PERSPECTIVE PROJECTIONS

1. Perspective Transformations and Projections
   a) Single point
   b) Two point
   c) Three Point
2. Vanishing points and trace points

➤ Generalized 4 x 4 transformation matrix in homogeneous coordinates

$$[T] = \begin{bmatrix} a & b & c & p \\ d & e & i & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix}$$

☐ Perspective transformations

Linear transformations – local scaling, shear, rotation reflection

Translations l, m, n along x, y, and z axis

Overall scaling

Perspective Transformation

If any of the first 3 elements in the last column of 4x4 transformation matrix is non-zero a perspective transformation results

$$[x \quad y \quad z \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad (rz+1)]$$

$$[x^* \quad y^* \quad z^* \quad 1] = \begin{bmatrix} \dfrac{x}{rz+1} & \dfrac{y}{rz+1} & \dfrac{z}{rz+1} & 1 \end{bmatrix}$$
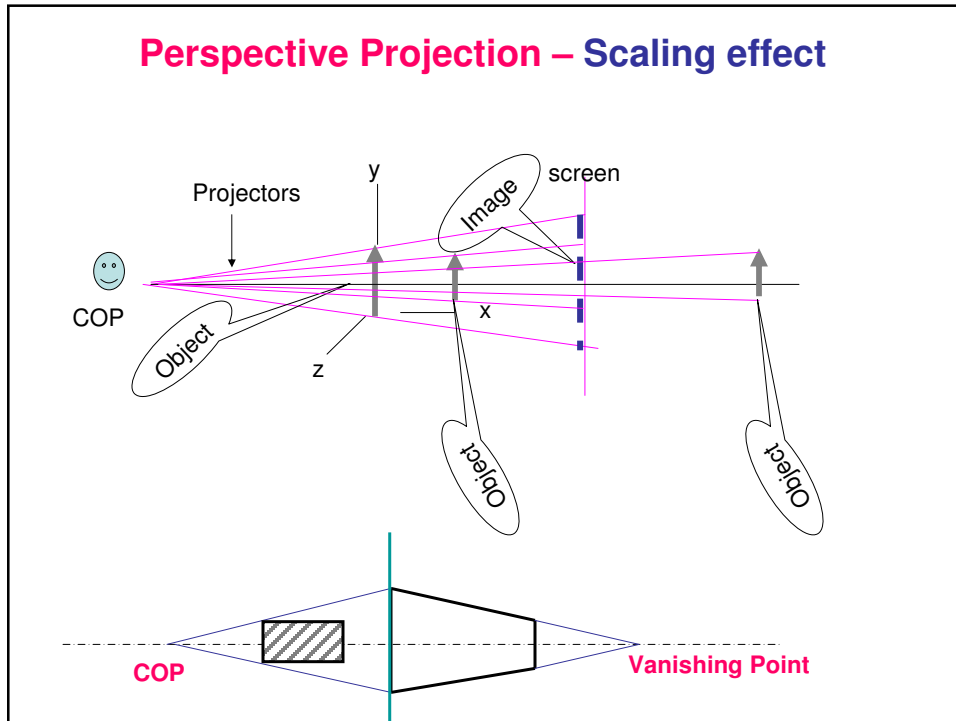
Perspective Projection

To obtain perspective projection, we project the results of perspective transformation on to a any of the orthographic projection planes, say, z=0 plane.

$$[x \quad y \quad z \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad 0 \quad (rz+1)]$$

$$[x^* \quad y^* \quad z^* \quad 1] = \begin{bmatrix} \dfrac{x}{rz+1} & \dfrac{y}{rz+1} & 0 & 1 \end{bmatrix}$$

Perspective Projection – Scaling effect

Projectors

y

screen

Image

COP

Object

x

z

Object

Object

COP

Vanishing Point

The Perspective Projection of a Point

Consider the following figure where using similar triangles we can write the transformed coordinates as:

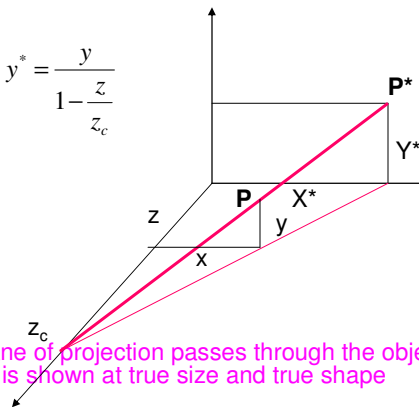$$\frac{x^*}{z_c} = \frac{x}{z_c - z} \qquad \Rightarrow x^* = \frac{x}{1 - \dfrac{z}{z_c}}$$

$$\frac{y^*}{\sqrt{x^{*2} + z_c^{*2}}} = \frac{y}{\sqrt{x^2 + (z_c - z)^2}} \qquad \Rightarrow y^* = \frac{y}{1 - \dfrac{z}{z_c}}$$
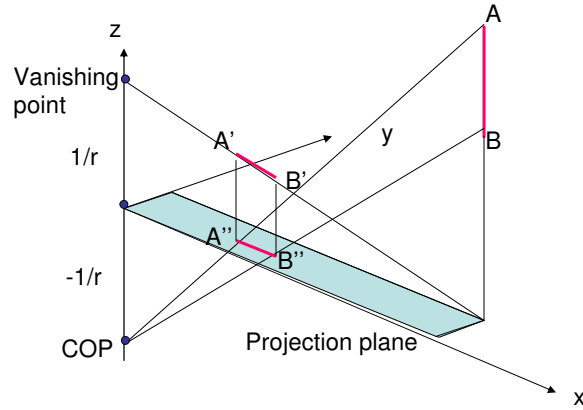
Let

$$r = -\frac{1}{z_c} \Rightarrow \frac{x}{1 + rz} ; \frac{y}{1 + rz}$$

The origin is unaffected. If the plane of projection passes through the object, then that section of the object is shown at true size and true shape

P*

Y*

z

P

X*

y

x

z_c

3

➢ Perspective projection of line parallel to z-axis

1. Perspective transformation of AB yields A'B'

2. Orthographic projection of A'B' gives the required projection A"B"on z=0 plane

1. The original line AB and transformed line A'B' intersect the z=0 plane at the same point on it.

2. The line A'B' intersects z axis at z=1/r

3. This point is called the **vanishing point**

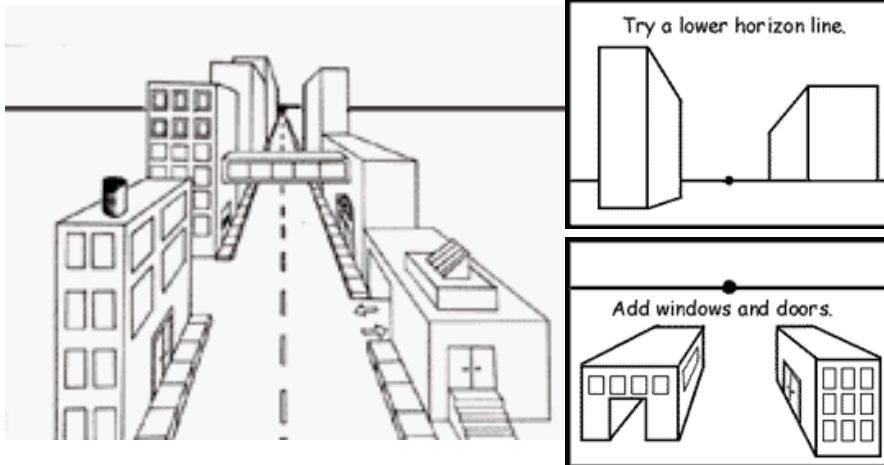# Perspective Transformation

The effect of the perspective transformation is to bring a point at infinity to a finite value in the 3D space

The entire semi-infinite positive space $0 \leq z \leq \infty$ is transformed to finite positive half-space $0 \leq z^* \leq 1/r$

A point at infinity can be shown to transform to a finite distance

$$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{r} & 1 \end{bmatrix}$$

> Perspective Transformations

Try a lower horizon line.

Add windows and doors.

A Single Point Perspective

A single point perspective transformation with respect to z-axis

$$[x \quad y \quad z \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x \quad y \quad z \quad (rz+1)]$$

$$[x^* \quad y^* \quad z^* \quad 1] = \begin{bmatrix} \dfrac{x}{rz+1} & \dfrac{y}{rz+1} & \dfrac{z}{rz+1} & 1 \end{bmatrix}$$
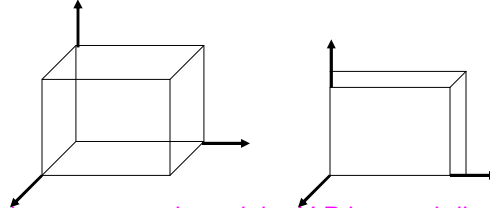
Now the perspective projection is obtained by concatenating the orthographic projection matrix

$$[T] = [P_t][P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[x^* \quad y^* \quad z^* \quad 1] = \begin{bmatrix} \dfrac{x}{rz+1} & \dfrac{y}{rz+1} & 0 & 1 \end{bmatrix}$$

A Single Point Perspective

A single point perspective transformation with respect to z-axis

The COP is on +ve z-axis and the V.P is equal distance away on –ve z-axis

Perspective Projection of a centered cube

A Single Point Perspective

A single point perspective transformation with respect to x-axis and y-axis are given below respectively

$$[T]=[P_t][P_z]=\begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
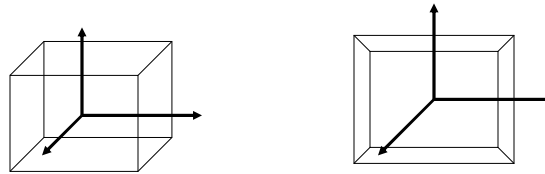
$$\begin{bmatrix} x^* & y^* & z^* & 1 \end{bmatrix}=\begin{bmatrix} \dfrac{x}{px+1} & \dfrac{y}{px+1} & 0 & 1 \end{bmatrix}$$

$$[T]=[P_t][P_z]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x^* & y^* & z^* & 1 \end{bmatrix}=\begin{bmatrix} \dfrac{x}{qy+1} & \dfrac{y}{qy+1} & 0 & 1 \end{bmatrix}$$

A Single Point Perspective

Numerical Example: A single point perspective transformation has to be performed on a unit cube from a center $z_c=10$ on the z-axis, followed by its projection on z=0 plane

Sol:Since $z_c=10$, $r=-1/ z_c = -1/10 = -0.1$

The perspective projection matrix for this problem can be written as

$$[U][T] = [U][P_r] = [U] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Two Point Perspective Projection

The two point perspective projection can directly be obtained on similar lines as:

$$[T] = [P_{pq}] = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

VP

$$\begin{bmatrix} x^* & y^* & z^* & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{x}{px+qy+1} & \dfrac{y}{px+qy+1} & 0 & 1 \end{bmatrix}$$
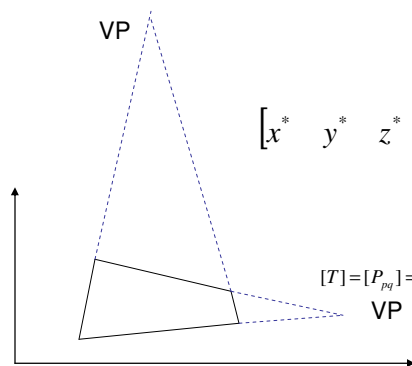
$$[T] = [P_{pq}] = [P_p][P_q] = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

VP

Three Point Perspective Projection

The three point perspective projection can be obtained on similar lines as:

$$[T] = [P_t][P_z] = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x^* & y^* & z^* & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{x}{px+qy+rz+1} & \dfrac{y}{px+qy+rz+1} & 0 & 1 \end{bmatrix}$$
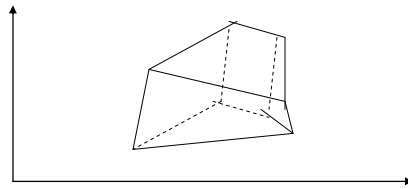
Some Techniques to produce Perspective Projections

Just applying the perspective transformation may not show all details. To view 3 faces of a cuboid, a translation, one or more rotations are used before applying the perspective projections.

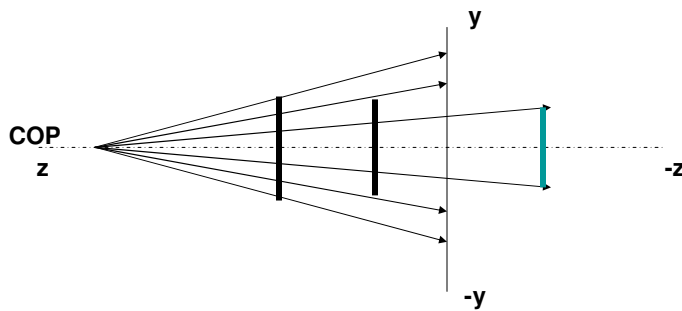**One translation followed by a single point Pers. Projn.**

$$[T] = [T_{xyz}][P_{rz}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ l & m & 0 & (1-n)/z_c \end{bmatrix}$$

Note that the scaling factor of (1-n)/Zc appearing in the above result. It is close to reality as the objects gets smaller away from the observerAs Zc → ∞, the scale effect disappears

One translation followed by a single point Perspective

The scale effect

$$[T]=[T_{xyz}][P_{rz}]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\l&m&n&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1/z_c\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1/z_c\\l&m&0&(1-n)/z_c\end{bmatrix}$$

One Rotation and Single Point Perspective

Consider the transformation matrix for rotation about the y-axis by an angle ϕ, followed by a single point perspective projection on the plane $z=0$ from a COP at $z=z_c$

$$[T]=[R_\phi][P_{rz}]=\begin{bmatrix}\cos\phi&0&-\sin\phi&0\\0&1&0&0\\\sin\phi&0&\cos\phi&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1/z_c\\0&0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}\cos\phi&0&-\sin\phi&\dfrac{\sin\phi}{z_c}\\0&1&0&0\\\sin\phi&0&\cos\phi&-\dfrac{\cos\phi}{z_c}\\0&0&0&1\end{bmatrix}$$

Two Point Perspective

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One Rotation and Single Point Perspective

- Thus a single rotation about a principal axis perpendicular to the one on which the COP lies is equivalent to the two point perspective transformation.

- However rotation about the same axis on which COP lies does not have this effect

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & \dfrac{\sin\phi}{z_c} \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & -\dfrac{\cos\phi}{z_c} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Two Point Perspective

Two Rotations and Single Point Perspective

Consider the transformation matrix for rotation about the y-axis by an angle ϕ, followed by rotation about the x-axis by an angle θ, and a single point perspective projection on the plane z=0 from a cop at z=z_c

$$[T] = [R_y][R_x][P_{rz}] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & \dfrac{\sin\phi\cos\theta}{z_c} \\ 0 & \cos\theta & 0 & -\dfrac{\sin\theta}{z_c} \\ \sin\phi & -\cos\phi\sin\theta & 0 & -\dfrac{\cos\phi\cos\theta}{z_c} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
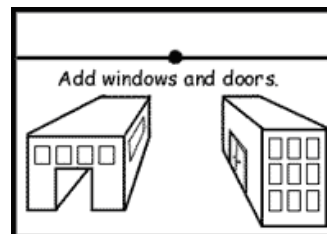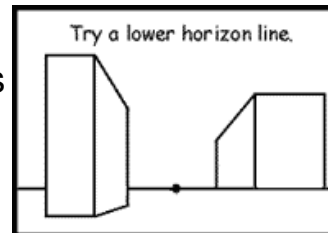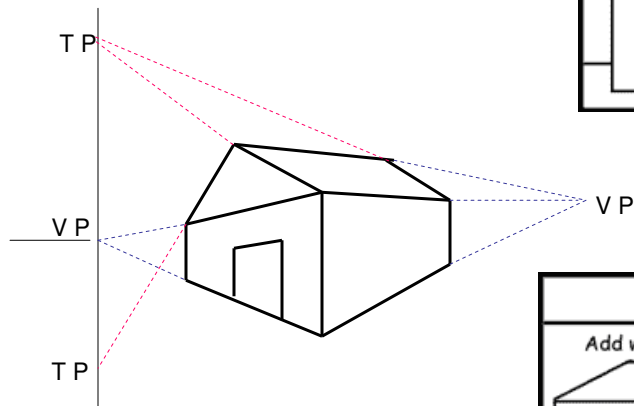
Three Point Perspective

Perspective Transformations: Vanishing Points

There are two methods of finding VPs

1. Finding intersection between any two parallel line after transformation and projection
2. Find the concatenated Perspective matrix. Use this to transform points at infinity

$$[VP][T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} VP_x \\ VP_y \\ VP_z \end{bmatrix}$$

Perspective Transformations:
Vanishing Points and Trace points

T P

V P

V P

T P

Try a lower horizon line.

Add windows and doors.

Perspective Transformations: Trace Points

These points result from the set of parallel lines on inclined planes.

We use the concatenated Perspective matrix to transform points at infinity in the direction of inclined edges

$$[VP][T] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} TP \\ TP \end{bmatrix}$$

Oblique Projections

Oblique projection results when parallel projectors from centre of projection at infinity intersect the plane of projection at an oblique angle.

Useful to show the general 3D shape of an object.

Two oblique projections are well known: Cavalier and Cabinet.

Cavalier Projection: It results when the angle of projection is 45º. In this projection all the three foreshortening factors are equal and the image appears too thick.

Cabinet Projection: It is an oblique projection with foreshortening factor for edges perpendicular to the plane of projection is half of that in other directions.

This projection corrects the thickening effect of cavalier projection.

Oblique Projection

The transformation matrix for an oblique projection is given below:

$$T_{ob} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The angle of projection is β and the angle projected z axis makes with the horizontal is α and *f* is the foreshortening factor,

$$\beta = \cot^{-1}(f=1) = 45° \qquad \text{Cavalier}$$

$$\beta = \cot^{-1}(f=\tfrac{1}{2}) = 63.43° \qquad \text{Cabinet}$$

$$\beta = \cot^{-1}(f=0) = 90° \qquad \text{Orthographic}$$

Non-planar Projections

Telescopic projection: As the telescope makes the distant objects appear closer and the microscope makes the smaller objects appear

Map Projection: This is also called the cartographic projection. These mappings are based on Gaussian curvature matching.

Display Projection: The computer graphic displays use a variety of 2D and 3D projections.