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# LADR 1A

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## 1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

### Problem 1

Suppose  $a$  and  $b$  are real numbers, not both 0. Find real numbers  $c$  and  $d$  such that

$$\frac{1}{a + bi} = c + di.$$

*Proof.* Multiply the numerator and denominator of the fraction by the conjugate of the denominator

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2}.$$

Expand the fraction

$$\frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

We see a form arise similar to that of  $c + di$ . Let

$$c = \frac{a}{a^2 + b^2}$$

$$d = -\frac{b}{a^2 + b^2}$$

and we're done. □

**2Q.** Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

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is a cube root of 1 (meaning that its cube equals 1).

**2A.** Use euler form to represent the expression as

$$\operatorname{cis}\left(\frac{2}{3}\pi\right).$$

Apply DeMoivre's theorem to cube the expression

$$\operatorname{cis}\left(\frac{2}{3}\pi\right)^3 = \operatorname{cis}\left(3 \cdot \frac{2}{3}\pi\right) = \cos(\pi) + \sin(\pi)i = 1.$$

Hence, it is indeed a cube root of 1.

**3Q.** Find two distinct square roots of  $i$ .

**3A.** Suppose there exists  $a, b \in \mathbb{R}$  where  $(a + bi)^2 = i$ . Expand the expression into

$$(a^2 - b^2) + (2ab)i = i.$$

We split this into 2 equalities

$$a^2 - b^2 = 0$$

$$2ab = 1.$$

There are 2 cases that satisfy the first equation:  $a = b$  and  $a = -b$ .

Case 1:  $a = b$

Then

$$\begin{aligned} 2a^2 &= 1 \\ a &= \pm\sqrt{\frac{1}{2}}. \end{aligned}$$

We yield 2 solutions in this case.

Case 2:  $a = -b$

Then

$$\begin{aligned} -2a^2 &= 1 \\ a &= \pm\sqrt{-\frac{1}{2}}. \end{aligned}$$

We yield 0 solutions in this case, since the condition we set in the beginning is that  $a \in \mathbb{R}$ . Hence, there are 2 and only 2 distinct roots of  $i$

$$\pm\sqrt{\frac{1}{2}}(1 + i).$$

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**4Q.** Show that  $\alpha + \beta = \beta + \alpha$  for all  $\alpha, \beta \in \mathbb{C}$ .

**4A.** Let  $\alpha = a + bi$  and  $\beta = c + di$  for  $a, b, c, d \in \mathbb{R}$ . Then

$$\alpha + \beta = a + bi + c + di = c + di + a + bi = \beta + \alpha.$$

**5Q.** Show that  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$  for all  $\alpha, \beta, \gamma \in \mathbb{C}$ .

**5A.** Trivial.

**6Q.** Show that  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$  for all  $\alpha, \beta, \gamma \in \mathbb{C}$ .

**6A.** Trivial.

**7Q.** Show that for every  $\alpha \in \mathbb{C}$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha + \beta = 0$ .

**7A.** Let  $\alpha = a_1 + a_2i$  for some  $a_1, a_2 \in \mathbb{R}$  and let  $\beta = -a_1 - a_2i$ . Then

$$\alpha + \beta = (a_1 - a_1) + (a_2 - a_2)i = 0,$$

proving existence.

Let  $\gamma \in \mathbb{C}$  where  $\alpha + \gamma = 0$ . Then

$$\gamma = \gamma + (\alpha + \beta) = (\gamma + \alpha) + \beta = 0 + \beta = \beta,$$

proving uniqueness.