Motivation and Problem Statement

Project Name: Shooting Method for a Two-point Boundary Value Problem
Team Members with Karin Zhang 8715195, Zoe Zhou 8713661, Xingyu Fang 3998754

Motivation

- Numerical solution for the second differential equation
- Linear differential equation
- Nonlinear differential equation
- Dirichlet boundary and mixed Dirichlet boundary
- Finite Difference Method to solve the same differential equation

Problem Statement

- Two-point Boundary Value Problem

$$y''(x) = f(x, y(x), y'(x)), \quad y(x_1) = y_1, \quad y(x_2) = y_2$$

- Initial value problem

$$y''(x) = f(x, y(x), y'(x)), \quad y(x_1) = y_1, \quad y'(x_2) = a$$

- Difference at $x = x_2$

$$F(a) = y(x_2; a) - y_2 = 0$$

- Find the root of $F(a) = 0$
  - Bisection method
  - Newton's method

Methods and Validation

- Linear Shooting Method

  Transform second order differential equation two first order differential equations

  $F(a) = y(x_2;a) - y_2$ is linear function ,we only need one iterations

- Mixed boundary condition

  $F(a) = a_2 y(x_2;a) + b_2 y'(x_2;a) - c_2$

- Nonlinear differential equation

  (1) Compute the values $\phi_1 = \phi(z_1), \phi_2 = \phi(z_2)$ where ϕ is a non-linear function denoting the relation on how the value $\tilde{y}(1)$ depend z;

  (2) Then, the next value z_3 could be computed by a secant step:

$$z_3 = z_2 + (y(1) - \phi_2)\frac{z_2 - z_1}{\phi_2 - \phi_1}.$$

  (3) One can then iterate and get values $z_4, z_5, \cdots$ until converges.for example, until

$$|\phi(z_n) - y(1)| < \text{tol}.$$

- Finite Difference Method

$$y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}, \quad y''(x_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Results

- Linear Shooting Method $y'' = y' + 2y + \cos(x),\ for\ \ 0 \le x \le \frac{\pi}{2}$

  $$y(x) = \lambda y_1(x) + (1-\lambda) y_2(x)$$

  Only need one iterations to get $y'(0) = 0.1$

- Mixed boundary condition $-u'' + 3u' = x(1-x), \quad u(0) = 0, \quad u(1) + u'(1) = 1.$

  Only need one iterations to get $y'(0) = 0.3009$

- Nonlinear differential equation

  $$y'' = (y')^2 - y + \ln(x) \quad 1 \le x \le 2 \quad , y(1) = 0, \quad y(2) = \ln 2.$$

  5steps $z_1 = 0.7, z_2 = 1.2, tol = 10^{-9}$

  1steps $z_1 = 0.5, z_2 = 1, tol = 10^{-9}$

- Finite Difference Method $y'' = y' + 2y + \cos(x),\ for\ \ 0 \le x \le \frac{\pi}{2}$

  $$-(1+\frac{h}{2})y_{i-1} + (2+2h^2)y_i - (1-\frac{h}{2})y_{i+1} = -h^2\cos(x_i)$$

  $$\begin{bmatrix} d_1 & c_1 & & & \\ a_2 & d_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-2} & d_{n-2} & c_{n-2} \\ & & & a_{n-1} & d_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 - a_1\alpha \\ b_2 \\ \vdots \\ b_{n-1} \\ b_{n-1} - c_{n-1}\beta \end{bmatrix}$$