Convex Hull

Expected Complexity

01-XA

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Complexity of Convex Hull

- Let P be a set of n points in the plane

  How many points are expected

  to lie on CH(P)?

- Point sets with different $\boxed{\text{distributions}}$

  will give quite different answers

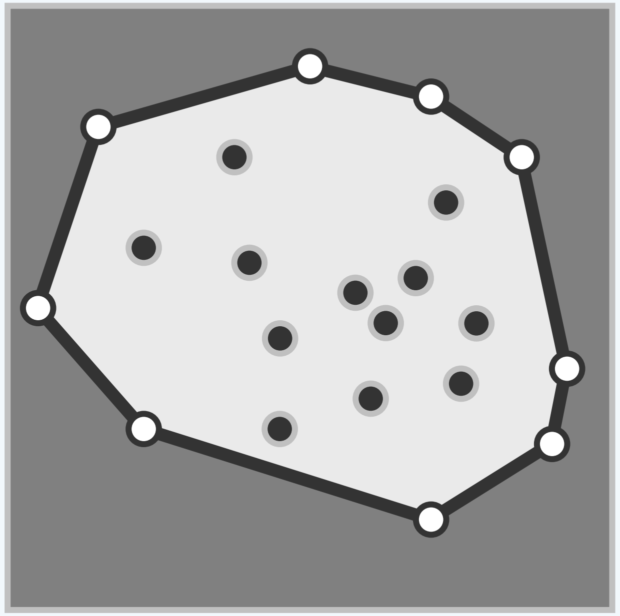
- As an example, this section will discuss the

  $\boxed{\text{uniform}}$ and $\boxed{\text{independent}}$ distribution

  inside a unit square (or affinely, a box)

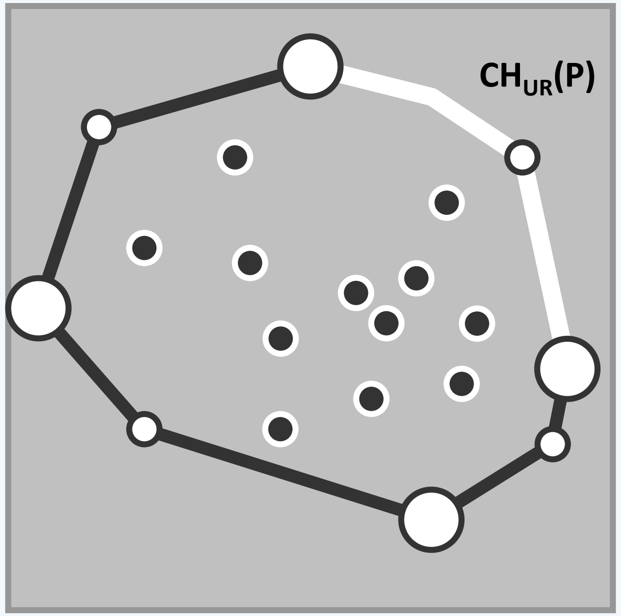
- We will show that

  the expected number of vertices on CH(P) is $\boxed{\mathcal{O}(\log n)}$



Quadrant Hulls

- Actually, instead of the entire hull, we will count only a quarter of CH(P)
- CH(P) is broken by its leftmost, rightmost, highest, and lowest vertices into (no more than) four parts, each of which is called a $\boxed{\text{quadrant hull}}$
- In terms of worst case complexity, to count the points on CH(P), it suffices to count any of the four quadrant hulls
- W.L.O.G., here we examine the upper-right one, which henceforth will be denoted as $CH_{UR}(P)$



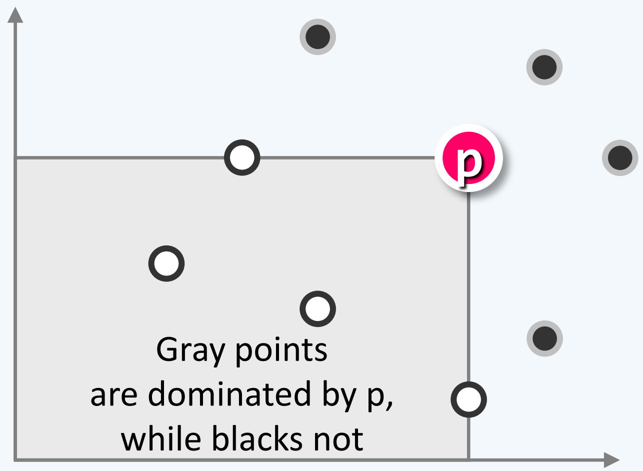
Domination

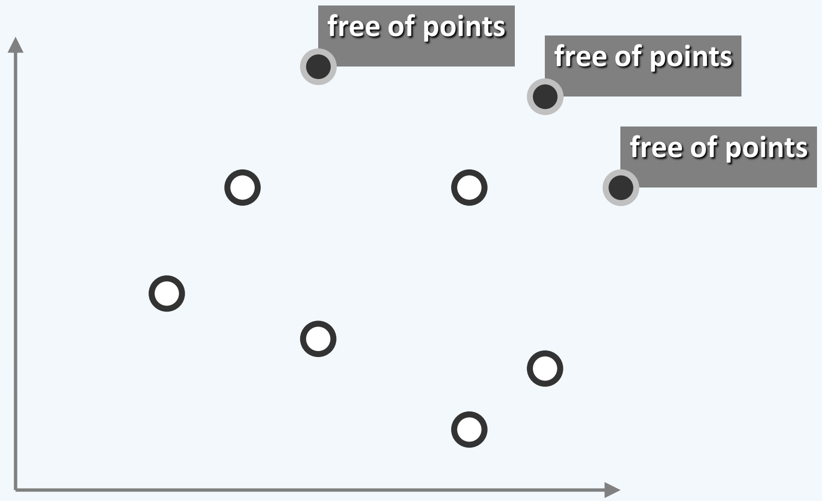
- Given two points $\boxed{p_1} = (x_1, y_1)$ and $\boxed{p_2} = (x_2, y_2)$ in the plane

  p_1 is said to be $\boxed{\text{dominated}}$ by p_2 if $\boxed{x_1 \le x_2}$ and $\boxed{y_1 \le y_2}$

- Ex: how to count dominations for any given P in $\boxed{\mathcal{O}(n\log n)}$ time?

  Note that there could be up to $\boxed{\Omega(n^2)}$ dominations





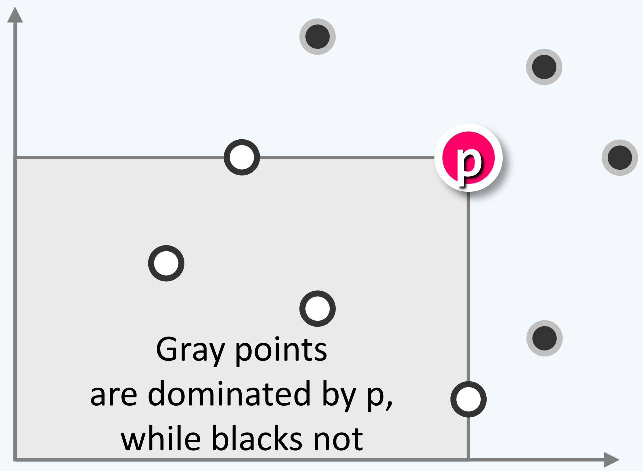
Maximal Point

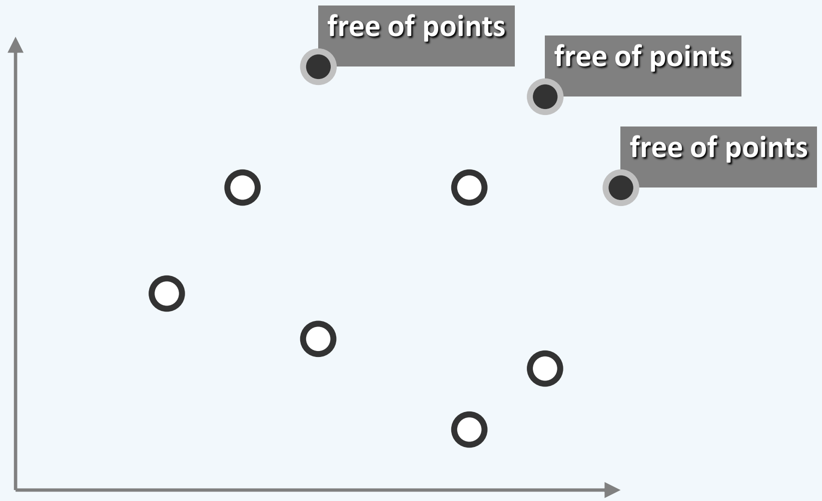
❖ In fact, we will count a super set of $CH_{UR}(P)$ ...

❖ Let P be a finite planar point set

A $\boxed{\text{maximal point}}$ of P is one $\boxed{\text{not}}$ dominated by any of the other points

❖ The set of all maxima of P is denoted as $\boxed{\text{MAX(P)}}$





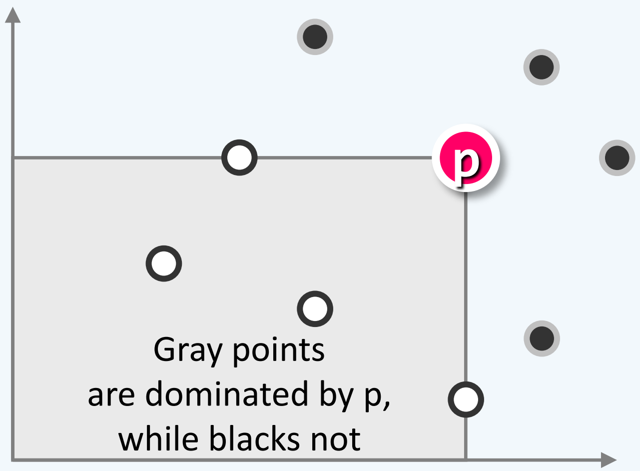
Maxima Set

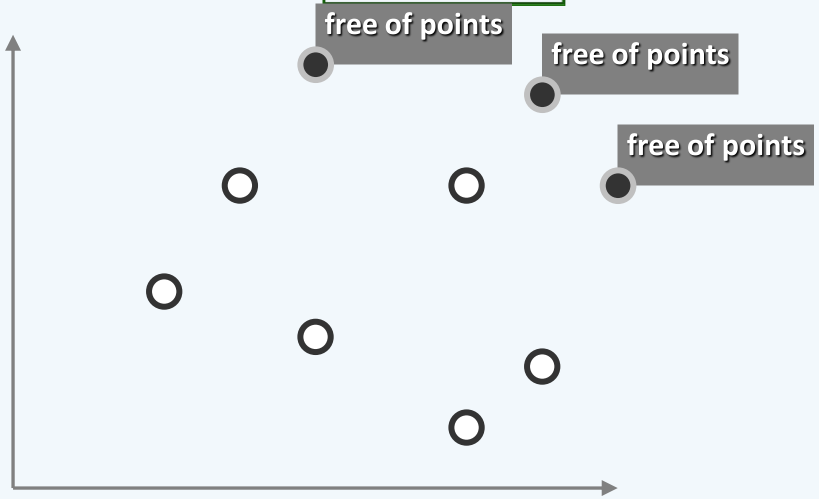
- Note that each vertex of $CH_{UR}(P)$ is a maximal point of P

  Therefore, $CH_{UR}(P)$ is a subset of MAX(P), and $\boxed{|CH_{UR}(P)| \leq |MAX(P)|}$ and ...

- The expected number of points on $CH_{UR}(P)$ (and hence on CH(P)) is bounded by the expected number of points in MAX(P)

- We will show next that |MAX(P)| is expected to be $\boxed{\mathcal{O}(\log n)}$

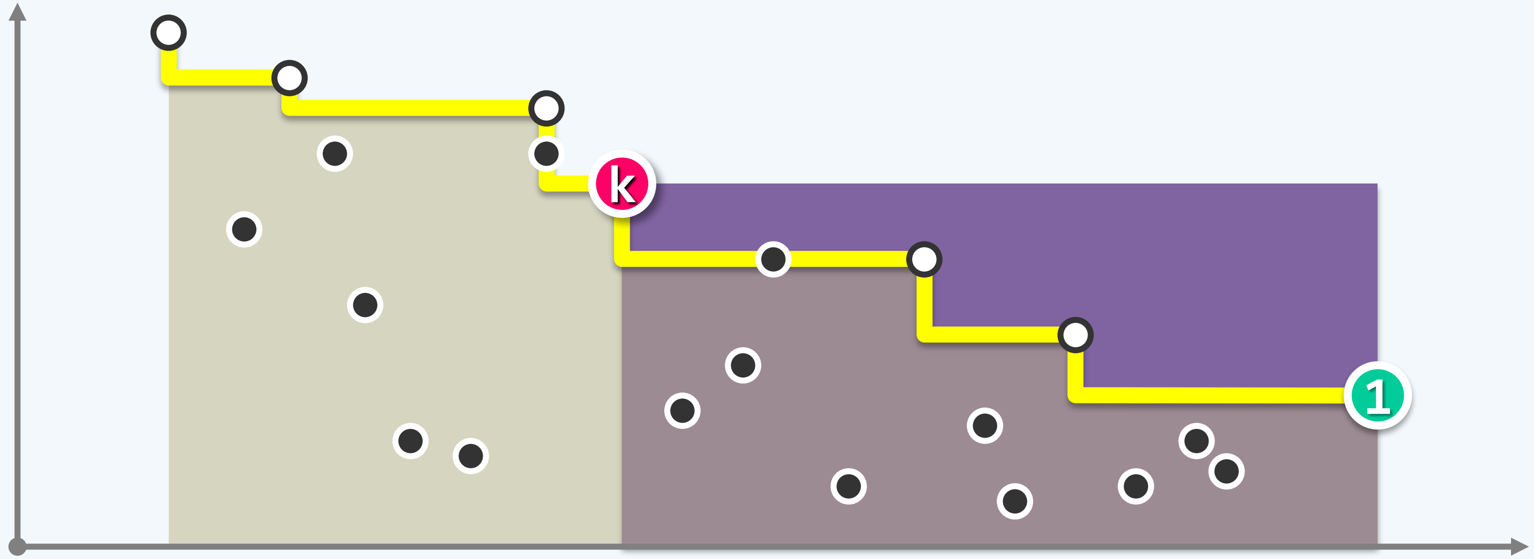




expected (|MAX(P)|) = $\mathcal{O}(\log n)$

- Number all points in P as { p_1, ..., $\boxed{p_n}$ }, from right to left
- Consider the point p_k ...
- Note that p_k is a $\boxed{\text{maximal}}$ point iff

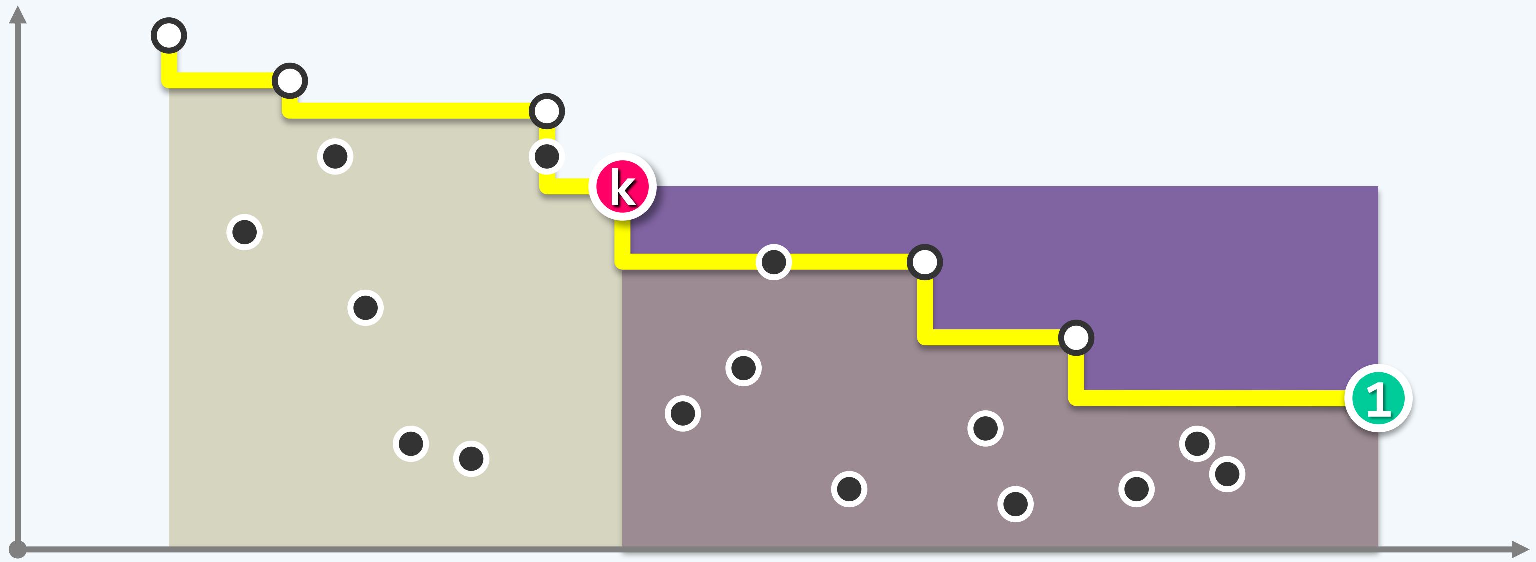
  p_k is the $\boxed{\text{highest}}$ among { p_1, ..., $\boxed{p_k}$ }



expected (|MAX(P)|) = $\mathcal{O}$(logn)

- By the assumption of uniform and independent distribution,
  this is true with a probability of $\boxed{1/k}$

- By $\boxed{\text{linearity of expectation}}$
  the sum of these expectations gives the expected number of maxima
  expected (|MAX(P)|) = 1/n + 1/(n-1) + ... + 1/3 + 1/2 + 1 = $\boxed{\mathcal{O}(\log n)}$



Expected Complexity & Expected Area

- For n points chosen uniformly and independently from the $\boxed{\text{unit square}}$,

  1) the expected $\boxed{\text{number of vertices}}$ of the convex hull is $\boxed{\mathcal{O}(\log n)}$
  2) the expected $\boxed{\text{area}}$ of the convex hull is $\boxed{\mathcal{O}(1 - \log(n)/n)}$

- For n points chosen uniformly and independently from the $\boxed{\text{unit disk}}$,

  1) the expected $\boxed{\text{number of vertices}}$ of the convex hull is $\boxed{\mathcal{O}(n^{1/3})}$
  2) the expected $\boxed{\text{area}}$ of the convex hull is $\boxed{\pi * \mathcal{O}(1 - n^{-2/3})}$

- For n points chosen uniformly and independently from a $\boxed{\text{triangle}}$,

  the expected $\boxed{\text{number of vertices}}$ of the convex hull is $\boxed{\mathcal{O}(\log n)}$

- For n points chosen uniformly and independently from a $\boxed{\text{convex k-gon}}$

  the expected $\boxed{\text{number of vertices}}$ of the convex hull is $\boxed{\mathcal{O}(k\log n)}$

Efron's Theorem, 1965

- B. Efron, *The convex hull of a random set of points*

  Biometrika, 52(3):331-343, 1965

- Let C be a compact convex set in the plane

- If the expected area of the convex hull of n points,

  chosen uniformly and independently from C, is

  $\boxed{\boxed{\mathcal{O}(1 - f(n))} * Area(C)}$, where $1 \geq f(n) \geq 0$, for $n \geq 0$,

  then the expected number of vertices of the convex hull is

  $\boxed{\mathcal{O}(n * f(n/2))}$