LADR 2C

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Dimension

2.35 Basis length doesn't depend on basis

Proof. Let B_1, B_2 be bases for V. Then, $len(B_1) \geq len(B_2) \wedge len(B_2) \geq len(B_1) \implies len(B_1) = len(B_2)$. □

2.36 Definition: dimension (dim V)

Length of any basis of V.

2.38 Dimension of a subspace $\leq$ the space

Proof. Trivial. □

2.39 Linearly independent list of right length is basis

2.42 Spanning list of right length is basis

2.43 Dimension of a sum

$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$

Proof. Intersection is a subspace w/ dimension m. We can extend it by j vectors to create a basis for U_1. Similarly, we can extend it by k vectors to create a basis for U_2. Finally, $\dim(U_1 + U_2) = (m + j) + (m + k) - m$. □

Problem 9

Suppose $v_1 \dots v_m$ is linearly independent in V and $w \in V$. Prove $\dim \operatorname{span}(v_1 + w \dots v_m + w) \geq m - 1$.

Proof. The given span is equivalent to the span of any linear combination of its spanning list. Thus, the given span is equivalent to $\operatorname{span}(v_1 - v_2, v_2 - v_3, \dots, v_m + w)$. Since $v_1, \dots, v_m$ is linearly independent and $v_1 - v_2, v_2 - v_3, \dots, v_{m-1} - v_m$ is a linear combination of $v_1, \dots, v_m$, it is also linearly independent. Thus, $\dim \operatorname{span}(v_1 - v_2, v_2 - v_3, \dots, v_{m-1} - v_m) = m - 1$, and $\dim \operatorname{span}(v_1 - v_2, v_2 - v_3, \dots, v_m + w) \geq m - 1$. $\square$

Problem 11

Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$ and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Proof. We've proven earlier that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

$$8 = 3 + 5 - \dim(U \cap W).$$

Since $\dim(U \cap W) = 0$, we have it that the intersection is $\{0\}$. We've also proven earlier that

$$U \cap W = \{0\} \implies U + W = U \oplus W.$$

$\square$

Problem 17

Prove or disprove:

$$\dim(U_1 + U_2 + U_3) = dim(U_1) + dim(U_2) + dim(U_3)$$

$$- \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_3 \cap U_1)$$

$$+ \dim(U_1 \cap U_2 \cap U_3)$$

Proof. Counterexample: $U_1 = \operatorname{span}\{(1, 0)\}, U_2 = \operatorname{span}\{(1, 1)\}, U_3 = \operatorname{span}\{(0, 1)\}$.

$$2 \neq 1 + 1 + 1 - 1 - 1 - 0 + 0.$$

$\square$