Linear Algebra - Hoffman & Kunze (Problems)

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First part

Section 3.5

6. Let m and n be positive integers and $\mathbb{F}$ a field. Let $f_1, \ldots, f_m$ be linear functionals on $\mathbb{F}^n$. For α in $\mathbb{F}^n$ define

$$T\alpha = (f_1(\alpha), \ldots, f_m(\alpha)).$$

Show that T is a linear transformation from $\mathbb{F}^n$ into $\mathbb{F}^m$. Then show that every linear transformation from F^n into F^m is of the above form, for some $f_1, \ldots, f_m$

Proof.
The transformation T is linear:

$$\begin{aligned}
T(\alpha + c\beta) &= (f_1(\alpha + c\beta), \ldots, f_m(\alpha + c\beta)) \\
&= (f_1(\alpha) + cf_1(\beta), \ldots, f_m(\alpha) + cf_m(\beta)) \\
&= (f_1(\alpha), \ldots, f_m(\alpha)) + c(f_1(\beta), \ldots, f_m(\beta)) \\
&= T\alpha + cT\beta.
\end{aligned}$$

We show that for every linear transformation T from $\mathbb{F}^n$ to $\mathbb{F}^m$ there exists $f_1, \ldots, f_m \in V^*$ such that $T\alpha = (f_1(\alpha), \ldots, f_m(\alpha))$. Let $(\alpha_1, \ldots, \alpha_n) := \alpha$ be a vector $\alpha \in \mathbb{F}^n$ represented in the canonical basis $\mathscr{B}$. Then we can write the linear transformation $T\alpha = (\sum_{i=1}^n a_{1i}\alpha_i, \ldots, \sum_{i=1}^n a_{mi}\alpha_i)$, thus we define the linear functionals:

$$f_1 := \sum_{i=1}^n a_{1i}\varphi^i,$$

$$\vdots$$

$$f_m := \sum_{i=1}^n a_{mi}\varphi^i;$$

where $\{\varphi_1, \ldots, \varphi^n\}$ is the basis dual to $\mathscr{B}$. Hence, the linear transformation T is

$$T = (f_1, \ldots, f_m).$$

□

Section 3.7

1. Let $\mathbb{F}$ be a field and let f be the linear functional on $\mathbb{F}^2$ defined by $f(x_1, x_2) = ax_1 + bx_2$. For each of the following linear operators T, let $g = T^t g$, and find $g(x_1, x_2)$.

(a) $T(x_1, x_2) = (x_1, 0)$;

(b) $T(x_1, x_2) = (-x_2, x_1)$;

(c) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$.

Solution.

(a)

$$\begin{aligned} g(x_1, x_2) &= T^t f(x_1, x_2) \\ &= f(T(x_1, x_2)) \\ &= f(x_1, 0) \\ &= a x_1. \end{aligned}$$

(b)

$$\begin{aligned} g(x_1, x_2) &= T^t f(x_1, x_2) \\ &= f(T(x_1, x_2)) \\ &= f(-x_2, x_1) \\ &= -a x_2 + b x_1. \end{aligned}$$

(c)

$$\begin{aligned} g(x_1, x_2) &= T^t f(x_1, x_2) \\ &= f(T(x_1, x_2)) \\ &= f(x_1 - x_2, x_1 + x_2) \\ &= (a + b) x_1 + (b - a) x_2. \end{aligned}$$

□

Section 8.1

2. Let V be a vector space over $\mathbb{F}$. Show that the sum of two inner products on V is an inner product on V. Is the difference of two inner products an inner product? Show that a positive multiple of an inner product is an inner product.

Proof.
Let $\langle\cdot|\cdot\rangle_1$ and $\langle\cdot|\cdot\rangle_2$ be inner products on V. In order to $\langle\cdot|\cdot\rangle$, defined by $\langle\alpha|\beta\rangle = \langle\alpha|\beta\rangle_1 + \langle\alpha|\beta\rangle_2$ for all α, $\beta \in V$, to be an inner product it must satisfy

(a) $\langle\alpha + \beta|\gamma\rangle = \langle\alpha|\gamma\rangle + \langle\beta|\gamma\rangle$,

(b) $\langle c\alpha|\beta\rangle = c\,\langle\alpha|\beta\rangle$,

(c) $\langle\beta|\alpha\rangle = \overline{\langle\alpha|\beta\rangle}$,

(d) $\langle\alpha|\alpha\rangle > 0$ if $\alpha \neq 0$.

Then

(a)

$$\begin{aligned} \langle\alpha + \beta|\gamma\rangle &= \langle\alpha + \beta|\gamma\rangle_1 + \langle\alpha + \beta|\gamma\rangle_2 \\ &= \langle\alpha|\gamma\rangle_1 + \langle\alpha|\gamma\rangle_2 + \langle\beta|\gamma\rangle_1 + \langle\beta|\gamma\rangle_2 \\ &= \langle\alpha|\gamma\rangle + \langle\beta|\gamma\rangle \end{aligned}$$

(b)

$$\begin{aligned}\langle c\alpha|\beta\rangle &= \langle c\alpha|\beta\rangle_1 + \langle c\alpha|\beta\rangle_2 \\ &= c\,\langle\alpha|\beta\rangle_1 + c\,\langle\alpha|\beta\rangle_2 \\ &= c(\langle\alpha|\beta\rangle_1 + \langle\alpha|\beta\rangle_2) \\ &= c\,\langle\alpha|\beta\rangle\end{aligned}$$

(c)

$$\begin{aligned}\langle\beta|\alpha\rangle &= \langle\beta|\alpha\rangle_1 + \langle\beta|\alpha\rangle_2 \\ &= \overline{\langle\alpha|\beta\rangle_1} + \overline{\langle\beta|\alpha\rangle_2} \\ &= \overline{\langle\alpha|\beta\rangle_1 + \langle\beta|\alpha\rangle_2} \\ &= \overline{\langle\alpha|\beta\rangle}\end{aligned}$$

(d)

$$\langle\alpha|\alpha\rangle = \langle\alpha|\alpha\rangle_1 + \langle\alpha|\alpha\rangle_2 > 0$$

The difference of two inner products is in general *not an inner product*. Indeed, let $\langle\cdot|\cdot\rangle_2 = \langle\cdot|\cdot\rangle_1$, then (d) isn't satisfied: $\langle\alpha|\alpha\rangle = \langle\alpha|\alpha\rangle_1 - \langle\alpha|\alpha\rangle_1 = 0$. A positive multiple of an inner product, defined by $\langle\alpha|\beta\rangle = \lambda\,\langle\alpha|\beta\rangle_1$ with $\lambda > 0$, is an inner product since it trivially holds (a), (b), (c) and (d). $\square$

Section 8.2

1. Consider $\mathbb{R}^4$ with the standard inner product. Let W be the subspace of $\mathbb{R}^4$ consisting of all vectors which are orthogonal to both $\alpha = (1, 0, -1, 1)$ and $\beta = (2, 3, -1, 2)$. Find a basis for W.

Solution.
Let $S := \operatorname{span}\{\alpha, \beta\}$ and W the subspace orthogonal to S. We know that $\dim W + \dim S = \dim V$, therefore a basis for W is a set of two linearly independent solutions to

$$\begin{pmatrix}1 & 0 & -1 & 1\\ 2 & 3 & -1 & 2\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\\ w_4\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Hence, we may choose $\{e_1, e_2\}$ as basis for W, where $e_1 = (2, -1, 3, 1)$ and $e_2 = (1, 1/3, -1, -2)$. $\square$

9. Let V be the subspace of $\mathbb{R}[x]$ of polynomials of degree at most 3. Equip V with the inner products

$$\langle f|g\rangle = \int_0^1 f(t)g(t)dt.$$

(a) Find the orthogonal complement of the subspace of scalar polynomials.

(b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$.

Solution.

(a) We shall answer this with the results of (b).

(b) Since $\dim V = 4$ we need to find

$$\begin{aligned}
\alpha_1 &= 1,\\
\alpha_2 &= x - \frac{\langle x|\alpha_1\rangle}{\|\alpha_1\|^2}\alpha_1,\\
\alpha_3 &= x^2 - \frac{\langle x^2|\alpha_1\rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle x^2|\alpha_2\rangle}{\|\alpha_2\|^2}\alpha_2,\\
\alpha_4 &= x^3 - \frac{\langle x^3|\alpha_1\rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle x^3|\alpha_2\rangle}{\|\alpha_2\|^2}\alpha_2 - \frac{\langle x^3|\alpha_3\rangle}{\|\alpha_3\|^2}\alpha_3,
\end{aligned}$$

Calculating

$$\begin{aligned}
\langle x|\alpha_1\rangle &= \frac{1}{2}, & \|\alpha_1\|^2 &= 1,\\
\alpha_2 &= x - \frac{1}{2}, & &\\
\langle x^2|\alpha_1\rangle &= \frac{1}{3}, & \|\alpha_2\|^2 &= \int_0^1 x^2 - x + \frac{1}{4}dx = \frac{1}{12},\\
\langle x^2|\alpha_2\rangle &= \int_0^1 x^3 - \frac{1}{2}x^2 dx = \frac{1}{12}, & &\\
\alpha_3 &= x^2 - x + \frac{1}{6}, & &\\
\langle x^3|\alpha_1\rangle &= \frac{1}{4}, & \|\alpha_3\|^2 &= \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx = \frac{1}{180},\\
\langle x^3|\alpha_2\rangle &= \int_0^1 x^4 - \frac{1}{2}x^3 dx = \frac{3}{40}, & &\\
\langle x^3|\alpha_3\rangle &= \int_0^1 x^5 - x^4 + \frac{1}{6}x^3 dx = \frac{1}{120}, & &\\
\alpha_4 &= x^3 - \frac{3}{4}x^2 + \frac{3}{5}x - \frac{1}{20}. & &
\end{aligned}$$

Thus, we get an orthogonal basis:

$$\begin{aligned}
\alpha_1 &= 1,\\
\alpha_2 &= x - \frac{1}{2},\\
\alpha_3 &= x^2 - x + \frac{1}{6},\\
\alpha_4 &= x^3 - \frac{3}{4}x^2 + \frac{3}{5}x - \frac{1}{20}.
\end{aligned}$$

Finally, the answer to (a) is: $W := \operatorname{span}\{\alpha_2, \alpha_3, \alpha_4\}$ is the orthogonal complement of the subspace of scalar polynomials. □

Section 8.3

5. Let V be a finite-dimensional inner product space and T a linear operator on V. If T is invertible, show that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Proof.
Since T is invertible, it follows:

$$\begin{aligned}\langle\alpha|\beta\rangle &= \left\langle T^{-1}T\alpha\middle|\beta\right\rangle, \\ &= \left\langle T\alpha\middle|(T^{-1})^*\beta\right\rangle, \\ &= \left\langle\alpha\middle|T^*(T^{-1})^*\beta\right\rangle.\end{aligned}$$

Hence, $T^*(T^{-1})^* = \mathbb{I}$, for which T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

□

Section 8.4

1. Find a unitary matrix which is not orthogonal, and find an orthogonal matrix which is not unitary.

Solution.
The matrix:

$$A := \begin{pmatrix} \frac{1}{\sqrt{3}} & i\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

is *unitary and not ortogonal.* Indeed:

$$\begin{aligned}A^*A &= \begin{pmatrix} \frac{1}{\sqrt{3}} & -i\sqrt{\frac{2}{3}} \\ -i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & i\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \\ &= \begin{pmatrix} \frac{1}{3}+\frac{2}{3} & i\frac{\sqrt{2}}{3} - i\frac{\sqrt{2}}{3} \\ -i\frac{\sqrt{2}}{3}+i\frac{\sqrt{2}}{3} & \frac{2}{3}+\frac{1}{3} \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ A^tA &= \begin{pmatrix} \frac{1}{\sqrt{3}} & i\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & i\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \\ &= \begin{pmatrix} \frac{1}{3}-\frac{2}{3} & i\frac{\sqrt{2}}{3} + i\frac{\sqrt{2}}{3} \\ i\frac{\sqrt{2}}{3}+i\frac{\sqrt{2}}{3} & -\frac{2}{3}+\frac{1}{3} \end{pmatrix}, \\ &= \begin{pmatrix} -\frac{1}{3} & 2i\frac{\sqrt{2}}{3} \\ 2i\frac{\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}.\end{aligned}$$

The matrix:

$$B := \begin{pmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{pmatrix},$$

is *orthogonal and not unitary.* Indeed:

$$\begin{aligned}
B^*B &= \begin{pmatrix} \sqrt{2} & -i \\ -i & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{pmatrix}, \\
&= \begin{pmatrix} 2+1 & i\sqrt{2}+i\sqrt{2} \\ -i\sqrt{2}-i\sqrt{2} & 1+2 \end{pmatrix}, \\
&= \begin{pmatrix} 3 & 2i\sqrt{2} \\ -2i\sqrt{2} & 3 \end{pmatrix}, \\
B^tB &= \begin{pmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & i \\ i & -\sqrt{2} \end{pmatrix}, \\
&= \begin{pmatrix} 2-1 & i\sqrt{2}-i\sqrt{2} \\ i\sqrt{2}-i\sqrt{2} & -1+2 \end{pmatrix}, \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

□

Section 8.5

2. Is a complex symmetric matrix self-adjoint? Is it normal?

Solution.
A complex symmetric matrx is in general *neither self-adjoint nor normal.* Counterexample, let $A \in M_2(\mathbb{C})$ be the symmetric matrix

$$A := \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

The matrix A is not self-adjoint, i.e. $A \neq A^*$:

$$A^* = \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \neq A.$$

The matrix A is not normal, i.e. $AA^* \neq A^*A$:

$$AA^* = 2\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \tag{1}$$

$$A^*A = 2\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \tag{2}$$

$$\neq AA^*. \tag{3}$$

□

10. Prove that every positive matrix is the square of a positive matrix.

Proof.
Let $A \in M_n(\mathbb{C})$ positive. Then, A is Hermitian and by the corollary of theorem 18 (section 8.5) there is a unitary matrix P such that $\Lambda := P^{-1}AP$ is diagonal. Therefore, we have $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_j > 0$ for all $j = 1, \dots, n$ since A is a positive the matrix.

Define $\Lambda^{1/2} := \operatorname{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ and $B := P\Lambda^{1/2}P^{-1}$, then B is a positive matrix and

$$\begin{aligned} B^2 &:= BB \\ &= P\Lambda^{1/2}P^{-1}P\Lambda^{1/2}P^{-1} \\ &= P\Lambda^{1/2}\Lambda^{1/2}P^{-1} \\ &= P\Lambda P^{-1} \\ &= A. \end{aligned}$$

□

Second part

Section 9.2

1. Which of the following functions f, defined on vectors $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in C^2, are (sesqui-linear) forms on C^2?

(a) $f(\alpha, \beta) = 1$.

(b) $f(\alpha, \beta) = (x_1 - \bar{y}_1)^2 + x_2\bar{y}_2$.

(c) $f(\alpha, \beta) = (x_1 + \bar{y}_1)^2 - (x_1 - \bar{y}_1)^2$.

(d) $f(\alpha, \beta) = x_1\bar{y}_2 - \bar{x}_2 y_1$.

Solution.

(a) No, $f(\alpha + \beta, \gamma) = 1 \neq f(\alpha, \gamma) + f(\beta, \gamma)$.

(b) No, $f(\alpha, c\beta + \gamma) \neq \bar{c}f(\alpha, \beta) + f(\alpha, \gamma)$.

(c) Yes,

$$\begin{aligned} f(\alpha + c\beta, \gamma) &= 4x_1\bar{z}_1 + 4cy_1\bar{z}_1 = f(\alpha, \gamma) + cf(\beta, \gamma) \\ f(\alpha, c\beta + \gamma) &= 4\bar{c}x_1\bar{y}_1 + 4x_1\bar{z}_1 = \bar{c}f(\alpha, \beta) + f(\alpha, \gamma) \end{aligned}$$

(d) No, $f(\alpha, c\beta + \gamma) = \bar{c}x_1\bar{y}_2 - c\bar{x}_2 y_1 + x_1\bar{z}_2 - \bar{x}_2 z_1 \neq \bar{c}f(\alpha, \beta) + f(\alpha, \gamma)$.

□

9. Let f be a non-degenerate form on a finite-dimensional space V. Show that each linear operator S has an 'adjoint relative to f', i.e., an operator S' such that

$$f(S\alpha, \beta) = f(\alpha, S'\beta) \quad \text{for all} \quad \alpha, \beta.$$

Proof.
We define

$$\ell : V \to V^* \ : \ \beta \to (\alpha \to f(S\alpha, \beta)),$$

and for each $\beta \in V$ we write $\ell(\beta) = \ell_\beta \in V^*$. Since f is non-degenerate and ℓ_β is a linear functional on V, by exercise 8 (section 9.2) there exists a unique $\beta' \in V$ such that

$$\ell_\beta(\alpha) = f(\alpha, \beta'),$$

i.e.

$$f(S\alpha, \beta) = f(\alpha, \beta').$$

Let S be the operator which maps β to β' ($f(S\alpha, \beta) = f(\alpha, S'\beta)$). It remains to prove that S is linear:

$$\begin{aligned} f(\alpha, S'(\beta + c\gamma)) &= f(S\alpha, \beta + c\gamma) && (4)\\ &= f(S\alpha, \beta) + cf(S\alpha, \gamma) && (5)\\ &= f(\alpha, S'\beta) + cf(\alpha, S'\gamma) && (6)\\ &= f(\alpha, S'\beta + cS'\gamma). && (7)\end{aligned}$$

Hence, $S'(\beta + c\gamma) = S'\beta + cS'\gamma$. □

Section 9.3

7. Give an example of an $n \times n$ matrix which has all its principal minors positive, but which is not a positive matrix.

Solution.
Consider the matrix

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The principal minors of A are

$$\Delta_1(A) = 1, \quad \Delta_2(A) = 1, \quad \Delta_3(A) = 1;$$

and A isn't a positive matrix since $A^t \neq A$. □

14. Prove that the product of two positive linear operators is positive if and only if they commute.

Proof.
Let T_1 and T_2 be positive operators, then $(T_i)^* = T_i$ and $\langle T_i\alpha|\alpha\rangle > 0$ for all $\alpha \neq 0$ in V. We show that if T_1T_2 is positive then $T_1T_2 = T_2T_1$:

Since T_1, T_2 and T_1T_2 are positive, it follows

$$\begin{aligned} T_1T_2 &= (T_1T_2)^* \\ &= T_2^*T_1^* \\ &= T_2T_1. \end{aligned}$$

Conversely, we show that if $T_1T_2 = T_2T_1$ then $(T_1T_2)^* = T_1T_2$ and $\langle T_1T_2\alpha|\alpha\rangle > 0$ for all $\alpha \neq 0$:

For the first part, $T_1T_2 = T_2T_1 = T_2^*T_1^* = (T_1T_2)^*$. For the second part, we have from the problem 10 of section 8.5 that on each basis the matrix of T_i is the square of a positive matrix. Let B be the operator $B^2 = T_2$, then $T_1BB = BBT_1 = BB^*T_1^* = BT_1B$, hence

$$\langle T_1BB\alpha|\alpha\rangle = \langle BT_1B\alpha|\alpha\rangle = \langle T_1B\alpha|B\alpha\rangle = \langle T_1\tilde{\alpha}|\tilde{\alpha}\rangle > 0.$$

□

Section 10.1

5. Describe the bilinear forms on R^3 which satisfy $f(\alpha, \beta) = -f(\beta, \alpha)$ for all α, β.

Solution.
We choose the canonical basis $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, and let:

$$A := (f(e_i, e_j)) = \begin{bmatrix} 0 & A_{12} & -A_{31} \\ -A_{12} & 0 & A_{23} \\ A_{31} & -A_{23} & 0 \end{bmatrix}$$

where $A_{ij} = f(e_i, e_j)$. Then, for vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ on $\mathbb{R}^3$ we get

$$\begin{aligned} f(\alpha, \beta) &= [\alpha]^t A[\beta] \\ &= A_{12}(\alpha_1\beta_2 - \alpha_2\beta_1) + A_{23}(\alpha_2\beta_3 - \alpha_3\beta_2) + A_{31}(\alpha_3\beta_1 - \alpha_3\beta_1). \end{aligned}$$

□

Section 10.2

5. Let q be the quadratic form on $\mathbb{R}^2$ given by

$$q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2, \qquad a \neq 0.$$

Find an invertible linear operator U on $\mathbb{R}^2$ such that

$$(U^\dagger q)(x_1, x_2) = ax_1^2 + \left(c - \frac{b^2}{a}\right) x_2^2,$$

Solution.
Completing the square,

$$\begin{aligned} q(x_1, x_2) &= ax_1^2 + 2bx_1x_2 + cx_2^2 \\ &= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(c - \frac{b^2}{a}\right) x_2^2 \\ &= q(U^{-1}U(x_1, x_2)), \end{aligned}$$

then, $U^{-1}(\tilde{x}_1, \tilde{x}_2) = \left(\tilde{x}_1 + \frac{b}{a}\tilde{x}_2, \tilde{x}_2\right)$. Therefore,

$$U(x_1, x_2) = \left(x_1 - \frac{b}{a}x_2, x_2\right).$$

Indeed

$$\begin{aligned} q(U(x_1, x_2)) &= a\left(x_1 + \frac{b}{a}x_2\right)^2 + \left(c - \frac{b^2}{a}\right) x_2^2 \\ &= (U^\dagger q)(x_1, x_2). \end{aligned}$$

□