LADR 1B

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Definition of Vector Space

1.18 Definition of addition, scalar multiplication on V

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1.19 Definition of a vector space

A vector space is a set V along with an addition on V and a scalar multiplication on V such that: commutativity, associativity, additive inverse, multiplicative identity, distributive properties all hold.

1.20 Definition of vector, point

Vectors or points refer to elements of a vector space.

Scalar multiplication in V depends on $\mathbb{F}$. When we need to be precise, we say that V is a vector space over $\mathbb{F}$. Usually it is obvious from context or irrelevant though.

1.21 Definition of real, complex vector spaces

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1.23 Notation $\mathbb{F}^S$

- $\mathbb{F}^S$ is the set of functions w/ domain set S and range set $\mathbb{F}$.
- (Fake explanation) Such a function has $|S|$ possible inputs and $|F|$ possible outputs. There are then $|F|^{|S|}$ functions that obey this rule which motivates the notation.
- More info here
- For $f, g, f+g \in \mathbb{F}^S$,

$$(f+g)(x) = f(x) + g(x)$$

- For $\lambda \in \mathbb{F}$ and $f, \lambda f \in \mathbb{F}^S$,

$$(\lambda f)(x) = \lambda f(x)$$

Problem 1

Prove that $-(-v) = v$ for every $v \in V$

Proof. We want to show that v is the additive inverse of $(-v)$. We have

$$(-v) + v = (-1)v + 1v = (-1+1)v = 0v = 0,$$

as desired. □

Problem 3

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Proof. Let $x = \frac{1}{3}(w - v)$. This

$$v + 3x = v + (w - v) = w,$$

proving existence. Let $y \in V$ where $v + 3y = w$. Then

$$v + 3y = v + 3x \iff 3y = 3x \iff y = x,$$

proving uniqueness. □

Problem 5

Show that the additive inverse condition on vector spaces (1.19) can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Proof. We show that the 2 statements are equivalent.
First, we prove the old condition implies the new condition. Assume that every $v \in V$ has an additive inverse. We have

$$0v + 0v = (0 + 0)v = 0v.$$

Adding the additive inverse of 0v on both sides yields $0v = 0$ as desired.
Second, we prove the new condition implies the old condition. Assume that $0v = 0$ for all $v \in V$. We have

$$v + (-1)v = (1 + (-1))v = 0v = 0.$$

Hence, every element has an additive inverse, as desired. □

Problem 6

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$?

Proof. For a set to be a vector space, it must follow 6 conditions: commutativity, associativity, additive identity, additive inverse, multiplicative identity, and distributive properties. We will try to find a counter-example to break one of these rules.

$$(\infty + (-\infty)) + 3 = 0 + 3 = 3$$

$$\infty + ((-\infty) + 3) = \infty + (-\infty) = 0$$

Since these 2 expressions are not equal to each other, we have shown that the rule of associativity is broken in the set given. Therefore, it is not a vector space over $\mathbb{R}$. □