CS70 Note 2: Proofs

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1 Proof Basics

A proof is a finite sequence of steps called logical deductions which establishes the truth of a statement. There are certain statements called axioms/postulates that we accept without proof. We apply the rules of logic to formulate complex truths.
Direct proof assume $P \ldots$ therefore Q.
Ex: x divisible by 9 IFF the sum of its digits are also divisible by 9.
Proof by contraposition assume $\neg Q \ldots$ therefore $\neg P$
Ex: Pigeonhole principle
Proof by contradiction assume $\neg P \ldots R \ldots \neg R$ therefore P
Ex: "There are infinitely many primes"
Proof by cases we don't know which of a set of possible cases is true, but we know at least one case is true. Prove the result in both cases then the general statement holds.
Ex: there exists irrational x, y so x^y is rational.

Proof. Demonstrate a single x, y pair works is sufficient. Let $x = \sqrt{2}$ and $y = \sqrt{2}$. Divide the proof into 2 cases, one case must be true.

- $\sqrt{2}^{\sqrt{2}}$ is rational
  Immediately yields our claim, rational.
- $\sqrt{2}^{\sqrt{2}}$ is irrational
  Try a different irrational x, $x = \sqrt{2}^{\sqrt{2}}$.

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$$

  which is rational.

$\square$

To me, this is oddly beautiful. We don't know what the actual (x, y) pair satisfied the claim.
Non-constructive proof is when we prove object X exists without revealing what X is.

2 Proof Errors

1. Do not assume the claim you aim to prove.
Ex: $-2 = 2 \implies (-2)^2 = (2)^2 \implies 4 = 4$ which is true. Therefore $-2 = 2$.
2. Don't forget the case where your variables are 0.
Ex: accidentally dividing by 0
3. Be careful when mixing negatives with inequalities.
Ex: you need to flip the inequality if multiply by a negative.