Sampling-based Motion Planning (Pre Lecture)

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Outline

Sampling-Based Motion Planning

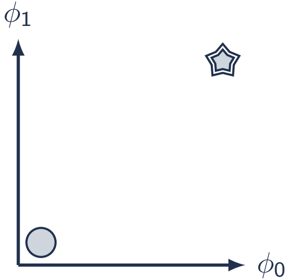
Metric Spaces

Rapidly-Exploring Random Trees (RRT)

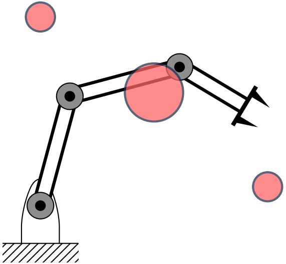
Probabilistic Roadmaps (PRM)

Overview of Sampling-based Motion Planning

1. Start from initial configuration
2. Sample new states in the configuration space
3. Add valid states to tree or graph
4. Terminate when:
    - We find a path to the goal in the tree/graph
    - We exhaust a timeout



Configuration Space



- Configuration Space $\mathcal{Q}$
- Obstacle Region: $\mathcal{Q}_{\text{obs}} = \{q \in \mathcal{Q} \mid \neg\texttt{is-valid}(q)\}$
- Free Space: $\mathcal{Q}_{\text{free}} = \{q \in \mathcal{Q} \mid \texttt{is-valid}(q)\}$
- Total Space:
  - $\mathcal{Q}_{\text{obs}} \cap \mathcal{Q}_{\text{free}} = \emptyset$
  - $\mathcal{Q}_{\text{obs}} \cup \mathcal{Q}_{\text{free}} = \mathcal{Q}$

Generally: no explicit representation of $\mathcal{Q}_{\text{obs}}$, $\mathcal{Q}_{\text{free}}$

Fundamental Abstractions

Configuration Space: Set of possible configurations

- $\mathcal{Q}$

Configuration Sampler: Generates candidate configurations

- $() \mapsto \mathcal{Q}$

Validity Checker: Is some configuration in $\mathcal{Q}_{\text{free}}$?

- $\underbrace{\mathcal{Q}}_{\text{sample}} \mapsto \underbrace{\{0,1\}}_{\text{valid?}}$

Nearest Neighbors: Closest point in tree/graph to new sample:

- $\underbrace{\mathcal{P}\left(\mathcal{Q}\right)}_{\text{tree}} \times \underbrace{\mathcal{Q}}_{\text{new sample}} \mapsto \underbrace{\mathcal{Q}}_{\text{nearest neighbor}}$

Sampling-based motion planning generalizes beyond robotics.

Outline

Definition: Metric Space

Definition: Metric Space

A **metric space** is a space $\mathcal{Q}$ equipped with a function $\rho : \mathcal{Q} \times \mathcal{Q} \mapsto \mathbb{R}$ that has the following properties for any $a, b, c \in \mathcal{Q}$

Nonnegative: $\rho(a, b) \geq 0$
(Distances are always greater than zero)

Reflexive: $(\rho(a, b) = 0) \iff (a = b)$
(Distance is zero only for identical elements)

Symmetric: $\rho(a, b) = \rho(b, a)$
(Distance from a to b is the same as from b to a)

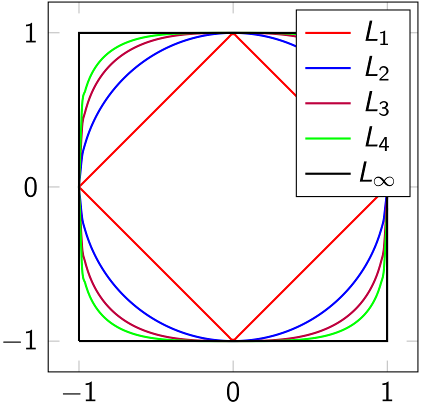
Triangle: $\rho(a, b) + \rho(b, c) \geq \rho(a, c)$
(a to b to c cannot be shorter than directly from a to c)

Example: Robot Joint Space and L_p metrics

$$\rho(x, x') = \left(\sum_{i=0}^{n-1} \left| x_i - x'_i \right|^p \right)^{\frac{1}{p}}$$

- **Space:**
  $\mathcal{Q} \subseteq \mathbb{R}^n$, the set of robot joint positions
- **Euclidean Distance:**
  $L_2(x, x') = \sqrt{\sum_{i=0}^{n-1} \left(x_i - x'_i\right)^2}$
- **Manhattan Distance:**
  $L_1(x, x') = \sum_{i=0}^{n-1} \left| x_i - x'_i \right|$
- **$\mathbf{L}_\infty$:**
  $L_\infty(x, x') = \max_{0 \leq i \leq n-1} \left(\left| x_i - x'_i \right| \right)$

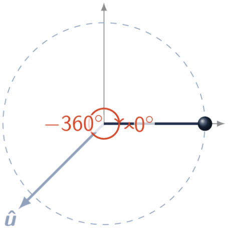
Unit "Circle"



Example: Rotation Metric

Quaternion Double-Cover

Axis-Angle



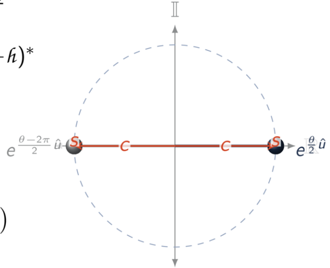
$$\exp\left(\tfrac{\theta}{2}\hat{u}\right) = \sin\tfrac{\theta}{2}\hat{u} + \cos\tfrac{\theta}{2}$$

$$h \otimes \vec{p} \otimes h^* = (-h) \otimes \vec{p} \otimes (-h)^*$$

$$\begin{aligned} \sin\tfrac{\theta}{2} &= -\sin\tfrac{\theta-2\pi}{2} \\ \cos\tfrac{\theta}{2} &= -\cos\tfrac{\theta-2\pi}{2} \end{aligned}$$

$$\begin{aligned} \exp\left(\tfrac{\theta}{2}\hat{u}\right) &= -\exp\left(\tfrac{\theta-2\pi}{2}\hat{u}\right) \\ \exp\left(\tfrac{\theta}{2}\hat{u}\right) &\sim \exp\left(\tfrac{\theta-2\pi}{2}\hat{u}\right) \end{aligned}$$

Quaternion



$$\begin{aligned} \theta &= 0.000 \\ \theta - 2\pi &= -6.280 \end{aligned}$$

$$\begin{aligned} & 0.000\ \hat{\boldsymbol{u}} + 1.000 \\ \sim\ & 0.000\ \hat{\boldsymbol{u}} + -1.000 \end{aligned}$$

Example: Rotation Metric

Distance Functions

$$\exp\left(\frac{\theta}{2}\hat{u}\right) = \sin\frac{\theta}{2}\hat{u} + \cos\frac{\theta}{2} \qquad \text{and} \qquad \exp\left(\frac{\theta}{2}\hat{u}\right) \sim \exp\left(\frac{\theta - 2\pi}{2}\hat{u}\right)$$

Quaternion Space

- L_p Norm:
  - $\min\left\{\|p - q\|, \|p + q\|\right\}$
  - Length of 4D line segment
- 4D angle:
  - $\min\left\{\cos^{-1}\left(q \bullet p\right), \cos^{-1}\left(-q \bullet p\right)\right\}$
  - Angle between 4D quaternions

Log Space

$$\min\left\{\|\ln\left(q \otimes p^*\right)\|, \|\ln\left(-q \otimes p^*\right)\|\right\}$$

- $p \otimes h_r = q \quad \rightsquigarrow \quad h_r = q \otimes p^*$
- Matrix: $\|\ln\left(\mathbf{PQ}^{-1}\right)\|$

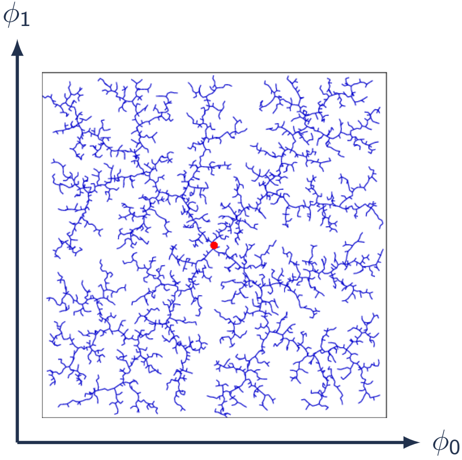
Exercise: Transformation Metric

Dual Quaternion: $\rho\left(h_1 + d_1\varepsilon, h_2 + d_2\varepsilon\right)$

Transformation Matrix: $\rho\left(\begin{bmatrix}\mathbf{R}_1 & \mathbf{v}_1 \\ \mathbf{0} & \mathbf{1}\end{bmatrix}, \begin{bmatrix}\mathbf{R}_2 & \mathbf{v}_2 \\ \mathbf{0} & \mathbf{1}\end{bmatrix}\right)$

Outline

RRT Illustration



http://msl.cs.uiuc.edu/rrt/

RRT Algorithm

Algorithm 1: Rapidly-Exploring Random Tree

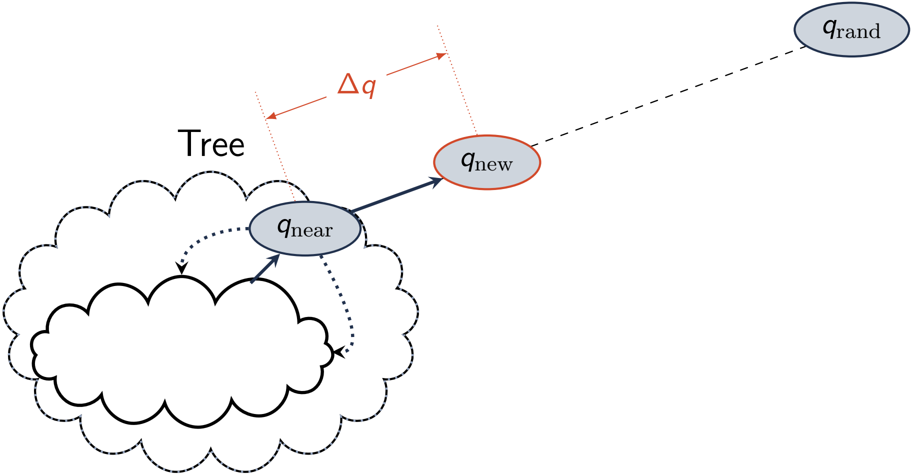
Input: $\phi_0 \in \mathcal{Q}$: start
Input: $\Phi_g \subseteq \mathcal{Q}$: goal set

$V \leftarrow \{\phi_0\}$;
$E \leftarrow \emptyset$;
for $k \leftarrow 0$ **to** `LIMIT` **do**
  $q_{\mathrm{rand}} \leftarrow$ `sample`$()$;
  $q_{\mathrm{near}} \leftarrow$ `nearest-neighbor`(V, q_{rand});
  $q_{\mathrm{new}} \leftarrow$ `new-conf`$(q_{\mathrm{near}}, q_{\mathrm{new}})$;
  if `valid`(q_{new}) **then**
    if $\exists q \in \Phi_g,$ `dist`$(q, q_{\mathrm{new}}) < \epsilon$ **then**
      return path from q_0 to q_{new};
    $V \leftarrow V \cup \{q_{\mathrm{new}}\}$;
    $E \leftarrow E \cup \{q_{\mathrm{near}} \rightarrow q_{\mathrm{new}}\}$;
return `TIMEOUT`;

Subroutines

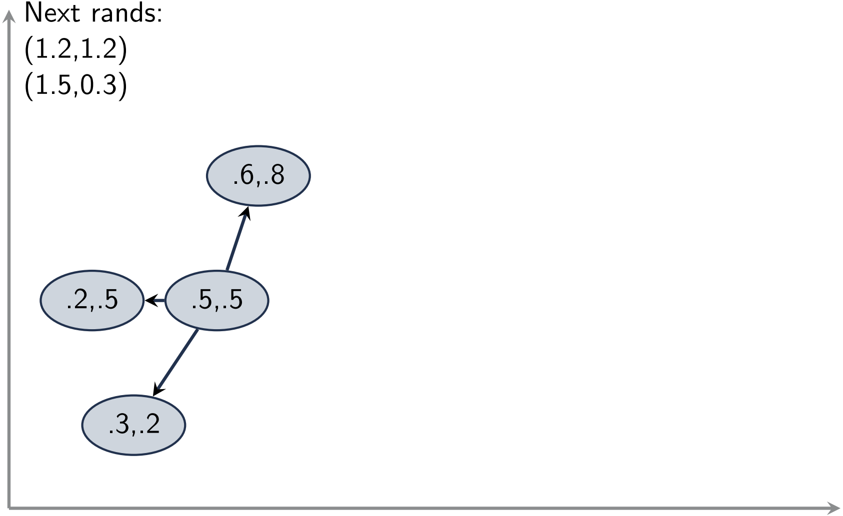
- `sample`
- `nearest-neighbor`
- `new-conf`
- `valid`
- `dist`

RRT Step Illustration



Exercise: RRT-Step

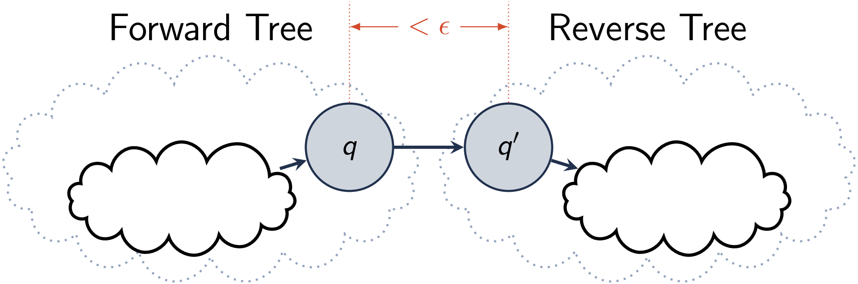
Next rands:
(1.2,1.2)
(1.5,0.3)



RRT-Connect

Bi-directional RRT

1. Construct one RRT, rooted at the start
2. Construct second RRT, rooted at the goal
3. Terminate when the two trees connect



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PRM Overview

Preprocessing: Construct the Roadmap

1. Sample new configurations
2. Connect to "neighboring" configurations in roadmap

Query: Search the Roadmap

1. Find path in roadmap via discrete search (e.g., A*)
2. Connect subsequent configurations with "local planner" (interpolate)

PRM Construction

Algorithm

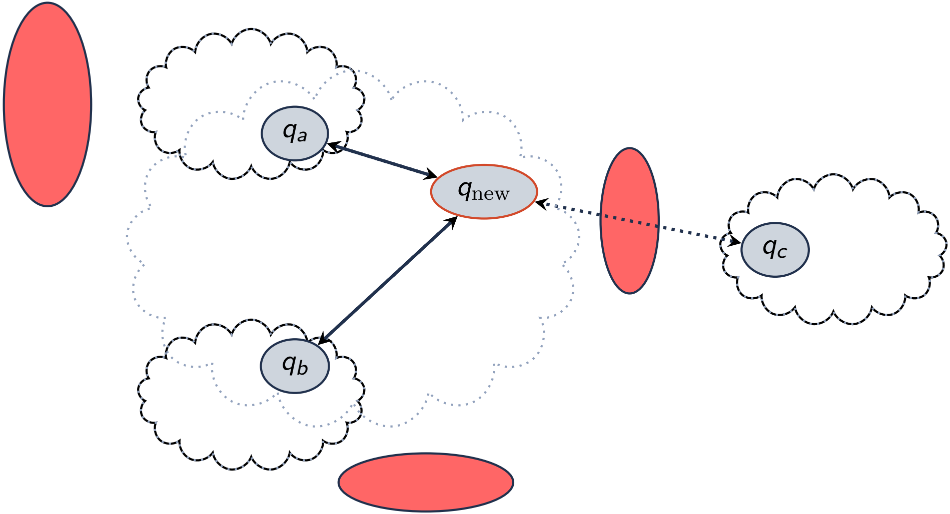
Algorithm 2: Construct Probabilistic Roadmap

$V \leftarrow \emptyset$;
$E \leftarrow \emptyset$;
$k \leftarrow 0$;
while $k < \texttt{LIMIT}$ **do**
    $q_{\text{new}} \leftarrow \texttt{sample}()$;
    if $\texttt{valid}(q_{\text{new}})$ **then**
        $V \leftarrow V \cup \{q_{\text{new}}\}$;
        $k \leftarrow k + 1$;
        foreach $q \in \texttt{neighborhood}(q_{\text{new}})$ **do**
            if $\neg\texttt{connected}(q, q_{\text{new}}) \wedge \texttt{connect}(q, q_{\text{new}})$ **then**
                $E \leftarrow E \cup \{q_{\text{new}} \rightarrow q_{\text{new}}\}$;
return (V, E);

- Neighbors:
  - K-nearest neighbors (KNN)
  - KNN from connected-components of V
  - Radius (within ϵ distance)
- Connected Components:
  - Optional: do we want different paths?
  - Efficiency: Track as edges are added
- Connect: valid path exists
  - E.g., linearly interpolate and check validity

PRM Construction

Illustration



Practical Considerations

- Nearest Neighbors: Kd-tree (partitioning) or brute force (for high-dof)
- Collision Checking: Bounding-boxes then decompose into triangles
- Paths: Smooth and Shortcut
- Hundreds of variations on RRT, PRM, and similar methods
- RRT-Connect usually works pretty well