Topological Mating of CRTs & Geometry of Brownian Excursion

GSAS Talk

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Outline

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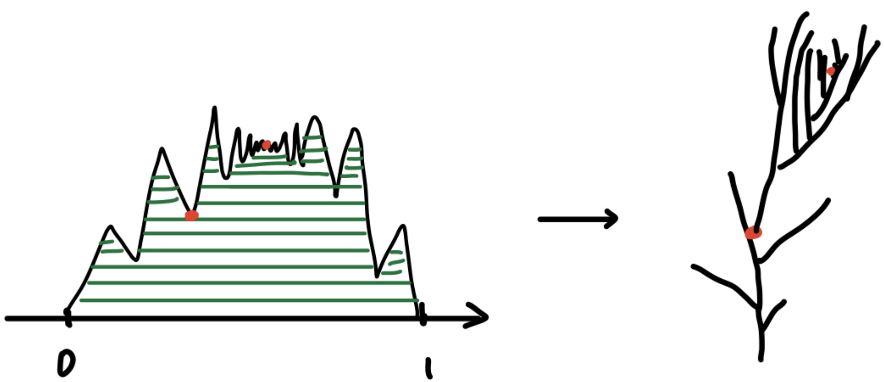
Continuum Random Tree

Given an **excursion** $e : [0,1] \to [0,+\infty)$, one can define an equivalence relation $\sim_e$ on $[0,1]$ (and thus on $\mathbb{S}^1$) such that

$$x \sim_e y \quad \Leftrightarrow \quad d_e(x,y) = e(x) + e(y) - 2 \min_{t \in [x,y]} e(t) = 0.$$

This induces a **topological tree** with metric $(T_e := [0,1]/_{\sim_e}, d_e)$.



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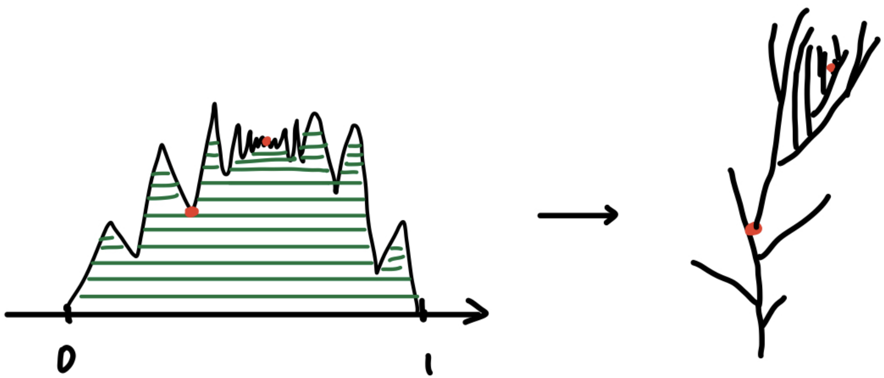

Brownian Excursion & Continuum Random Tree

The **Continuum Random Tree** $\mathcal{T}$ is the tree induced by a **Brownian excursion** (defined later).

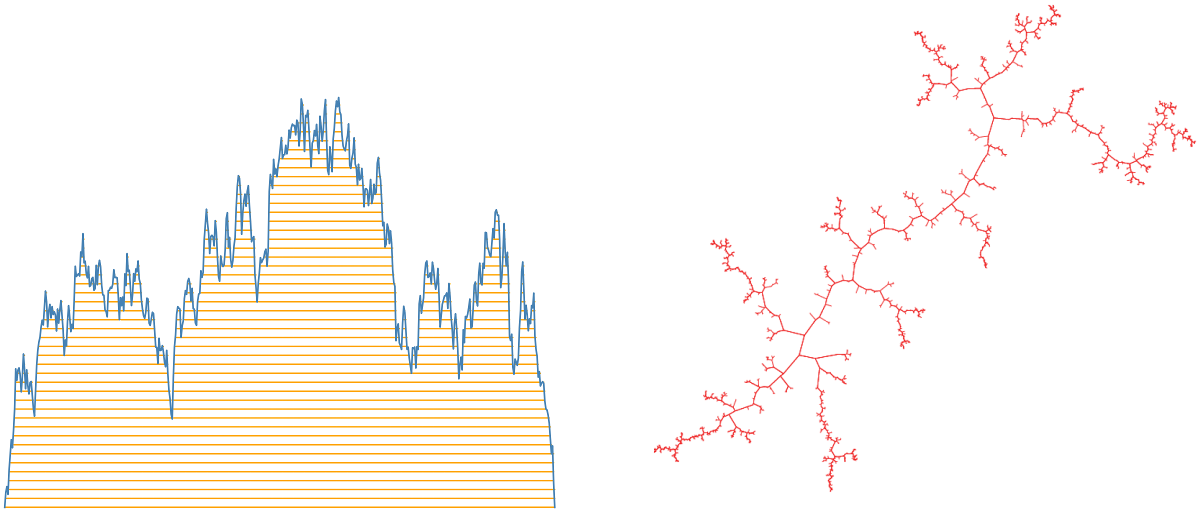

Figure 1: Brownian Excursion & Continuum Random Tree

Mating of Two CRTs

Given two independent Brownian excursion E_t, E'_t, the **mating of two CRTs** can be defined using equivalence relation as follows:

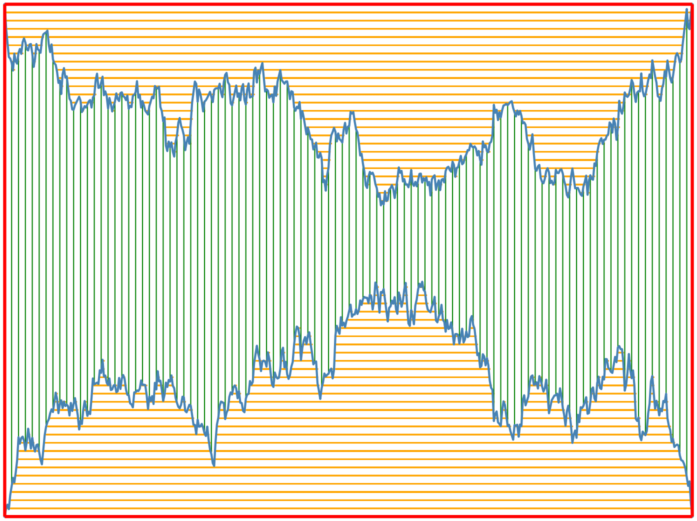

Figure 2: Mating of two CRTs

Statement

Theorem (Duplantier, Miller & Sheffeld 2014)

1. *With probability one, the mating of two independent CRTs will produce a (topological)* **sphere** *with a* **conformal structure**;
2. *The conformal structure induces the* **Liouville Quantum Gravity (LQG) measure** *on* $\mathbb{S}^2$.

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Moore's Theorem

To check that the equivalence relation actually gives a sphere. We will use the famous **Moore's theorem**.

Theorem (Moore, 1925)

Let $\sim$ be a closed equivalence relation on $\mathbb{S}^2$ such that

1. *Each equivalence class is* **connected***;*
2. *Each equivalence class has* **connected complement***;*
3. *There are more than one equivalence classes.*

Then the quotient space $\mathbb{S}^2/\sim$ is **homeomorphic** *to $\mathcal{S}^2$.*

What do equivalent class look like?

Proposition (Equivalence Classes for Mating of CRTs)

With probability one, all the equivalence classes are of one of the following forms:

1. *The boundary of the rectangle;*
2. *A* **single vertical segment***;*
3. *A* **single horizontal segment** *with* **two vertical segments** *connecting to its end points;*
4. *A* **single horizontal segment** *and* **two vertical segment** *connecting to its end points, along with* **one additional vertical segment** *in between.*

In particular, except for the boundary, no equivalent classes contain more than one horizontal segment.

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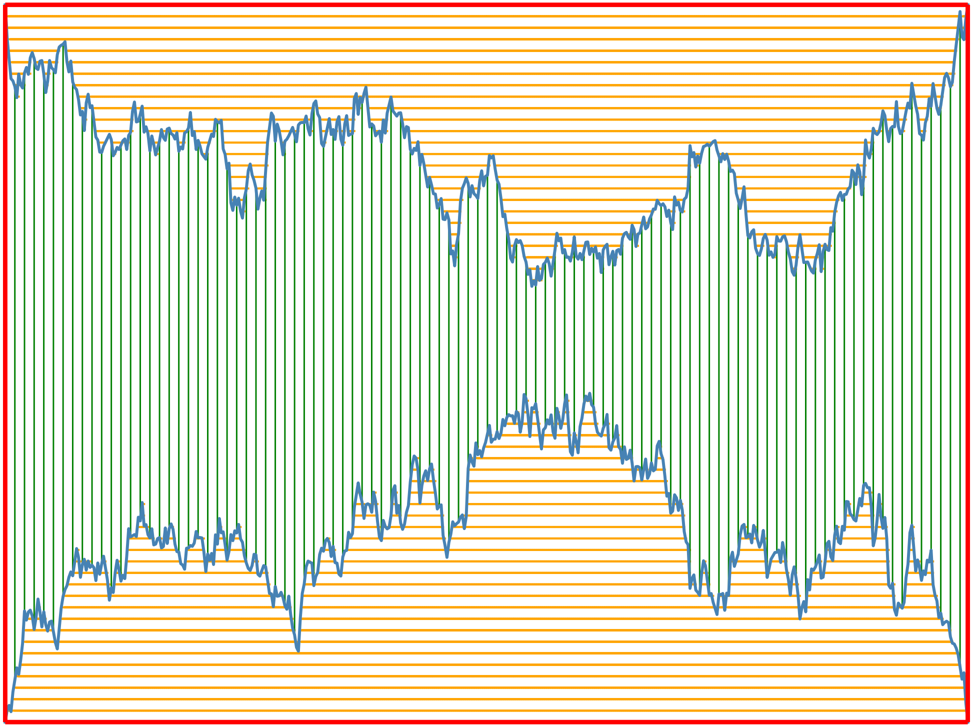

Local Half Extrema

Given a curve $\gamma \in C([0,1])$, we use L_γ to denote the set of **local left maxima** of γ. That is,

$$L_\gamma := \{t \in (0,1] : \gamma(t) > \max_{s \in (t-\varepsilon, t)} \gamma(s) \quad \text{for some } \varepsilon > 0\}.$$

We also define the **local right maxima** set R_γ similarly.

For our two independent Brownian excursions E_t, E'_t, the proposition is equivalent to that

$$(L_E \cup R_E \cup L_{-E} \cup R_{-E}) \cap (L_{E'} \cup R_{E'} \cup L_{-E'} \cup R_{-E'}) = \emptyset.$$

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Definition

Given i.i.d **coin tosses** $X_n \in \{-1, 1\}$, we define **random walk** $S_t := \sum_{n=0}^{\lfloor t \rfloor} X_n$.

Brownian motion is the almost sure **compact convergence** limit

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Geometry of Brownian Motion

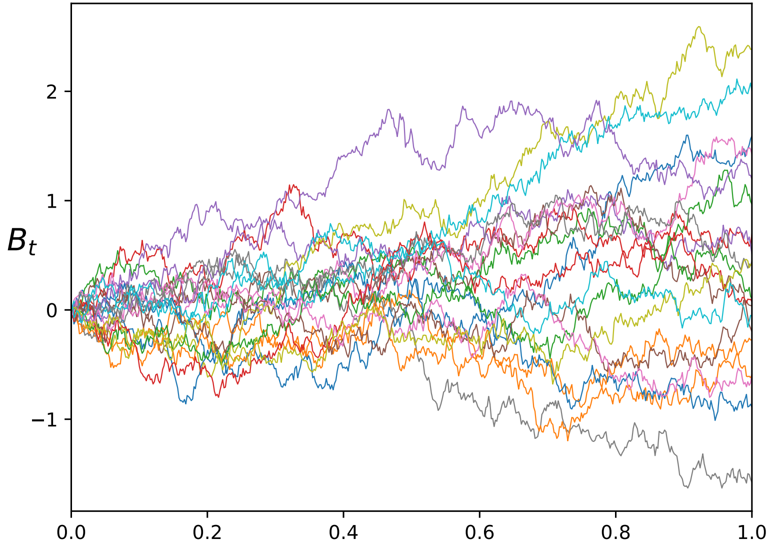

Figure 3: Simulation of Brownian Motion

B_t is almost surely **Hölder continuous** and **nowhere differentiable**. The graph of B_t has dimension $3/2$.

Bizzare Geometry of Brownian Motion

Perfect Zero Set

The zero set $\{t \in [0,1] : B_t = 0\}$ is homeomorphic to **Cantor set**.

No Point of Increase

Brownian motion has no **point of increase**. That is, there does not exist $t \in \mathbb{R}$ such that $\max_{s\in(t-\varepsilon,t)} B_s \leq B_t \leq \min_{s\in(t,t+\varepsilon)} B_s$ for some $\varepsilon > 0$.

Dimension Doubling

Let $B^d(t)$ be d-dimension Brownian motion. Then for any Borel set A, we have

$$\dim B^d(A) = 2\dim A.$$

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Definitions of Brownian Excursion

Brownian excursion $(E_t)_{0\leq t\leq 1}$ can be understood as Brownian motion $(B_t)_{0\leq t\leq 1}$ conditioned on

$$B_0 = B_1 = 0, B_t > 0 \quad \text{for} \quad 0 < t < 1.$$

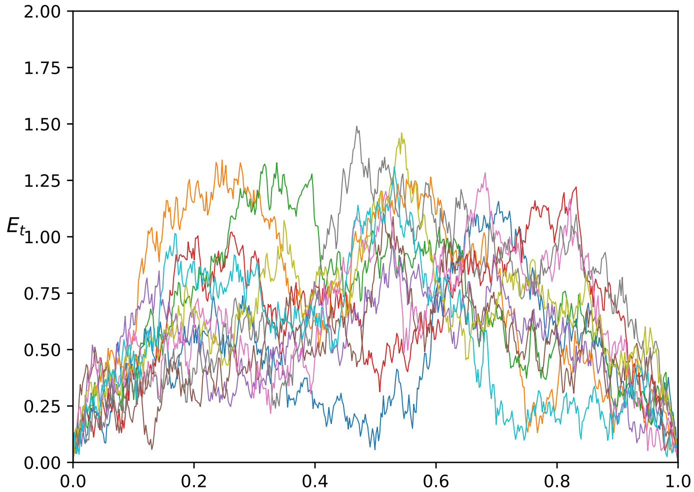

Figure 4: Simulation of Brownian Excursion

Equivalent Definitions of Brownian Excursion

This definition is hard to work with as the event has **zero probability**. Alternatively, we can define $(E_t)_{0\leq t\leq 1}$ as

1. Rescaled $|B_t|$ restricted to a **maximal positive interval**; or
2. $||Br_t^{(3)}||_2$ for 3d **Brownian bridge** $Br_t^{(3)}$; or
3. $(1-t)X_{t/(1-t)}$ for **Bessel process** X_t of order 3; or
4. Solution to the stochastic differential equation

$$dE_t = dB_t + \left(\frac{1}{E_t} - \frac{E_t}{1-t}\right) dt.$$

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Outline of the Proof

1. Prove that **no local half extrema** of two independent Brownian motion match;
2. Show that Brownian excursion **locally "looks like"** Brownian motion.

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Lévy's Reflection Theorem

Recall that L_γ is the set of local left maxima of the curve γ. We would like to show that almost surely

$$L_B \cap L_{B'} = \emptyset.$$

The key ingredient will be the following famous result by Lévy:

Theorem (Drawdown Process)

Let $M_t := \sup_{s\in[0,t]} B_s$ be the **maximum process** *of Brownian motion B_t. Then we have*

$$(M_t - B_t)_{t\in[0,\infty)} \overset{d}{=} (|B_t|)_{t\in[0,\infty)}.$$

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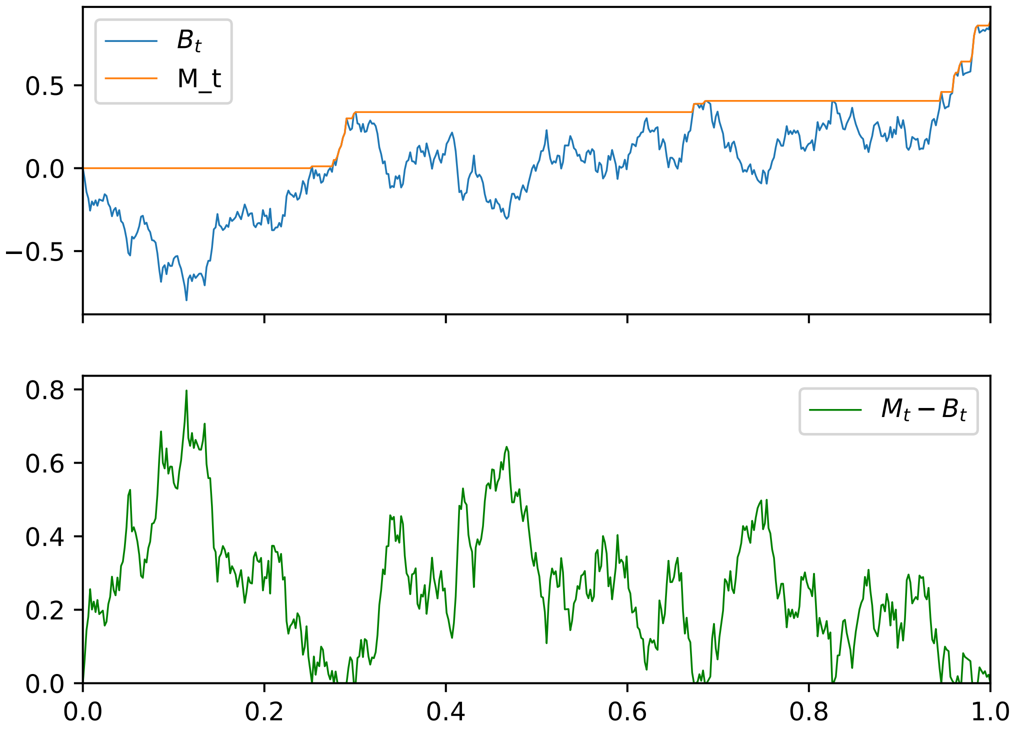

Figure 5: Brownian Dropdown

No Local Half Extrema Match - BM Case

Set $M_t^r := \sup_{s\in[r,t]} B_s$. We then have

$$L_B = \bigcup_{q\in[0,1)\cap\mathbb{Q}} \{t \in (q,1] : M_t^q - B_t = 0\}.$$

Thus it is suffice to check that for any $q, q' \in [0,1)$, the event

$$\{t \in (q \vee q', 1] : M_t^q - B_t = {M'}_t^{q'} - B'_t = 0\} = \emptyset$$

has probability zero, where ${M'}_t^{q'}$ is similarly defined.

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No Local Half Extrema Match - BM Case

Since we have

$$(M^q - B, {M'}^{q'} - B') \stackrel{d}{=} (|B - B_q|, |B' - B'_{q'}|),$$

this follows from the fact that (B_t, B'_t) visit $(B_q, B'_{q'})$ after time $q \vee q'$ with **probability zero** by the **Markov property**.

Absolute Continuity

How to go from $L_B \cap L_{B'}$ to $L_E \cap L_{E'}$?

Definition

Given two random objects X, Y on space $(S, \mathcal{S})$, we say X is **absolutely continuous** with respect to Y if for any event A we have

$$P(Y \in A) = 0 \quad \Rightarrow \quad P(X \in A) = 0.$$

1. The things that won't happen for Y will **not take place** for X either;
2. We can't tell X apart from Y with certainty.

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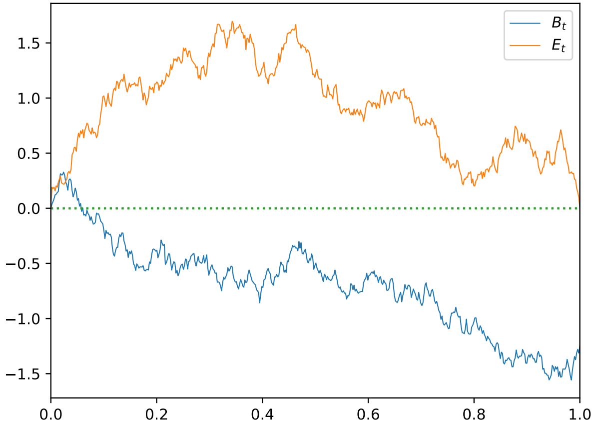
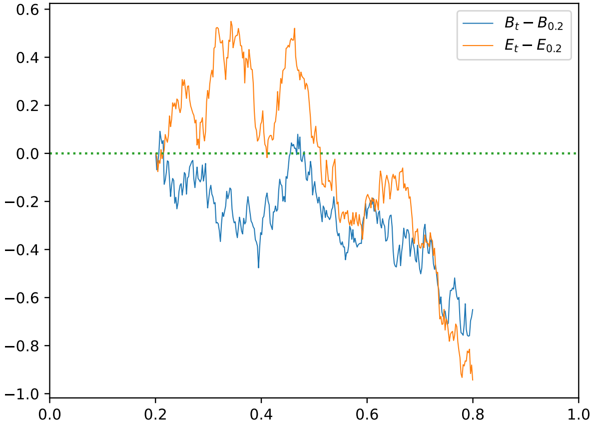
Local Equivalence

We want to show that E_t is absolutely continuously to B_t **locally**.





Girsanov Transform

Theorem (Girsanov Transform for Langevin Equations)

Given measurable function $\mu(t,x)$, if Langevin SDE

$$dX_t = \mu(t, X_t)dt + dB_t$$

has a **unique strong solution**, *and*

$$\exp\left(\int_0^t \mu(s, B_s)dB_s - \frac{1}{2}\int_0^t \mu(s, B_s)^2 ds\right)$$

is **uniformly integrable**. *Then X_t and B_t are* **mutually absolute continuous**.

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Pathwise Uniqueness for SDE

For Brownian excursion E_t, we have SDE

$$dE_t = \mu(t, E_t)dt + dB_t,$$

with $\mu(t,x) = \frac{1}{x} - \frac{x}{1-t}$. We can modify μ with

$$\mu^\varepsilon(t,x) = \frac{1}{x \vee \varepsilon} - \frac{x}{1-t}.$$

μ^ε is **Lipschitz** away from $t = 1$. Note that $\mu = \mu^\varepsilon$ when $x > \varepsilon$.

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Pathwise Uniqueness for SDE

If we restrict our process to time interval $[\delta, T] \Subset (0,1)$, then given initial condition $E^{\varepsilon}_{\delta} = x > 0$, SDE

$$dE^{\varepsilon}_t = \mu^{\varepsilon}(t, E^{\varepsilon}_t)dt + dB_t,$$

has **a unique strong solution** and

$$(E^{\varepsilon}_t - x)_{[\delta,T]} \equiv (B_t - B_{\delta})_{[\delta,T]}.$$

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$$(E_t^\varepsilon - x)_{[\delta,T]} \equiv (B_t - B_\delta)_{[\delta,T]}.$$

Finishing up the Proof

The last ingredient is to show that E_t is a **strong solution** to its SDE.

This follows from the fact that $E_t = (1-t)\sqrt{Y_{t/(1-t)}}$, where Y_t is **squared Bessel-3 process** satisfying **strongly** the SDE

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Finishing up the Proof

Now consider the event

$$A^{\varepsilon}_{\delta,T} := \{X_t(\omega) : X_t(\omega) \geq \varepsilon \quad \text{for all} \quad t \in [\delta, T]\}.$$

Conditioned on $X_\delta(\omega) = x > 2\varepsilon$, this is an event with **positive probability** for B_t and thus for E^{ε}_t.

Note that on $A^{\varepsilon}_{\delta,T}$, E_t and E^{ε}_t satisfy the same SDE and thus are **indistinguishable**. It follows that conditioned on $A^{\varepsilon}_{\delta,T}$,

$$P^B_x \equiv P^E_x.$$

for all $x > 2\varepsilon$. Here, the processes are defined on $[\delta, T]$.

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Now consider the event

$$A^{\varepsilon}_{\delta,T} := \{X_t(\omega) : X_t(\omega) \geq \varepsilon \quad \text{for all} \quad t \in [\delta, T]\}.$$

Conditioned on $X_\delta(\omega) = x > 2\varepsilon$, this is an event with **positive probability** for B_t and thus for E^{ε}_t.

Note that on $A^{\varepsilon}_{\delta,T}$, E_t and E^{ε}_t satisfy the same SDE and thus are **indistinguishable**. It follows that conditioned on $A^{\varepsilon}_{\delta,T}$,

$$P^B_x \equiv P^E_x.$$

for all $x > 2\varepsilon$. Here, the processes are defined on $[\delta, T]$.

Finishing up the Proof

Since E_t always **stays positive**, we have

$$P_x^E(\cup_{\varepsilon>0} A^{\varepsilon}_{\delta,T}) = 1.$$

It follows that

Theorem (Local Absolute Continuity)

For any $\delta < t_1 < t_2 < T$, we set $\mathcal{F}_{t_1}^{t_2}$ to be the **σ-algebra** *generated by $(\omega \mapsto \omega(t))_{t\in[t_1,t_2]}$. Then restricted to $\mathcal{F}_{t_1}^{t_2}$, we have*

$$P_x^E \ll P_x^B.$$

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