Triangulation

Lower & Upper Bounds of Tetrahedralization

03-J

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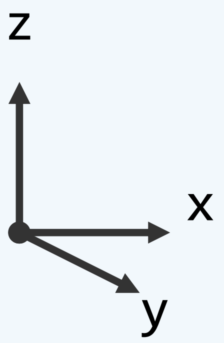
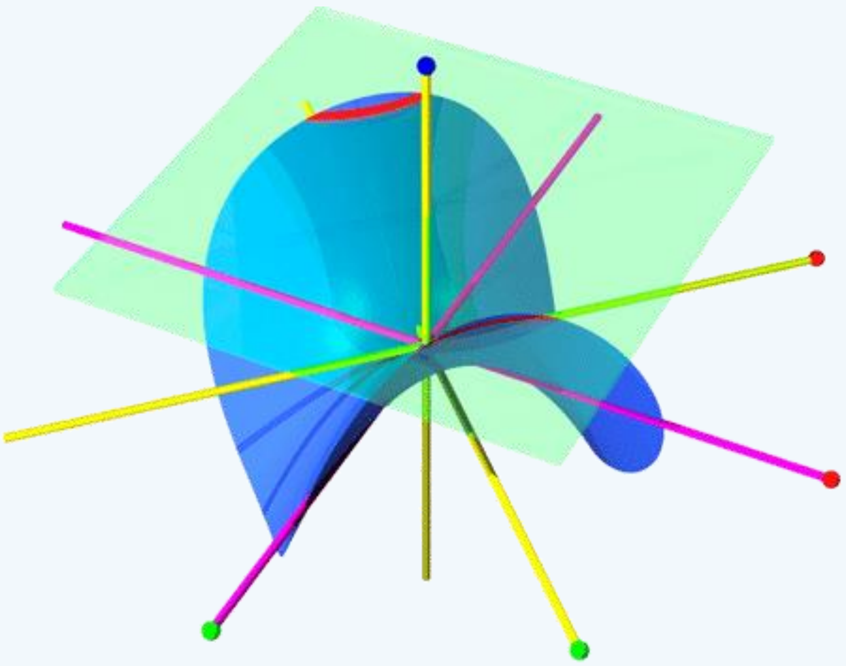
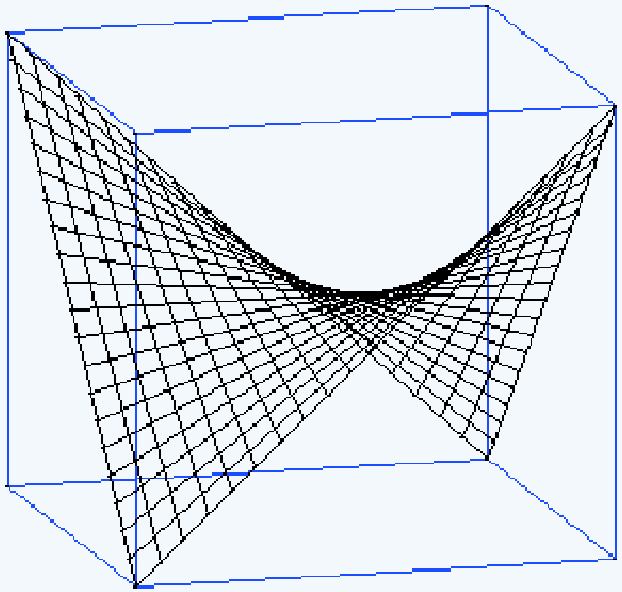
Chazelle's Lower Bound

- Using Steiner points, we can triangulate every polyhedron. But ...
- How many Steiner points are needed to triangulate a polyhedron? And how many tetrahedra will a polyhedron be partitioned into?
- If the goal is to cut the polyhedron into pieces as $\boxed{\text{few}}$ as possible, how well can you do? i.e., how many tetrahedra are necessary in the worst cases?
- [Chazelle, 1984] The tetrahedralization (using Steiner points) of a polyhedron with $\boxed{n}$ vertices consists of $\boxed{\Omega(n^2)}$ tetrahedra in the worst cases
- The lower bound is achieved by Chazelle's polyhedron ...

Hyperbolic paraboloid



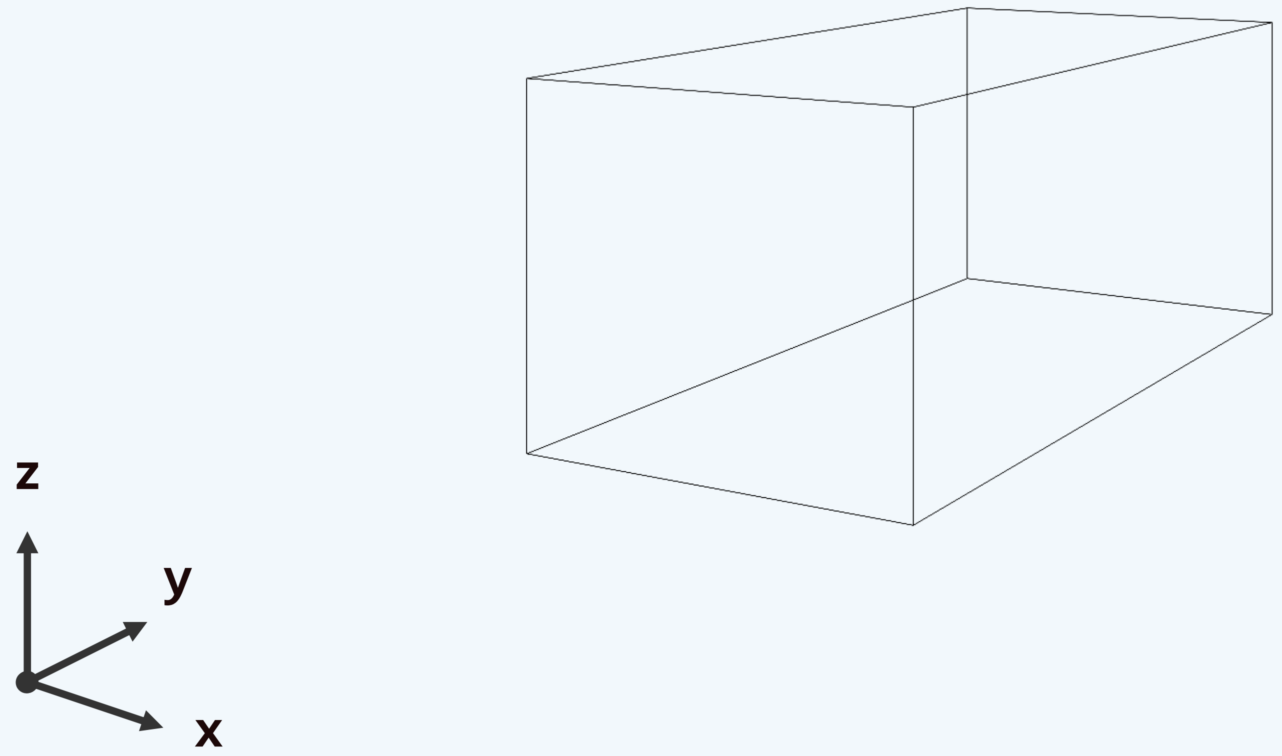




$z = yx$

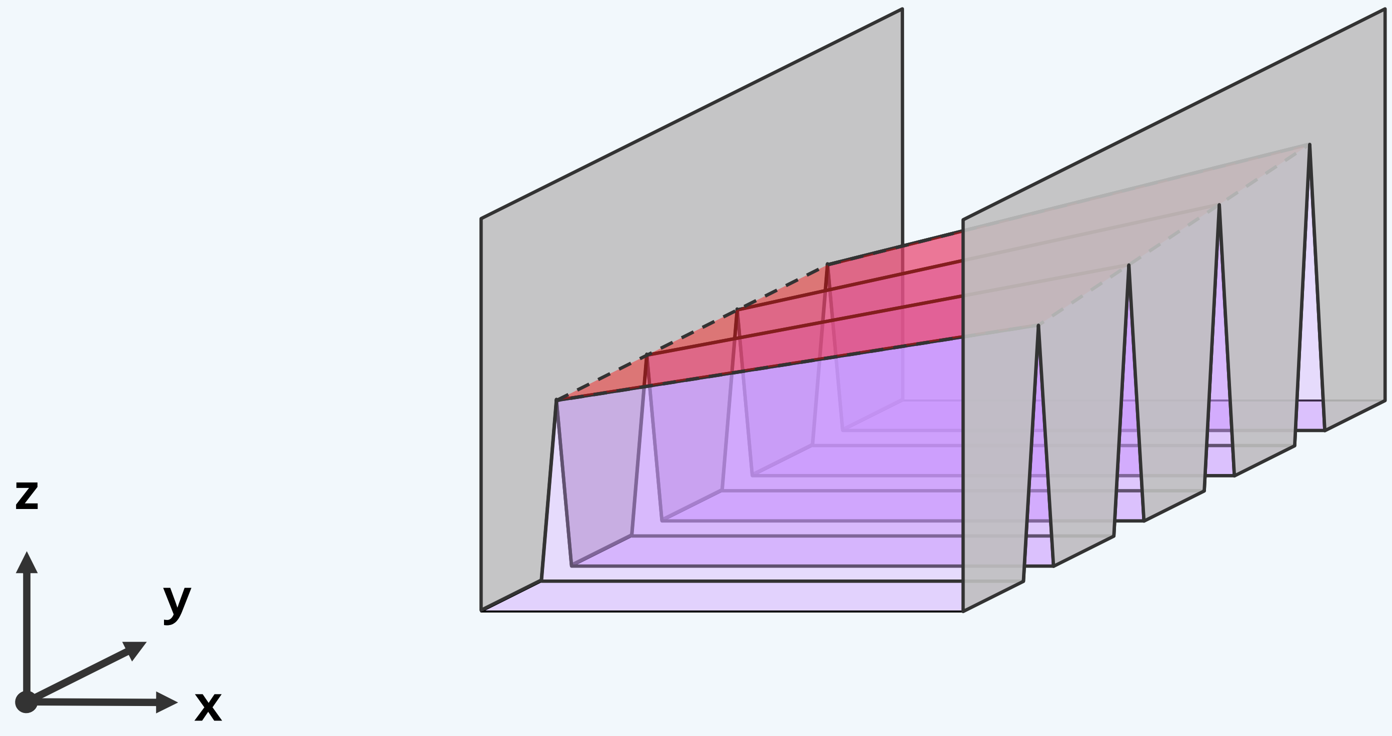
Constructing Chazelle's Polyhedron

Start with an orthogonal cube



Constructing Chazelle's Polyhedron

Cut $\boxed{k}$ thin notches into the top face, parallel to the $\boxed{xz}$-plane



Constructing Chazelle's Polyhedron

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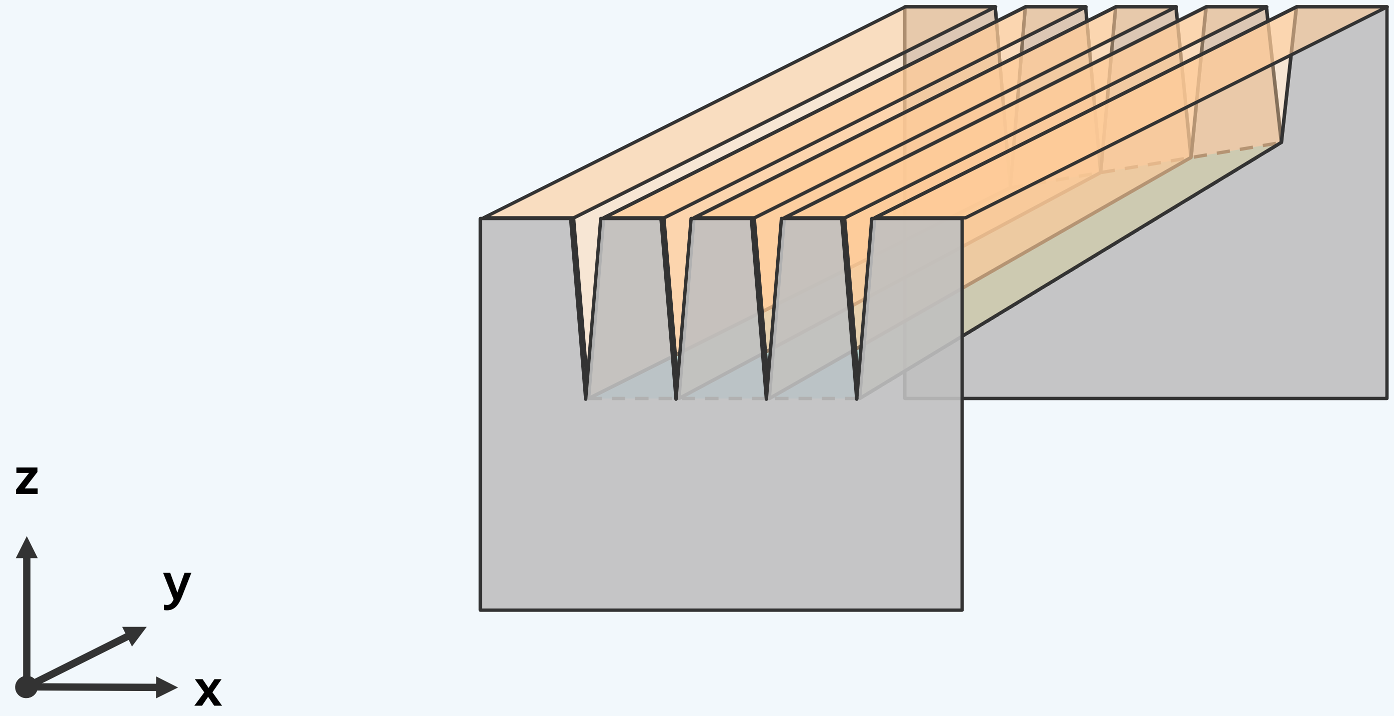


Chazelle's Polyhedra - $\Omega(n^2)$

- [Thomas, 1962]

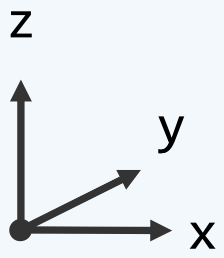
  A hyperbolic paraboloid can be generated by two sets of orthogonal lines.

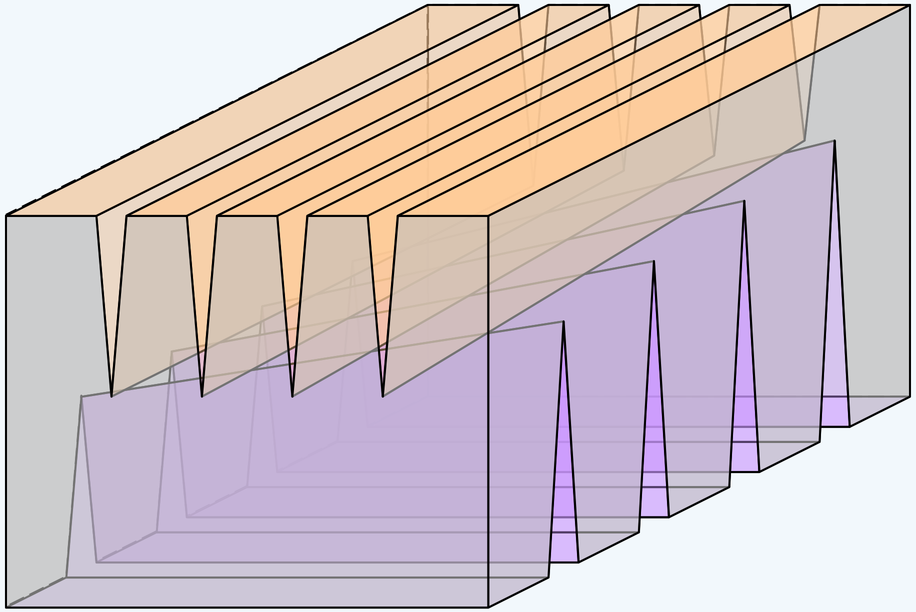
- So, the k $\boxed{\text{top}}$ edges can be chosen

  to lie on the surface $\boxed{z = xy \boxed{+} \varepsilon}$

  the k $\boxed{\text{bottom}}$ edges can be chosen

  to lie on the surface $\boxed{z = xy \boxed{-} \varepsilon}$



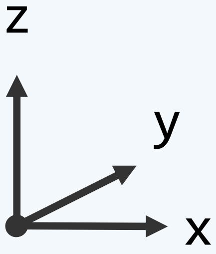


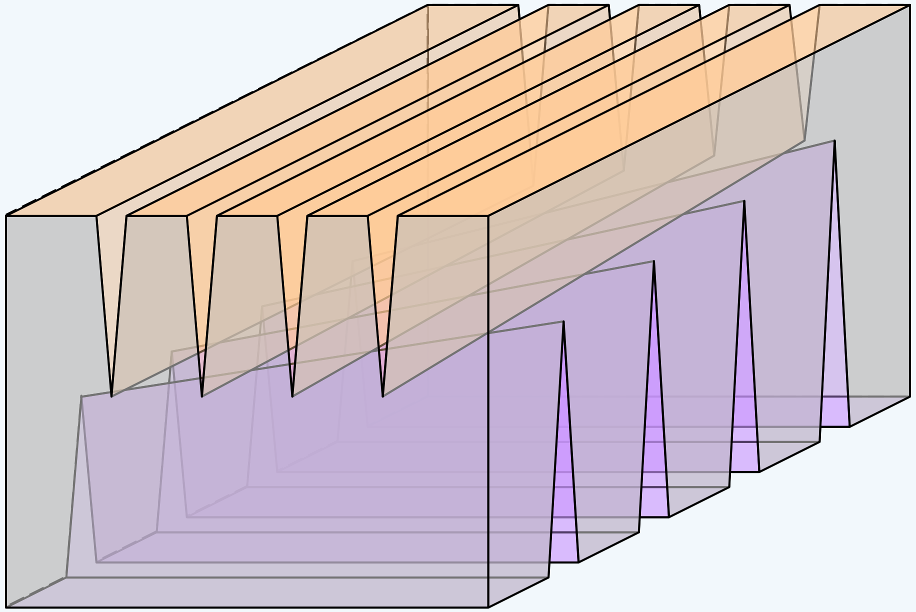
Chazelle's Polyhedra - $\Omega(n^2)$

❖ Chazelle proved that

1)The intersection of the warped shape between the two hyperbolic paraboloids with any convex subset of the polyhedron can only have such $\boxed{\text{a small volume}}$ that

2) $\boxed{\Omega(n^2)}$ pieces are necessary to make up the volume of the shape





Chazelle's Polyhedra - $\Omega(n^2)$

❖ More precisely,

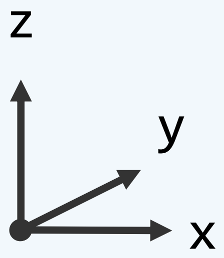
[Chazelle, 1984] proved that ...

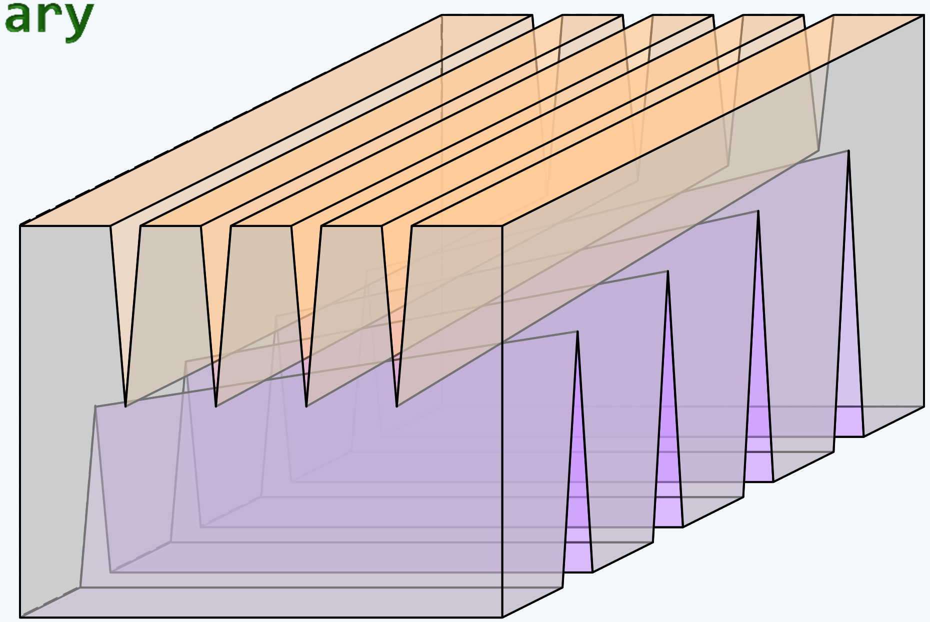
At least $\boxed{\Omega(n^2)}$ convex pieces are necessary

in any convex partition

of Chazelle's polyhedron

with $\boxed{n}$ vertices





Upper Bound - Algorithms

❖ Every polyhedron with $\boxed{n}$ vertices

can be triangulated into $\boxed{\mathcal{O}(n^2)}$ tetrahedra with Steiner points

❖ The upper bound was "improved" later by Chazelle and Palios,

by taking into consideration $\boxed{r}$, the number of reflex edges

❖ [Chazelle & Palios, 1989]//again, $\boxed{r}$ can be as large as $\boxed{n}$ in the worst cases

Every polyhedron with $\boxed{n}$ vertices

can be triangulated into $\boxed{\mathcal{O}(n + r^2)}$ tetrahedra with $\boxed{\text{Steiner points}}$,

where $\boxed{r}$ is the number of $\boxed{\text{reflex edges}}$ on the original polyhedron

❖ Since this complexity depends

not only on the input $\boxed{\text{size}}$ but also on the polyhedron's $\boxed{\text{shape}}$,

you may want to call it $\boxed{\text{input sensitive}}$, or more precisely, $\boxed{\text{shape sensitive}}$

Beyond 3-Dimension

- R. P. Stanley

  Decompositions of Rational Convex Polytopes

  Ann. Discrete Math., 6:333-342, 1980

- I. M. Gelfand, M. M. Kapranov, and A. V. Zelvinsky

  Newton Polytopes of the Classical Discriminant and Resultant

  Adv. Math., 84:237-254, 1990

- J. E. Goodman and J. Pach

  Cell Decomposition of Polytopes by Bending

  Israel J. Math., 64:129-138, 1988

- M. Haiman

  A Simple and Relatively Efficient Triangulation of the n-Cube

  Discrete Comput. Geom., 6:287-289, 1991