Rotation (Pre Lecture)

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Spring 2018

Outline

The "Imaginary" Number

$$\hat{\imath}^2 = -1$$

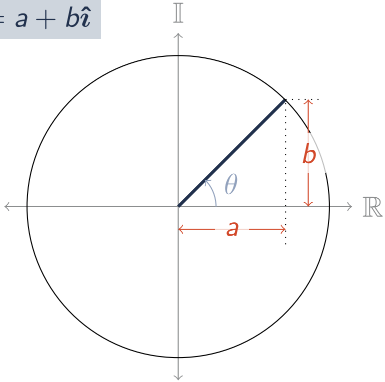
$$c = \overbrace{\underbrace{a}_{\text{real}} + \underbrace{b\hat{\imath}}_{\text{imaginary}}}^{\text{complex number}}$$

Complex Algebra

$$
\begin{aligned}
a &= a_r + a_i \hat{\imath} \\
b &= b_r + b_i \hat{\imath} \\
\textbf{Addition:} \quad (a_r + a_i \hat{\imath}) + (b_r + b_i \hat{\imath}) &= (a_r + b_r) + (a_i + b_i)\hat{\imath} \\
\textbf{Multiplication:} \quad (a_r + a_i \hat{\imath})(b_r + b_i \hat{\imath}) &= (a_r b_r - a_i b_i) + (a_r b_i + a_i b_r)\hat{\imath}
\end{aligned}
$$

Complex Plane

$$c = a + b\hat{\imath}$$



Euler's Formula

Theorem: Euler's Formula

$$e^{\theta \hat{\imath}} = \cos\theta + \hat{\imath}\sin\theta$$

- Exponential Properties:

$$\text{zero:} \quad e^0 = 1$$
$$\text{derivative:} \quad f^{(n)}\left(e^x\right) = e^x$$

- Maclaurin Series: $f(x) = \frac{f(0)}{0!}\left(x^0\right) + \frac{f'(0)}{1!}\left(x^1\right) + \frac{f''(0)}{2!}\left(x^2\right) + \ldots$
  - $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$
- $\hat{\imath}^0 = 1 \quad \hat{\imath}^1 = \hat{\imath} \quad \hat{\imath}^2 = -1 \quad \hat{\imath}^3 = -\hat{\imath}$
  $\hat{\imath}^4 = 1 \quad \ldots$

Euler's Formula

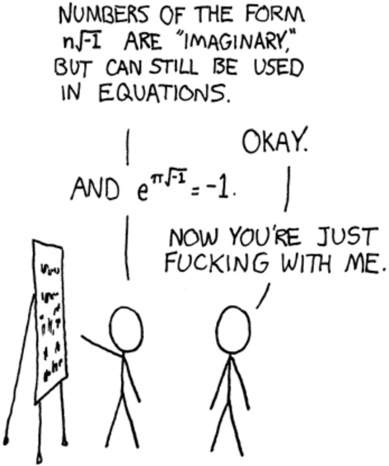
Proof

Proof Outline

$$
\begin{aligned}
e^{\theta\hat{\imath}} &= \frac{1}{0!} + \frac{(\theta\hat{\imath})^1}{1!} + \frac{(\theta\hat{\imath})^2}{2!} + \frac{(\theta\hat{\imath})^3}{3!} + \frac{(\theta\hat{\imath})^4}{4!} + \frac{(\theta\hat{\imath})^5}{5!} + \dots \\
&= \frac{1}{0!} + \frac{\theta\hat{\imath}}{1!} - \frac{\theta^2}{2!} - \frac{\theta^3\hat{\imath}}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5\hat{\imath}}{5!} + \dots \\
&= \left(\frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)\hat{\imath} \\
&= \cos\theta + \hat{\imath}\sin\theta
\end{aligned}
$$

$\square$

Exercise: Euler's Formula



https://xkcd.com/179/

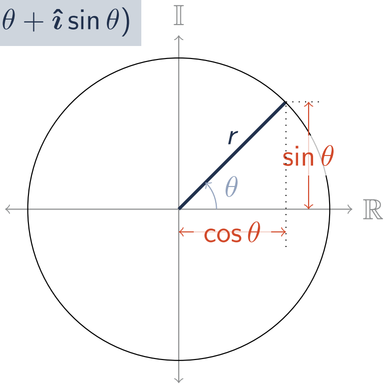
Proof

1. $e^{\pi\sqrt{-1}}$

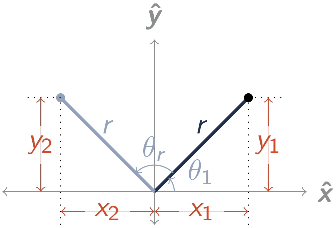
Complex Plane

Redux

$$re^{\theta\hat{\imath}} = r(\cos\theta + \hat{\imath}\sin\theta)$$



Planar Rotations



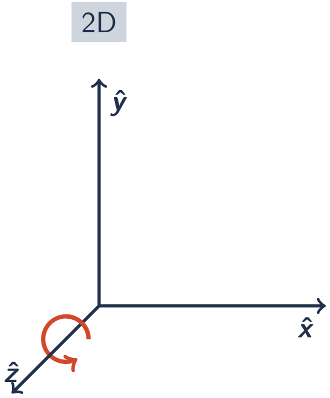
Rectangular / Matrix	**Polar / Complex**
$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = r \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}$	$re^{\theta_1 \hat{\imath}} = x_1 + y_1 \hat{\imath}$
$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = r \begin{bmatrix} \cos(\theta_1 + \theta_r) \\ \sin(\theta_1 + \theta_r) \end{bmatrix}$	$re^{(\theta_1 + \theta_r)\hat{\imath}} = x_2 + y_2 \hat{\imath}$
$= r \begin{bmatrix} \cos\theta_1 \cos\theta_r - \sin\theta_1 \sin\theta_r \\ \cos\theta_1 \sin\theta_r + \sin\theta_1 \cos\theta_r \end{bmatrix}$	
$= \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & +\cos\theta_r \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$	$\left(re^{\theta_1 \hat{\imath}}\right) e^{\theta_r \hat{\imath}} = x_2 + y_2 \hat{\imath}$

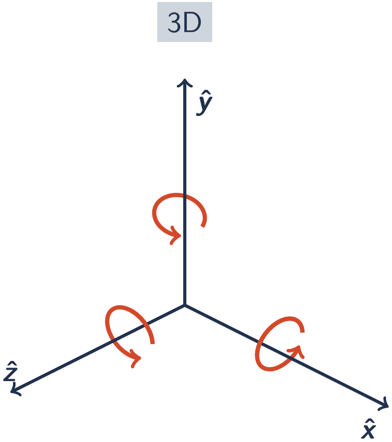
Computational Issues

- Sounds complicated. Why not just use angles, sin, and cos?
  1. Efficiency: sin/cos are expensive to compute.
     Multiplies and adds (matrix/complex) are cheaper
  2. Generalization to 3D
- Floating Point Error:
  - $\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & +\cos\theta_1 \end{bmatrix} \overset{\texttt{fp}}{*} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & +\cos\theta_2 \end{bmatrix} = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 \\ c_1s_2 + s_1c_2 & c_1c_2 - s_1s_2 \end{bmatrix} + \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$
  - Normalize Rotation Matrix: Gram-Schmidt process
  - $(\cos\theta_1 + \hat{\imath}\sin\theta_1) \overset{\texttt{fp}}{*} (\cos\theta_2 + \hat{\imath}\sin\theta_2) = (c_1c_2 - s_1s_2) + (c_1s_2 + s_1c_2)\,\hat{\imath} + e_r + \hat{\imath}e_i = \tilde{c} + \hat{\imath}\tilde{s}$
  - Normalize complex: $\tilde{c} + \hat{\imath}\tilde{s} \quad \rightsquigarrow \quad \frac{\tilde{c}+\hat{\imath}\tilde{s}}{\sqrt{\tilde{c}^2+\tilde{s}^2}}$

Outline

Geometric Intuition

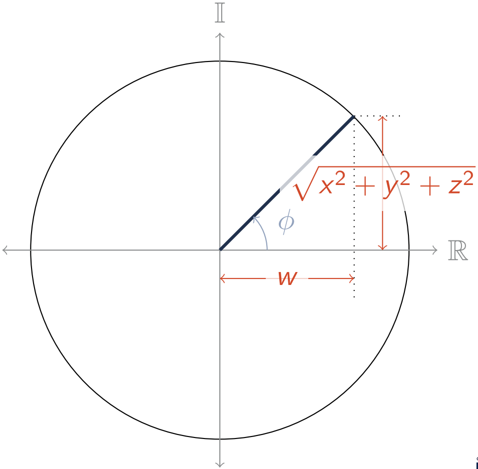




The Quaternion Axiom

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1$$

$$\hbar = \overbrace{\underbrace{w}_{\text{scalar/real}} + \underbrace{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}_{\text{vector/imaginary}}}^{\text{quaternion}}$$



Why not use three angles?

Conventions: 6 varying axis + 6 fixed axis:

- Euler Angles (varying-axis): (z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)
- Tait-Bryan (fixed-axis): (z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)

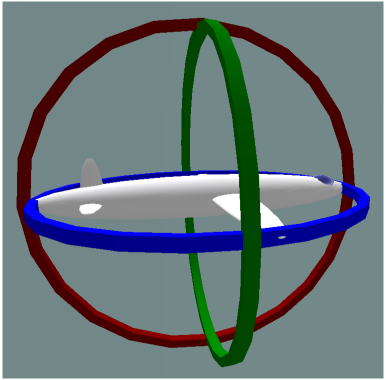
Sequential Rotations: Sequence of rotations is no longer addition of angles.
Axes change with each rotation.

Singularities: Aligned axes can remove a degree-of-freedom ("gimbal lock")

Euler Angle Singularities

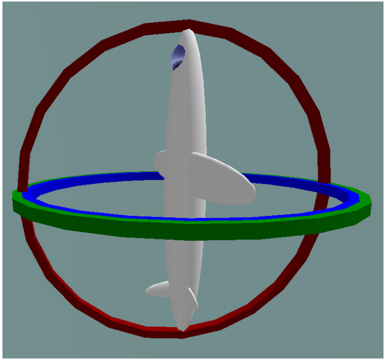
Gimbal Lock

OK



https://commons.wikimedia.org/wiki/File:No_gimbal_lock.png

Singularity



https://commons.wikimedia.org/wiki/File:Gimbal_lock.png

The Quaternion Axiom

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1$$

$$\hbar = \underbrace{w}_{\text{scalar}} + \underbrace{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}_{\text{vector}}$$

Exercise: Multiplication of Quaternion Units

$\hat{\imath}\hat{\jmath}$	$\hat{\jmath}\hat{k}$	$\hat{\imath}\hat{k}$	$\hat{\jmath}\hat{\imath}$	$\hat{k}\hat{\jmath}$	$\hat{k}\hat{\imath}$
$\hat{\imath}\hat{\jmath}\hat{k} = -1$ $\hat{\imath}\hat{\jmath}\hat{k}^2 = -\hat{k}$ $-\hat{\imath}\hat{\jmath} = -\hat{k}$ $\hat{\imath}\hat{\jmath} = \hat{k}$					

Multiplication of Quaternion Units

Summary

$*$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	$\hat{\imath}^2 = -1$	$\hat{\imath}\hat{\jmath} = \hat{k}$	$\hat{\imath}\hat{k} = -\hat{\jmath}$
$\hat{\jmath}$	$\hat{\jmath}\hat{\imath} = -\hat{k}$	$\hat{\jmath}^2 = -1$	$\hat{\jmath}\hat{k} = \hat{\imath}$
$\hat{k}$	$\hat{k}\hat{\imath} = \hat{\jmath}$	$\hat{k}\hat{\jmath} = -\hat{\imath}$	$\hat{k}^2 = -1$

Example: Complex Multiplication

1. $(a_w + a_x \hat{\imath})(b_w + b_x \hat{\imath})$
2. $(a_w + a_x \hat{\imath})b_w + (a_w + a_x \hat{\imath})b_x \hat{\imath}$
3. $a_w b_w + a_x b_w \hat{\imath} + a_w b_x \hat{\imath} + a_x b_x \hat{\imath}^2$
4. $(a_w b_w - a_x b_x) + (a_x b_w + a_w b_x)\hat{\imath}$

Exercise: Quaternion Multiplication

1. $(a_w + a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \otimes (b_w + b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k})$

Complex Multiplication Matrix

Multiplication:

$$(a_w + a_x\hat{\imath}) \otimes (b_w + b_x\hat{\imath}) = a_w b_w - a_x b_x + (a_x b_w + a_w b_x)\hat{\imath}$$

Multiplication Matrix:

$$\begin{bmatrix} a_w & -a_x \\ a_x & a_w \end{bmatrix} \begin{bmatrix} b_w \\ b_x \end{bmatrix} = \begin{bmatrix} a_w b_w - a_x b_x \\ a_x b_w + a_w b_x \end{bmatrix}$$

Quaternion Multiplication Matrix

Matrix Form:

$$\left(x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}} + w\right) = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

Multiplication:

$$a \otimes b = \begin{matrix} (a_w b_x - a_z b_y + a_y b_z + a_x b_w)\hat{\imath} \\ +(a_z b_x + a_w b_y - a_x b_z + a_y b_w)\hat{\jmath} \\ +(-a_y b_x + a_x b_y + a_w b_z + a_z b_w)\hat{\boldsymbol{k}} \\ +(-a_x b_x - a_y b_y - a_z b_z + a_w b_w) \end{matrix}$$

Multiplication Matrix:

$$\begin{bmatrix} a_w & -a_z & a_y & a_x \\ a_z & a_w & -a_x & a_y \\ -a_y & a_x & a_w & a_z \\ -a_x & -a_y & -a_w & a_w \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \\ b_w \end{bmatrix} = \begin{bmatrix} a_w b_x - a_z b_y + a_y b_z + a_x b_w) \\ a_z b_x + a_w b_y - a_x b_z + a_y b_w \\ -a_y b_x + a_x b_y + a_w b_z + a_z b_w \\ -a_x b_x - a_y b_y - a_z b_z + a_w b_w \end{bmatrix}$$

Use: quaternions within a larger system of linear equations

Exercise: Quaternion Multiplication Commutativity

Quaternion Norm, Inverse, and Conjugate

Norm

$$\|q\| = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2}$$

Inverse

$$q^{-1} = \frac{\left(-q_x\hat{\imath} + -q_y\hat{\jmath} + -q_z\hat{\boldsymbol{k}} + q_w\right)}{q_x^2 + q_y^2 + q_z^2 + q_w^2}$$

$$q \otimes q^{-1} = -1$$

Conjugate

$$q^* = \left(-q_x\hat{\imath} + -q_y\hat{\jmath} + -q_z\hat{\boldsymbol{k}} + q_w\right)$$

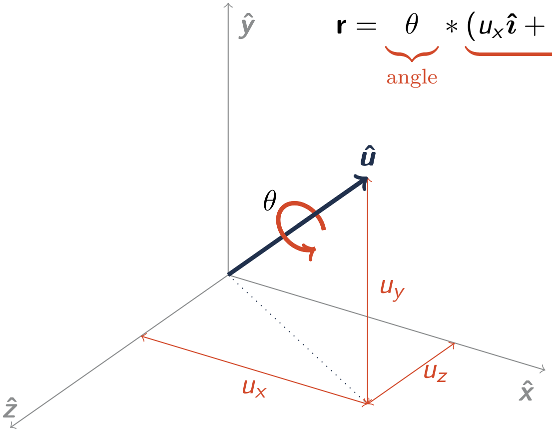
When $\|q\| = 1$, then $q^* = q^{-1}$ and $q \otimes q^* = 1$

Quaternion Algebra

$$p = p_w + p_x \hat{\imath} + p_y \hat{\jmath} + p_z \hat{k}$$
$$q = q_w + q_x \hat{\imath} + q_y \hat{\jmath} + q_z \hat{k}$$

Axis-Angle and Rotation Vector

$$\mathbf{r} = \underbrace{\theta}_{\text{angle}} * \underbrace{\left(u_x \hat{\boldsymbol{\imath}} + u_y \hat{\boldsymbol{\jmath}} + u_z \hat{\boldsymbol{k}}\right)}_{\text{axis}}$$



Any 3D rotation: an angle θ about an axis $\hat{\boldsymbol{u}}$

Quaternion Rotations

Complex:

$$(x_1 + y_1\hat{\boldsymbol{\imath}}) = e^{\theta\hat{\boldsymbol{\imath}}}(x_0 + y_0\hat{\boldsymbol{\imath}})$$

Quaternion:

$$\begin{aligned}(x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}} + z_1\hat{\boldsymbol{k}}) &= \exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right)(x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\hat{\boldsymbol{k}})\exp\left(-\frac{\theta}{2}\hat{\boldsymbol{u}}\right) \\ &= \exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right)(x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\hat{\boldsymbol{k}})\left(\exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right)\right)^*\end{aligned}$$

- Pre/Post multiply to keep point in vector/imaginary part
- Each contributes half the angle

Quaternion Exponential

Complex:

$$e^{\theta \hat{\boldsymbol{\imath}}} = \hat{\boldsymbol{\imath}} \sin \theta + \cos \theta$$

Pure Quaternion:

$$\exp\left(\frac{\mathbf{r}}{2}\right) = \exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right) = \sin\frac{\theta}{2}\left(u_x \hat{\boldsymbol{\imath}} + u_y \hat{\boldsymbol{\jmath}} + u_z \hat{\boldsymbol{k}}\right) + \cos\frac{\theta}{2}$$

General Quaternion:

$$\exp\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} + w\right) = \exp\left(\vec{v} + w\right) = e^{w}\left(\frac{\sin\|v\|}{\|v\|}\vec{v} + \cos\|v\|\right)$$

Exercise: Axis-Angle to Quaternion

1. $\theta = \pi$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{\imath}}$
   $\exp\left(\frac{\pi}{2}\hat{\boldsymbol{\imath}}\right) \quad \rightsquigarrow \quad \sin\frac{\pi}{2}\hat{\boldsymbol{\imath}} + \cos\frac{\pi}{2} \quad \rightsquigarrow \quad \hat{\boldsymbol{\imath}}$
2. $\theta = \frac{\pi}{2}$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{k}}$
3. $\theta = -\frac{3\pi}{2}$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{k}}$
4. $\theta = 0$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{\imath}}$

Velocities and Derivatives

Angular Velocity: $\boldsymbol{\omega} = \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$

Quaternion Derivative:

- $\dot{\hslash} = \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$
  $= \frac{1}{2} \boldsymbol{\omega} \otimes \hslash$
- $\boldsymbol{\omega} = 2\, \dot{\hslash} \otimes \hslash^*$

Product Rule: $\frac{d}{dt}\left(a(t) \otimes b(t)\right) = \left(\dot{a}(t) \otimes b(t)\right) + \left(a(t) \otimes \dot{b}(t)\right)$

Acceleration:

- $\ddot{\hslash} = \frac{1}{2}\left(\dot{\boldsymbol{\omega}} \otimes \hslash + \boldsymbol{\omega} \otimes \dot{\hslash}\right)$
- $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = 2\left(\ddot{\hslash} \otimes \hslash^* + \dot{\hslash} \otimes \dot{\hslash}^*\right)$

Integration

Quaternions as Linear ODE

$$\frac{d}{dt}\hbar = \frac{1}{2}\boldsymbol{\omega} \otimes \hbar$$

$$\hbar_1 = \exp\left(\frac{\boldsymbol{\omega}\Delta t}{2}\right) \otimes \hbar_0$$

Note 0: William Rowan Hamilton

1805-1865

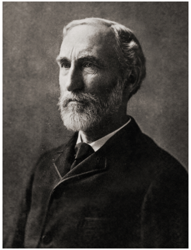
Brougham (Broom) Bridge, Dublin

Note 1: Gibb's Vector Analysis

$$a \otimes b = \begin{matrix} (a_y b_z - a_z b_y + a_w b_x + b_w a_x)\hat{\imath}+ \\ (a_z b_x - a_x b_z + a_w b_y + b_w a_y)\hat{\jmath}+ \\ (a_x b_y - a_y b_x + a_w b_z + b_w a_z)\hat{\mathbf{k}}+ \\ (a_w b_w - a_x b_x - a_y b_y - a_z b_z) \end{matrix} = \begin{pmatrix} a_v \times b_v + a_w b_v + b_v a_v \\ a_w b_w - a_v \bullet b_v \end{pmatrix}$$

$$\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{\mathbf{k}}\right) \times \left(b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{\mathbf{k}}\right) = \begin{matrix} (a_y b_z - a_z b_y)\ \hat{\imath} \\ +\ (a_z b_x - a_x b_z)\ \hat{\jmath} \\ +\ (a_x b_y - a_y b_x)\ \hat{\mathbf{k}} \end{matrix}$$

$$\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{\mathbf{k}}\right) \bullet \left(b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{\mathbf{k}}\right) = a_x b_x + a_y b_y + a_z b_z$$



J. Willard Gibbs
1839-1903

Rebuttal: Quaternions are the best analytical and computational construct for rotations.

Summary

Complex Numbers
- Definition
- Rotations

Quaternions
- Definitions
- 3D Rotations