LADR 2A

Lucas Zheng

$\mathbb{R}^n$ and $\mathbb{C}^n$

2.3 Definition: linear combination

A linear combination of a list of vectors in V is a vector:

$$a_1 v_1 + \cdots + a_m v_m,$$

where the scalars $\in \mathbb{F}$.

2.5 Definition: span

$$\text{span}(v_1, \ldots, v_m) = \{\text{all possible linear combinations}\}$$

$$\text{span}() = \{0\}$$

2.7 Span is the smallest containing subspace

Proof. $\text{span}(v_1, \ldots, v_m) \subset$ any subspace of V containing $v_1, \ldots, v_m$ and span itself is a subspace of V. Therefore the smallest. $\square$

2.8 Definition: spans

If $\text{span}(v_1, \ldots, v_m) = V$, we say that $v_1, \ldots, v_m$ spans V.

2.10 Definition: finite-dimensional vector space

A vector space is finite-dimensional if some list of vectors in it spans the space. Every list has finite length (by definition).

2.11 Definition polynomial $\mathcal{P}(\mathbb{F})$

A function $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients $\mathbb{F}$ if there exists $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + ... + a_m z^m$$

for all $z \in \mathbb{F}$.
$\mathcal{P}(\mathbb{F})$ is the set of all polys w/ coefficients in $\mathbb{F}$.

2.12 Definition degree of a polynomial

Exists scalars $a_0, \ldots, a_m \in \mathbb{F}$ w/ $a_m \neq 0$ so:

$$p(z) = a_0 + \cdots + a_m z^m.$$

The poly $= 0$ has degree $-\infty$.

2.13 Definition: $\mathcal{P}_m(\mathbb{F})$

All polynomials w/ degree $\leq m$.

infinite-dimensional vector space

Not finite-dimensional.

2.17 Definition: linear independent

$v_1, \ldots, v_m$ is linearly independent IFF each vector in span$(v_1, \ldots, v_m)$ has only 1 representation as a linear combination of $v_1, \ldots, v_m$. The most convenient condition that enables this is that there is only 1 (the trivial solution) to make 0.

2.19 Definition: linear dependent

Not linearly independent.

2.21 Definition: Linear Dependent Lemma

Given a linear dependent list of vectors, you can remove a vector without changing the span of the list.

2.23 len(linearly independent list) $\leq$ len(spanning list)

Trivial

Problem 17

Suppose $p_j(2) = 0$ for each j. Prove $p_0 \dots p_m$ is not linearly independent in $\mathcal{P}_m(F)$.

Proof. Safely assume no 0 polynomials, or else it would be linearly dependent. Factor out $(x - 2)$ from all the polynomials into $q_0 \dots q_m$ in $\mathcal{P}_{m-1}(F)$. Easy to see that this list of polynomials are not linearly independent in the new space.
(Better) Define polynomial $q \equiv 1$. Then, $q \in \operatorname{span}(p_0 \dots p_m)$,

$$q(2) = a_0 p_0(2) + \cdots + a_m p_m(2).$$

Contradiction, 1 can't equal 0. Therefore $p_0 ... p_m$ can't be linearly independent. $\square$