LADR 1A

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$\mathbb{R}^n$ and $\mathbb{C}^n$

Notes

1.1 Definition of complex numbers

1.3 Properties of complex arithmetic

Commutativity, associativity, identities, additive inverse, multiplicative inverse, distributive property.

1.5 Definition of subtraction, division in $\mathbb{C}$

1.8 Definition of list, length

1.10 Definition of $\mathbb{F}^n$

$\mathbb{F}^n$ is the set of all lists of length n, with all list elements from $\mathbb{F}$. We call x_j the j^{th} coordinate of $(x_1, \ldots, x_n)$.

1.12 Definition of addition in $\mathbb{F}^n$

1.13 Commutativity of addition in $\mathbb{F}^n$

1.14 Definition of 0

0 is the list of length n whose coordinates are all 0.

1.16 Additive inverse in $\mathbb{F}^n$

1.16 Scalar multiplication in $\mathbb{F}^n$

Exercises

Problem 1

Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$\frac{1}{a+bi} = c + di.$$

Proof. Multiply the numerator and denominator of the fraction by the conjugate of the denominator

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}.$$

Expand the fraction

$$\frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

We see a form arise similar to that of $c + di$. Let

$$c = \frac{a-bi}{a^2+b^2}$$

$$d = -\frac{b}{a^2+b^2}$$

and we're done. $\square$

Problem 2

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Proof. Use euler form to represent the expression as

$$\text{cis}(\frac{2}{3}\pi).$$

Apply DeMoivre's theorem to cube the expression

$$\text{cis}(\frac{2}{3}\pi)^3 = \text{cis}(3 \cdot \frac{2}{3}\pi) = \cos(\pi) + \sin(\pi)i = 1.$$

Hence, it is indeed a cube root of 1. $\square$

Problem 3

Find two distinct square roots of i.

Proof. Suppose there exists $a, b \in \mathbb{R}$ where $(a+bi)^2 = i$. Expand the expression into

$$(a^2 - b^2) + (2ab)i = i.$$

We split this into 2 equalities

$$a^2 - b^2 = 0$$

$$2ab = 1.$$

There are 2 cases that satisfy the first equation: $a = b$ and $a = -b$.
Case 1: $a = b$
Then

$$2a^2 = 1$$
$$a = \pm\sqrt{\frac{1}{2}}.$$

We yield 2 solutions in this case.
Case 2: $a = -b$
Then

$$-2a^2 = 1$$
$$a = \pm\sqrt{-\frac{1}{2}}.$$

We yield 0 solutions in this case, since the condition we set in the beginning is that $a \in \mathbb{R}$. Hence, there are 2 and only 2 distinct roots of i

$$\pm\sqrt{\frac{1}{2}}(1+i).$$

□

Problem 4

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha = a + bi$ and $\beta = c + di$ for $a, b, c, d \in \mathbb{R}$. Then

$$\alpha + \beta = a + bi + c + di = c + di + a + bi = \beta + \alpha.$$

□

Problem 5

Show that $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ for all $\alpha,\beta,\gamma\in\mathbb{C}$.

Proof. Trivial. $\square$

Problem 6

Show that $(\alpha\beta)\gamma=\alpha(\beta\gamma)$ for all $\alpha,\beta,\gamma\in\mathbb{C}$.

Proof. Trivial. $\square$

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Problem 7

Show that for every $\alpha\in\mathbb{C}$, there exists a unique $\beta\in\mathbb{C}$ such that $\alpha+\beta$ $=0$.

Proof. Let $\alpha=a_1+a_2i$ for some $a_1,a_2\in\mathbb{R}$ and let $\beta=-a_1-a_2i$. Then

$$\alpha+\beta=(a_1-a_1)+(b_1-b_1)i=0,$$

proving existence.
Let $\gamma\in\mathbb{C}$ where $\alpha+\gamma=0$. Then

$$\gamma=\gamma+(\alpha+\beta)=(\gamma+\alpha)+\beta=0+\beta=\beta,$$

proving uniqueness. $\square$