Study Material of B.Sc. III Semester

Electricity and Magnetism

(Reference Book: Electricity and Magnetism by Ahmad & Lal)

Unit-I

The Gauss's theorem

The Gauss's law or theorem is just an alternative way of stating the Coulomb's law of electrostatic forces. It provides an easy and simple method of calculating the intensity of electric field due to charged bodies.

Let us consider a hypothetical closed surface N enclosing a point charge (+q) (Fig. 2). Let *ds* be a small element of the surface and let the vector $dS^{\wedge}$ represent it in magnitude and direction (the direction of $dS^{\wedge}$ is to be that of the outward drawn normal on *dS)*. Then ($E^{\wedge}$.$dS^{\wedge}$) gives the total number of lines of force originating from $+q$ and passing through the element dS in perpendicular direction, i.e., it gives the normal flux through *dS.* The total normal flux through the whole closed surface *S* will be given by:

$$\int_s E^{\wedge}.dS^{\wedge} = \int_s \frac{1}{4\pi\varepsilon 0} \left|\frac{q}{r^2}\right| r^{\wedge}.n^{\wedge}\, dS) \qquad \text{(taking k=1 for air)}$$

$$= \frac{q}{4\pi\varepsilon 0}\left(\int_s \frac{ds\cos\theta}{r^2}\right) \ldots\ldots\ldots\ldots\ldots(1)$$

where θ is the angle between the vector *E* and the outward drawn normal to the surface *dS* as indicated in fig. 1, and the integration is over the whole of the closed surface.

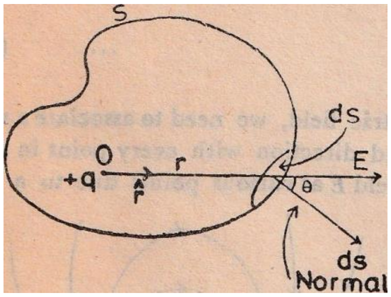


fig 1

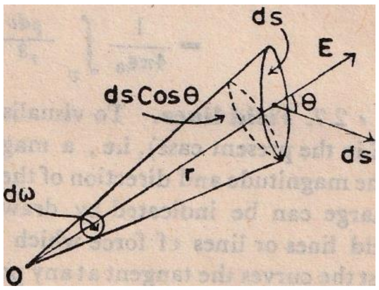


fig 2

Now *dS* cosθ is the projection of *dS* normal to the radius vector as shown in fig. 2. When this normal component is divided by r^2 we get the solid angle $d\omega$ subtended by *dS* at q, i. e., The integral on the right-hand side of this expression gives the complete solid angle subtended by the whole surface *S* at q which is equal to the ratio of the area of a sphere surrounding the point occupied by q

Now from definition of solid angle, $ds\cos\theta/r^2 = d\omega$; so

$$\int_s E^{\wedge}.dS^{\wedge}= \frac{q}{4\pi\varepsilon 0}\left(\int_s d\omega\right)\ldots\ldots\ldots\ldots.(2)$$

$$\int_s d\omega = \frac{4\pi r^2}{r^2} = 4\pi \ldots\ldots\ldots\ldots\ldots.(3)$$

$$\int_s E^{\wedge}.dS^{\wedge}= \frac{q}{4\pi\varepsilon 0} * 4\pi = \frac{q}{\varepsilon 0}\ldots\ldots(4)$$

If there are a number of charger enclosed in surface S, then the total normal flux:

$$\int_s E^{\wedge}.dS^{\wedge}= \frac{(q1+q2+q3+\cdots)}{\varepsilon 0} = \frac{Q}{\varepsilon 0}\ldots\ldots(5)$$

where Q is the total charge (algebraic sum of the charges) contained in the volume enclosed by the surface *S*. Equation (5) is the Gauss's law or theorem in **Integral form** and states that the "surface integral of the normal component of the electric field over any closed surface is equal to the algebraic sum of the charges enclosed by the surface divided by ε_0 .

When the charge is distributed over a volume, if ρ be the density of the charge then the total charge inside the closed surface enclosing the volume will be:

$$Q= \int_v \rho \, dv\ldots.(6)$$

From (5) in eqn (6)

$$\int_s E^{\wedge}.dS^{\wedge}= \int_v \rho \, dv/ \varepsilon_0\ldots\ldots.(7)$$

But according to divergence theorem $\int_s E^{\wedge}.dS^{\wedge}=\int_v \text{div } E \, dv\ldots.(8)$

From eqn.(7) and (8) div $E = \rho/ \varepsilon_0$ or

$$\textbf{div } \mathbf{E}= \frac{\boldsymbol{\rho}}{\boldsymbol{\varepsilon 0}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\mathbf{(9)}$$

Eqn no (9) is the **differential form** of Gauss's theorem and is a fundamental equation of electrostatics. It states that "the divergence of electric field E at any point is proportional to the charge density at that point".

Applications of Gauss Theorem

1. Field due to uniform charged sphere

Outside

Let we have a sphere filled with uniform distribution of charge (like nucleus). Let us find the electric field E at a point P outside the surface of the sphere. Considering an imaginary sphere concentric with the given sphere of charge and passing through the point P. The field at P and everywhere on the imaginary sphere will be perpendicular to its surface due to symmetry. The outward normal flux through the spherical surface S is,

$$\int_s E^{\wedge}.dS^{\wedge} = \int_s E_n\, dS$$

where E_n is the normal (or radial) component of $E^{\wedge}$. As E_n is constant everywhere on the sphere,

$$\int_s E^{\wedge}.dS^{\wedge} = E_n \int_s dS = 4\pi R^2 E \ldots\ldots..(1)$$

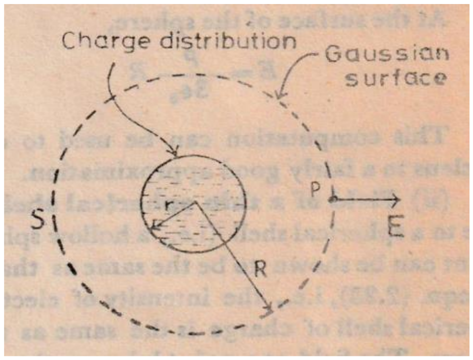


Fig 1

from Gauss theorem we have $\int_s E^{\wedge}.dS^{\wedge} = \frac{Q}{\varepsilon 0}$ ………………...(2)

so we have $4\pi R^2 E = \frac{Q}{\varepsilon 0}$

$$E = \frac{Q}{4\pi\varepsilon 0 R^2} = \left(\frac{1}{4\pi\varepsilon 0}\right)\frac{Q}{R^2} \ldots\ldots\ldots\ldots(3)$$

Which is the same as the electric field at a distance R due to a point charge Q at the centre of the sphere, i.e., the electric field due to a sphere of charge Q at an outside point is the same as if the whole charge is concentrated at its centre.

Inside the sphere

Let us now find the electric field at a point P inside the sphere of charge (fig 2).

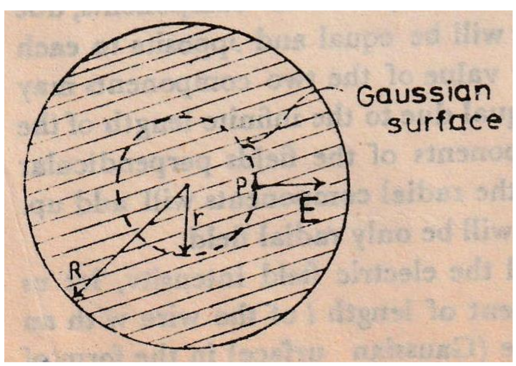


Fig 2

Let ρ be the charge per unit volume. To find the field at P at a distance r from the center, we again imagine a spherical Gaussian surface of radius r< R, concentric with the sphere of charge as shown in fig (2). The electric field will be constant everywhere on this sphere and will be perpendicular to its surface, i.e., will be radial due to symmetry. The outward normal flux through the surface of this imaginary sphere will be:

$$\int_s E^{\wedge}.dS^{\wedge} = E_n \int_s dS = 4\pi r^2 E \ldots..(1)$$

Now the total charge enclosed by the Gaussian surface Q=Volume enclosed by it x charge density

$$Q = \frac{4\,\pi r^3 \rho}{3}$$

According to Gauss's theorem $\int_s E^{\wedge}.dS^{\wedge} = \frac{Q}{\varepsilon 0} = \frac{4\,\pi r^3 \rho}{3\varepsilon 0}$ ……………….(2)

From eqn (1) & (2) $4\pi r^2 E = \frac{4\,\pi r^3 \rho}{3\varepsilon 0}$ …………(3)

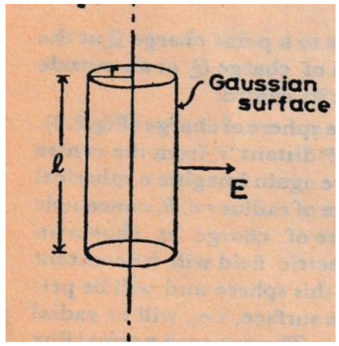
$$E = \frac{\rho}{3\varepsilon 0} r \ldots\ldots\ldots\ldots(4)$$

At the surface of the sphere $E = \frac{\rho}{3\varepsilon 0} R \ldots\ldots..(5)$ (since at surface r=R)

This computation can be used to obtain the electric field inside an atomic nucleus to a fairly good approximation.

2. Field of a thin Spherical shell of charge. The electric field intensity due to a spherical shell (i.e., a hollow sphere of thin wall) of charge at an outside point can be shown to be the same as that due to a uniform sphere of charge given by eqn. (3), i.e., the intensity of electric field at an outside point due to a thin spherical shell of charge is the same as that of a point charge at the centre of the sphere. The field at a point lying on the shell will a1so be given by eqn. (3) with R=r. However since a Gaussian surface lying inside the shell encloses no charge, the field at a point inside the shell will be zero.

3. Field of a line charge (or uniformly charged wire). Suppose we have an infinitely long straight uniformly charged wire or rod. If we neglect its thickness, it can be characterized by the amount of charge it carries per unit length λ. The electric field will be directed radially outward from the wire. There will be electric field perpendicular to the length of the wire only because parallel components to the wire will be cancelled. Let us surround a segment of the wire "l" with a cylinder of radius "r" and length "l" which is Gaussian surface in the form of a cylinder as shown in figure.



Fig

The outward normal flux through the cylindrical surface is,

$$\int_s E\ .dS= E\ 2\pi rl.......(1)$$

where E is the normal component of the field .The charge inside the

Gaussian surface is "l λ". Therefore according to Gauss's theorem,

$$\int_s E\ .dS = \frac{Q}{\varepsilon 0} = \frac{l\lambda}{\varepsilon 0}\ ....(2)$$

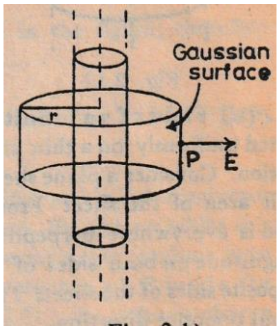
from (1) and (2); $E\ 2\pi rl = \frac{l\lambda}{\varepsilon 0}$

$$E= \frac{\lambda}{2\pi r\varepsilon 0}..................(3)$$

This shows that the intensity of electric field of a line charge is inversely proportional to the distance r from the line charge.

4. Field of a uniformly charged hollow cylinder

The field of uniformly charged hollow cylinder of infinite length (Fig) can be treated in a manner analogous to that of a long charged wire discussed above. The field at an outside point will be given by eqn.



Fig

$$E=\frac{\lambda}{2\pi r\varepsilon 0}.....(1)$$

Where λ is the charge per unit length of the cylinder and r is the distance of the given point from the axis of the cylinder. If the point P lies infinitely close to the surface of the cylinder, then

$$\lambda = 2\ r\ \sigma\ \pi,$$

where σ is the surface density of the cylinder (since charge on unit length of the cylinder is λ and surface area of unit length of cylinder is 2 π r).

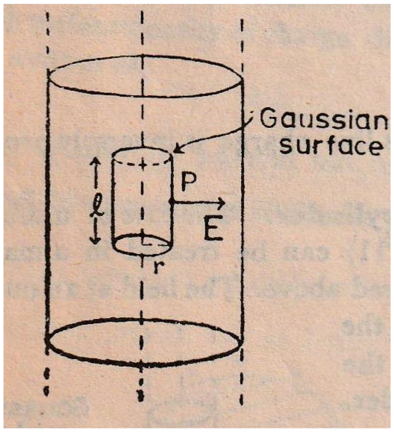
Therefore,

$$\mathbf{E}=\frac{\mathbf{2\pi r\sigma}}{\mathbf{2\pi r\varepsilon 0}}=\frac{\boldsymbol{\sigma}}{\boldsymbol{\varepsilon 0}}\ ....(2)$$

The field at an inside point will be zero as a Gaussian surface through this point will include no charge.

5. Field of a cylinder of uniform charge

If charge is uniformly distributed throughout the volume of an infinitely long cylinder, then the electric field at an outside point can be shown to be given by the eqn. below



Fig

$$E=\frac{\lambda}{2\pi r\varepsilon 0}\ ....(1)$$

The electric field at a point lying inside the cylinder can be determined as follows:

Considering a Gaussian surface in the form of a cylinder of length I, coaxial with the cylinder of charge and passing through the point P at which the field is to be determined. The field will be normal to the curved surface and will have equal magnitude at all point lying on it. The outward normal flux through the cylindrical Gaussian surface is,

$$\int_s E^{\wedge}.dS^{\wedge} = E\, 2\pi rl \text{.......}(2)$$

Now the total charge Q enclosed by the Gaussian surface is "$\pi r^2 l\rho$" where ρ is the volume charge density. According to Gauss's Law,

$$\int_s E^{\wedge}.dS^{\wedge} = \frac{Q}{\varepsilon 0} = \frac{\pi r^2 l\rho}{\varepsilon 0} \text{........}(3)$$

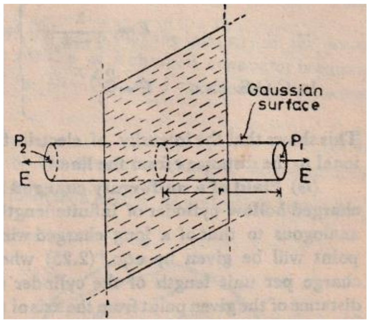
from (2) and (3) we have, $E\, 2\pi rl = \frac{\pi r^2 l\rho}{\varepsilon 0}$

$$\mathbf{E} = \frac{\mathbf{r\rho}}{\mathbf{2\varepsilon 0}} \text{...............}(4)$$

So the electric field intensity is directly proportional to the distance of the point from the axis of cylinder.

6. Field of an infinite Sheet of charge

Consider a plane sheet of charge infinite in extent. Let d be the charge per unit area of the sheet From considerations o[symmetry it can be said that the field is everywhere perpendicular to the plane and the field must have the same magnitude on both sides of the sheet. Let P_1 and P_2 be two equidistant points on opposite sides of the sheet. The field at these points will have the same magnitude E but opposite direction.



Fig

The intensity of electric field due to the sheet of charge may be determined by considering a cylinder that cuts the sheet, with P_1 on one side and P_2, on the other side (fig) as Gaussian surface. The end-faces of the cylinder parallel to the sheet will have equal area A. The field is normal to these faces and is parallel to the curved surface of the cylin.ler. Theref.ere the curved surface does not contribute to the outward flux. The total outward flux is equal to the sum of the contributions from the two end faces, i.e.

$$\int_s E^{\wedge}.dS^{\wedge} = EA + EA = 2\, EA \text{.......}(1)$$

Total charge enclosed by the cylinder is Q= σA

From Gauss law $\int_s E^{\wedge}.dS^{\wedge} = \frac{Q}{\varepsilon 0} = \frac{\sigma A}{\varepsilon 0}$ ....(2)

Equating the two eqns. $2EA = \frac{\sigma A}{\varepsilon 0}$

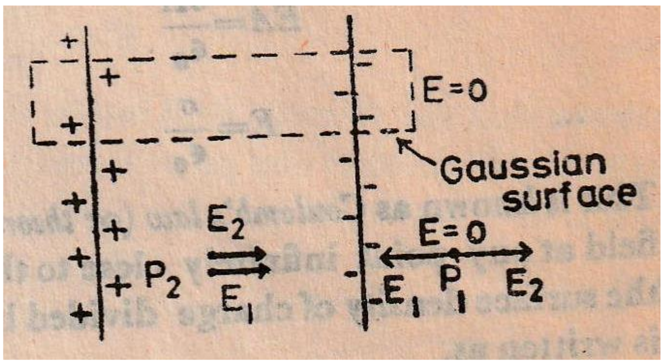
$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This shows that the field intensity is independent of the distance of the point from the charge sheet.

7. **Field of two sheets of charge**

The electric field intensity due to two plane parallel infinite of charge with equal and opposite charge densities $+\sigma$ and $-\sigma$ can be found with the help of Gauss theorem. Since the electric field due to a single charge sheet is $\sigma/2\varepsilon_0$. The total electric field at an outside point P1 is zero and at an inside point P2 it is $E=\sigma/\varepsilon_0$ due to the fact that directions of E_1 and E_2 due to two sheets is opposite at P_1 and towards right at P_2. This can also be calculated by considering a Gaussian surface in form of a box which contains both sheets as shown in figure below. Total charge in the box seeing from outside point is zero so from Gauss law total electric field at an outside point is zero. And at an inside point total field is

$$E=E_1+E_2= \frac{\sigma}{2\varepsilon_0}+\frac{\sigma}{2\varepsilon_0}=\frac{\sigma}{\varepsilon_0}$$

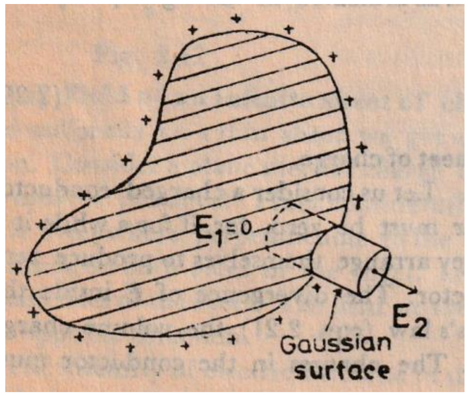


Fig

8. Field of a charged conductor

The field inside the conductor must be zero, for, if for a while it is not zero, the free electrons will move until they arrange themselves to produce zero electric field everywhere inside the conductor. The divergence of E inside the conductor is therefore also zero and by Gauss's law the volume charge density inside the conductor will be zero. The charges in the conductor must move to the surface of the conductor.

When the conductor is charged, the charges reside on its surface where there are strong forces to stop them from leaving the surface. The electric field just outside the surface of a conductor must be normal to the surface, because, if it has a tangential component the electrons will move along the surface. In other words the field lines must always be normal to the surface of the conductor which is an equipotential surface. The field outside a charged conductor can be found with the help of Gauss's theorem. Let there be a small cylindrical box lying half inside and half outside the conductor as Gaussian surface as shown in fig below.



Fig

Only the end-face of the box lying outside the conductor will contribute to the total normal flux of E through the box at the curved surface of the box is parallel to the field and the field inside the conductor is zero. Therefore the total outward flux is given by:

$$\int_s \hat{E}.dS = EA \text{..........(1)}$$

where A is the surface area of the end face of the cylinder. The total charge enclosed by the Gaussian surface is σA, where σ is the surface charge density. Therefore from Gauss theorem,

$$\int_s \hat{E}.dS = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \text{......(2)}$$

from eqn (1) and (2) we have $EA = \frac{\sigma A}{\varepsilon_0}$

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0} \text{................(3)}$$

This is known as Coulomb's law and states that the intensity of the electric field at any point infinitely close to the surface of a charged conductor is equal to the surface density of charge divided by ε_0.
In the case of a charged non-conductor, the field close to the surface is $\sigma/2\varepsilon_0$ as we have seen in application of single charge sheet.