LADR 3B

Lucas Zheng

Null Spaces and Ranges

3.12 Definition: null space, $\operatorname{null} T$

For $T \in \mathcal{L}(V, W), \operatorname{null} T = \{v \in V : Tv = 0\}$

3.15 Definition: injective (aka one to one)

A function $T : V \to W$ is injective if

$$Tu = Tv \implies u = v$$

It may be easier to think of the contrapositive. In english: T is injective if it maps distinct inputs to distinct outputs.

3.16 Injectivity is equivalent to null space = {0}

($\to$) Since injective, $T(x) = 0$ is only true for one x value. $T(x) = T(0)+T(x) \implies T(0) = 0$. That x must be 0. Therefore, injective $\implies$ $\operatorname{null} T = \{0\}$.
($\leftarrow$) $T(u) = T(v) \implies T(u - v) = 0$. Since $\operatorname{null} T = \{0\}$, $u - v = 0 \implies u = v$.

3.17 Definition: range

For $T : V \to W$, $\operatorname{range} T$ is subset of W w/ form Tv.

$$\operatorname{range} T = \{Tv : v \in V\}$$

3.19 Range is a subspace

$$T(0) = 0 \implies 0 \in \operatorname{range} T$$
$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$$
$$T(\lambda v) = \lambda T v = \lambda w$$

3.20 Definition: surjective (aka onto)

A function $T : V \to W$ is called surjective if its range equals W.

3.22 Fundamental Theorem of Linear Maps

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. Let $u_1, \ldots, u_m$ is a basis of $\operatorname{null} T$. Extend this to a basis of V:

$$u_1, \ldots, u_m, v_1, \ldots, v_n.$$

Then, $\dim V = m + n$. Now, we need to show $\operatorname{range} T = n$ and finite dimensional. Let

$$v \in V, v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n.$$

$$Tv = b_1 T v_1 + \cdots + b_n T v_n.$$

Indeed, $\operatorname{range} T = n$ and finite dimensional. □

3.23 A map to a smaller dimensional space is not injective

$\dim V > \dim W \implies T : V \to W$ is not injective.

Proof.

$$\dim V > \dim W \implies \dim V - \dim W \geq 1$$
$$\dim V - \dim \operatorname{range} T \geq 1 \implies \dim \operatorname{null} T \geq 1$$

Then, $\operatorname{null} T \neq \{0\}$, and thus T is not injective. □

3.24 A map to a larger dimensional space is not surjective

$\dim V < \dim W \implies T : V \to W$ is not surjective.

Proof.

$$\dim V - \dim W < 0$$

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \implies \dim V - \dim \operatorname{range} T \geq 0$$

$$\dim \operatorname{range} T \leq \dim V < \dim W \implies \dim \operatorname{range} T < \dim W$$

Since the dimensions of $\operatorname{range} T$ and W are different, they are not the same vector space. Therefore, T is not surjective. □

3.25 Linear system of equations

n variables, m equations:

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$

$$\vdots$$

$$\sum_{k=1}^{n} A_{m,k} x_k = c_m$$

$T(x_1, \ldots, x_n) = (\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k) = (c_1, \ldots, c_m) \implies T : \mathbb{F}^n \to \mathbb{F}^m$

3.26 Definition: Homogeneous system of linear equations

System of linear equations where constant on RHS of each equation is zero.
Property (similar to 3.23): If more variables than equations, their are multiple solutions that lead to 0 (not injective).

3.29 Definition: Inhomogeneous

Not homogeneous, vector of constants is non-zero.
Property (similar to 3.24): If less variables than equations, there are constants with no corresponding solutions (not surjective).

Problem 1

Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Proof.

$$T(e_1, e_2, e_3, e_4, e_5) = (e_1, e_2, 0, 0, 0)$$

□

Problem 2

Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\operatorname{range} S \subset \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

Proof. Since $\operatorname{range} S$ is a subset of $\operatorname{null} T$, $T(S(v)) = 0$ for all $v \in V$. Then the composed linear map $TS = 0$. Then,

$$(ST)^2 = S(TS)T = S(0)T = 0$$

. □

Problem 5

Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Proof. Notice that the given condition implies:

$$\forall v \in \mathbb{R}^4, T(T(v)) = 0$$

and that nothing outside of $x = T(v)$ for $T(x)$ yields 0. What about a linear map that shifts the vector to the left by 2?

$$T(x_1, x_2, x_3, x_4) = x_3, x_4, 0, 0$$

$$\operatorname{null} T = \operatorname{range} T = \mathbb{R}^2 \times \{0\}^2$$

□

Problem 6

Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

Proof. By Fundamental Theorem of Linear Maps,

$$\dim T = \dim \operatorname{range} T + \dim \operatorname{null} T.$$

Since both the range and null space are the same it follows that

$$\dim \operatorname{range} T = \dim \operatorname{null} T.$$

Then, $\dim \operatorname{range} T = \frac{\dim T}{2} = 2.5$, which is impossible. $\square$

Problem 7

Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace.

Proof. Let S, T be not injective linear maps.

$$S(v_1) \neq 0, S(v_2) = 0, T(v_1) = 0, T(v_2) \neq 0$$

$$(S + T)(v_1) = S(v_1) + T(v_1) = S(v_1) \neq 0$$

$\square$

Problem 8

Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace.

Proof.

$$S(v_1, v_2) = (v_1, 0), T(v_1, v_2) = (0, v_2) \implies (S + T)(v_1, v_2) = (v_1, v_2)$$

$\square$

Problem 31

Give an example of two linear maps $T_1, T_2 : \mathbb{R}^5 \to \mathbb{R}^2$ that have the same null space but are not scalar multiples of each other.

Proof.

$$T_1(v_1, v_2, v_3, v_4, v_5) = (v_1, v_2, 0, 0, 0)$$

$$T_2(v_1, v_2, v_3, v_4, v_5) = (v_2, v_1, 0, 0, 0)$$

$$e_1 \neq \lambda e_2$$

□