LADR 1C

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Subspaces

1.32 Definition subspace

A subset U of V is called a subspace of V if U is a vector space.

1.34 Conditions of subspace

- additive identity: $0 \in U$
- closed under addition: $u, w \in U \implies u + w \in U$
- closed under scalar multiplication: $a \in F \wedge u \in U \implies au \in U$

1.35 Examples

Solved them on paper separately.

1.36 Definition sum of subsets

$$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$$

1.40 Definition direct sum

The sum $U_1 + ... + U_m$ is a direct sum if each element in the sum can be written in only one way as a sum $u_1 + ... u_m$ where $u_j \in U_j$.

1.44 Condition for a direct sum

IFF there is only 1 way to create 0: each $u_j = 0$.

Proof. (Me)
Let there be multiple ways to make 0.

$$x = U_1(x) + ... + U_m(x) = U_1(x + 0) + ... + U_m(x + 0)$$

Then there are multiple ways to make x.
Conversely, let there be multiple ways to make x.

$$U_1(x) + ... + U_m(x) = U_1(y) + ... + U_m(y)$$

$$U_1(x - y) + ... + U_m(x - y) = 0$$

Therefore, multiple ways to make x IFF multiple ways to make 0. Finally, 1 way to make x IFF 1 way to make 0. □

Proof. (Book)

$$v = u_1 + ... + u_m = v_1 + ... + v_m$$

$$0 = (u_1 - v_1) + ... + (u_m - v_m)$$

If there is only 1 way to make 0, $u_j = v_j$ is a solution and the only solution. □

1.45 Direct sum of 2 subspaces

IFF $U \cap W = \{0\}$

Proof. (Simple) If more elements in the intersection, then multiple ways to make 0. If multiple ways to make 0, then need at least one other element in the intersection. □

Problem 1

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

Proof. $(0, 0, 0)$ exists. $(x_1, x_2, x_3) + (y_1, y_2, y_3) \implies (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3)$ exists. $c(x_1, x_2, x_3) \implies (cx_1) + 2(cx_2) + 3(cx_3)$ exists. □

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

Proof. We can easily see this is not closed under addition. □

(c) similar to (b)
(d) similar to (a)

Problem 3

Show that the set of differential real-valued functions on $(-4, 4)$ where $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Proof.

$$f(x) = 0 \implies f'(-1) = 0 = 3f(2)$$

$$(f + g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f + g)(2)$$

$$c(f'(-1)) = c(3f(2)) \implies (cf')(-1) = 3(cf)(2)$$

□

Problem 5

Is $\mathbb{R}^2$ a subspace of $\mathbb{C}^2$?

Proof. Not closed under multiplication: $\forall x \in \mathbb{R} : ix \notin \mathbb{R}$ □