LADR 3A

Lucas Zheng

The Vector Space of Linear Maps

3.2 Definition: linear map

Aka linear transformation, a linear map from V to W is a function $T : V \to W$ which follows:

- additivity: $(\forall u, v \in V); T(u + v) = Tu + Tv$
- homogeneity: $(\forall \lambda \in F, v \in V); T(\lambda v) = \lambda(Tv)$

Special:

- zero: $0v = 0$. The LHS 0 is a function. The LHS 0 is the additive identity of W.
- identity: $Iv = v$. Identity map is the function that maps each element to itself.

Notation: the set of all linear maps from V to W is denoted as $\mathcal{L}(V, W)$.

3.6 Definition: addition, scalar multiplication on $\mathcal{L}(V, W)$

$$(S + T)(v) = Sv + Tv \text{ and } (\lambda T)(v) = \lambda(Tv)$$

$\mathcal{L}(V, W)$ is a vector space!

3.8 Definition: Product of Linear Maps

If $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W), u \in U$, then

$$ST \in \mathcal{L}(U, W); (ST)(u) = S(Tu)$$

3.9 Algebraic properties of products of linear maps

- associativity $(T_1T_2)T_3 = T_1(T_2T_3)$
- identity $TI = IT = T$
- distributive $(S_1+S_2)T = S_1T+S_2T$ and $S(T_1+T_2) = ST_1+ST_2$

Not commutative!

Problem 9

Give an example of a function $\phi : \mathbb{C} \to \mathbb{C}$ that has additivity but not homogeneity.

Proof. Let $\phi(a+bi) = a - bi$ (flip over x axis). It is easy to see that this is additive. Let $\lambda = \text{cis}(\pi/3)$ (60 degree CCW rotation).
Flip + 60 degree rotation is not equal to 60 degree rotation + flip. □

Problem 10

Let U be a subspace of V. $Tv = \begin{cases} Sv & v \in U \\ 0 & v \in V, v \notin U \end{cases}$. Prove T is not a linear map on V.

Proof. Let $u \in U$ and $v \in V, v \notin U$. Then

$$T(u) + T(v) = S(u)$$

$$\neq T(u+v) = S(u+v).$$

Since additivity does not hold, T is not a linear map. □

Problem 14

Prove there exists $S, T \in \mathcal{L}(V,V)$ where $ST \neq TS$.

Proof. We can come up with a similar solution to problem 9. A rotation function and flip function are not commutative. □