2$ can be written in the form

$$p_i(x) = (a_i x + b_i)p_{i-1}(x) + c_i p_{i-2}(x), i \geq 2. \tag{4}$$

a_i, b_i, c_i are constants.

4 Question 4 20 pt

Consider using a DFT to interpolate the function $f(x) = \log(x+1)$ on the interval $[0, 2\pi]$.
(a) Construct and plot the interpolant on $[0, 2\pi]$ for $l = 16$ and $l = 32$. Explain why the results look unsatisfactory.
(b) Consider an even extension of $f(x)$, defining

$$g(t) = \begin{cases} f(t), & 0 \le t < 2\pi, \\ f(4\pi - t), & 2\pi \le t < 4\pi. \end{cases}$$

Apply DFT interpolation to $g(t)$ and plot the results on $[0, 2\pi]$. Find maximum errors for $l = 16$ and $l = 32$. Are they better than before? Why?

5 Question 5 20 pt

Given equidistant data (x_i, y_i), $i = 0, 1, \ldots, n$, with $n = 2l - 1$ and $x_i = \frac{2\pi i}{n+1}$, prove the following discrete orthogonality,

$$\frac{2}{n+1} \sum_{i=0}^{n} \cos(kx_i) \cos(jx_i) = \begin{cases} 0, & k \neq j, \\ 1, & 0 < k = j < l, \\ 2, & k = j = 0, \text{ or } k = j = l, \end{cases}$$

$$\frac{2}{n+1} \sum_{i=0}^{n} \sin(kx_i) \sin(jx_i) = \begin{cases} 0, & k \neq j, \\ 1, & 0 < k = j < l, \end{cases}$$

$$\frac{2}{n+1} \sum_{i=0}^{n} \sin(kx_i) \cos(jx_i) = 0$$