09-C

Arrangement

Arrangement: Complexity

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Arrangement Complexity

- For $A(H)$ an arrangement of n hyperplanes in E^d
  1) how many k-faces could $A(H)$ have?
  2) how many incidences could $A(H)$ have?

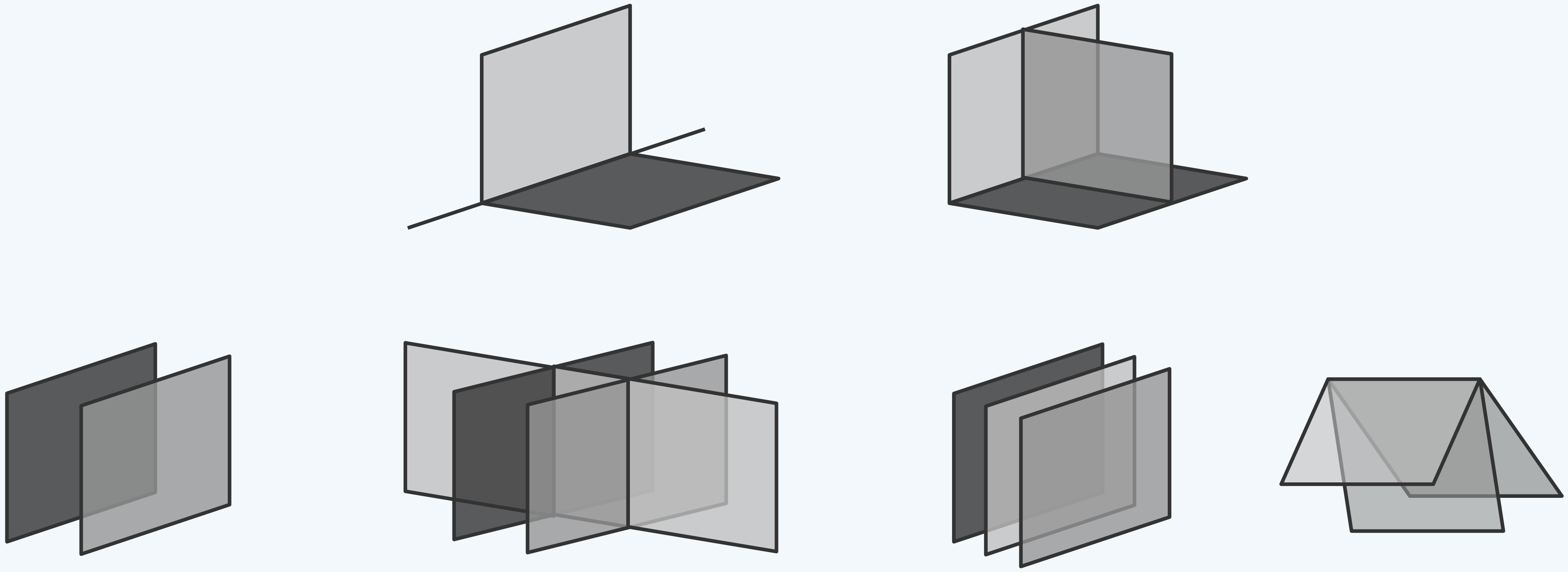
- For $0 \leq k \leq d$, let
  1) $f_k(H) = |\{ \text{k-faces of } A(H) \}|$
  2) $i_k(H) = |\{ \text{incidences between k-face and (k+1)-faces of } A(H) \}|$

- For n and d two positive integers, let
  1) $f_k^{(d)}(n) = \max \{ f_k(G) \mid A(G) \text{ is an arrangement in } \mathcal{E}^d, |G| = n \}$, and
  2) $i_k^{(d)}(n) = \max \{ i_k(G) \mid A(G) \text{ is an arrangement in } \mathcal{E}^d, |G| = n \}$

- Problems: $f_k^{(d)}(n) = ?$ $i_k^{(d)}(n) = ?$ //upper bounds

Simple Arrangements

- An arrangement A(H) of n hyperplanes in E^d is called simple if any $\boxed{d - k}$ of the hyperplanes intersect in a common k-flat

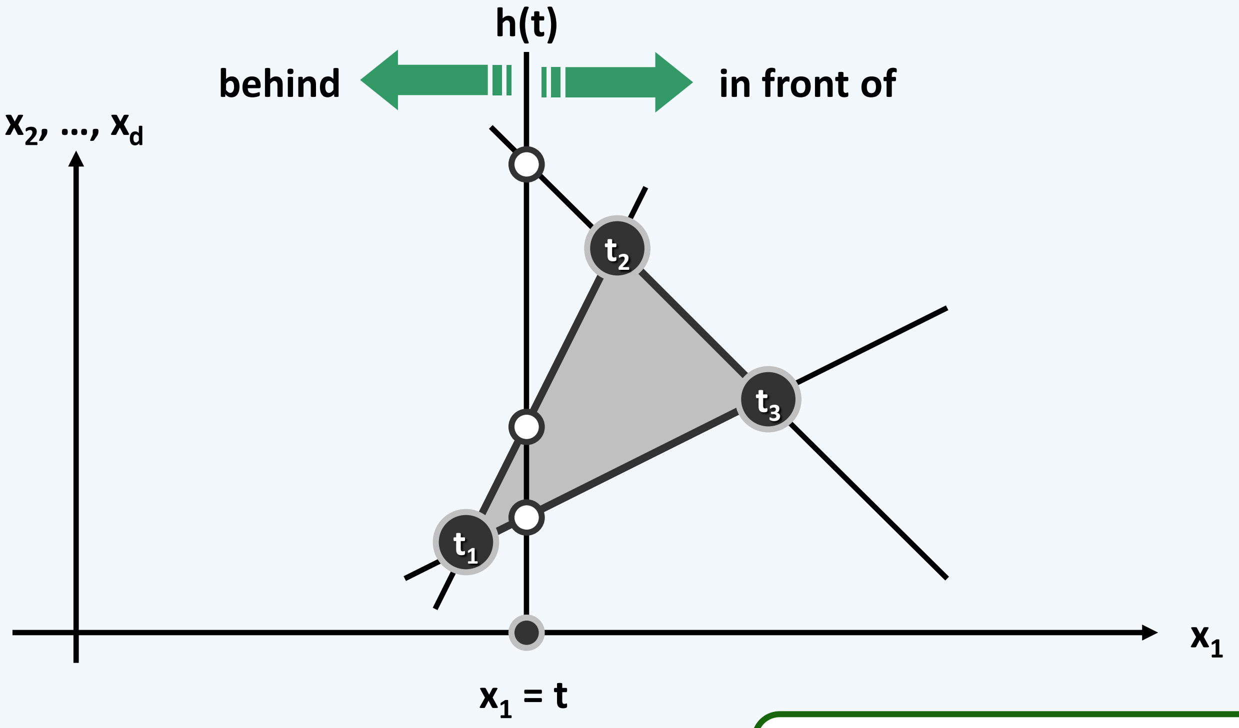


Complexity of Simple Arrangements

❖ If A(H) is a simple arrangement of n hyperplanes in $\mathcal{E}^d$, then

1) $f_k(H) = \sum_{i=0}^{k} \binom{d-i}{k-i} \binom{n}{d-i}$, for $0 \le k \le d$, and

2) $i_k(H) = 2*(d - k)*f_k(H)$, for $0 \le k \le d - 1$



Complexity of General Arrangements

❖ For any integers $d \geq 1$ and $n \geq 1$, we have

1) $f_k^{(d)}(n) = \sum_{i=0}^{k} \binom{d-i}{k-i} \binom{n}{d-i}$, for $0 \leq k \leq d$,

2) $i_k^{(d)}(n) = 2*(d-k)*f_k^{(d)}(n)$, for $0 \leq k \leq d-1$, and

3) $i_k(H) = i_k^{(d)}(n)$ and $f_k(H) = f_k^{(d)}(n)$ iff $A(H)$ is simple

❖ Buck's Counting (1943)

1) $f_k^{(d)}(n) = (n, d-k) \sum_{i=0}^{k} \binom{n-d+k}{i}$, for $0 \leq k \leq d$, and

2) $f_k(H) = f_k^{(d)}(n)$ iff $A(H)$ is simple