Some Math Equations

1. Composite numbers

Highly Composite Numbers		
Digits	**Number**	**Divisors**
1	6	4
2	60	12
3	840	32
4	7560	64
5	83160	128
6	720720	240
7	8648640	448
8	73513440	768
9	735134400	1344
10	6983776800	2304
11	97772875200	4032
12	963761198400	6720
13	9316358251200	10752
14	97821761637600	17280
15	866421317361600	26880
16	8086598962041600	41472
17	74801040398884800	64512
18	897612484786617600	103680

Binomial Identities

1. **Symmetry Rule**

$$\binom{n}{k} = \binom{n}{n-k}$$

2. **Factoring In**

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

3. **Sum Over k**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

4. **Sum Over n**

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

5. **Sum Over n and k**

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

6. **Sum of the Squares**

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

7. **Weighted Sum**

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}$$

8. **Connection with Fibonacci Numbers**

$$\binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-k}{k} + \cdots + \binom{0}{n} = F_{n+1}$$

Combinatorics

Binomial Coefficient Identities

1. **Fibonacci Binomial Identity**

$$\sum_{0 \le k \le n} \binom{n-k}{k} = \text{Fib}_{n+1}$$

2. **Symmetry Rule**

$$\binom{n}{k} = \binom{n}{n-k}$$

3. **Pascal's Recurrence**

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

4. **Absorption Identity**

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

5. **Factoring In**

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

6. **Sum of Binomial Coefficients**

$$\sum_{i=0}^{n}\binom{n}{i} = 2^n$$

7. **Sum of Even Binomial Coefficients**

$$\sum_{i\geq 0}\binom{n}{2i} = 2^{n-1}$$

8. **Sum of Odd Binomial Coefficients**

$$\sum_{i\geq 0}\binom{n}{2i+1} = 2^{n-1}$$

9. **Alternating Sum Identity**

$$\sum_{i=0}^{k}(-1)^i\binom{n}{i} = (-1)^k\binom{n-1}{k}$$

10. **Hockey-Stick Identity**

$$\sum_{i=0}^{k}\binom{n+i}{i} = \sum_{i=0}^{k}\binom{n+i}{n} = \binom{n+k+1}{k}$$

11. **Weighted Linear Sum**

$$\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$$

12. **Weighted Quadratic Sum**

$$\sum_{i=1}^{n} i^2\binom{n}{i} = (n+n^2)2^{n-2}$$

13. **Vandermonde Convolution**

$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

14. **Upward Hockey-Stick Identity**

$$\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}$$

15. **Sum of Squared Binomial Coefficients**

$$\sum_{i=0}^{k}\binom{k}{i}^2 = \binom{2k}{k}$$

16. **Chu-Vandermonde Special Case**

$$\sum_{i=1}^{n}\binom{n}{i}\binom{n-1}{i-1} = \binom{2n-1}{n-1}$$

17. **Double Subset Selection**

$$\sum_{k=q}^{n}\binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

18. **Generalized Binomial Theorem**

$$\sum_{i=0}^{n} k^i\binom{n}{i} = (k+1)^n$$

19. **Half Binomial Sum**

$$\sum_{i=0}^{n}\binom{2n}{i} = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}$$

20. **Squared Central Binomial Sum**

$$\sum_{i=0}^{n}\binom{2n}{i}^2 = \frac{1}{2}\left(\binom{4n}{2n} + \binom{2n}{n}^2\right)$$

Common Summation Formulas

1. **Sum of First n Integers**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

2. **Sum of Squares**

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

3. **Sum of Cubes**

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. **Sum of Fourth Powers**

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

5. **Sum of Fifth Powers**

$$\sum_{k=1}^{n} k^5 = \frac{n(n+1)(2n+1)(3n^2+3n-1)(4n^3+6n^2-1)}{30}$$

6. **Sum of Odd Numbers**

- Sum of odd numbers $\leq n$:

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} k = \left\lfloor \frac{n+1}{2} \right\rfloor^2$$

7. **Sum of Even Numbers**

- Sum of even numbers $\leq n$:

$$\sum_{\substack{k=1 \\ k \text{ even}}}^{n} k = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$$

8. **Finite Geometric Series**

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad (a \neq 1)$$

9. **Weighted Geometric Series (Linear)**

$$\sum_{k=0}^{n} k a^k = \frac{a[1-(n+1)a^n + n a^{n+1}]}{(1-a)^2}$$

10. **Weighted Geometric Series (Quadratic)**

$$\sum_{k=0}^{n} k^2 a^k$$

$$= \frac{a[(1+a)-(n+1)^2 a^n + (2n^2+2n-1)a^{n+1} - n^2 a^{n+2}]}{(1-a)^3}$$

11. **Binomial Theorem**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

12. **Alternating Binomial Sum**

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

13. **Weighted Binomial Sum**

$$\sum_{k=0}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$$

14. **Harmonic Series**

$$\sum_{k=1}^{n} \frac{1}{k} = H_n \approx \ln n + \gamma$$

15. **Sum of Reciprocal Squares**

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}$$

16. **Fibonacci Series Sum**

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

17. **Telescoping Series**

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}$$

18. **Sine Taylor Series**

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin x$$

19. **Cosine Taylor Series**

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \cos x$$

20. **Exponential Taylor Series**

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

21. **Geometric Series (Base 2)**

$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2$$

22. **Finite Geometric Series (General)**

$$\sum_{k=1}^{n} x^k = \frac{x(x^n - 1)}{x - 1}, \quad x \neq 1$$

$$\sum_{k=1}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1$$

23. **Weighted Geometric Series (Base 2)**

$$\sum_{k=1}^{n} 2^k \cdot k = 2^{n+1} \cdot n - 2$$

24. **Exponential Series with Constant Exponent**

$$\sum_{k=1}^{n} 2^{km} = \frac{2^m(2^{mn} - 1)}{2^m - 1}, \quad m \text{ constant}$$

Euler Totient Function Properties

1. **Multiplicative Property**
   - If $\gcd(m, n) = 1$, then $\phi(m \cdot n) = \phi(m) \cdot \phi(n)$

2. **Closed Form Formula**

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

3. **Prime Power Case**
   - For prime p and $k \geq 1$:

$$\phi(p^k) = p^{k-1}(p-1) = p^k \left(1 - \frac{1}{p}\right)$$

4. **Jordan Totient Generalization**

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right)$$

   - $J_1(n) = \phi(n)$ counts $(k+1)$-tuples coprime with n

5. **Sum of Jordan Totients**

$$\sum_{d|n} J_k(d) = n^k$$

6. **Divisor Sum**

$$\sum_{d|n} \phi(d) = n$$

7. **Möbius Inversion Formulas**

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}$$

$$\phi(n) = \sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right)$$

8. **Divisibility Property**

$$a \mid b \Rightarrow \phi(a) \mid \phi(b)$$

9. **Exponent Divisibility**

$$n \mid \phi(a^n - 1) \quad \text{for } a, n > 1$$

10. **General Product Formula**

$$\phi(mn) = \phi(m)\phi(n) \frac{d}{\phi(d)} \quad \text{where } d = \gcd(m, n)$$

11. **Special Cases**
   - For even numbers:

$$\phi(2m) = \begin{cases} 2\phi(m) & \text{if } m \text{ is even} \\ \phi(m) & \text{if } m \text{ is odd} \end{cases}$$

- Power formula:
$$\phi(n^m) = n^{m-1}\phi(n)$$

12. **LCM-GCD Relationship**

$$\phi(\mathrm{lcm}(m,n)) \cdot \phi(\mathrm{gcd}(m,n)) = \phi(m) \cdot \phi(n)$$

13. **Parity Property**

- $\phi(n)$ is even for $n \geq 3$
- If n has r distinct odd prime factors, then $2^r \mid \phi(n)$

14. **Reciprocal Sum**
$$\sum_{d|n} \frac{\mu^2(d)}{\phi(d)} = \frac{n}{\phi(n)}$$

15. **Sum of Coprimes**
$$\sum_{\substack{1\leq k\leq n \\ \mathrm{gcd}(k,n)=1}} k = \frac{1}{2}n\phi(n) \quad \text{for } n > 1$$

16. **Radical Property**
$$\frac{\phi(n)}{n} = \frac{\phi(\mathrm{rad}(n))}{\mathrm{rad}(n)}$$

- where $\mathrm{rad}(n) = \prod_{p|n} p$

17. **Bounds**

- Lower bound: $\phi(m) \geq \log_2 m$
- Iterated totient bound: $\phi(\phi(m)) \leq \frac{m}{2}$

18. **Exponent Reduction**

- For $x \geq \log_2 m$:

$$n^x \mod m = n^{\phi(m)+x \mod \phi(m)} \mod m$$

19. **GCD Sum**
$$\sum_{\substack{1\leq k\leq n \\ \mathrm{gcd}(k,n)=1}} \mathrm{gcd}(k-1,n) = \phi(n)d(n)$$

- Also holds for $\mathrm{gcd}(a \cdot k - 1, n)$ when $\mathrm{gcd}(a,n) = 1$

20. **Non-uniqueness**

- For every n there exists $m \neq n$ with $\phi(m) = \phi(n)$

21. **Weighted Sum**
$$\sum_{i=1}^{n} \phi(i) \left\lfloor \frac{n}{i} \right\rfloor = \frac{n(n+1)}{2}$$

22. **Odd-indexed Sum**
$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} \phi(i) \left\lfloor \frac{n}{i} \right\rfloor = \sum_{k\geq 1} \left[\frac{n}{2^k}\right]^2$$

- where $[\cdot]$ denotes rounding

23. **Double Sum**
$$\sum_{i=1}^{n}\sum_{j=1}^{n} ij[\mathrm{gcd}(i,j)=1] = \sum_{i=1}^{n} \phi(i)i^2$$

24. **Average Value**

- The average of coprimes of n less than n is $\dfrac{n}{2}$

Fibonacci

1. **Definition**

- $F_0 = 0,\ F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

2. **Combinatorial Formula**
$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$$

3. **Binet's Formula**
$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

4. **Sum of First n Fibonacci Numbers**
$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

5. **Sum of Odd-indexed Fibonacci Numbers**
$$\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$$

6. **Sum of Even-indexed Fibonacci Numbers**

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1$$

7. **Sum of Squares**

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

8. **Cassini's Identity**

$$F_m F_{n+1} - F_{m-1} F_n = (-1)^n F_{m-n}$$

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n(F_{n+1} + F_{n-1})$$

9. **Addition Formulas**

$$F_m F_n + F_{m-1} F_{n-1} = F_{m+n-1}$$

$$F_m F_{n+1} + F_{m-1} F_n = F_{m+n}$$

10. **Fibonacci Test**
    - A natural number n is Fibonacci iff $5n^2 + 4$ or $5n^2 - 4$ is a perfect square
11. **Divisibility Property**
    - Every k-th Fibonacci number is a multiple of F_k
12. **GCD Property**

$$\text{gcd}(F_m, F_n) = F_{\text{gcd}(m,n)}$$

13. **Coprimality**
    - Any three consecutive Fibonacci numbers are pairwise coprime
14. **Periodicity Modulo n**
    - Fibonacci sequence modulo n is periodic with Pisano period $\leq 6n$

GCD and LCM Properties

1. **Basic GCD Properties**

$$\text{gcd}(a, 0) = a$$

$$\text{gcd}(a, b) = \text{gcd}(b, a \mod b)$$

   - Every common divisor of a and b divides $\text{gcd}(a, b)$

2. **Linear Combination Property**

$$\text{gcd}(a + m \cdot b, b) = \text{gcd}(a, b) \quad \text{for any integer } m$$

3. **Multiplicative Property**
   - If $\text{gcd}(a_1, a_2) = 1$, then $\text{gcd}(a_1 a_2, b) = \text{gcd}(a_1, b) \cdot \text{gcd}(a_2, b)$
4. **GCD-LCM Product Identity**

$$\text{gcd}(a, b) \cdot \text{lcm}(a, b) = |a \cdot b|$$

5. **GCD-LCM Distributive Laws**

$$\text{gcd}(a, \text{lcm}(b, c)) = \text{lcm}(\text{gcd}(a, b), \text{gcd}(a, c))$$

$$\text{lcm}(a, \text{gcd}(b, c)) = \text{gcd}(\text{lcm}(a, b), \text{lcm}(a, c))$$

6. **Exponent GCD Property**

$$\text{gcd}(n^a - 1, n^b - 1) = n^{\text{gcd}(a,b)} - 1$$

7. **Totient Sum Representation**

$$\text{gcd}(a, b) = \sum_{\substack{k|a \\ k|b}} \phi(k)$$

8. **Counting GCD Values**

$$\sum_{i=1}^{n} [\text{gcd}(i, n) = k] = \phi\left(\frac{n}{k}\right)$$

9. **Sum of GCDs**

$$\sum_{k=1}^{n} \text{gcd}(k, n) = \sum_{d|n} d \cdot \phi\left(\frac{n}{d}\right)$$

10. **Exponential GCD Sum**

$$\sum_{k=1}^{n} x^{\text{gcd}(k,n)} = \sum_{d|n} x^d \cdot \phi\left(\frac{n}{d}\right)$$

11. **Reciprocal GCD Sum**

$$\sum_{k=1}^{n} \frac{1}{\text{gcd}(k, n)} = \frac{1}{n} \sum_{d|n} d \cdot \phi(d)$$

12. **Weighted Reciprocal GCD Sum**

$$\sum_{k=1}^{n} \frac{k}{\text{gcd}(k,n)} = \frac{1}{2}\sum_{d|n} \phi(d)$$

13. **Modified GCD Sum**

$$\sum_{k=1}^{n} \frac{n}{\text{gcd}(k,n)} = 2\sum_{k=1}^{n} \frac{k}{\text{gcd}(k,n)} - 1 \quad (n > 1)$$

14. **Coprime Pairs Count**

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[\text{gcd}(i,j)=1] = \sum_{d=1}^{n} \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^2$$

15. **Sum of GCD Pairs**

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\text{gcd}(i,j) = \sum_{d=1}^{n} \phi(d) \left\lfloor \frac{n}{d} \right\rfloor^2$$

16. **Coprime Pairs Product Sum**

$$\sum_{i=1}^{n}\sum_{j=1}^{n} i \cdot j[\text{gcd}(i,j)=1] = \sum_{i=1}^{n} \phi(i) i^2$$

17. **LCM Pairs Sum**

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\text{lcm}(i,j) = \sum_{l=1}^{n} \left(\frac{(1+\lfloor n/l \rfloor)\cdot \lfloor n/l \rfloor}{2} \right)^2 \sum_{d|l} \mu(d) l d$$

18. **Multiple GCD-LCM Relationship**

$$\text{gcd}(\text{lcm}(a,b), \text{lcm}(b,c), \text{lcm}(a,c))$$
$$= \text{lcm}(\text{gcd}(a,b), \text{gcd}(b,c), \text{gcd}(a,c))$$

19. **Array GCD Property**

$$\text{gcd}(A_L, A_{L+1}, \ldots, A_R) = \text{gcd}(A_L, A_{L+1} - A_L, \ldots, A_R - A_{R-1})$$

20. **LCM Sum Formula**

- $\text{SUM} = \sum_{k=1}^{n} \text{lcm}(k,n)$

$$\text{SUM} = \frac{n}{2}\left(\sum_{d|n} \phi(d) \cdot d + 1 \right)$$

Geometric Series

1. **Standard Geometric Series**

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r| < 1$$

2. **Geometric Series with First Term a**

$$\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}, \quad |r| < 1$$

3. **Alternating Geometric Series**

$$\sum_{k=0}^{\infty} (-1)^k r^k = \frac{1}{1+r}, \quad |r| < 1$$

4. **Series Starting at $k=1$**

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}, \quad |r| < 1$$

5. **Weighted Geometric Series (Linear)**

$$\sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2}, \quad |r| < 1$$

6. **Weighted Geometric Series (Quadratic)**

$$\sum_{k=1}^{\infty} k^2 r^k = \frac{r(1+r)}{(1-r)^3}, \quad |r| < 1$$

7. **Weighted Geometric Series (Cubic)**

$$\sum_{k=1}^{\infty} k^3 r^k = \frac{r(1+4r+r^2)}{(1-r)^4}, \quad |r| < 1$$