Convex Hull

Beyond 3 Dimension

- Upper Bound

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Upper Bound Theorem

- In $\boxed{1970}$, McMullen & Shephard proved their Upper Bound Theorem

  P. McMullen and G. C. Shepard,

  Convex Polytopes and the Upper Bound Conjecture

  Cambridge University Press, Cambridge, 1971

- A d-polytope with n vertices

  1) has at most $2 \times C(n, \lfloor d/2 \rfloor) = \boxed{\mathcal{O}(n^{\lfloor d/2 \rfloor})}$ facets, and

  2) at most $2^{d+1} \times C(n, \lfloor d/2 \rfloor) = \boxed{\mathcal{O}(n^{\lfloor d/2 \rfloor})}$ faces in total

- This is regarded as a main achievement

  in the modern theory of convex polytopes

- In fact, this upper bound is achieved by cyclic polytopes

Algorithms in Higher Dimensions - Even Dimensions

- Representations of higher dimensional CH
  - $\boxed{\text{Vertex}}$ description
  - $\boxed{\text{Facet}}$ description
  - $\boxed{\text{Double}}$ description
  - $\boxed{\text{Lattice}}$ description
  - $\boxed{\text{Boundary}}$ description
- [R. Seidel, 1981] Using the beneath-beyond algorithm, the convex hull of n points in $\mathcal{E}^d$ can be constructed in time $\boxed{\mathcal{O}(n^{\lceil d/2 \rceil})}$
- By Ziegler's lower bound, Seidel's algorithm is worst-case optimal in $\boxed{\text{even}}$ dimensions

Algorithms in Higher Dimensions - Any Fixed Dimension

- [B. Chazelle, 1993]

  If the dimension d is considered constant,

  then given S,

  each of the $\boxed{\text{3 combinatorial descriptions}}$ of conv(S) can be computed

  in time $\boxed{\mathcal{O}(\ n\log n + n^{\lfloor d/2 \rfloor})}$

  using space $\boxed{\mathcal{O}(\ n^{\lfloor d/2 \rfloor})}$

- This is asymptotically worst-case optimal

- Again, by Ziegler's lower bound,

  Chazelle's algorithm is worst-case optimal in any $\boxed{\text{fixed}}$ dimension

Algorithms in Higher Dimensions - Output Sensitivity

❖ [D. Avis & K. Fukuda, 1992]

1) Given S,

   a $\boxed{\text{boundary description}}$ of conv(S) can be computed

   in time $\boxed{\mathcal{O}(\ d * n * M\ )}$

   using space $\boxed{\mathcal{O}(\ d * n\ )}$,

   where M is the size of the boundary description produced

2) If S is non-degenerate, then

   each of the $\boxed{\text{3 combinatorial descriptions}}$ of P

   can be computed in time $\boxed{\mathcal{O}(\ d^{\mathcal{O}(1)} * n * M\ )}$,

   where M is the size of the respective description

Algorithms in Higher Dimensions - Output Sensitivity

- [D. R. Chand, S. S. Kapur, 1970] & [G. F. Swart, 1985]

  Given S,

  the $\boxed{\text{lattice description}}$ of conv(S) can be computed

  in time and space polynomial in $\boxed{d}$, $\boxed{n}$, and the size of the output

- [Seidel]

  Is there an algorithm that,

  given S,

  computes the $\boxed{\text{double description}}$ of conv(S)

  in time polynomial in $\boxed{d}$, $\boxed{n}$, and the size of the double description?