Euclidean Transformation (Pre Lecture)

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CSCI-498/598 RPM, Colorado School of Mines

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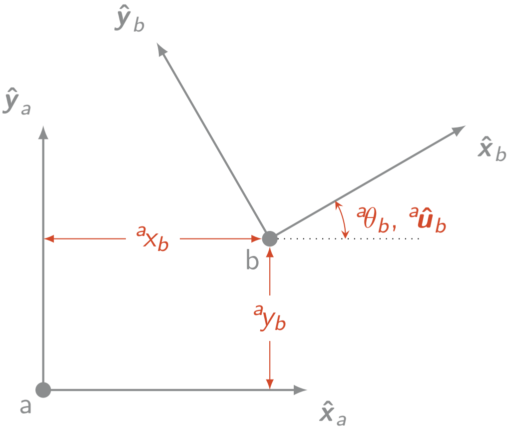
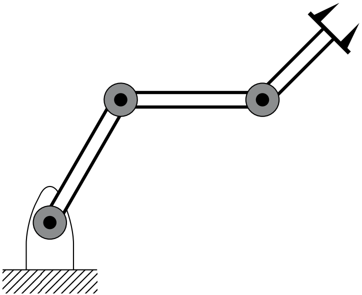
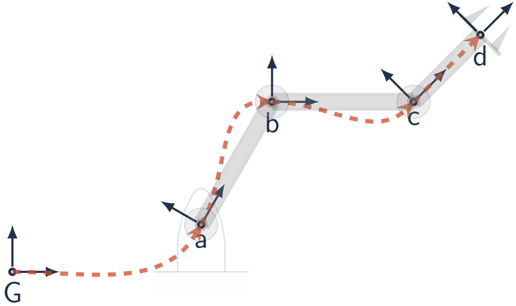
Outline

Local Coordinate Frames

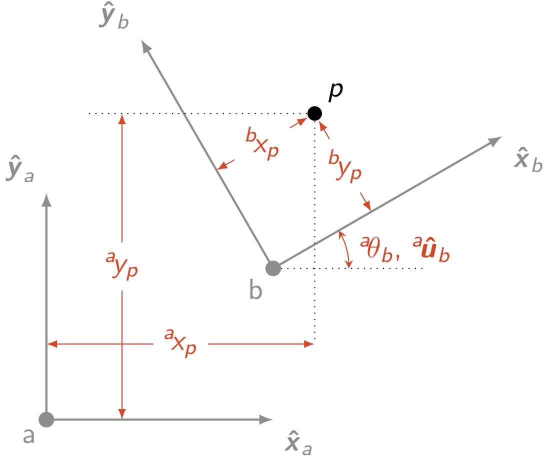


Robots are Local Frames

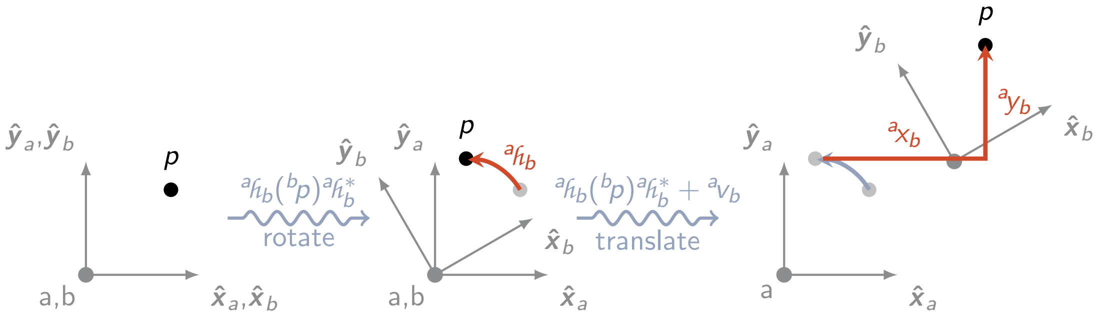




Transformations



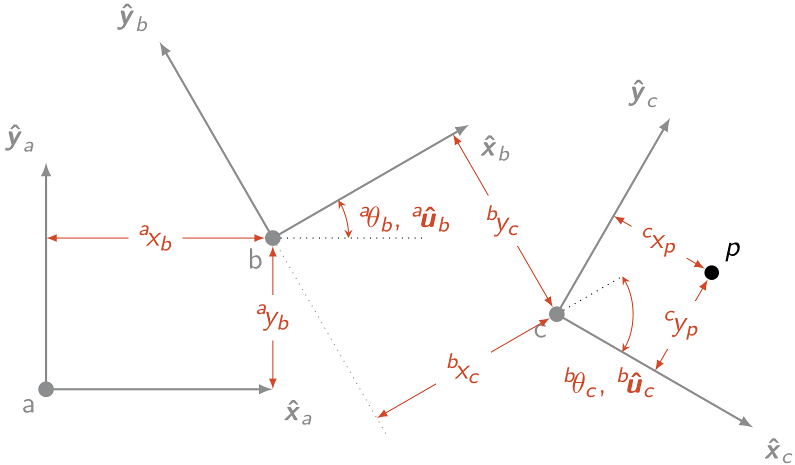
Transforming a Point



$$ {}^a p = \underbrace{\left({}^a\hbar_b\right) \otimes \left({}^b p\right) \otimes \left({}^a\hbar_b\right)^*}_{\text{rotation}} + \underbrace{{}^a v_b}_{\text{translation}} $$

Chaining Transforms

Geometric Illustration



Chaining Transforms

Algebraic Solution

- Transform ${}^{c}p$ to ${}^{b}p$: $\quad {}^{b}p = \left({}^{b}h_c\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}h_c\right)^* + {}^{b}v_c$
- Transform ${}^{b}p$ to ${}^{a}p$: $\quad {}^{a}p = \left({}^{a}h_b\right) \otimes \left({}^{b}p\right) \otimes \left({}^{a}h_b\right)^* + {}^{a}v_b$
- Transform ${}^{b}p$ to ${}^{a}p$:

1. $${}^{a}p = \left({}^{a}h_b\right) \otimes \overbrace{\left(\left({}^{b}h_c\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}h_c\right)^* + {}^{b}v_c\right)}^{{}^{b}p} \otimes \left({}^{a}h_b\right)^* + {}^{a}v_b$$
2. $${}^{a}p = \left(\left({}^{a}h_b\right) \otimes \left({}^{b}h_c\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}h_c\right)^* + \left({}^{a}h_b\right) \otimes {}^{b}v_c\right) \otimes \left({}^{a}h_b\right)^* + {}^{a}v_b$$
3. $${}^{a}p = \left({}^{a}h_b\right) \otimes \left({}^{b}h_c\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}h_c\right)^* \otimes \left({}^{a}h_b\right)^* + \left({}^{a}h_b\right) \otimes {}^{b}v_c \otimes \left({}^{a}h_b\right)^* + {}^{a}v_b$$
4. $${}^{a}p = \underbrace{\left({}^{a}h_b\right) \otimes \left({}^{b}h_c\right)}_{{}^{a}h_c} \otimes \left({}^{c}p\right) \otimes \underbrace{\left({}^{a}h_b \otimes {}^{b}h_c\right)^*}_{{}^{a}h_c} + \underbrace{\left({}^{a}h_b\right) \otimes {}^{b}v_c \otimes \left({}^{a}h_b\right)^* + {}^{a}v_b}_{{}^{a}v_c}$$

- ${}^{a}h_c = \left({}^{a}h_b \otimes {}^{b}h_c\right)$ and ${}^{a}v_c = \left({}^{a}h_b\right) \otimes {}^{b}v_c \otimes \left({}^{a}h_b\right)^* + {}^{a}v_b$

Outline

Dual Axiom

$$\varepsilon^2 = 0 \qquad \wedge \qquad \varepsilon \neq 0$$

Example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Dual Numbers

$$\varepsilon^2 = 0 \qquad \land \qquad \varepsilon \neq 0$$

$$\tilde{n} = \overbrace{\underbrace{n_r}_{\text{real}} + \underbrace{n_d \varepsilon}_{\text{dual}}}^{\text{dual number}}$$

Dual Multiplication

1. $\tilde{a} \otimes \tilde{b}$
2. $= (a_r + a_d\varepsilon) \otimes (b_r + b_d\varepsilon)$
3. $= a_r(b_r + b_d\varepsilon) + a_d\varepsilon(b_r + b_d\varepsilon)$
4. $= a_r b_r + a_r b_d\varepsilon + a_d b_r\varepsilon + a_d b_d\varepsilon^2$
5. $= a_r b_r + a_r b_d\varepsilon + a_d b_r\varepsilon + \cancel{a_d b_d\varepsilon^2} \rightarrow 0$
6. $= a_r b_r + (a_r b_d + a_d b_r)\varepsilon$

Dual Conjugate

- $(r + d\varepsilon)^{\bullet} = r - d\varepsilon$
- Multiplication by conjugate:
    1. $\quad (r + d\varepsilon)\,(r - d\varepsilon)$
    2. $= r^2 + rd\varepsilon - rd\varepsilon$
    3. $= r^2$

Cancels the dual part

Dual Number Taylor Series

Taylor Series: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$

Dual Number Taylor Series: Evaluate Taylor series at the real part:

1. $f(a+b\varepsilon) = f(a) + \frac{f'(a)}{1!}(b\varepsilon) + \frac{f''(a)}{2!}(b\varepsilon)^2 + \frac{f'''(a)}{3!}b\varepsilon)^3 + \ldots$
2. $f(a+b\varepsilon) = f(a) + \frac{f'(a)}{1!}(b\varepsilon) + \cancelto{0}{\frac{f''(a)}{2!}(b\varepsilon)^2} + \cancelto{0}{\frac{f'''(a)}{3!}b\varepsilon)^3} + \cancelto{0}{\ldots}$
3. $f(a+b\epsilon) = f(a) + bf'(a)\varepsilon$

Higher-order dual terms cancel

Exercise: Dual Number Transcendental Functions

Exponential: $e^{r+d\varepsilon}$

Logarithm: $\ln(r + d\varepsilon)$

Sine: $\sin(r + d\varepsilon)$

Cosine: $\cos(r + d\varepsilon)$

Square Root: $\sqrt{r + d\varepsilon}$

Dual Quaternions

$$\overbrace{\boxed{r + d\varepsilon}}^{\text{Dual Number}} \qquad \overbrace{\boxed{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} + w}}^{\text{Quaternion}}$$

$$\left.\boxed{\begin{aligned} &\left(r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} + r_w\right) + \left(d_x\hat{\boldsymbol{\imath}} + d_y\hat{\boldsymbol{\jmath}} + d_z\hat{\boldsymbol{k}} + d_w\right)\varepsilon \\ &= (r_x + d_x\varepsilon)\,\hat{\boldsymbol{\imath}} + (r_y + d_y\varepsilon)\,\hat{\boldsymbol{\jmath}} + (r_z + d_z\varepsilon)\,\hat{\boldsymbol{k}} + (r_w + d_w\varepsilon) \end{aligned}}\right\}\text{Dual Quaternion}$$

8 factors for the combinations of real, quaternion, and dual parts.

Dual Quaternion Multiplication

- $a = (a_r + a_d\varepsilon) = \left(\left(a_{rx}\hat{\boldsymbol{\imath}} + a_{ry}\hat{\boldsymbol{\jmath}} + a_{rz}\hat{\boldsymbol{k}} + a_{rw}\right) + \left(a_{dx}\hat{\boldsymbol{\imath}} + a_{dy}\hat{\boldsymbol{\jmath}} + a_{dz}\hat{\boldsymbol{k}} + a_{dw}\right)\varepsilon\right)$
- $b = (b_r + b_d\varepsilon) = \left(\left(b_{rx}\hat{\boldsymbol{\imath}} + b_{ry}\hat{\boldsymbol{\jmath}} + b_{rz}\hat{\boldsymbol{k}} + b_{rw}\right) + \left(b_{dx}\hat{\boldsymbol{\imath}} + b_{dy}\hat{\boldsymbol{\jmath}} + b_{dz}\hat{\boldsymbol{k}} + b_{dw}\right)\varepsilon\right)$
- $a \otimes b =$
  1. $(a_r + d_d\varepsilon) \otimes (b_r + b_d\varepsilon)$
  2. $(a_r \otimes b_r) + (a_r \otimes b_d + a_d \otimes b_r)\,\varepsilon$

Dual Quaternion Conjugates

Quaternion Conjugate:

$$(h + d\varepsilon)^* = h^* + d^*\varepsilon$$

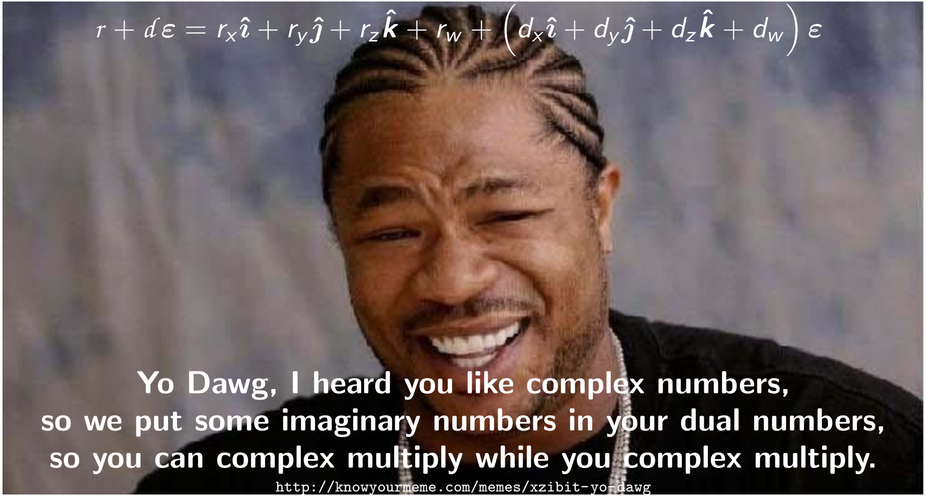
Dual Conjugate:

$$(h + d\varepsilon)^\bullet = h - d\varepsilon$$

Joint Conjugate:

$$(h + d\varepsilon)^\diamond = ((h + d\varepsilon)^*)^\bullet = h^* - d^*\varepsilon$$

Hypercomplex Numbers



Dual Quaternions Transformations

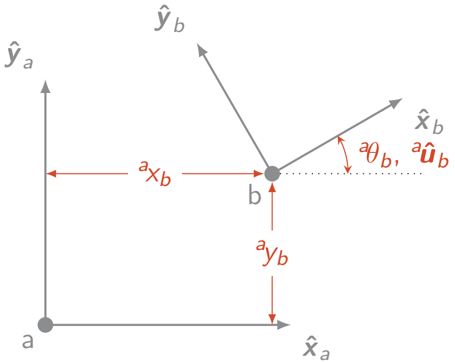
Illustration

Rotation: ${}^{a}\mathcal{h}_{b} = \exp\left(\frac{1}{2}\theta\hat{\boldsymbol{u}}\right)$

Translation: ${}^{a}\mathcal{v}_{b} = {}^{a}x_{b}\hat{\boldsymbol{\imath}} + {}^{a}y_{b}\hat{\boldsymbol{\jmath}} + {}^{a}z_{b}\hat{\boldsymbol{k}}$

Transform: ${}^{a}\mathcal{S}_{b} = \left({}^{a}\mathcal{h}_{b}\right) + \left(\frac{1}{2}{}^{a}\mathcal{v}_{b} \otimes {}^{a}\mathcal{h}_{b}\right)\varepsilon$

- $\mathcal{d} = \frac{1}{2}\mathcal{v} \otimes \mathcal{h}$
- $\mathcal{v} = 2\mathcal{d} \otimes \mathcal{h}^*$



Dual Quaternions Transformations

Algebra

$$\text{Rotation:} \quad {}^a p = {}^a h_b \otimes \overbrace{\left(p_x \hat{\imath} + p_y \hat{\jmath} + p_z \hat{\boldsymbol{k}}\right)}^{\text{point}} \otimes \left({}^a h_b\right)^*$$

$$\text{Transform:} \quad {}^a p = {}^a S_b \otimes \overbrace{\left(1 + \left(p_x \hat{\imath} + p_y \hat{\jmath} + p_z \hat{\boldsymbol{k}}\right) \varepsilon\right)}^{\text{point}} \otimes \left({}^a S_b\right)^{\diamond}$$

1. $= \left(h + d\,\varepsilon\right)\left(1 + p\varepsilon\right)\left(h + d\,\varepsilon\right)^{\diamond}$
2. $= \left(h + \left(d + hp\right)\varepsilon\right)\left(h^* - d^*\varepsilon\right)$
3. $= hh^* + \left(\left(d + hp\right) h^* - hd^*\right)\varepsilon$
4. $= 1 + (\underbrace{hph^*}_{\text{rotate}} + \underbrace{dh^* - hd^*}_{\text{translate}})\varepsilon$

$$\text{Translation Check:} \quad {}^a p = \left({}^a h_b\right) \otimes \left({}^b p\right) \otimes \left({}^a h_b\right)^* + {}^a v_b$$

$${}^a v_b = dh^* - hd^* = \frac{1}{2} vhh^* - h\left(\frac{1}{2} vh\right)^* = \frac{1}{2} vhh^* - \frac{1}{2} hh^* v^* = \frac{1}{2} v + \frac{1}{2} v$$

Transformation Formula

Simplified

Point: ${}^{b}p = p_x \hat{\imath} + p_y \hat{\jmath} + p_z \hat{\boldsymbol{k}}$

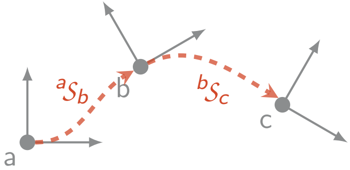
Transform: ${}^{a}\mathcal{S}_b = \mathscr{h} + \mathscr{d}\boldsymbol{\varepsilon}$

Result:

$$
\begin{aligned}
{}^{a}p &= {}^{a}\mathcal{S}_b \otimes \left(1 + {}^{b}p\boldsymbol{\varepsilon}\right) \otimes \left({}^{a}\mathcal{S}_b\right)^{\diamond} \\
&= \left(\mathscr{h} \otimes {}^{b}p + 2\mathscr{d}\right) \otimes \mathscr{h}^{*}
\end{aligned}
$$

Dual Quaternion Chaining

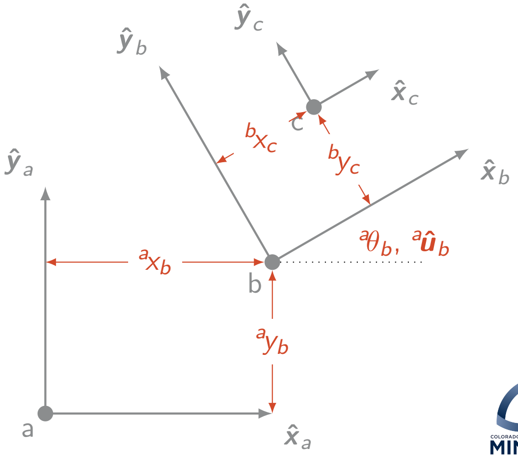
- $$\begin{aligned} {}^{a}S_{c} &= \left({}^{a}S_{b} \otimes {}^{b}S_{c} \right) = \left(\left({}^{a}h_{b} + {}^{a}d_{b}\varepsilon \right) \otimes \left({}^{b}h_{c} + {}^{b}d_{c}\varepsilon \right) \right) \\ &= \left(\left({}^{a}h_{b} \otimes {}^{b}h_{c} \right) + \left({}^{a}h_{b} \otimes {}^{b}d_{c} + {}^{a}d_{b} \otimes {}^{b}h_{c} \right) \varepsilon \right) \end{aligned}$$
- Transform Multiply:
  1. $${}^{a}S_{c} = \left({}^{a}h_{b} + \tfrac{1}{2}\, {}^{a}v_{b}\, {}^{a}h_{b}\varepsilon \right) \otimes \left({}^{b}h_{c} + \tfrac{1}{2}\, {}^{b}v_{c}\, {}^{b}h_{c}\varepsilon \right)$$
  2. $$= \underbrace{\left({}^{a}h_{b}\, {}^{b}h_{c} \right)}_{\text{rotation}} + \underbrace{\frac{1}{2} \left({}^{a}h_{b}\, {}^{b}v_{c}\, {}^{b}h_{c} + {}^{a}v_{b}\, {}^{a}h_{b}\, {}^{b}h_{c} \right)\varepsilon}_{\text{translation}}$$
- Extract Translation: $v = 2d \otimes h^{*}$
  1. $${}^{a}v_{c} = 2 \left(\tfrac{1}{2} \left({}^{a}h_{b}\, {}^{b}v_{c}\, {}^{b}h_{c} + {}^{a}v_{b}\, {}^{a}h_{b}\, {}^{b}h_{c} \right) \right) \otimes \left({}^{a}h_{b}\, {}^{b}h_{c} \right)^{*}$$
  2. $${}^{a}v_{c} = \left({}^{a}h_{b}\, {}^{b}v_{c}\, {}^{b}h_{c} + {}^{a}v_{b}\, {}^{a}h_{b}\, {}^{b}h_{c} \right) \otimes \left({}^{b}h_{c} \right)^{*} \left({}^{a}h_{b} \right)^{*}$$
  3. $${}^{a}v_{c} = \left({}^{a}h_{b}\, {}^{b}v_{c}\, {}^{b}h_{c} \left({}^{b}h_{c} \right)^{*} \left({}^{a}h_{b} \right)^{*} + {}^{a}v_{b}\, {}^{a}h_{b}\, {}^{b}h_{c} \left({}^{b}h_{c} \right)^{*} \left({}^{a}h_{b} \right)^{*} \right)$$
  4. $${}^{a}v_{c} = {}^{a}h_{b} \otimes {}^{b}v_{c} \otimes \left({}^{a}h_{b} \right)^{*} + {}^{a}v_{b}$$



Dual Quaternion Transformation as Chaining

Illustration

- ${}^a\mathcal{S}_b = \mathcal{h} + \mathcal{d}\varepsilon$
- ${}^b p = {}^b x_c \hat{\imath} + {}^b y_c \hat{\jmath} + {}^b z_c \hat{\boldsymbol{k}}$
- ${}^b\mathcal{S}_c = 1 + \frac{1}{2}{}^b p \varepsilon$
- Chain Transforms:
  1. ${}^a\mathcal{S}_c = {}^a\mathcal{S}_b \otimes {}^b\mathcal{S}_c$
  2. $\quad = (\mathcal{h} + \mathcal{d}\varepsilon) \otimes \left(1 + \frac{1}{2}{}^b p \varepsilon\right)$
  3. $\quad = \mathcal{h} + \left(\mathcal{d} + \frac{1}{2}\mathcal{h} \otimes {}^b p\right)\varepsilon$
- Extract Point: $v = 2\mathcal{d} \otimes \mathcal{h}^*$
  1. ${}^a v = 2\left(\mathcal{d} + \frac{1}{2}\mathcal{h} \otimes {}^b p\right) \otimes \mathcal{h}^*$
  2. $\quad = \left(2\mathcal{d} + \mathcal{h} \otimes {}^b p\right) \otimes \mathcal{h}^*$



Dual Quaternion Exponential

Quaternion: $\mathcal{h} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} + w$

$$\phi = \sqrt{x^2 + y^2 + z^2}$$
$$e^{\mathcal{h}} = e^{w}\left(\frac{\sin\phi}{\phi}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right) + \cos\phi\right)$$

Dual Quaternion: $\mathcal{S} = \left(r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} + r_w\right) + \left(d_x\hat{\boldsymbol{\imath}} + d_y\hat{\boldsymbol{\jmath}} + d_z\hat{\boldsymbol{k}} + d_w\right)\varepsilon$

$$\tilde{\phi} = \sqrt{(r_x + d_x\epsilon)^2 + (r_z + d_z\epsilon)^2 + (r_z + d_z\epsilon)^2}$$
$$e^{\mathcal{S}} = e^{r_w + d_w\varepsilon}\left(\frac{\sin\tilde{\phi}}{\tilde{\phi}}\left(\left(r_x + d_x\varepsilon\right)\hat{\boldsymbol{\imath}} + \left(r_y + d_y\varepsilon\right)\hat{\boldsymbol{\jmath}} + \left(r_z + d_z\varepsilon\right)\hat{\boldsymbol{k}}\right) + \cos\tilde{\phi}\right)$$

Dual Quaternion Exponential

Derivation: $\tilde{\phi}$

1. $\tilde{\phi} = \sqrt{(r_x + d_x\epsilon)^2 + (r_z + d_z\epsilon)^2 + (r_z + d_z\epsilon)^2}$
2. $\tilde{\phi} = \sqrt{\left(r_x^2 + r_y^2 + r_z^2\right) + 2\left(r_x d_x + r_y d_y + r_z d_z\right)\varepsilon}$
3. $\tilde{\phi} = \sqrt{r_x^2 + r_y^2 + r_z^2} + \frac{r_x d_x + r_y d_y + r_z d_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}\varepsilon$
4. $\tilde{\phi} = \phi + \frac{\gamma}{\phi}\varepsilon$
5. $\cos\tilde{\phi} = \cos\phi - \frac{\gamma}{\phi}\sin(\phi)\varepsilon = c - \frac{\gamma}{\phi}s\varepsilon$
6. $\sin\tilde{\phi} = \sin\phi + \frac{\gamma}{\phi}\cos(\phi)\varepsilon = s + \frac{\gamma}{\phi}c\varepsilon$

Dual Quaternion Exponential

Derivation: $\frac{\sin\tilde{\phi}}{\tilde{\phi}}$

1. $\frac{\sin\tilde{\phi}}{\tilde{\phi}}$
2. $\frac{\sin(\phi)+\frac{\gamma}{\phi}\cos(\phi)\varepsilon}{\phi+\frac{\gamma}{\phi}\varepsilon}$
3. $= \left(\frac{\sin(\phi)+\frac{\gamma}{\phi}\cos(\phi)\varepsilon}{\phi+\frac{\gamma}{\phi}\varepsilon}\right)\left(\frac{\phi-\frac{\gamma}{\phi}\varepsilon}{\phi-\frac{\gamma}{\phi}\varepsilon}\right)$
4. $= \frac{\sin(\phi)\phi+\left(\phi\cos(\phi)\frac{\gamma}{\phi}-\sin(\phi)\frac{\gamma}{\phi}\right)\varepsilon}{\phi^2}$
5. $= \frac{\sin(\phi)}{\phi} + \gamma\left(\frac{\cos(\phi)-\frac{\sin(\phi)}{\phi}}{\phi^2}\right)\varepsilon$
6. $= \left(1 - \frac{\phi^2}{6} + \frac{\phi^4}{120} + \ldots\right) + \left(-\frac{1}{3} + \frac{\phi^2}{30} - \frac{\phi^4}{840} + \ldots\right)\varepsilon$

Dual Quaternion Exponential

Derivation

1. $e^{\mathcal{S}} = e^{r_w + d_w \varepsilon} \left(\frac{\sin \tilde{\phi}}{\tilde{\phi}} \left(\vec{r}_v + \vec{d}_v \varepsilon \right) + \cos \tilde{\phi} \right)$
2. $\left(e^{r_w} + d_w e^{r_w} \varepsilon\right) \left(\left(\frac{s}{\phi} + \gamma \left(\frac{c - \frac{s}{\phi}}{\phi^2} \right) \varepsilon \right) \left(\vec{r}_v + \vec{d}_v \varepsilon \right) + c - \frac{\gamma}{\phi} s \varepsilon \right)$
3. $\left(e^{r_w} + d_w e^{r_w} \varepsilon\right) \left(\frac{s}{\phi} \vec{r}_v + \frac{s}{\phi} \vec{d}_v \varepsilon + \frac{c - \frac{s}{\phi}}{\phi^2} \gamma \vec{r}_v \varepsilon + c - \frac{\gamma}{\phi} s \varepsilon \right)$
4. $\left(e^{r_w} + d_w e^{r_w} \varepsilon\right) \left(\left(\frac{s}{\phi} \vec{r}_v + c \right) + \left(\frac{s}{\phi} \vec{d}_v + \frac{c - \frac{s}{\phi}}{\phi^2} \gamma \vec{r}_v - \frac{s}{\phi} \gamma \right) \varepsilon \right)$

Dual Quaternion Logarithm

Quaternion: $\mathcal{h} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} + w = \vec{v} + w$

$$\phi = \operatorname{atan2}\left(\|v\|, w\right)$$

$$\ln \mathcal{h} = \frac{\phi}{\|v\|}\vec{v} + \ln\|\mathcal{h}\|$$

Dual Quaternion: $\mathcal{S} = \mathcal{h} + \mathcal{d}\varepsilon$

$$(\ln S)_{\mathrm{real}} = \frac{\phi}{\|\mathcal{h}_v\|}\vec{\mathcal{h}}_v + \ln\|\mathcal{h}\|$$

$$\gamma = \vec{\mathcal{h}}_v \bullet \vec{\mathcal{d}}_v$$

$$\alpha = \frac{\mathcal{h}_w - \frac{\phi}{\|\mathcal{h}_v\|}\|\mathcal{h}\|^2}{\|\mathcal{h}_v\|^2}$$

$$(\ln S)_{\mathrm{dual}} = \frac{\gamma\alpha - \mathcal{d}_w}{\|\mathcal{h}\|^2}\vec{\mathcal{h}}_v + \frac{\phi}{\|\mathcal{h}_v\|}\vec{\mathcal{d}}_v + \frac{\gamma + \mathcal{h}_w \mathcal{d}_w}{\|\mathcal{h}\|^2}$$

Dual Quaternion Logarithm

Taylor Series

1. $\alpha = \frac{h_w - \frac{\phi}{\|h_v\|}\|h\|^2}{\|h_v\|^2}$
2. $= \frac{h_w}{\|h_v\|^2} - \frac{\phi\|h\|^2}{\|h_v\|^3}$
3. $= \frac{h_w\|h\|^2}{\|h_v\|^2\|h\|^2} - \frac{\phi\|h\|^3}{\|h_v\|^3\|h\|}$
4. $= \frac{1}{\|h\|}\left(\frac{h_w}{\|h\|}\frac{\|h\|^2}{\|h_v\|^2} - \phi\frac{\|h\|^3}{\|h_v\|^3}\right)$
   - 4.1 $\frac{h_w}{\|h\|} = \cos\phi$
   - 4.2 $\frac{h_v}{\|h\|} = \sin\phi$
5. $= \frac{1}{\|h\|}\left(\frac{\cos\phi}{\sin^2(\phi)} - \frac{\phi}{\sin^3(\phi)}\right)$
6. $= \frac{1}{\|h\|}\left(-\frac{2}{3} - \frac{1}{5}\phi^2 - \frac{17}{420}\phi^4 - \frac{29}{4200}\phi^6 + \ldots\right)$

Velocity and Derivatives

Quaternion Derivative: $\dot{h} = \frac{1}{2}\boldsymbol{\omega} \otimes h$

Dual Quaternion Derivative: $\dot{\mathcal{S}} = \frac{d}{dt}\left(h + \left(\frac{1}{2}v \otimes h\right)\varepsilon\right)$

1. $\dot{\mathcal{S}} = \dot{h} + \frac{d}{dt}\left(\frac{1}{2}v \otimes h\right)\varepsilon$
2. $\dot{\mathcal{S}} = \dot{h} + \frac{1}{2}\left(\dot{v} \otimes h + v \otimes \dot{h}\right)\varepsilon$
3. $\dot{\mathcal{S}} = \frac{1}{2}\left(\omega \otimes h + \left(\dot{v} \otimes h + v \otimes \left(\frac{1}{2}\omega \otimes h\right)\right)\varepsilon\right)$

Product Rule: ${}^{a}\mathcal{S}_{c} = {}^{a}\mathcal{S}_{b} \otimes {}^{b}\mathcal{S}_{c}$

1. $\frac{d}{dt}{}^{a}\mathcal{S}_{c} = \frac{d}{dt}\left({}^{a}\mathcal{S}_{b} \otimes {}^{b}\mathcal{S}_{c}\right)$
2. $\frac{d}{dt}{}^{a}\mathcal{S}_{c} = \frac{d}{dt}\left({}^{a}\mathcal{S}_{b}\right) \otimes {}^{b}\mathcal{S}_{c} + {}^{a}\mathcal{S}_{b} \otimes \frac{d}{dt}\left({}^{b}\mathcal{S}_{c}\right)$

Twist

Factorization of the Dual Quaternion Derivative

- $\dot{\mathcal{S}} = \left(\frac{1}{2}\left(\omega \otimes h + \left(\dot{v} \otimes h + \frac{1}{2} v \otimes \omega \otimes h\right)\varepsilon\right)\right)$

  $\rightsquigarrow \quad \left(\frac{1}{2}\Omega \otimes \left(h + \left(\frac{1}{2} v \otimes h\right)\varepsilon\right)\right)$

  1. $= \frac{1}{2}\left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2} v \otimes \omega\right) \otimes h\right)\varepsilon\right)$
  2. $= \frac{1}{2}\left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2} v \times \omega + \frac{1}{2} v \bullet \omega\right) \otimes h\right)\varepsilon\right)$
  3. $= \frac{1}{2}\left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2}\omega \times v + \frac{1}{2}\omega \bullet v + v \times \omega\right) \otimes h\right)\varepsilon\right)$
  4. $= \frac{1}{2}\left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2}\omega \otimes v + v \times \omega\right) \otimes h\right)\varepsilon\right)$
  5. $= \frac{1}{2}\left(\omega \otimes h + \left(\frac{1}{2}\omega \otimes v \otimes h + \left(\dot{v} + v \times \omega\right) \otimes h\right)\varepsilon\right)$
  6. $= \frac{1}{2}\left(\omega \otimes h + \left(\omega \otimes \left(\frac{1}{2} v \otimes h\right) + \left(\dot{v} + v \times \omega\right) \otimes h\right)\varepsilon\right)$
  7. $= \frac{1}{2}\left(\omega + \left(\dot{v} + v \times \omega\right)\varepsilon\right) \otimes \left(h + \frac{1}{2} v \otimes h\varepsilon\right)$
  8. $\dot{\mathcal{S}} = \frac{1}{2}\Omega \otimes \mathcal{S}$

- $\Omega = \omega + \left(\dot{v} + v \times \omega\right)\varepsilon$

Integration

Dual Quaternions as Linear ODE

- $\frac{d}{dt}\mathcal{S} = \frac{1}{2}\,\Omega \otimes \mathcal{S}$
- $\mathcal{S}_1 = \exp\left(\frac{\Omega \Delta t}{2}\right) \otimes \mathcal{S}_0$

Outline

Local Frames

Dual Quaternions

Other Representations

Kinematic Chains and Trees

Rotation Matrices

Quaternion Multiplication:

$$p \otimes q = \underbrace{\begin{bmatrix} p_w & -p_z & p_y & p_x \\ p_z & p_w & -p_x & p_y \\ -p_y & p_x & p_w & p_z \\ -p_x & -p_y & -p_z & p_w \end{bmatrix}}_{\mathbf{P}_L} \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix} = \underbrace{\begin{bmatrix} q_w & q_z & -q_y & q_x \\ -q_z & q_w & q_x & q_y \\ q_y & -q_x & q_w & q_z \\ -q_x & -q_y & -q_z & q_w \end{bmatrix}}_{\mathbf{Q}_R} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

Quaternion Rotation:

$$\hbar \otimes v \otimes \hbar^* = (\mathbf{H}_L)(v \otimes \hbar^*) = \underbrace{(\mathbf{H}_L)(\mathbf{H^*}_R)}_{\mathbf{R}} \begin{bmatrix} v_x & v_y & v_z & 0 \end{bmatrix}^T$$

$$\mathbf{R} = \begin{bmatrix} -h_z^2 - h_y^2 + h_x^2 + h_w^2 & 2h_x h_y - 2h_z h_w & 2h_x h_z + 2h_y h_w \\ 2h_z h_w + 2h_x h_y & -h_z^2 + h_y^2 - h_x^2 + h_w^2 & 2h_y h_z - 2h_x h_w \\ 2h_x h_z - 2h_y h_w & 2h_y h_z + 2h_x h_w & h_z^2 - h_y^2 - h_x^2 + h_w^2 \end{bmatrix}$$

Transformation Matrices

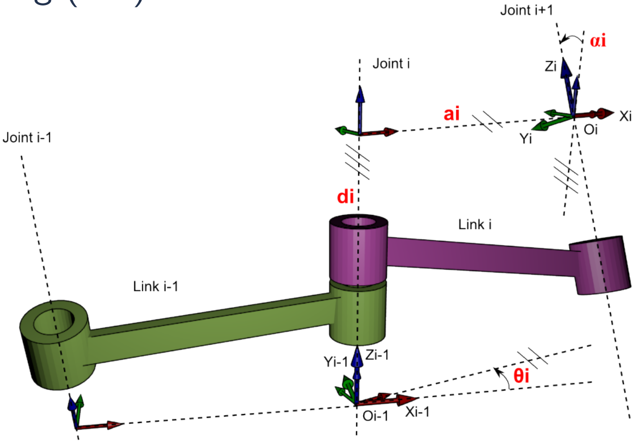
Transformation

- ${}^{a}p = {}^{a}h_{b} \otimes {}^{b}p \otimes ({}^{a}h_{b})^{*} + {}^{a}v_{b}$
- $= {}^{a}\mathbf{R}_{b} \begin{bmatrix} {}^{b}x \\ {}^{y}x \\ {}^{z}x \end{bmatrix} + \begin{bmatrix} ({}^{a}v_{b})_{x} \\ ({}^{a}v_{b})_{y} \\ ({}^{a}v_{b})_{z} \end{bmatrix}$
- $= \overbrace{\begin{bmatrix} {}^{a}\mathbf{R}_{b} & {}^{a}\mathbf{v}_{b} \\ 0 & 1 \end{bmatrix}}^{{}^{a}\mathbf{T}_{b}} \begin{bmatrix} {}^{b}x \\ {}^{y}x \\ {}^{z}x \\ 1 \end{bmatrix}$
- ${}^{a}\mathbf{p} = \left({}^{a}\mathbf{T}_{b}\right)\left({}^{b}\mathbf{p}\right)$

Chaining

- ${}^{a}\mathbf{p} = \overbrace{\left({}^{a}\mathbf{T}_{b}\right)\left({}^{b}\mathbf{T}_{c}\right)}^{{}^{a}\mathbf{T}_{c}}\left({}^{c}\mathbf{p}\right)$
- Chain:
  1. ${}^{a}\mathbf{T}_{c} = \left({}^{a}\mathbf{T}_{b}\right)\left({}^{b}\mathbf{T}_{c}\right)$
  2. ${}^{a}\mathbf{T}_{c} = \begin{bmatrix} {}^{a}\mathbf{R}_{b} & {}^{a}\mathbf{v}_{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{b}\mathbf{R}_{c} & {}^{b}\mathbf{v}_{c} \\ 0 & 1 \end{bmatrix}$
  3. ${}^{a}\mathbf{T}_{c} = \begin{bmatrix} \left({}^{a}\mathbf{R}_{b}\right)\left({}^{b}\mathbf{R}_{c}\right) & \left({}^{a}\mathbf{R}_{b}\right)\left({}^{b}\mathbf{v}_{c}\right) + {}^{a}\mathbf{v}_{b} \\ 0 & 1 \end{bmatrix}$

Denavit-Hartenberg (DH) Parameters



https://commons.wikimedia.org/wiki/File:Classic-DHparameters.png

Computationally inefficient and analytically awkward.

What about joints and links?

- Not part of equations per se
- Varying Transforms:
  Revolute Joint: ${}^{i}\mathcal{S}_{i+1}(\theta) = \exp\left(\frac{\theta}{2}{}^{i}\hat{\boldsymbol{u}}_{i+1}\right) + \left(\frac{1}{2}\left({}^{i}v_{i+1}\right) \otimes \exp\left(\frac{\theta}{2}{}^{i}\hat{\boldsymbol{u}}_{i+1}\right)\right)\varepsilon$
  Prismatic Joint: ${}^{j}\mathcal{S}_{j+1}(\ell) = {}^{j}h_{j+1} + \left(\frac{\ell}{2}\left({}^{j}\hat{\boldsymbol{u}}_{j+1}\right) \otimes {}^{j}h_{j+1}\right)\varepsilon$
- Fixed Transforms: ${}^{k}\mathcal{S}_{k+1}$
- 3D Meshes: sets of faces/triangles

Computational Issues

	Storage	Chain Transforms	Transform Point
Quaternion + Vector	7 elements	31 mul., 30 add.	15 mul., 18 add.
Dual Quaternion	8 elements	48 mul., 40 add.	24 mul., 21 add.
Transformation Matrix	12 elements	36 mul., 27 add.	9 mul., 9 add.

Singularities may appear in ln, exp, etc. Usually defined in the limit / can use Taylor series.

Which Representation Should I Use?

Analysis: Dual Quaternion and/or Matrix

- Linear operations

Chaining: Quaternion + Vector

- Fewest operations to chain
- Numerically stable / easy to normalize

Transforming: Matrix

- Fewest operations to transform

Filtering / Estimation: Quaternion + Vector or Dual Quaternion

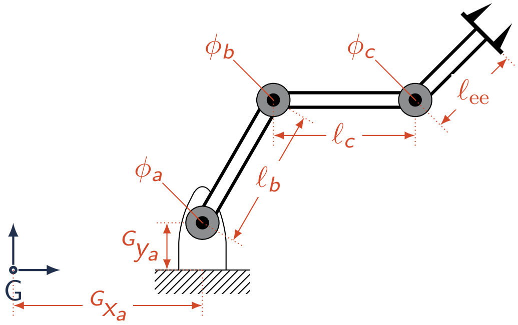
- Numerically stable / easy to normalize

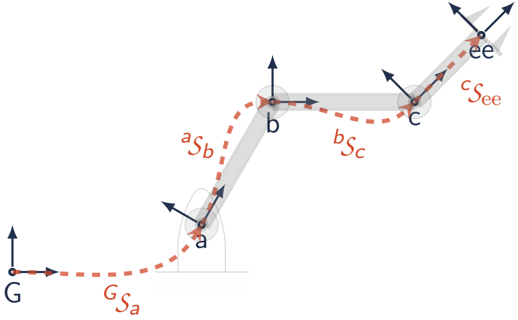
Humans: Axis-Angle and/or Euler Angles

- Easier to visualize angles than sin/cos

Outline

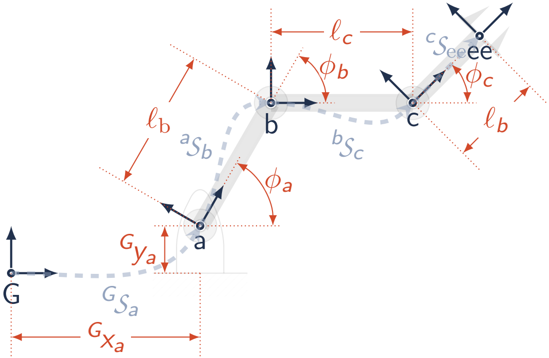
Serial Manipulator





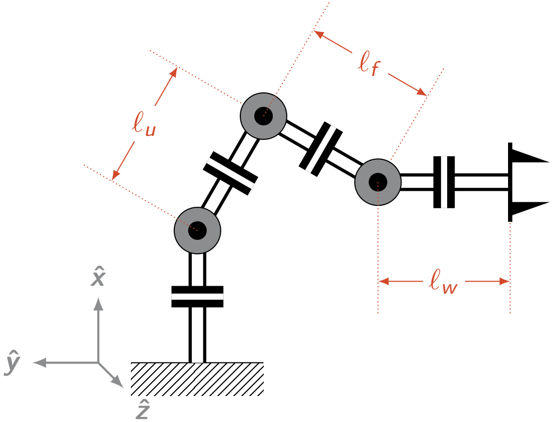
Serial Manipulator

Transforms



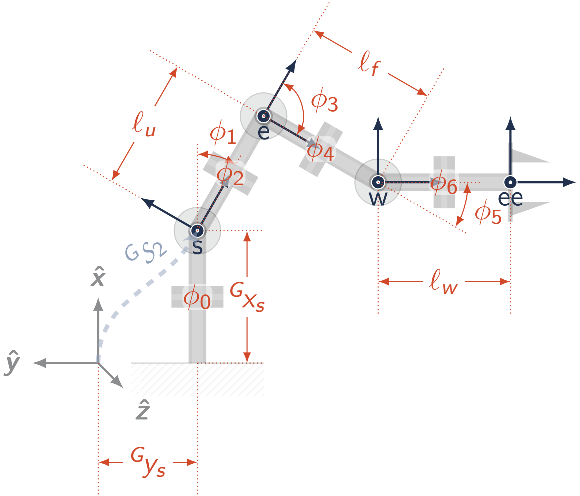
- **Relative:** $\mathcal{S} = h + \frac{1}{2} v \otimes h\varepsilon$
  - ${}^G\mathcal{S}_a = \exp\left(\frac{\phi_a}{2}\hat{\boldsymbol{k}}\right) + \frac{1}{2}{}^G v_a \otimes \exp\left(\frac{\phi_a}{2}\hat{\boldsymbol{k}}\right)\varepsilon$
  - ${}^a\mathcal{S}_b = \exp\left(\frac{\phi_b}{2}\hat{\boldsymbol{k}}\right) + \frac{1}{2}\ell_b\hat{\boldsymbol{\imath}} \otimes \exp\left(\frac{\phi_b}{2}\hat{\boldsymbol{k}}\right)\varepsilon$
  - ${}^b\mathcal{S}_c = \exp\left(\frac{\phi_c}{2}\hat{\boldsymbol{k}}\right) + \frac{1}{2}\ell_c\hat{\boldsymbol{\imath}} \otimes \exp\left(\frac{\phi_c}{2}\hat{\boldsymbol{k}}\right)\varepsilon$
  - ${}^c\mathcal{S}_{\mathrm{ee}} = 1 + \frac{1}{2}\ell_{\mathrm{ee}}\hat{\boldsymbol{\imath}}\varepsilon$
- **Absolute:** ${}^G\mathcal{S}_n = {}^G\mathcal{S}_m \otimes {}^m\mathcal{S}_n$
  - ${}^G\mathcal{S}_b = {}^G\mathcal{S}_a \otimes {}^a\mathcal{S}_b$
  - ${}^G\mathcal{S}_c = {}^G\mathcal{S}_b \otimes {}^b\mathcal{S}_c$
  - ${}^G\mathcal{S}_{\mathrm{ee}} = {}^G\mathcal{S}_c \otimes {}^c\mathcal{S}_{\mathrm{ee}}$

Anthropomorphic arm



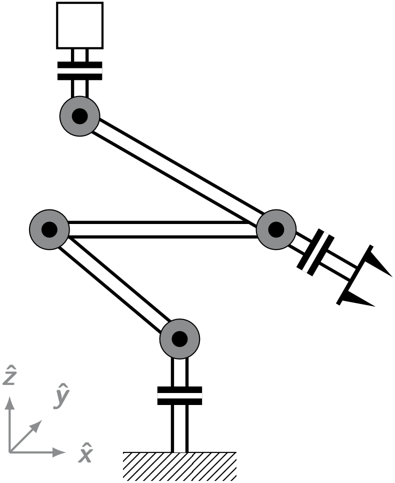
Anthropomorphic arm

Transforms



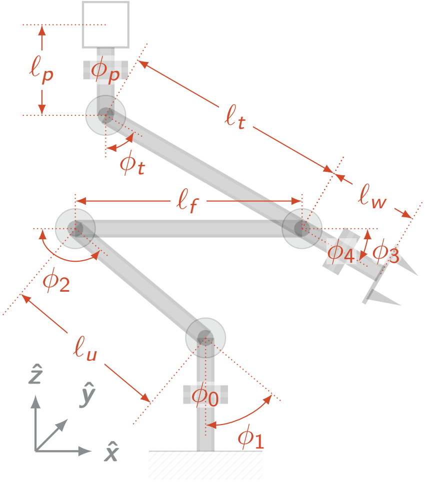
Packbot

http://endeavorrobotics.com/products



Packbot

Transforms



Implementation Notes

Fixed Frame: ${}^{\mathrm{parent}}\mathcal{S}_{\mathrm{id}} = \mathcal{S}$

- parent, id: Label
- transform $\mathcal{S}$

Revolute Frame: ${}^{\mathrm{parent}}\mathcal{S}_{\mathrm{id}}(\theta) = \exp\left(\frac{\theta\hat{\boldsymbol{u}}}{2}\right) + \left(\frac{1}{2}\exp\left(\frac{\theta\hat{\boldsymbol{u}}}{2}\right) \otimes v\right)\boldsymbol{\varepsilon}$

- parent, id: Label
- axis of rotation ($\hat{\boldsymbol{u}}$)
- fixed translation (v)

Prismatic Frame: ${}^{\mathrm{parent}}\mathcal{S}_{\mathrm{id}}(\ell) = \hbar + \left(\frac{1}{2}\ell\hat{\boldsymbol{u}} \otimes \hbar\right)\boldsymbol{\varepsilon}$

- parent, id: Label
- fixed rotation ($\hbar$)
- axis of translation ($\hat{\boldsymbol{u}}$)

Scene/Robot: A set of frames

Summary