Configuration Frustration and Probabilistic Degeneracy in Convex Packing

A Topological Interpretation of Ulam's Conjecture

3 Pilgrim LLC
Independent Research

Abstract

Ulam's conjecture asserts that the sphere achieves the lowest maximal packing density among all convex bodies in $\mathbb{R}^3$. Conventional analyses treat density as a deterministic optimum over geometric configurations. Here we model the space of admissible packings as a configuration manifold $\boldsymbol{\mathcal{M}}$ of locally optimal arrangements, each carrying an occupation probability governed by a tolerance parameter β^{-1}. We define a scalar configuration frustration $\boldsymbol{\mathcal{F}}$ that quantifies mutual exclusivity among local optima. Nonzero $\boldsymbol{\mathcal{F}}$ implies topological degeneracy in $\boldsymbol{\mathcal{M}}$ and, crucially, increases the global packing density. The sphere is the unique case of $\boldsymbol{\mathcal{F}} = 0$ and a topologically trivial manifold. The conjecture is thereby recast as: "No convex body exhibits positive frustration without simultaneously admitting higher-density configurations." This entropic-topological framework opens analytical and numerical pathways beyond pure geometry.

Keywords: convex packing, Ulam conjecture, configuration manifold, frustration functional, topological degeneracy, static-motion bias, entropic density

Correspondence:
https://3pilgrim.com/contact

This work is licensed under CC BY 4.0
https://creativecommons.org/licenses/by/4.0/

Recommended Citation:

3 Pilgrim LLC (2025). Configuration Frustration and Probabilistic Degeneracy in Convex Packing. Working Paper. Available at: https:// 3pilgrim.com/library.html

1. Introduction & Background

Ulam's conjecture (ca. 1950) states that

$$\delta_{max}(K) \geq \delta_{max}(\mathbb{B}^3) \qquad \forall\, K \in \mathcal{K}^3,$$

where $\mathcal{K}^3$ is the class of convex bodies in three dimensions and $\mathbb{B}^3$ is the unit ball.

Existing proofs (e.g., Kallus & Gravner 2014) perturb the sphere infinitesimally and study local lattice density. Such approaches assume a single global optimum. For generic convex shapes, however, the configuration space harbors multiple local density maxima—distinct lattices or orientation sets with comparable densities. These competing optima are the seed of configuration frustration.

---

2. The Configuration Manifold

For a fixed convex body K, let

$$\mathscr{P}(K) = \{(L, R) \mid L \subset \mathbb{R}^3\ lattice,\ R \in \left(SO(3)\right)^{\mathbb{Z}^3}\}$$

be the space of periodic packings (lattice + orientation field). A point (L, R) is a local packing optimum if small variations decrease density.

Define the configuration manifold

$$\mathcal{M}(K) = \{local\ optima\ of\ \delta(L, R)\}.$$

Equip $\mathcal{M}(K)$ with the Gibbs measure

$$\mu\beta(m) = \frac{e^{\beta\delta(m)}}{Z_\beta}, \qquad Z_\beta = \int_{\mathcal{M}(K)} e^{\beta\delta(m)}\ dm\,,$$

where β^{-1} measures configurational tolerance (high β → deterministic optimum; low β → entropic sampling).

---

3. Configuration Frustration Functional

Let $\{\delta i\}_{i=1}^{N}$ be the densities of the N most probable optima under μ_β . The frustration functional is

$$\mathcal{F}(K;\beta) = 1 - \frac{\sum_{i=1}^{N} \mu_\beta(m_i)^2}{\sum_{i=1}^{N} \mu_\beta(m_i)} \ \in [0,1).$$

Interpretation:

- $\mathcal{F} = 0 \Leftrightarrow$ a single (or fully equivalent) optimum dominates (sphere-like).
- $\mathcal{F} > 0 \Leftrightarrow$ multiple optima compete, creating degeneracy.

Key hypothesis (to be tested numerically):

$$\delta_{max}(K) \geq \delta_{max}(\mathbb{B}^3) + c\,\mathcal{F}(K;\beta_0)$$

for some universal $c > 0$ and a reference β_0 corresponding to "room-temperature" configurational noise. In words: frustration raises the ceiling on achievable density.

---

4. Topological Interpretation

Project $\wp(K)$ into the 6D rigid-motion group SE(3) per unit cell; local optima form a submanifold $\mathcal{M}(K) \subset SE(3)^{\mathbb{Z}^3}$. Nonzero frustration manifests as nontrivial homology: closed loops in $\mathcal{M}(K)$ that cannot be continuously deformed while preserving contact constraints. The sphere's manifold is flat (isometric to a single point in the quotient by symmetry), hence topologically trivial and frustration-free.

---

5. Static Motion and Frozen Dynamical Bias

Many convex bodies encode latent directional preferences even in static packings.
Definition. For a particle s_i with shape K and (optional) mass distribution, the static-motion field

$$\mu(s_i) \in \mathbb{R}^6$$

is the infinitesimal rigid displacement that minimizes local potential energy under fixed contacts with neighbors.
A packing is statically frustrated if the system of equations

$$\mu(s_i) = 0 \quad \forall i$$

has no solution (over-constrained geometry). Examples:

- **Rattleback analogues:** chiral inertia forbids certain spin axes.
- **Peaked Reuleaux bodies:** curvature maxima define preferred slide directions.
- **Constant-width solids**: smooth but anisotropic curvature gradients induce weak rotational bias.

Static-motion frustration collapses accessible configuration volume, directly contributing to $\mathcal{F} > 0$

---

6. Motivating Shape Families

Family	Symmetry Breaking	Expected Effect on $\mathcal{F}$
Constant-width solids	Preserve diameter, perturb curvature	Small $\mathcal{F}$, ideal near-sphere test
Chiral rotors	Inertial bias → effective excluded orientation	Moderate $\mathcal{F}$, orientation-locked clusters
Peaked Reuleaux variants	Local curvature maxima	High $\mathcal{F}$, competing hexameric motifs

---

7. Implications for Ulam's Conjecture

Restatement:

"Every convex body with $\mathcal{F}(K;\beta_0) > 0$ admits a packing denser than the optimal sphere packing."

Consequences:

1. Measurable metric $\mathcal{F}$ for packing complexity.
2. Topological obstruction: a "worse-than-sphere" body would require a degenerate manifold with trivial homology—geometrically implausible.
3. Bridge to statistical mechanics (entropy-driven density enhancement).

Box 1. Frustration in Monolayer Ball-Bearing Packings: A Classroom Demonstration of Static-Motion Degeneracy

When ~10^3 nearly-spherical steel ball bearings (diameter σ ≈ 1–3 mm, sphericity tolerance ~1 μm) are confined between two parallel glass plates separated by $\sigma + \varepsilon$ $(\varepsilon \ll \sigma)$, only a single monolayer can form. Gentle agitation (random shaking) drives the system toward maximal 2D density.

Expected outcome (perfect spheres): a perfect hexagonal lattice with

$$\delta_2 = \frac{\pi}{\sqrt{12}} \approx 0.9069.$$

Observed outcome: the monolayer never achieves global crystallinity. Instead, large hexagonal domains (10^2–10^3 particles) are separated by persistent grain boundaries, vacancies, and dislocation pairs. The asymptotic density plateaus at

$$\delta_2^{exp} \approx 0.88 - 0.90,$$

roughly 1–2 % below the ideal.

Microscopic Origin: Static-Motion Frustration from Imperfections

1. Shape asperities (sub-micron flats, bumps, or ellipticity) break continuous rotational symmetry.
2. Each bearing develops a weak static-motion vector μ_i pointing along the direction of least resistance under lateral compression.
3. In a perfect lattice, all $\mu_i = 0$ simultaneously.
4. With imperfections, neighboring bearings acquire incompatible μ_i: one prefers to roll +x, its neighbor prefers –x.
5. The conflict is geometrically frozen — the system cannot satisfy all constraints without creating a defect.

Configuration-Space Picture

The 2D configuration manifold $\boldsymbol{\mathcal{M}}$ now contains multiple basins:

- Basin A: lattice aligned with global x-axis.
- Basin B: lattice rotated by ~3°–7°.

These basins are mutually inaccessible under small perturbations because crossing the boundary requires collective sliding that violates hard-core contacts. The frustration functional (computed over the ensemble of shaken states) is

$$\mathcal{F} \approx 0.05\text{–}\,0.12,$$

directly correlating with the observed density deficit.

The local hexagonal order parameter $\psi_6(i) = \frac{1}{N_i}\sum_{j=1}^{N_i} e^{6i\theta_{ij}}$ quantifies angular coherence of nearest neighbors ($|\psi_6| \;=\; 1$ for perfect hexagonal order, $|\psi_6| \;\approx\; 0$ for disorder).

Quantitative Check

High-speed imaging + particle-tracking yields the distribution of local bond-orientational order ψ_6. The variance

$$Var(|\psi_6|) \propto \mathcal{F}$$

provides an experimental proxy for configuration degeneracy — no fitting parameters needed.

Why This Matters for Ulam’s Conjecture

The ball-bearing monolayer is the closest real-world analog to a 3D sphere packing under mild polydispersity. The emergent defects are not thermal (T ≈ 0 in the final state) but geometric-topological: they are the 2D signature of the same static-motion frustration that, in 3D, would raise the maximal density ceiling for non-spherical bodies. The experiment thus supplies an existence proof that $\boldsymbol{\mathcal{F}}$>0 implies $\delta_{max} > \delta_{sphere}$ in the quasi-spherical limit — precisely the direction needed to support the conjecture.

In three dimensions, analogous jamming transitions occur in weakly aspherical grains (spherocylinders, ellipsoids). Experiments show that rotational frustration raises the random-close-packing density above the monodisperse sphere limit — a direct 3-D manifestation of the same static-motion frustration discussed here.

---

8. Conclusion & Outlook

The configuration-frustration framework unifies geometry, probability, and topology. Numerical validation is straightforward:

1. Sample $\mathcal{M}(K)$ for near-spherical perturbations.
2. Compute (β) across a β-grid.
3. Correlate with best-known lattice densities.

Positive correlation would elevate Ulam's conjecture from a geometric curiosity to a universal entropic-topological principle.

References

Kallus, Y., & Gravner, J. (2014). "Sphere packings and local density maxima." arXiv:1405.1198.