Stable Ramsey's theorem and measure

Damir D. Dzhafarov
University of Chicago

University of Hawaiʻi at Mānoa
20 August 2009

Ramsey's theorem

Definition

Fix $n, k \in \mathbb{N}$ and let X be an infinite set.

1. $[X]^n = \{Y \subset X : |Y| = n\}$.
2. A *k-coloring of exponent n* is a map $f : [X]^n \to k = \{0, \ldots, k-1\}$.
3. A set H is homogeneous for f if it is infinite and $f \upharpoonright [H]^n$ is constant.

Ramsey's theorem

Definition

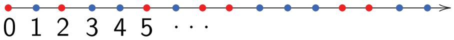
Fix $n, k \in \mathbb{N}$ and let X be an infinite set.

1. $[X]^n = \{Y \subset X : |Y| = n\}$.
2. A *k-coloring of exponent n* is a map $f : [X]^n \to k = \{0, \ldots, k-1\}$.
3. A set H is homogeneous for f if it is infinite and $f \restriction [H]^n$ is constant.

Ramsey's theorem

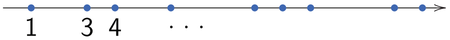
Fix $n, k \in \mathbb{N}$. Every coloring $f : [\mathbb{N}]^n \to k$ has a homogeneous set.

Ramsey's theorem



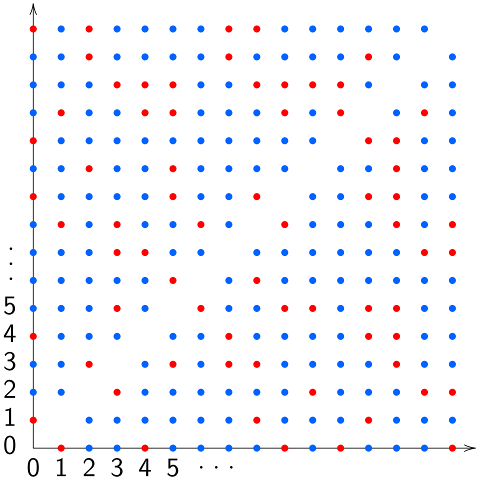
A coloring $[\mathbb{N}]^1 \to 2$.

Ramsey's theorem



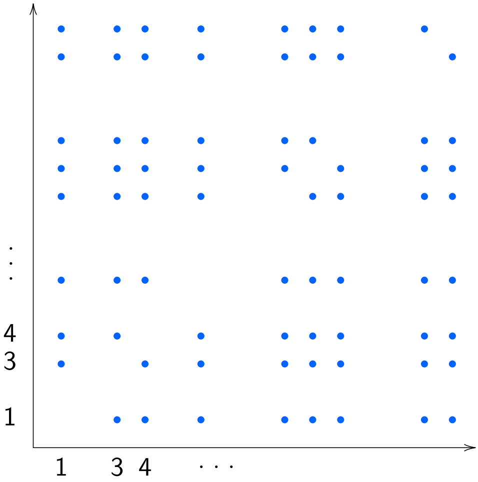
$H = \{1, 3, 4, 6, 9, 10, 11, 14, 15, \dots\}$ is homogeneous.

Ramsey's theorem



A coloring $[\mathbb{N}]^2 \to 2$.

Ramsey's theorem



$H = \{1, 3, 4, 6, 9, 10, 11, 14, 15, \dots\}$ is homogeneous.

Ramsey's theorem

Corollary (Chain Antichain Condition)

Every partial ordering of $\mathbb{N}$ *has an infinite chain or infinite antichain.*

Ramsey's theorem

Corollary (Chain Antichain Condition)

Every partial ordering of $\mathbb{N}$ has an infinite chain or infinite antichain.

Proof.

Let $\preceq$ be a partial ordering of $\mathbb{N}$. Define $f : [\mathbb{N}]^2 \to 2$ by

$$f(\{x,y\}) = \begin{cases} 0 & \text{if } x \text{ and } y \text{ are comparable under } \preceq \\ 1 & \text{otherwise} \end{cases}.$$

Apply Ramsey's theorem to get a homogeneous set H for f.

If f has value 0 on $[H]^2$ then H is a chain under $\preceq$, and if f has value 1 on $[H]^2$ then H is an antichain. $\square$

Ramsey's theorem

Moral

Complete disorder is impossible: in any configuration or arrangement of objects, however complicated or disorganized, some amount of structure and regularity is necessary.

Computability theory

- Okay, but how easy is it to describe this regularity?

Computability theory

- Okay, but how easy is it to describe this regularity?
- Computability-theoretic approach: let's try to gauge the complexity, with respect to some hierarchy, of homogeneous sets.

Computability theory

- Okay, but how easy is it to describe this regularity?
- Computability-theoretic approach: let's try to gauge the complexity, with respect to some hierarchy, of homogeneous sets.
- We restrict to *computable* colorings, i.e., $f : [\mathbb{N}]^n \to k$ for which there is an algorithm to compute $f(\{x_1, \ldots, x_n\})$ given $x_1, \ldots, x_n \in \mathbb{N}$.

Computability theory

Theorem (Specker, 1969)

There exists a comp. $f : [\mathbb{N}]^2 \to 2$ *with no computable homogeneous set.*

Definition (Turing jump)

Let X be a set.

1. The *(Turing) jump* of X is the set X' defined as

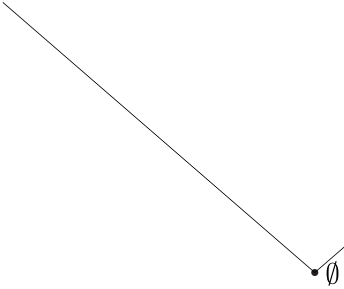
$$\{(e, x) : \text{the } e\text{th program with access to } X \text{ halts on input } x\}.$$

2. For $n \geq 1$, the $(n+1)$st jump of X is the jump of the nth.

Computability theory

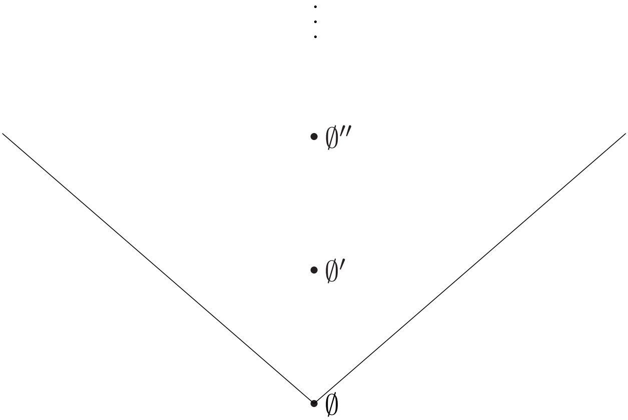
$\bullet\, \emptyset$

Turing hierarchy ($\leq_T$)

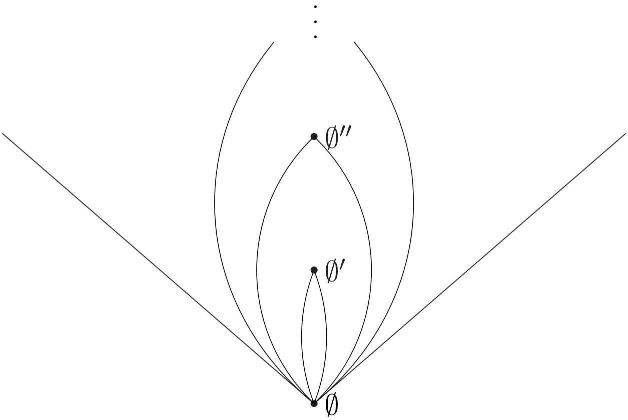


Turing hierarchy ($\leq_T$)

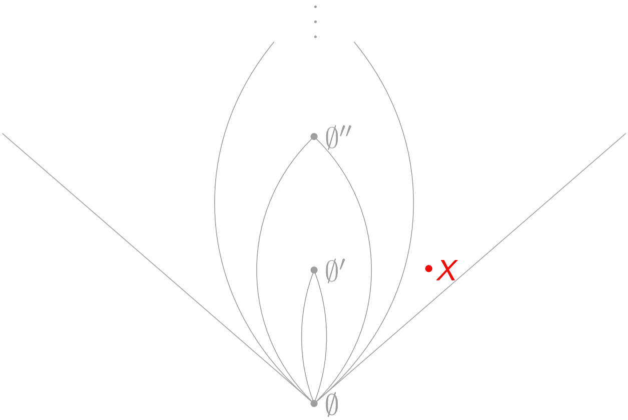
Computability theory



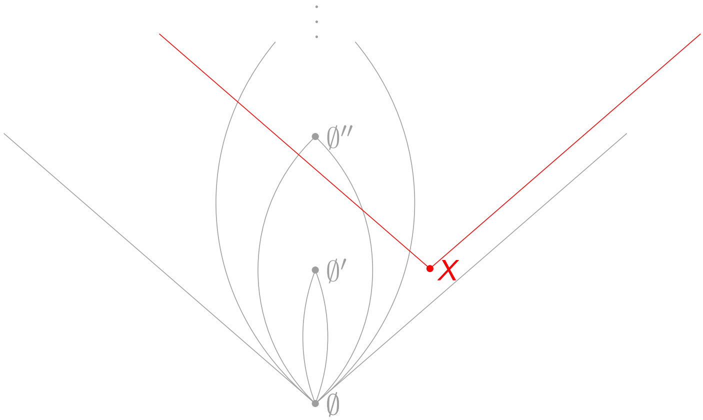
Turing hierarchy ($\leq_T$)



Turing hierarchy ($\leq_T$)

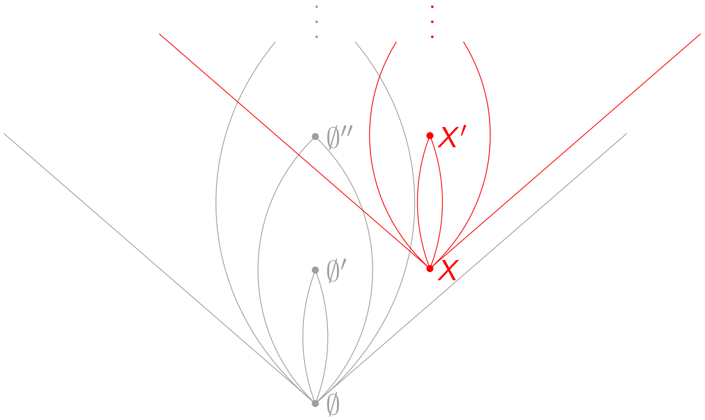


Turing hierarchy ($\leq_T$)



Turing hierarchy ($\leq_T$)

Computability theory



Turing hierarchy ($\leq_T$)

Theorem (Jockusch, 1972)

For each $n \geq 3$, there exists a computable $f : [\mathbb{N}]^n \to 2$ such that $H \geq_T \emptyset'$ (in fact, $H \geq_T \emptyset^{(n-2)}$) for every homogeneous set H of f.

Computability theory

Theorem (Jockusch, 1972)

For each $n \geq 3$, there exists a computable $f : [\mathbb{N}]^n \to 2$ such that $H \geq_T \emptyset'$ (in fact, $H \geq_T \emptyset^{(n-2)}$) for every homogeneous set H of f.

- This reduces the investigation to the $n = 2$ case.

Computability theory

Theorem (Jockusch, 1972)

For each $n \geq 3$, there exists a computable $f : [\mathbb{N}]^n \to 2$ such that $H \geq_T \emptyset'$ (in fact, $H \geq_T \emptyset^{(n-2)}$) for every homogeneous set H of f.

- This reduces the investigation to the $n = 2$ case.

Theorem (Seetapun, 1995)

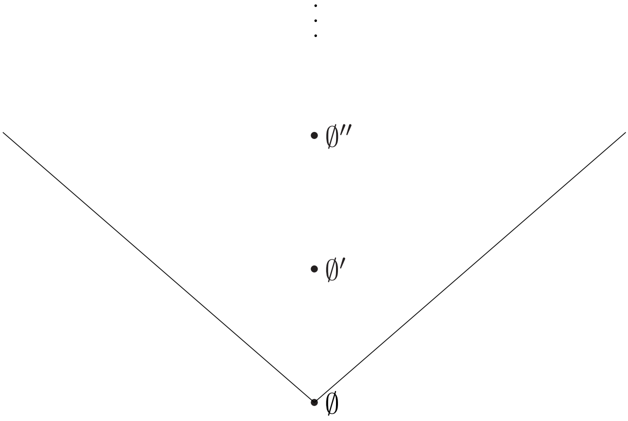
For every noncomputable set C, every computable $f : [\mathbb{N}]^2 \to k$ has a homogeneous set $H \not\geq_T C$.

Definition (Low and low$_2$ sets)

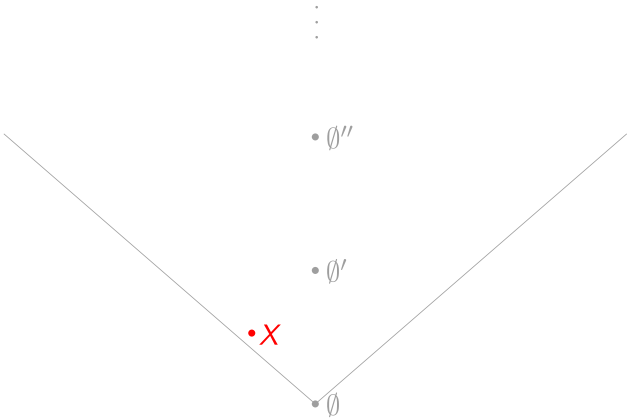
A set X is

1. *low* if $X' \leq_T \emptyset'$;
2. *low$_2$* if $X'' \leq_T \emptyset''$.

Computability theory

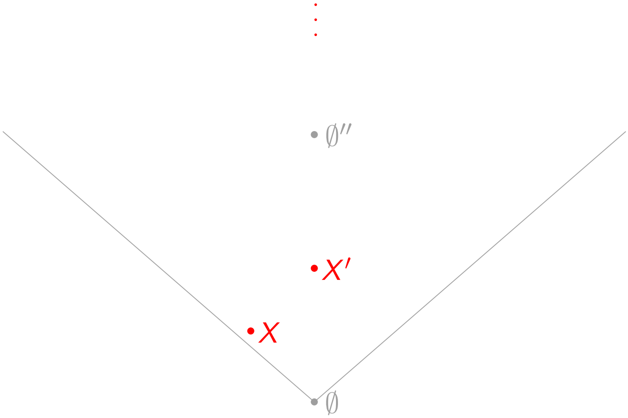


A low set.



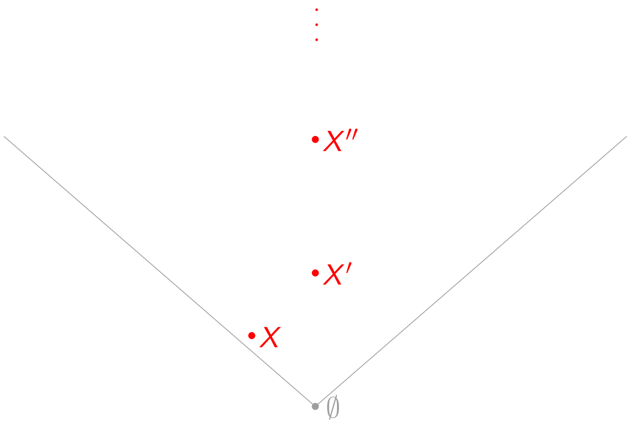
A low set.

Computability theory

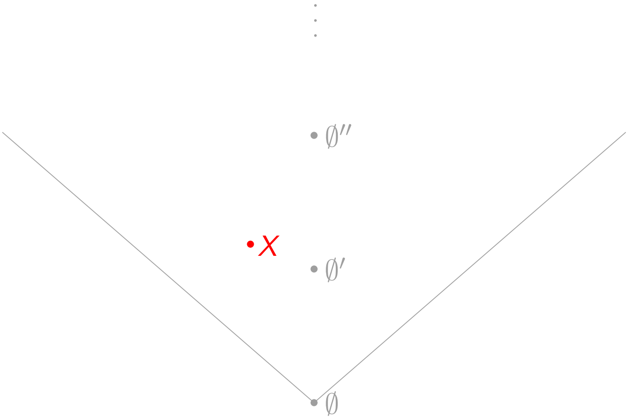


A low set.

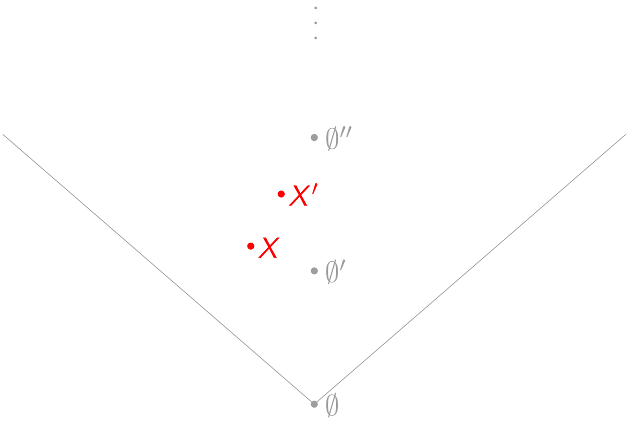
Computability theory



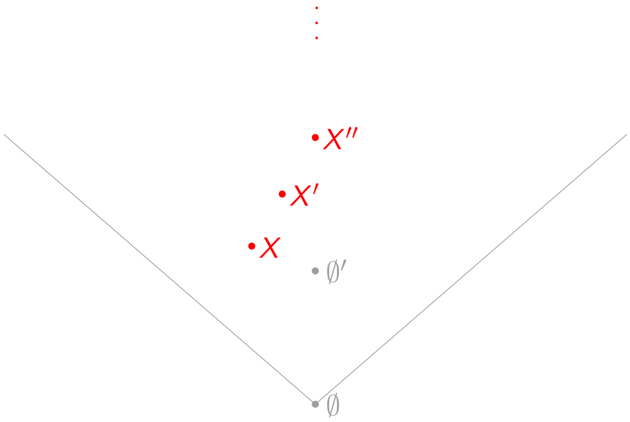
A low set.



A low_2 set.

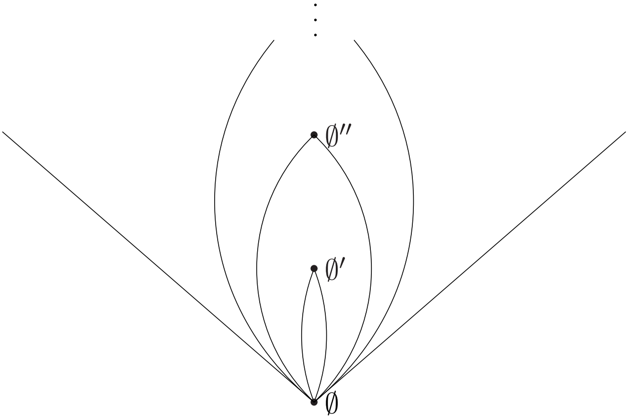


A low_2 set.

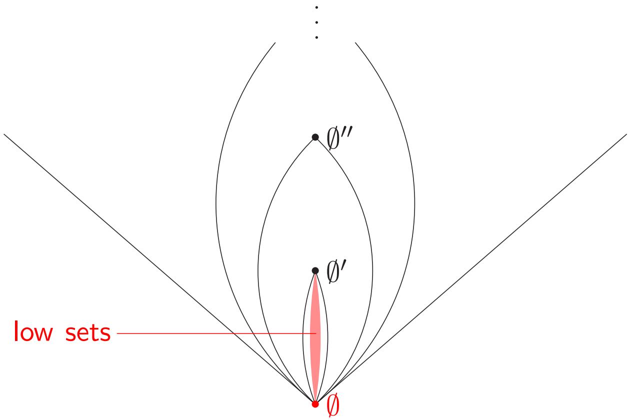


A low_2 set.

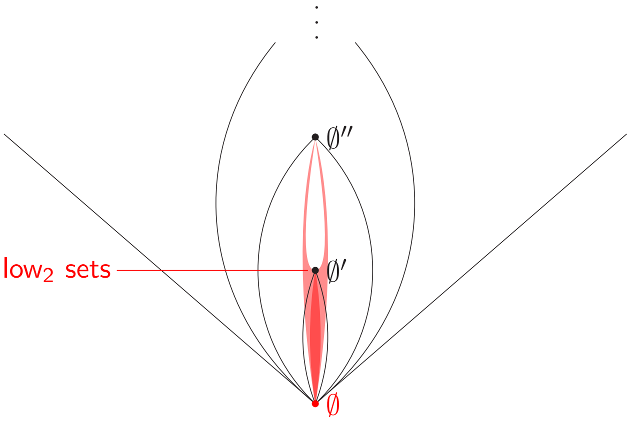
Computability theory



Lowness hierarchy.



Lowness hierarchy.



Lowness hierarchy.

Computability theory

Theorem (Jockusch, 1972)

There exists a comp. $f : [\mathbb{N}]^2 \to 2$ with no low hom. set (or even $\leq_T \emptyset'$).

Computability theory

Theorem (Jockusch, 1972)

There exists a comp. $f : [\mathbb{N}]^2 \to 2$ *with no low hom. set (or even* $\leq_T \emptyset'$*).*

Theorem (Cholak, Jockusch, and Slaman, 2001)

Every computable $f : [\mathbb{N}]^2 \to k$ *has a low*$_2$ *homogeneous set* H.

Computability theory

Theorem (Jockusch, 1972)

There exists a comp. $f : [\mathbb{N}]^2 \to 2$ *with no low hom. set (or even* $\leq_T \emptyset'$*).*

Theorem (Cholak, Jockusch, and Slaman, 2001)

Every computable $f : [\mathbb{N}]^2 \to k$ *has a low*$_2$ *homogeneous set* H*.*

Question (Cholak, Jockusch, and Slaman, 2001)

Can cone avoidance be added to this result?

Computability theory

Theorem (Dzhafarov and Jockusch, 2009)

For every noncomputable set C, every computable $f : [\mathbb{N}]^2 \to k$ has a low$_2$ homogeneous set that does not compute C.

Stable Ramsey's theorem

Definition

A coloring $f : [\mathbb{N}]^2 \to k$ is *stable* if for all x, $\lim_{y\to\infty} f(\{x, y\})$ exists.

Stable Ramsey's theorem

Definition

A coloring $f : [\mathbb{N}]^2 \to k$ is *stable* if for all x, $\lim_{y \to \infty} f(\{x, y\})$ exists.

Theorem (Jockusch, 1972)

Every computable stable $f : [\mathbb{N}]^2 \to k$ has a homogeneous set $H \leq_T \emptyset'$.

Stable Ramsey's theorem

Definition

A coloring $f : [\mathbb{N}]^2 \to k$ is *stable* if for all x, $\lim_{y\to\infty} f(\{x,y\})$ exists.

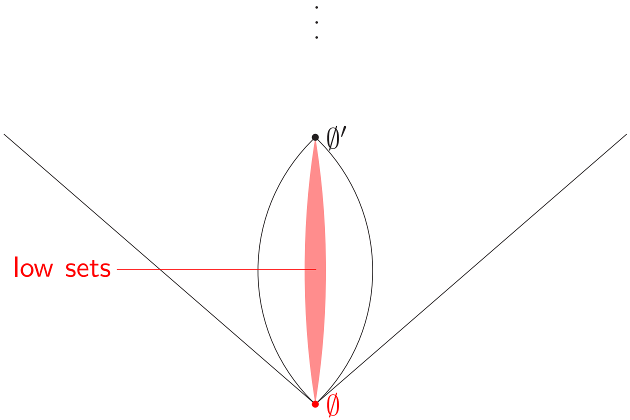
Theorem (Jockusch, 1972)

Every computable stable $f : [\mathbb{N}]^2 \to k$ has a homogeneous set $H \leq_T \emptyset'$.

Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There exists a computable stable $f : [\omega]^2 \to 2$ with no low hom. set.

Stable Ramsey's theorem



Stable Ramsey's theorem

Theorem (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman, 2008)

Every computable stable $f : [\omega]^2 \to 2$ has an homogeneous set $H <_T \emptyset'$.

Stable Ramsey's theorem

Theorem (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman, 2008)

Every computable stable $f : [\omega]^2 \to 2$ has an homogeneous set $H <_T \emptyset'$.

Theorem (Mileti, 2005)

There is no single set

- $X <_T \emptyset'$
- *or* X *low*$_2$

that computes a homogeneous set for every comp. stable $f : [\omega]^2 \to 2$.

Δ^0_2 measure

- Is there a way of telling which of the above results are "typical"?

Δ^0_2 measure

- Is there a way of telling which of the above results are "typical"?
- Measure-theoretic approach: define a notion of measure or nullity on the class of sets computable in $\emptyset'$.

Δ^0_2 measure

Definition (Martingales)

1. A *martingale* is a function M : {finite binary strings} $\to \mathbb{Q}^{\geq 0}$ such that for every string σ,

$$M(\sigma) = \frac{M(\sigma 0) + M(\sigma 1)}{2}.$$

Δ^0_2 measure

Definition (Martingales)

1. A *martingale* is a function M : {finite binary strings} $\to \mathbb{Q}^{\geq 0}$ such that for every string σ,

$$M(\sigma) = \frac{M(\sigma 0) + M(\sigma 1)}{2}.$$

2. M *succeeds* on a set X if

$$\limsup_{n\to\infty} M(X \upharpoonright n) = \infty.$$

Δ^0_2 measure

Theorem (Ville, 1939)

A class $\mathscr{C}$ of reals has Lebesgue measure zero if and only if there is a martingale that succeeds on every $X \in \mathscr{C}$.

Δ^0_2 measure

Theorem (Ville, 1939)

A class $\mathscr{C}$ of reals has Lebesgue measure zero if and only if there is a martingale that succeeds on every $X \in \mathscr{C}$.

Definition (Δ^0_2 nullity)

A class $\mathscr{C}$ of reals has Δ^0_2 *measure zero* if there is a martingale M computable in $\emptyset'$ that succeeds on every $X \in \mathscr{C}$.

Δ^0_2 measure

- Reasonable notion of measure on the class of sets computable in $\emptyset'$:
  - The class of all sets computable in $\emptyset'$ does not have Δ^0_2 measure zero.
  - For each set X computable in $\emptyset'$, the singleton $\{X\}$ has Δ^0_2 measure zero.

Δ^0_2 measure

- Reasonable notion of measure on the class of sets computable in $\emptyset'$:
  - The class of all sets computable in $\emptyset'$ does not have Δ^0_2 measure zero.
  - For each set X computable in $\emptyset'$, the singleton $\{X\}$ has Δ^0_2 measure zero.
- Gives us a notion of measure on the class of computable stable colorings.

Δ^0_2 measure

Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There exists a computable stable $f : [\omega]^2 \to 2$ with no low homogeneous set.

Δ^0_2 measure

Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)

There exists a computable stable $f : [\omega]^2 \to 2$ *with no low homogeneous set.*

Theorem (Hirschfeldt and Terwijn, 2008)

The class of computable stable colorings with no low homogeneous set has Δ^0_2 *measure zero.*

Δ^0_2 measure

Theorem (Mileti, 2005)

There is no single set $X <_T \emptyset'$ that computes a homogeneous set for every computable stable $f : [\omega]^2 \to 2$.

Δ_2^0 measure

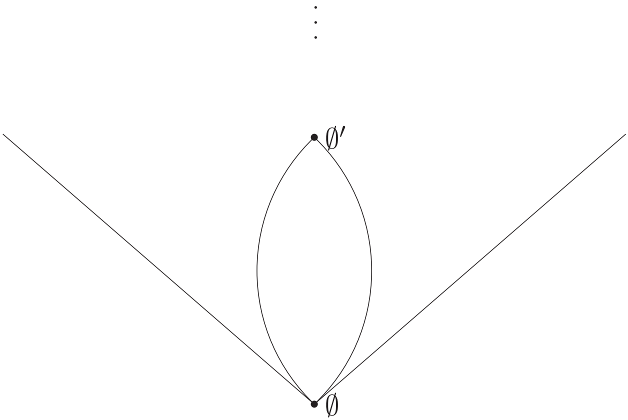
Theorem (Mileti, 2005)

There is no single set $X <_T \emptyset'$ that computes a homogeneous set for every computable stable $f : [\omega]^2 \to 2$.

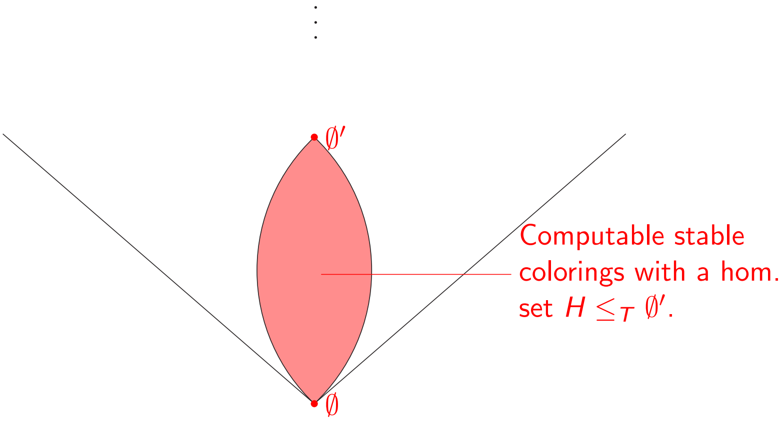
Theorem (Dzhafarov, submitted)

For each set $X <_T \emptyset'$, the class of computable stable colorings having a homogeneous set $H \leq_T X$ has Δ_2^0 measure zero.

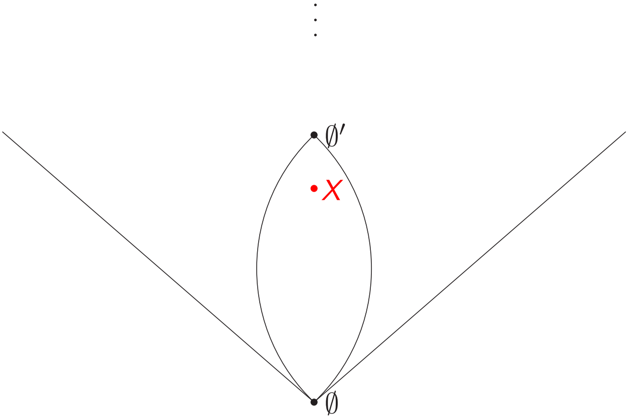
Δ^0_2 measure

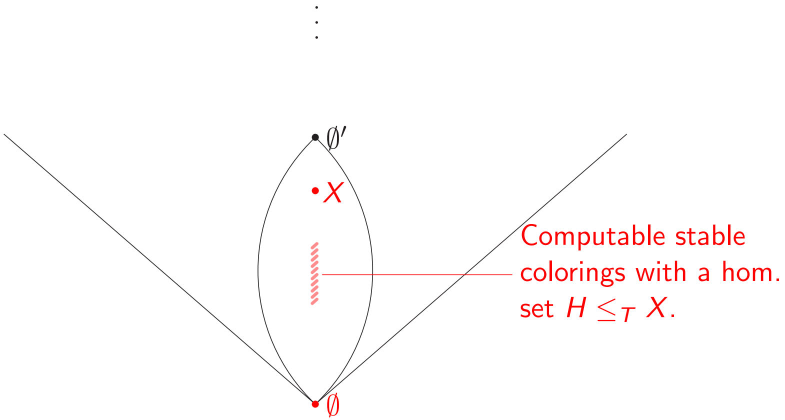


Δ^0_2 measure



Δ^0_2 measure




$\emptyset'$
X
$\emptyset$
Computable stable colorings with a hom. set $H \leq_T X$.

Δ^0_2 measure

Theorem (Mileti, 2005)

There is no single low$_2$ set that computes a homogeneous set for every computable stable $f : [\omega]^2 \to 2$.

Δ^0_2 measure

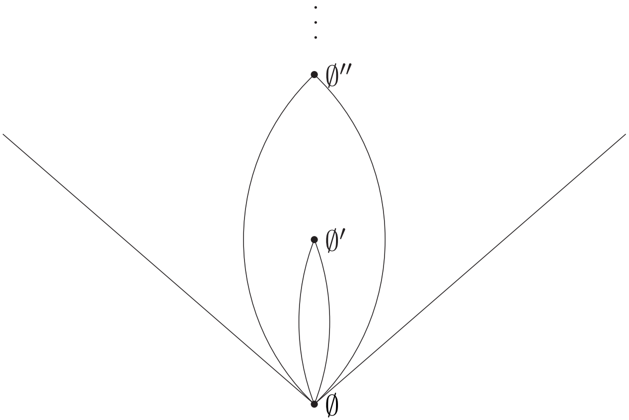
Theorem (Mileti, 2005)

There is no single low$_2$ set that computes a homogeneous set for every computable stable $f : [\omega]^2 \to 2$.

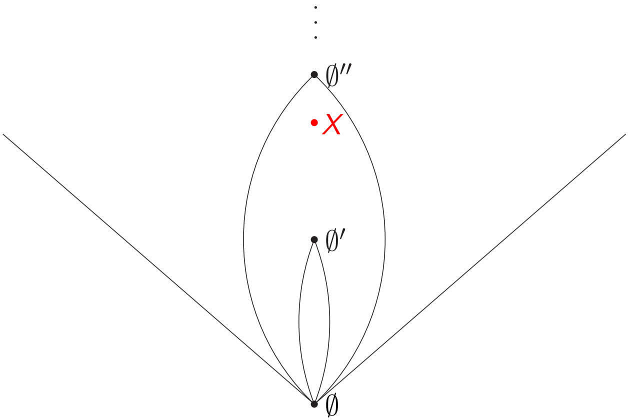
Theorem (Dzhafarov, submitted)

There is a set $X \leq_T 0''$ such that the class of computable stable colorings having a hom. set $H \leq_T X$ does not have Δ^0_2 measure zero but is not equal to the class of all computable stable colorings.

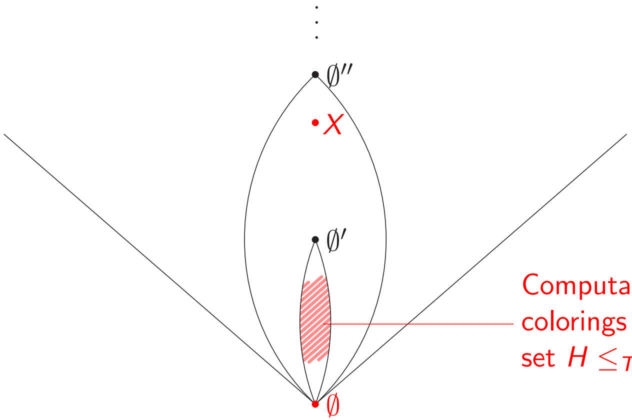
Δ^0_2 measure



Δ^0_2 measure

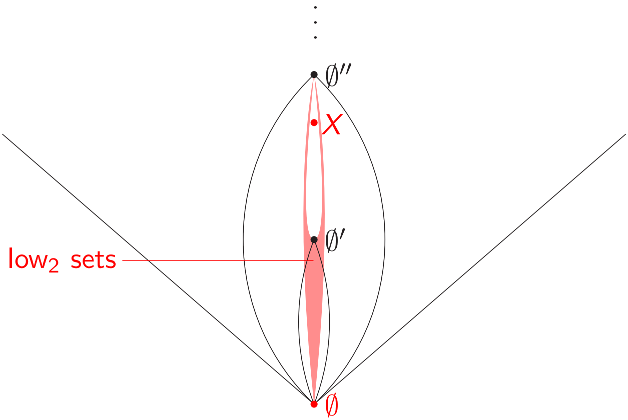


Δ^0_2 measure



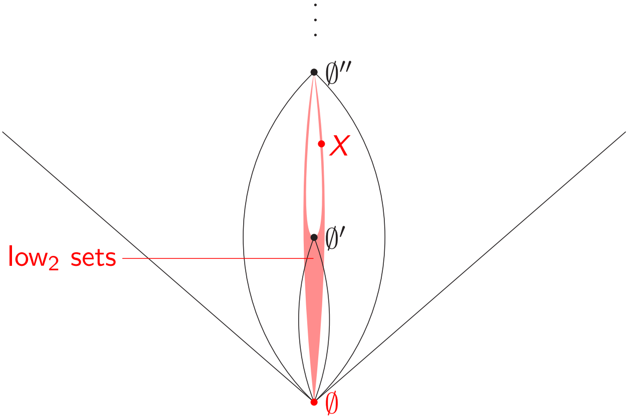
Computable stable colorings with a hom. set $H \leq_T X$.

Δ^0_2 measure



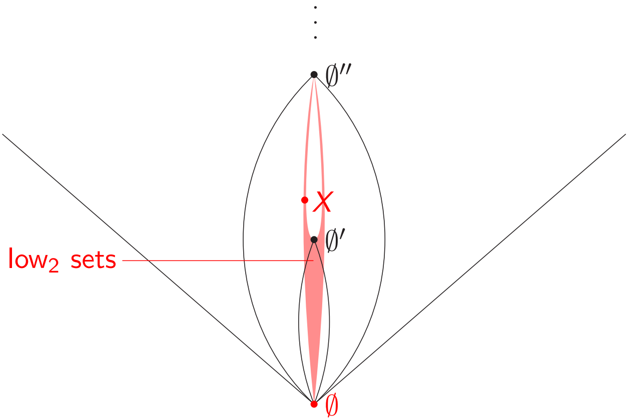
Open Question: Can X be chosen low_2?

Δ^0_2 measure



Open Question: Can X be chosen low_2?

Δ^0_2 measure



Open Question: Can X be chosen low_2?

Questions

- Is there a notion of measure that would allow for a similar analysis of Ramsey's theorem proper?

Questions

- Is there a notion of measure that would allow for a similar analysis of Ramsey's theorem proper?
- What is the situation if instead of Δ^0_2 measure we use an effective notion of dimension?

Thank you for your attention.