Convex Hull

Beyond 3 Dimension

- 4D

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$\mathcal{O}(n\log n)$: Upper Bound For Higher Dimensions?

- From the $\boxed{\text{theoretical}}$ point of view,

  two important questions in Combinatorial Geometry are:

  - What's the $\boxed{\text{largest number}}$ of faces/vertices/edges/.../facets of a d-polyt

  - Which polytopes achieve this largest number?

- From the $\boxed{\text{algorithmic}}$ point of view, we have seen that

  both $\boxed{\text{2D}}$ and $\boxed{\text{3D}}$ convex hulls can be constructed in $\boxed{\mathcal{O}(n\log n)}$ time

- Based on these facts, a natural conjecture would be

  $\boxed{\mathcal{O}(n\log n)}$ time is still sufficient

  for constructing $\boxed{\text{higher dimensional}}$ convex hulls

- We will, unfortunately, see from the followings that this is $\boxed{\text{not}}$ true

How Complex Could A d-Polytope Be?

- It can be deduced from Euler's formula that convex polytope in $\boxed{\mathcal{E}^3}$ has a linear number of edges and faces
- Specifically, $\boxed{3*(n-2)}$ edges and $\boxed{2*(n-2)}$ faces
  e.g., a tetrahedron has n = $\boxed{4}$ vertices, $\boxed{6}$ edges and $\boxed{4}$ faces
- However, we will show that the convex hull of n points in $\boxed{\mathcal{E}^4}$ may has as many as $\boxed{\Omega(n^2)}$ edges
- This fact immediately implies that we can't hope to construct $\boxed{\text{4D CH}}$ in less than $\boxed{\Omega(n^2)}$ time in the worst cases
- Generally, the CH of n points in $\boxed{\mathcal{E}^d}$ may have as many as $\boxed{\Omega(n^{\lfloor d/2 \rfloor})}$ facets
- So one can't expect an algorithm for computing convex hulls in $\boxed{\mathcal{E}^d}$ in less than $\boxed{\Omega(n^{\lfloor d/2 \rfloor})}$ time in the worst cases

Moment Curve

- parametric curve:

$$\gamma^{(d)} = \{ (t, t^2, t^3, \ldots, t^d) \mid t \in R \}$$

- To imagine the shape of the MC in $\mathcal{E}^d$,

  try to draw the unit parabola in the plane:

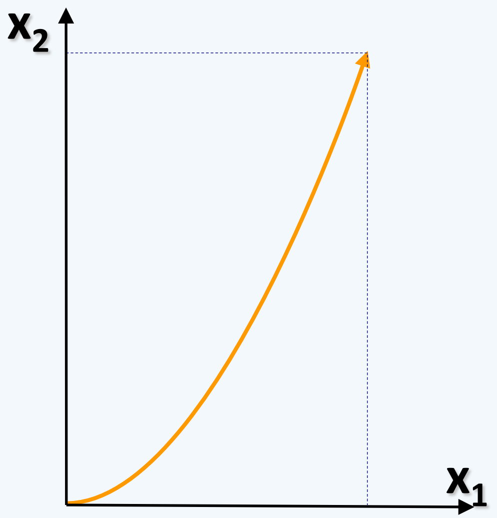
$$\gamma^{(2)} = \{ (t, t^2) \mid t \in R \}$$

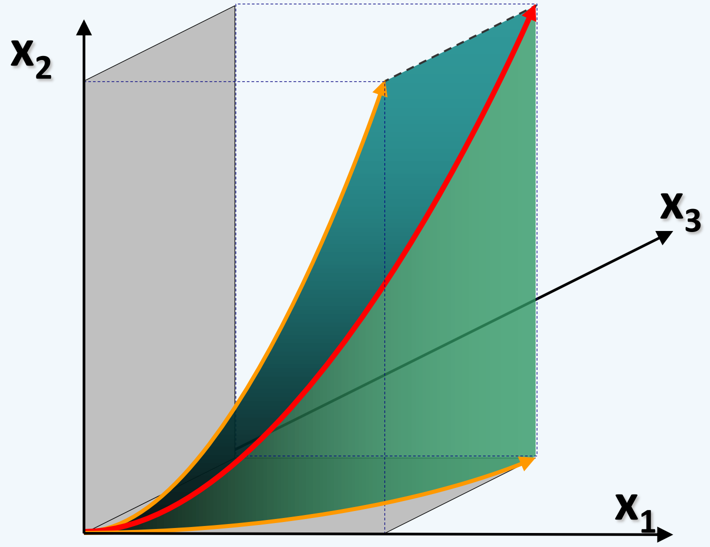
- Each point $p^{(d)}(t) = (t, t^2, t^3, \ldots, t^d)$

  is given by the single parameter $\boxed{t}$

- When the context is clear,

  $\gamma^{(d)}$ / $p^{(d)}(t)$ will be simplified as γ / $p(t)$





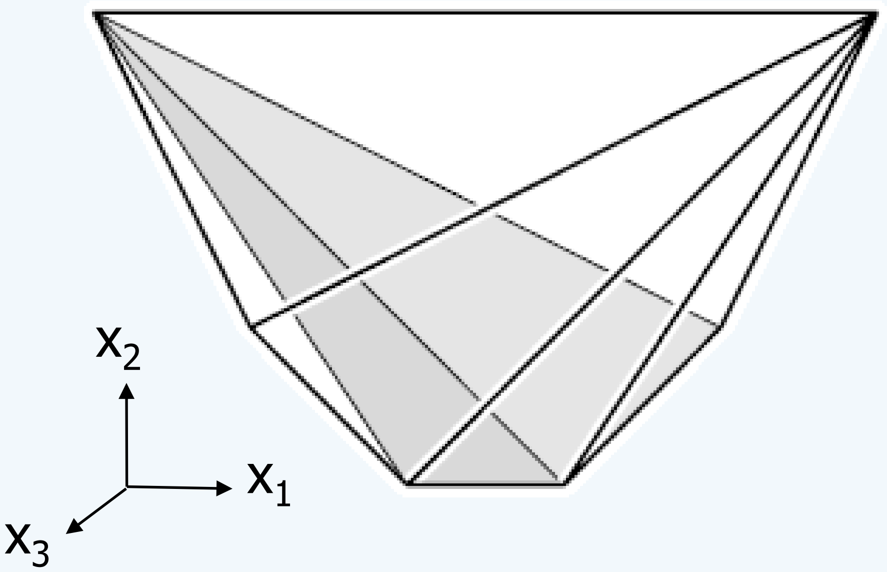
Cyclic Polytopes

- For any n distinct reals $u_1 < u_2 < u_3 < \ldots < u_n$

  there is a set of n distinct points on γ,

  $$S = \{ p_i = p(u_i) \mid 1 \le i \le n \}$$

- cyclic polytope in $\mathcal{E}^d$: $CP(\{ u_1, u_2, u_3, \ldots, u_n \}) = conv(S)$

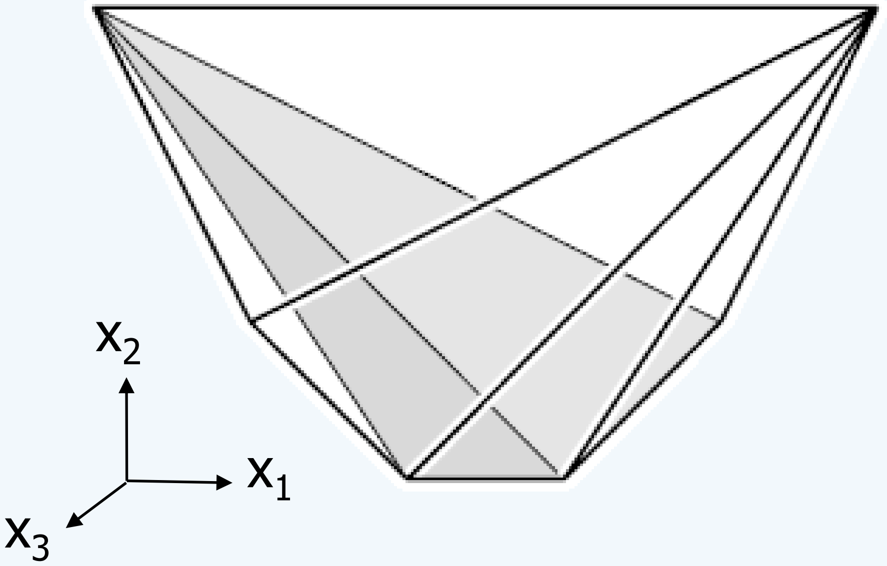


A cyclic polytope with 6 vertices in $\mathcal{E}^3$

Note that when projected to the x_1x_2-plane, all the 6 vertices lie on the parabola $x_2 = x_1^2$

Moment Curve & Cyclic Polytope in 4D

- Consider the 4D MC: $\gamma^{(4)}(t) = \{ (t, t^2, t^3, t^4) \mid t \in R \}$
- Let $S = \{ p_i = p(u_i) \mid 1 \le i \le n \}$ be a set of n distinct points on $\gamma^{(4)}$ and CP(S) be the corresponding cyclic polytope
- Surprisely, it can be proved that every segment between these points is an EE of CP(S)



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Tangent Hyperplane

- Now, for $\boxed{\text{any fixed}}$ i and j, consider the polynomial

$$P(t) \;=\; \boxed{(t - u_i)^2 \times (t - u_j)^2}$$

- This polynomial can be rewritten as

$$P(t) \;=\; \boxed{t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0}, \quad \text{where } a_{0\sim3} \in R$$

- It is easy to see that 1) $P(t) \geq 0$ for any t; and

  2) $P(t) = 0$ iff $t = u_i$ or u_j

- At the same time, $h : \boxed{x_4 + a_3 x_3 + a_2 x_2 + a_1 x_1 + a_0 = 0}$ ...... (1)

  is a hyperplane in $\mathcal{E}^4$

- We observe that

  1) at any point p on γ, equation (1) evaluates to a non-negative value; and

  2) the equation evaluates to zero iff $p = p_i$ or p_j

Edges on Convex Hull

- The above facts mean that $\gamma^{(4)}$

  1) lies on the $\boxed{\text{same}}$ side of hyperplane h, and

  2) $\boxed{\text{touches}}$ h at and only at p_i and p_j

- That means

  1) all points, except p_i and p_j, lie strictly on the $\boxed{\text{same}}$ side of h

  2) segment $p_i p_j$

     - is on the boundary of CP(S), and hence

     - defines an $\boxed{\text{edge}}$ of CH(S)

Edges on Convex Hull

- Since i and j were chosen $\boxed{\text{arbitrarily}}$, it can be concluded that

  1) every pair of points in S

     defines a (distinct) edge on CH(S) and asymptotically,

  2) the convex hull of n points in $\mathcal{E}^4$ has C(n, 2) = $\boxed{\Theta(n^2)}$ edges

- That's why we cannot hope to devise an algorithm which

  constructs a 4D convex hull

  in less than $\boxed{\Omega(n^2)}$ time in the worst cases

- This is quite different from the cases of 1D, 2D, and 3D convex hulls