# Predicate Calculus

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# Outline

# Predicate Calculus Parts

- Propositional Logic plus....
- **Objects**: instead of just Boolean values,
  - Finite set of values
  - Enumerated type
- **Predicates**: Function from objects to Booleans
  - $P : \mathcal{O} \times \ldots \times \mathcal{O} \mapsto \mathbb{B}$
- **Functions**: Function from objects to objects
  - $F : \mathcal{O} \times \ldots \times \mathcal{O} \mapsto \mathcal{O}$
- **Quantifiers**: express properties on collections
  - *Universal*: all items have the property
  - *Existential*: at least one item has the property

# Logical Predicate

$$f : \mathcal{O} \times \ldots \times \mathcal{O} \mapsto \mathbb{B}$$

- Boolean-valued function
- N-ary arguments
- Examples:

| English | | Logic |
|---|---|---|
| "x is happy" | $\rightsquigarrow$ | $\texttt{happy}\,(x)$ |
| "The suitcase contains a bomb." | $\rightsquigarrow$ | $\texttt{contains}\,(\texttt{suitcase},\ \texttt{bomb})$ |
| "$x$ is less than $y$." | $\rightsquigarrow$ | $\texttt{less}\,(x,\ y)$ |

# Exercise: Logical Predicates

## Material Properties

Objects: $\{\text{wood}, \text{glass}, \text{steel}\}$

Predicates:
- $\texttt{transparent}\,(?x)$
- $\texttt{flammable}\,(?x)$

Statements:
- $\texttt{transparent}\,(\text{wood}) =$
- $\texttt{transparent}\,(\text{glass}) =$
- $\texttt{transparent}\,(\text{steel}) =$
- $\texttt{flammable}\,(\text{wood}) =$
- $\texttt{flammable}\,(\text{glass}) =$
- $\texttt{flammable}\,(\text{steel}) =$

# Exercise: Logical Predicates

## Material Properties—Binary Predicate

Predicate: $\texttt{denser}\,(?x, ?y)$

Properties:

Irreflexive: $a \not> a$

Antisymmetric: $((a > b) \implies \neg(b > a)$

Transitive: $((a > b) \wedge (b > c)) \implies (a > c)$

# Logical Function

$$f : \mathcal{O} \times \ldots \times \mathcal{O} \mapsto \mathcal{O}$$

- Function from objects to objects
- Examples:

| English | | Logic |
|---|---|---|
| "The capital of Colorado is Denver." | $\rightsquigarrow$ | $\texttt{capital}\,(\texttt{colorado}) = \texttt{denver}$ |
| "Water is liquid." | $\rightsquigarrow$ | $\texttt{phase}\,(\texttt{water}) = \texttt{liquid}$ |
| "The suitcase contains a bomb." | $\rightsquigarrow$ | $\texttt{contents}\,(\texttt{suitcase}) = \texttt{bomb}$ |

# Example: Logical Functions

## Material Properties

Objects: $M : \{\texttt{wood}, \texttt{steel}, \texttt{helium}, \texttt{water}\}$
$H : \{\texttt{solid}, \texttt{liquid}, \texttt{gas}\}$

Functions: $\texttt{phase}_{\text{stp}} : M \mapsto H$

Statements:

- $\texttt{phase}_{\text{stp}}(\texttt{wood}) =$
- $\texttt{phase}_{\text{stp}}(\texttt{steel}) =$
- $\texttt{phase}_{\text{stp}}(\texttt{helium}) =$
- $\texttt{phase}_{\text{stp}}(\texttt{water}) =$

# Logical Predicates vs. Functions

## Predicate

- $f : \mathcal{O} \times \ldots \times \mathcal{O} \mapsto \mathbb{B}$
- Boolean-valued

## Function

- $f : \mathcal{O} \times \ldots \times \mathcal{O} \mapsto \mathcal{O}$
- Object-valued

# Quantifiers

## Universal ($\forall$)

- Holds for every element
- "For all x, ..."
- $\forall x,\ \phi(x)$

## Existential ($\exists$)

- Holds for at least one element
- "There exists an x such that ..."
- $\exists x,\ \phi(x)$

# Example: Quantifiers

## Material Properties

- "There exists a gas that is flammable."
  $\exists x,\ (\texttt{phase}_{\text{stp}}(x) = \texttt{gas}) \wedge \texttt{flammable}(x)$
- "All metals are not insulators."
  $\forall x,\ \texttt{metal}(x) \implies \neg\texttt{insulator}(x)$

# Exercise: Quantifiers

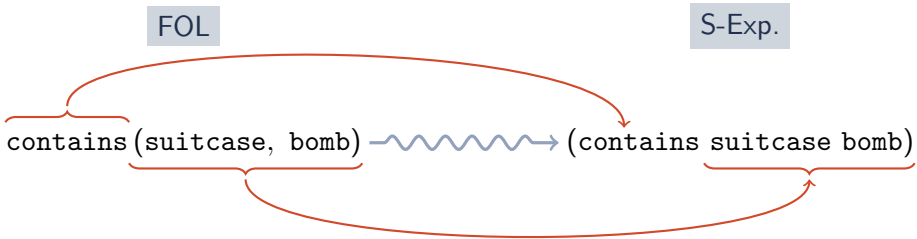
## Material Properties

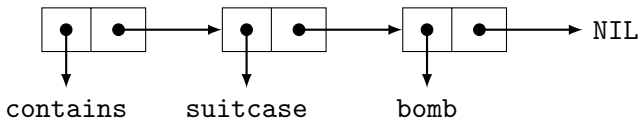
- "Some non-metal is not an insulator."
- "Every noble gas is a gas and is transparent."
- "All gasses are transparent."

# Grammar for First Order Logic

$$
\begin{array}{rcl}
\langle \text{Sentence} \rangle & \rightarrow & \langle \text{AtomicSentence} \rangle \\
& | & \langle \text{Sentence} \rangle \langle \text{Connective} \rangle \langle \text{Sentence} \rangle \\
& | & [(] \, \langle \text{Sentence} \rangle \, [)] \\
& | & [\neg] \, \langle \text{Sentence} \rangle \\
& | & \langle \text{Quantifier} \rangle \, [\text{Variable}] \, [,] \, \langle \text{Sentence} \rangle \\
\langle \text{AtomicSentence} \rangle & \rightarrow & [\text{Predicate}] \, [(] \, \langle \text{TermList} \rangle \, [)] \mid \langle \text{Term} \rangle = \langle \text{Term} \rangle \\
\langle \text{TermList} \rangle & \rightarrow & \langle \text{Term} \rangle \mid \langle \text{Term} \rangle \, [,] \, \langle \text{TermList} \rangle \\
\langle \text{Term} \rangle & \rightarrow & [\text{Constant}] \mid [\text{Variable}] \mid [\text{Function}] \, [(] \, \langle \text{TermList} \rangle \, [)] \\
\langle \text{Connective} \rangle & \rightarrow & [\wedge] \mid [\vee] \mid [\implies] \mid [\iff] \\
\langle \text{Quantifier} \rangle & \rightarrow & [\forall] \mid [\exists]
\end{array}
$$

# First Order Logic as S-Expressions



# Outline

# First Order vs. Propositional Logic

*Every sentence in first-order logic*
*can be converted into an equivalent sentence*
*in propositional logic*

(modulo functions)

# Example: Propositionalization

Objects: $\{\texttt{boulder}, \texttt{denver}, \texttt{golden}\}$

Predicates: $\texttt{capital}\,(?x)$

Sentences:

- $\overbrace{\left(\texttt{capital}\,(\texttt{boulder})\right)}^{\textit{predicate}} \rightsquigarrow \overbrace{\left(\texttt{capital-boulder}\right)}^{\textit{proposition}}$
- $\left(\forall x\ \texttt{capital}\,(x)\right) \rightsquigarrow \left(\texttt{capital-boulder} \wedge \texttt{capital-denver} \wedge \texttt{capital-golden}\right)$
- $\left(\exists x\ \texttt{capital}\,(x)\right) \rightsquigarrow \left(\texttt{capital-boulder} \vee \texttt{capital-denver} \vee \texttt{capital-golden}\right)$

# Exercise: Propositionalization

Objects: $\{\texttt{methane}, \texttt{nitrogen}, \texttt{water}\}$

Predicates:
- $\texttt{gas}\,(?x)$
- $\texttt{liquid}\,(?x)$
- $\texttt{flammable}\,(?x)$

Sentences:

- $\Big(\forall x,\ \texttt{gas}\,(x) \iff \neg\texttt{liquid}\,(x)\Big) \rightsquigarrow$

# Exercise: Propositionalization

continued

Objects: $\{\texttt{methane}, \texttt{nitrogen}, \texttt{water}\}$

Predicates:

- $\texttt{gas}\,(?x)$
- $\texttt{liquid}\,(?x)$
- $\texttt{flammable}\,(?x)$

Sentences:

- $\Big(\exists x,\ \texttt{gas}\,(x) \wedge \texttt{flammable}\,(x)\Big) \rightsquigarrow$

# Propositionalization Caveats

Dimensionality:

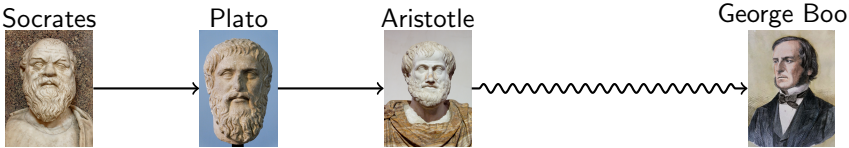
- $\left(\phi : \mathcal{O} \mapsto \mathbb{B}\right) \rightsquigarrow |\mathcal{O}|$ propositional terms
- $\left(\phi : \mathcal{O} \times \mathcal{O} \mapsto \mathbb{B}\right) \rightsquigarrow |\mathcal{O}^2|$ propositional terms
- $\left(\phi : \mathcal{O}^k \mapsto \mathbb{B}\right) \rightsquigarrow |\mathcal{O}^k|$ propositional terms

Functions:

- $\texttt{father} : \mathcal{O} \mapsto \mathcal{O}$
- $\ldots\texttt{father}\left(\texttt{father}\left(\texttt{father}\left(\texttt{john}\right)\right)\right)$

# Outline

# Historical Note: Aristotle's Syllogisms




## Syllogism

Major Premise All men are mortal.

Minor Premise Socrates is a man.

Conclusion Therefore, Socrates is mortal.

## Propositional

$$(\overbrace{p_1}^{\text{major}} \wedge \overbrace{p_2}^{\text{minor}}) \implies \overbrace{p_3}^{\text{conclusion}}$$

## First-Order

$$\Big(\overbrace{(\texttt{man}(m) \implies \texttt{mortal}(m))}^{\text{major}} \wedge \overbrace{(\texttt{man}(\text{Socrates}))}^{\text{minor}}\Big) \implies \texttt{mortal}(\text{Socrates})$$