Numerical Solutions of 1D Wave Equation with Boundary Value

Math 104A Final Project

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1.Introduction and Problem Formulation

The (two-way) wave equation is a second-order linear partial differential equation for the description of waves or standing wave fields – as they occur in classical physics – such as mechanical waves (e.g. water waves, sound waves and seismic waves) or electromagnetic waves (including light waves). It arises in fields like acoustics, electromagnetism, and fluid dynamics. Single mechanical or electromagnetic waves propagating in a pre-defined direction can also be described with the first-order one-way wave equation, which is much easier to solve and also valid for inhomogeneous media. The general expression of the one-dimensional wave equation is as follows:

$$\frac{\partial^2 \mu}{\partial t^2} = a\frac{\partial^2 \mu}{\partial x^2} + f(x, t)$$

Where $\mu(x, t)$ is the wave equation with respective to time and space, a is speed, $f(x, t)$ is bias . This equation is typically described as having only one space dimension x, because the only other independent variable is the time t. Nevertheless, the dependent variable u may represent a second space dimension, if, for example, the displacement u takes place in y direction, as in the case of a string that is located in the xy plane. Since the wave equation is a second-order differential equation with bias, it usually does not have a general or fixed formula to calculate the actual expression of the wave equation. In this report, we are going to fit the wave equation by numerical analysis, and specifically we will accomplish these two main tasks.:

- How to estimate the partial derivatives numerically and to ensure that the error range is small.
- How to construct a set of sparse system of linear equations and solve them numerically to find the solutions of the linear equations.

By solving the above tasks, we can demonstrate that by using numerical analysis, we can fit the wave equation well within a reasonable range of error, thus effectively helping us to explore the properties of the wave equation.

2.Methods

- *Numerical Differentiation*

In solving the wave equation, the first problem we need to solve is how to estimate the second-order partial derivative. First, We use Taylor's formula to construct two sets of terms with second-order partial derivatives,

$$\begin{cases} \mu(x, t+h) = \mu(x, t) + \frac{\partial \mu}{\partial t}\cdot h + \frac{\partial^2 \mu}{\partial t^2}\cdot h^2 + o(h^2) \\ \mu(x, t-h) = \mu(x, t) - \frac{\partial \mu}{\partial t}\cdot h + \frac{\partial^2 \mu}{\partial t^2}\cdot h^2 + o(h^2) \end{cases}$$

By adding them together, we get:

$$\mu(x,\ t+h)+\mu(x,\ t-h)=2\mu(x,\ t)+2\frac{\partial^2\mu}{\partial t^2}\cdot\ h^2+o(h^2)$$

Thus we get the estimation formula for the second-order partial derivative::

$$\frac{\partial^2\mu}{\partial t^2}(x,t)=\frac{\mu(x,\ t+h)-2\mu(x,\ t)+\mu(x,\ t-h)}{2}+o(h^2)$$

As well as we get the estimation formula for $\frac{\partial^2\mu}{\partial x^2}$ is:

$$\frac{\partial^2\mu}{\partial x^2}(x,t)=\frac{\mu(x+j,\ t)-2\mu(x,\ t)+\mu(x-j,\ t)}{2}+o(j^2)$$

- *Gaussian elimination method for sparse systems*

First we have chosen a suitable step size h and j to get a sparse system of linear equation :

$$\left\{\begin{array}{l}\frac{-2*\mu(j,\ h)+\mu(j,\ 2h)}{h^2}=\frac{-2\mu(j,\ h)+\mu(2j,\ h)}{j^2}+f(j,h)\\ \dots\\ \frac{\mu(x_0\ t_0+h)-2\mu(x_0,\ t_0)+\mu(x_0,\ t_0-h)}{h^2}=\frac{\mu(x_0+j,\ t_0)-2\mu(x_0,\ t_0)+\mu(x_0-j,\ t_0)}{j^2}+f(x_0,\ t_0)\\ \dots\\ \frac{-2*\mu(1-j,\ 1-h)+\mu(1-j,\ 1-2h)}{h^2}=\frac{-2\mu(1-j,\ 1-h)+\mu(1-2j,\ 1-h)}{j^2}+f(1-j,1-h)\end{array}\right.$$

Then we change the number of equation in the system $(n-1)*(n-1)$ to matrix, we get:

$$AW=b$$

$$\left\{\begin{array}{l}W=[\mu(h,\ j),\mu(h,\ 2j),\dots,\mu(h,\ (n-1)j),\dots,\mu((n-1)h,\ j),\mu((n-1)h,\ 2j),\dots,\mu((n-1)h,\ (n-1)j)]\\ b==[f(h,\ j),f(h,\ 2j),\dots,f(h,\ (n-1)j),\dots,f((n-1)h,\ j),f((n-1)h,\ 2j),\dots,f((n-1)h,\ (n-1)j)]\end{array}\right.$$

It is easy to see that the coefficient matrix of the equation is extremely sparse, so if we directly use the Gaussian elimination method to estimate the solution of the equation, it will bring a lot of redundant calculations. Here we use a modified Gaussian elimination method

:

Gaussian Elimination for a sparse linear system

Give $A\in R^{(n-1)*(n-1)}$ and $b\in R^{(n-1)}$

for $i=1,2,\dots,(n-1)*(n-1)$ **do**

  for $j=1,2,\dots,(n-1)*(n-1)$ **do**

    if $a_{j,i}\neq 0$ do

      $m_j=\frac{a_{j,i}}{a_{i,i}}$

      $a_{j,i,\dots,(n-1)*(n-1)}=a_{j,i,\dots,(n-1)*(n-1)}-m_j\cdot\ a_{i,i,\dots,(n-1)*(n-1)}$

      $b_j=b_j-m_j\cdot\ b_i$

    end

  end

end

$$W_{(n-1)*(n-1)} = b_{(n-1)*(n-1)}$$

for $i = (n-1)*(n-1) - 1$ **do**

$$W_i = \frac{b_i - \sum_{j=i+1}^{(n-1)*(n-1)} a_{i,j} \cdot W_j}{a_{i,i}}$$

end

$W \in R^{(n-1)}$ is the solution.

We can use the above code to find a numerical approximation method for the wave equation. The final error will be a second-order error because the method we use is an approximation method for partial derivatives.

3. Result

Here we presuppose that the actual solution of the wave equation is:

$$\mu(x,t) = \sin(x(1-x)t(1-t))$$

Thus we get:

$$\begin{cases} \dfrac{\partial^2\mu}{\partial t^2} = -\dfrac{\partial^2\mu}{\partial x^2} + f(x,t), (x,t) \in D = [0,1]*[0,1] \\ \mu(x,0) = (-2t(1-t) - 2x(1-x))\cos(x(1-x)t(1-t)) + (-(1-2x)^2(t(1-t))^2 - (1-2t)^2(x(1-x))^2)\sin x(1-x)t(1-t) \\ \mu = 0 \text{ on } \partial D, \partial D = \{0,1\}*[0,1] \cup [0,1]*\{0,1\} \end{cases}$$

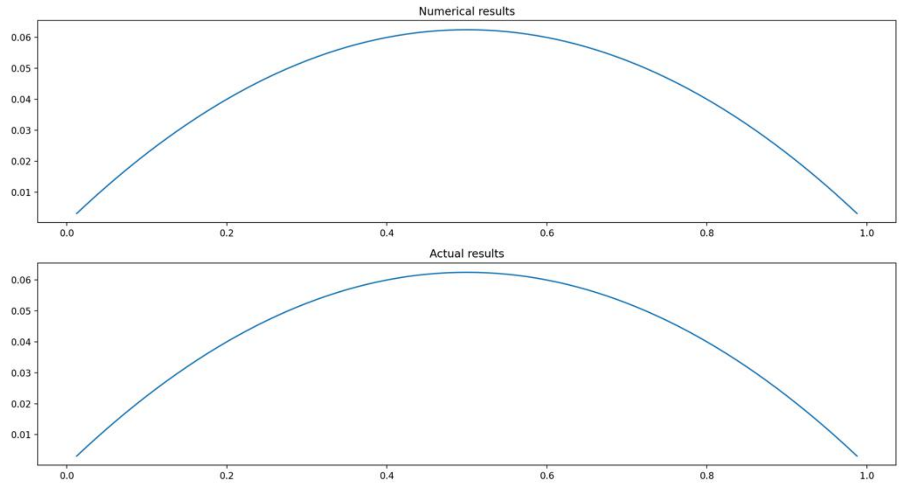
∂D is boundary values that to make sure our solution is unique.
In order to see the result, first thing we do is to verify that our method is a second order estimation method, the error is estimated as follows:

$$error = \quad \mu(x,t) - \mu(x,t)|_\infty$$

Then we use infinite norm to select the largest error term from all errors, and we get :

h	j	error
$\frac{1}{10}$	$\frac{1}{10}$	1.178e-06
$\frac{1}{20}$	$\frac{1}{20}$	2.280e-7
$\frac{1}{40}$	$\frac{1}{40}$	6.922e-8
$\frac{1}{80}$	$\frac{1}{80}$	1.325e-8

It shows that the error is a second order error, which is $error = o(h_2)$, and the estimate of this error is very small. Also we plot what the wave looks like at a given time t, where $t = 0.5, h = \frac{1}{80}$:



We can see from the graph above that our method has small error and perfectly fit the shape of wave function.。

4. Conclusion

In this project, we solve the problem of estimating the wave equation by numerical analysis, by introducing numerical partial derivatives to obtain a set of linear sparse equations, and later using an improved Gaussian elimination method to solve the sparse system of linear equations set.

Our results show that our method can estimate the value of the wave equation perfectly with a small error, which proves that numerical analysis also has good applications in physics.

For the further study, we can try to use better coefficient system structures to achieve higher-dimensional wave equations and apply them in various fields.