LADR Chapter 1

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1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Problem 1

Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$\frac{1}{a+bi} = c + di.$$

Proof. Multiply the numerator and denominator of the fraction by the conjugate of the denominator

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}.$$

Expand the fraction

$$\frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

We see a form arise similar to that of $c + di$. Let

$$c = \frac{a-bi}{a^2+b^2}$$

$$d = -\frac{b}{a^2+b^2}$$

and we're done. $\square$

Problem 2

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Proof. Use euler form to represent the expression as

$$\text{cis}(\frac{2}{3}\pi).$$

Apply DeMoivre's theorem to cube the expression

$$\text{cis}(\frac{2}{3}\pi)^3 = \text{cis}(3 \cdot \frac{2}{3}\pi) = \cos(\pi) + \sin(\pi)i = 1.$$

Hence, it is indeed a cube root of 1. □

Problem 3

Find two distinct square roots of i.

Proof. Suppose there exists $a, b \in \mathbb{R}$ where $(a+bi)^2 = i$. Expand the expression into

$$(a^2 - b^2) + (2ab)i = i.$$

We split this into 2 equalities

$$a^2 - b^2 = 0$$

$$2ab = 1.$$

There are 2 cases that satisfy the first equation: $a = b$ and $a = -b$.
Case 1: $a = b$
Then

$$2a^2 = 1$$
$$a = \pm\sqrt{\frac{1}{2}}.$$

We yield 2 solutions in this case.
Case 2: $a = -b$
Then

$$-2a^2 = 1$$
$$a = \pm\sqrt{-\frac{1}{2}}.$$

We yield 0 solutions in this case, since the condition we set in the beginning is that $a \in \mathbb{R}$. Hence, there are 2 and only 2 distinct roots of i

$$\pm\sqrt{\frac{1}{2}}(1+i).$$

□

Problem 4

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha = a + bi$ and $\beta = c + di$ for $a, b, c, d \in \mathbb{R}$. Then

$$\alpha + \beta = a + bi + c + di = c + di + a + bi = \beta + \alpha.$$

□

Problem 5

Show that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Trivial. □

Problem 6

Show that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Trivial. □

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Problem 7

Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta$ $= 0$.

Proof. Let $\alpha = a_1 + a_2 i$ for some $a_1, a_2 \in \mathbb{R}$ and let $\beta = -a_1 - a_2 i$. Then

$$\alpha + \beta = (a_1 - a_1) + (b_1 - b_1)i = 0,$$

proving existence.
Let $\gamma \in \mathbb{C}$ where $\alpha + \gamma = 0$. Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = 0 + \beta = \beta,$$

proving uniqueness. □