Assignment 2

ME685A

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Problem 1

Given Problem: To find a cubic interpolating polynomial given four data points of the form $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) using Monomial Lagrange and Newton basis.

(a) Monomial Basis: The monomial basis is $\{1, x, x^2, x^3\}$. We will assume the polynomial as $a_0 + a_1x + a_2x^2 + a_3x^3$
Thus, we have four equations:

$$\begin{aligned}
a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 = y_0\\
a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 = y_1\\
a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3 = y_2\\
a_0 + a_1x_3 + a_2x_3^2 + a_3x_3^3 = y_3
\end{aligned}$$

Which can put in a matrix form as: $Mx = y$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} \mathrm{x}_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathrm{y}_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Using Singularity Robust solution, the equation was solved

(b) Newton's Basis: The Newton's basis is

$$\{1, x - x_0, (x - x_0)(x - x_1), (x - x_0)(x - x_1)(x - x_2)\}$$

.
Now, we have the interpolating function as:
$y(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) + \ldots$

Clearly, $c_0 = f(x_0)$.To find c_1, subtract c_0 from both sides and divide the result by $x - x_0$:

$$\tfrac{y(x)-c_0}{x-x_0} = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) + \ldots$$

.
Both the above relations look same only for a different function and with indices shifted. The step $c_k \leftarrow \dfrac{c_k - c_0}{x_k - x_0}$ is the algorithmic version of

replacing y with the left side of 2nd relation, which we will call y_1. After that, the process repeats: c_1 is already found, and we move on to

$$\frac{y_1(x) - c_1}{x - x_1} = c_2 + c_3(x - x_2) + \dots$$

The step $a_k \leftarrow \dfrac{c_k - c_1}{x_k - x_1}$ amounts to replacing y_1.

(c) Lagrange Basis: The lagrange basis is :

$$g_i(x) := \prod_{j=1, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

From observation, we can say the coefficients for interpolating polynomial will be $\{y_0,\ y_1,\ y_2\ ,\ y_3\}$

For the data set (-1,4), (0,3), (2,7), (8,11), the following coefficients(in order) were obtained:

1. Monomial Basis: 3.000, 0.259, 1.130, -0.130
2. Newton's Basis: 4.000, -1.000, 1.000, -0.130
3. Lagrange Basis: 4.0, 3.0, 7.0, 11.0

Problem 2

Here we have to find the derivative using Richardson Extrapolation
We have:

$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + ah^2 + bh^4 + \dots$$

Since the approximation of the derivative is a function of h, call it $F(h)$ and the actual derivative F.
So, we have:

$$F(h) = F + c_1 x^2 + O(h^4)$$

Now,

$$F(\alpha h) = F + c_1 (\alpha x)^2 + O(h^4)$$

Eliminate c to get

$$F_1(h) = F(\alpha h) - \frac{\alpha^2 F(h)}{1-\alpha^2} + O(h^4)$$

So, $F_1(h)$ gives a better estimate of the derivative
Calculation using analytic differentiation:
The given function is:

$$\frac{\sin(2x+\frac{\pi}{3})\sqrt{3x^2+2x-4}}{\ln(2x+4)}$$

Now differentiating it:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)}{\ln\left(2x+4\right)}\right]$$

$$= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)\right]\cdot\ln\left(2x+4\right) - \sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(2x+4\right)\right]}{\ln^2\left(2x+4\right)}$$

$$= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)\right]\cdot\ln\left(2x+4\right) - \sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(2x+4\right)\right]}{\ln^2\left(2x+4\right)}$$

$$= \frac{\ln\left(2x+4\right)\left(\frac{(3\cdot 2x+2\cdot 1+0)\sin\left(2x+\frac{\pi}{3}\right)}{2\sqrt{3x^2+2x-4}} + (2\cdot 1+0)\sqrt{3x^2+2x-4}\cos\left(2x+\frac{\pi}{3}\right)\right) - \frac{2\sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)}{2x+4}}{\ln^2\left(2x+4\right)}$$

$$= \frac{\ln\left(2x+4\right)\left(\frac{(6x+2)\sin\left(2x+\frac{\pi}{3}\right)}{2\sqrt{3x^2+2x-4}} + 2\sqrt{3x^2+2x-4}\cos\left(2x+\frac{\pi}{3}\right)\right) - \frac{2\sqrt{3x^2+2x-4}\sin\left(2x+\frac{\pi}{3}\right)}{2x+4}}{\ln^2\left(2x+4\right)}$$

Putting x=2:
The value of derivative using numerical differentiation is 0.36613216574
The value of derivative using analytic differentiation is 0.366132165716
So we get almost same values using analytic and numerical differentiation techniques

Problem 3

Given Problem: To find 8^{th} degree interpolating polynomial and cubic piece-wise interpolating by choosing 9 points from a given function in a given domain

8^{th} degree interpolating Polynomial: We take the basis as $\{1, x, x^2, ..., x^7, x^8\}$
Thus, we'll have nine equations:

$$a_0 + a_1x_0 + ... + a_7x_0^7 + a_8x_0^8 = y_0$$
$$a_0 + a_1x_1 + ... + a_7x_1^7 + a_8x_1^8 = y_1$$
....
$$a_0 + a_1x_7 + ... + a_7x_7^7 + a_8x_7^8 = y_7$$
$$a_0 + a_1x_8 + ... + a_7x_8^7 + a_8x_8^8 = y_8$$

Which can put in a matrix form as: $Mx = y$

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^7 & x_0^8 \\ 1 & x_1 & \dots & x_1^7 & x_1^8 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_7 & \dots & x_7^7 & x_7^8 \\ 1 & x_8 & \dots & x_8^7 & x_8^8 \end{bmatrix} \begin{bmatrix} \mathrm{x}_0 \\ x_1 \\ \dots \\ \dots \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} \mathrm{y}_0 \\ y_1 \\ \dots \\ \dots \\ y_7 \\ y_8 \end{bmatrix}$$

Using Singularity Robust solution, the equation was solved

Piece-wise cubic interpolation: Let's say we have n+1 data points, so we'll take n piece-wise polynomials for interpolation: $S_0, S_1, S_2 \dots, S_n$, where :

$$S_i = a_i + b_ih_i + c_i{h_i}^2 + d_i{h_i}^3$$

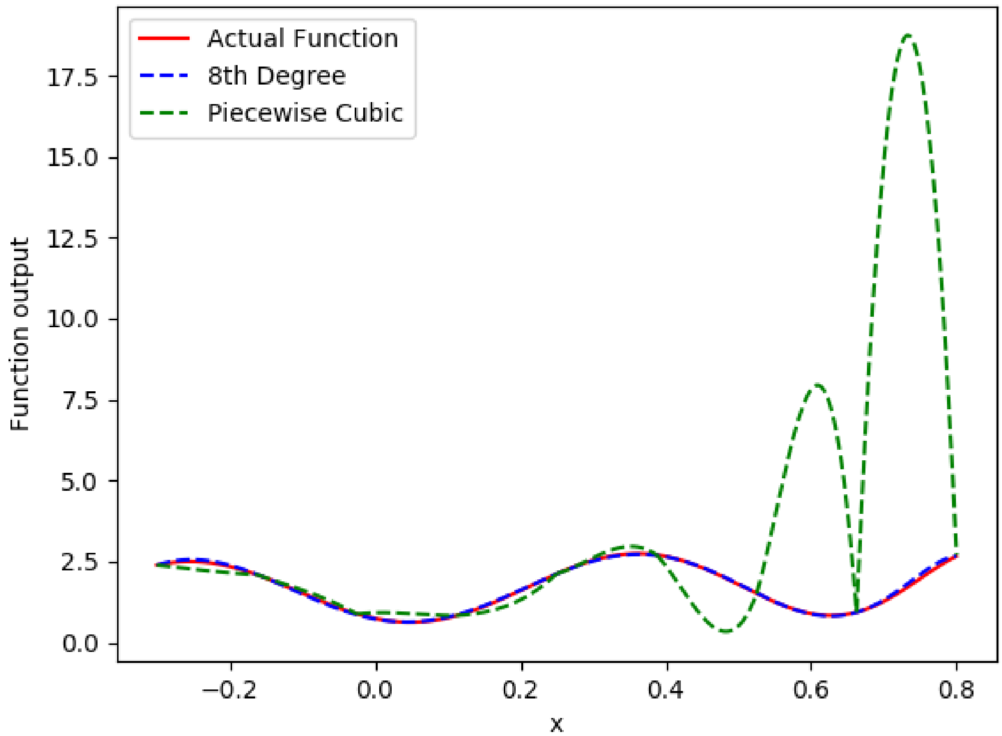
where $h_i = x - x_i$
Now, $x = x_i \implies h_i = 0$, so $S_i(x_i) = a_i = y_i$
I did the point sampling in such a way that all the x_is are equidistant from each other, so we define $h = x_1 - x_0$
Now, $S_{i-1}(h) = y_i = y_{i-1} + b_{i-1}h + c_{i-1}h^2 + d_{i-1}h^3 \dots (1)$

$$\text{Again, } S_i'(0) = S_{i-1}'(h) \implies b_i = b_{i-1} + 2c_{i-1}h + 3d_{i-1}h^2 \dots (2)$$
$$\text{and, } S_i'(0) = S_{i-1}'(h) \implies 2c_{i-2} + 6d_{i-1}h = 2c_i$$
$$\implies d_{i-1} = \frac{c_i - ci-1}{3h}$$
$$\text{So, } d_i = \frac{c_{i+1} - c_i}{3h}$$
$$\text{from the first equation, } b_{i-1} = \frac{y_i - y_{i-1} - c_{i-1}h^2 - d_{i-1}h^3}{h} \dots (3)$$
$$\text{So, } b_i = \frac{y_{i+1} - y_i - c_ih^2 - d_ih^3}{h} \text{ (shifting 'i' by 1)}$$

Putting above two equations in (2), we get :

$$2c_{i-1}h = b_i = b_{i-1} - 3d_{i-1}h^2$$
$$2c_{i-1}h = \frac{y_{i+1} - yi}{h} - \frac{y_i - y_{i-1}}{h} - c_ih_i + c_{i-1}h_{i-1} - d_ih_i^2 + d_{i-1} - (c_i - c_{i-1})$$
$$\text{Which on solving, gives : } c_{i-1} + 4c_i + c_{i+1} = \frac{3(y_{i+1} - 2y_i + y_{i-1})}{h^2}$$

This equation can be put in matrix and the coefficients can be obtained
Here are the obtained solutions:



Problem 4

For method of false position, we employ secant method and maintaining a bracket over root at every iteration
So the iterations will be like : $x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$
and at every step, we choose x_k *or* x_{k-1} with x_{k+1} depending on whoever brackets the solution
For Newton-Raphson Method, the iterations will be:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

For initial value 1.42 and 1.44, the newton raphson method showed convergence
But for 1.46, the values were diverging and thus newton raphson fails for initial value 1.46

Problem 6

We have:

$$f(x) = 2{x_1}^2 - {x_1}^4 + \frac{{x_1}^6}{6} + x_1 x_2 + \frac{{x_2}^2}{2}$$

For stationary we find its gradient and set it to zero

$$\implies \nabla f = [4x_1 - 4{x_1}^3 + {x_1}^5 + x_2 \qquad x_1 + x_2]^T = 0$$

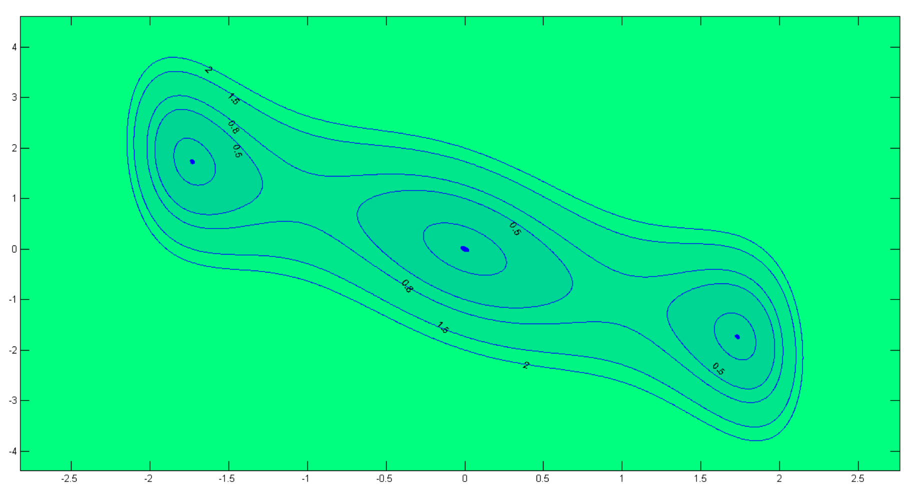
Which gives:

$$x_1 = -\sqrt{3}, x_2 = \sqrt{3} \quad x_1 = \sqrt{3}, x_2 = -\sqrt{3} \quad x_1 = -1, x_2 = 1 \quad x_1 = 1, x_2 = -1 \quad x_1 = 0, x_2 = 0$$

$$\text{Hessian Matrix, } H = \begin{bmatrix} 4 - 12{x_1}^2 + 5{x_1}^4 & 1 \\ 1 & 1 \end{bmatrix}$$

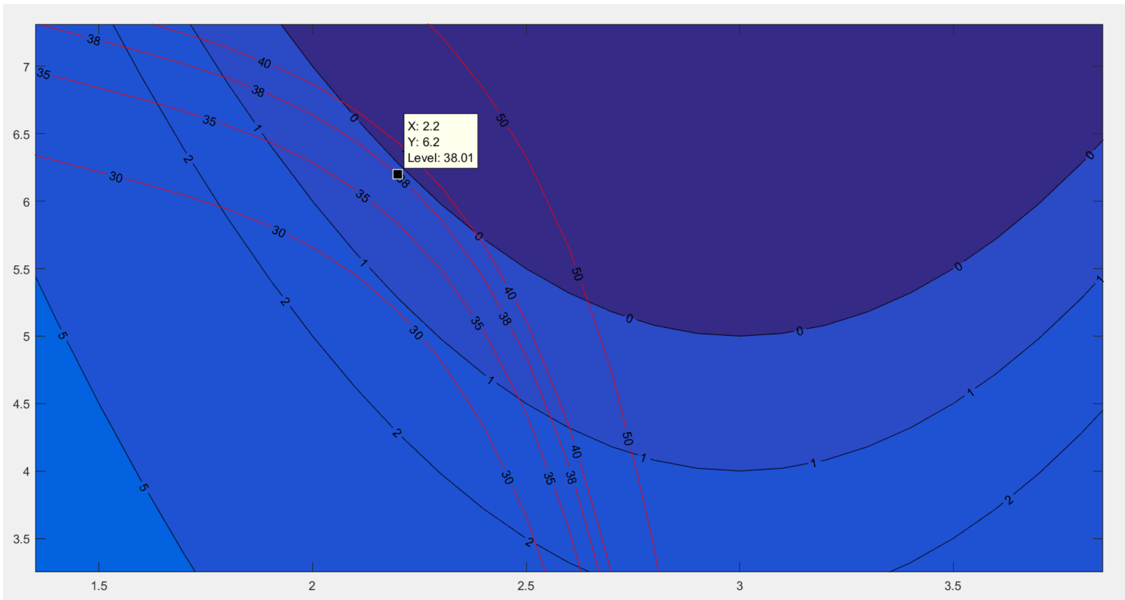
Now we'll find eigen values or use Sylvester's criterion to determine whether H is positive definite or negative definite, if H is positive definite, then its a Minima, if H is negative definite, then its a Maxima, else its a saddle point
For $(-\sqrt{3}, \sqrt{3})$, H is +ve definite(all leading Minors are +ve) $\implies$ Minima
For $(\sqrt{3}, -\sqrt{3})$, H is +ve definite(All leadning Minors are +ve) $\implies$ Minima
For $(-1, 1)$, H is indefinite(eigen values are -3.236, 1.236) $\implies$ Saddle
For $(1, -1)$, H is indefinite(eigen values are -3.236, 1.236) $\implies$ Saddle
For $(0, 0)$, H is +ve definite(All leadning Minors are +ve) $\implies$ Minima

Here are the contours for f



There are three minimas and 2 saddle points.

The contours of f and g together shows that minima is around (2.2, 6.2) and the value of f is around 38


X: 2.2
Y: 6.2
Level: 38.01
1.5
2
2.5
3
3.5
3.5
4
4.5
5
5.5
6
6.5
7