LADR 1C

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Subspaces

1.32 Definition subspace

A subset U of V is called a subspace of V if U is a vector space.

1.34 Conditions of subspace

- additive identity: $0 \in U$ (Note: this is simply the easiest way to check U has at least 1 element in it. Closure under scalar multiplication already implies a 0 element exists given a non-empty space.)
- closed under addition: $u, w \in U \implies u + w \in U$
- closed under scalar multiplication: $a \in F \wedge u \in U \implies au \in U$

1.35 Examples

Solved them on paper separately.

1.36 Definition sum of subsets

$$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$$

1.40 Definition direct sum

The sum $U_1 + ... + U_m$ is a direct sum if each element in the sum can be written in only one way as a sum $u_1 + ...u_m$ where $u_j \in U_j$.

1.44 Condition for a direct sum

IFF there is only 1 way to create 0: each $u_j = 0$.

Proof. (Me)
Let there be multiple ways to make 0.

$$x = U_1(x) + ... + U_m(x) = U_1(x+0) + ... + U_m(x+0)$$

Then there are multiple ways to make x.
Conversely, let there be multiple ways to make x.

$$U_1(x) + ... + U_m(x) = U_1(y) + ... + U_m(y)$$

$$U_1(x-y) + ... + U_m(x-y) = 0$$

Therefore, multiple ways to make x IFF multiple ways to make 0. Finally, 1 way to make x IFF 1 way to make 0. □

Proof. (Book)

$$v = u_1 + ... + u_m = v_1 + ... + v_m$$

$$0 = (u_1 - v_1) + ... + (u_m - v_m)$$

If there is only 1 way to make 0, $u_j = v_j$ is a solution and the only solution. □

1.45 Direct sum of 2 subspaces

IFF $U \cap W = \{0\}$

Proof. (Simple) If more elements in the intersection, then multiple ways to make 0. If multiple ways to make 0, then need at least one other element in the intersection. □

Problem 1

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ (b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$ (c) similar to (b)
(d) similar to (a)

Proof. (a) $(0,0,0)$ exists. $(x_1, x_2, x_3) + (y_1, y_2, y_3) \implies (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3)$ exists. $c(x_1, x_2, x_3) \implies (cx_1) + 2(cx_2) + 3(cx_3)$ exists. □

Proof. (b) We can easily see this is not closed under addition. ☐

Problem 3

Show that the set of differential real-valued functions on $(-4, 4)$ where $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Proof.

$$f(x) = 0 \implies f'(-1) = 0 = 3f(2)$$
$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f+g)(2)$$
$$c(f'(-1)) = c(3f(2)) \implies (cf')(-1) = 3(cf)(2)$$

☐

Problem 5

Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$?

Proof. Not closed under multiplication: $\forall x \in \mathbb{R} : ix \notin \mathbb{R}$ ☐

Problem 7

Give an example of a subset U of $\mathbb{R}^2$ where U is closed under addition and additive inverses.

Proof. Since it's closed under addition, it's closed under integer scalar multiplication. However, we need it to be not closed under float scalar multiplication to not be a subspace. The subset $U = \mathbb{Z}^2$ fits the conditions nicely. ☐

Problem 8

Closed under scalar multiplication, but not a subset?

Proof.

$$\{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$$

☐

Problem 9

Set of periodic functions $\mathbb{R} \to \mathbb{R}$ subspace of $\mathbb{R}^{\mathbb{R}}$?

Proof. Let $f(x)$ be a function with period 2: $\sin(2\pi/2)$ and $g(x)$ be a function with period $\sqrt{2}$: $\sin(2\pi/\sqrt{2})$. In order for $(f+g)(x)$ to be periodic as well, there must be infinite x where $f(x)$ and $g(x)$ are at their maximas. There must be a pair of integers a, b where $a \cdot 2 = b \cdot \sqrt{2}$. This is impossible, and thus by contradiction, the set is not closed under addition and thus not a subspace. $\square$

Problem 10

Prove $U_1 \cap U_2$ is a subspace of V.

Proof. First, we prove non-empty intersection.

$$0 \in U_1 \wedge 0 \in U_2 \implies 0 \in U_1 \cap U_2$$

Next, we prove closed under addition. Assume $x, y \in U_1 \cap U_2 \wedge x + y \notin U_1 \cap U_2$. However, by definition of subspace, $x + y \in U_1 \wedge x + y \in U_2$. Thus contradiction; truly, $x + y \in U_1 \cap U_2$. Similarly, we can prove closed under multiplication. $\square$

Problem 11

Prove that the intersection of every collection of subspaces of V is a subspace of V.

Proof. Let $\mathfrak{C}$ denote some collection of subspaces of V. Let

$$U = \bigcap_{W \in \mathfrak{C}} W.$$

Solution should be very similar to problem 10. $\square$

Problem 12

Prove $U_1 \cup U_2$ is a subspace IFF one is contained in the other.

Proof. WLOG, if one subspace contains the other, let the superset be U_1 and $U_1 \cup U_2 = U_1$.
Forward direction is trivial: U_1 is a subspace.
Conversely, we prove $U_1 \cup U_2$ is closed under addition IFF $U_1 \supset U_2$. Let

$$x \in U_1 \wedge x \notin U_2 \wedge y \in U_2 \wedge y \notin U_1$$

$$\implies x + y \notin U_1 \wedge x + y \notin U_2$$
$$\implies x + y \notin U_1 \cap U_2.$$

By contraposition: $x + y \in U_1 \cup U_2 \implies U_1 \supset U_2$. Thus, we have proved IFF. □

Problem 13

Prove $U_1 \cup U_2 \cup U_3$ is a subspace IFF one contains the other 2.

Proof. Similar to (12) □

Problem 15

What is $U + U$?

Proof. By definition,

$$U + U = \{u_1 + u_2 : u_1 \in U, u_2 \in U\}$$

Since U is closed under addition,

$$\implies u_1 + u_2 \in U \implies U + U = U$$

□

Problem 16

Is addition on subspaces of V commutative?

Proof. Trivial: yes □

Problem 17

Associative?

Proof. Trivial: yes □

Problem 25

Let U_e denote the set of real-valued even functions and let U_o denote the set of real-valued odd functions. Prove $U_e \oplus U_o = \mathbb{R}^{\mathbb{R}}$.

Proof. Let

$$h \in \mathbb{R}^{\mathbb{R}}, f(x) = \frac{h(x) + h(-x)}{2}, g(x) = \frac{h(x) - h(-x)}{2}.$$

$f(-x) = f(x)$, and is thus an even function. $g(-x) = -g(x)$, and is thus an odd function. $f(x) + g(x) = h(x)$, and thus every function $h \in \mathbb{R}^{\mathbb{R}}$ can be represented as the sum of a even and odd function. $\square$

Below are some points that led me to the solution. Even and odd functions are free functions for $x \geq 0$ but have a fixed behavior for $x < 0$. Is it possible to prove that all functions $\mathbb{R}^{\mathbb{R}}$ can be represented as the sum of an even and odd function? This would be the same thing as saying, for $x \geq 0$, $h(x) = f(x) + g(x)$ and $h(-x) = f(x) - g(x)$.