Convex Hull

Beyond 3 Dimension

- Lower Bound

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Lower Bound Theorem

- For higher dimensions, a challenging question could be
  how many faces can a convex polytope with n vertices in $\mathcal{E}^d$ have?
- G. M. Ziegler presented an answer to this question in 1994
  Lectures on Polytopes. Vol. 152 of *Graduate Texts in Mathematics*
  Springer-Verlag, New York, $\boxed{1994}$
- [Ziegler's Theorem]
  An n-vertex polytope in $\mathcal{E}^d$ can have $\boxed{\Omega(n^{\lfloor d/2 \rfloor})}$ facets
- This implies that
  to construct a d-dimensional CH, we need $\boxed{\Omega(n^{\lfloor d/2 \rfloor})}$ time in worst cases
- Let's prove this theorem next ...

Affine Independency of Points on γ

- Claim:

  every d+1 distinct points on the MC γ are $\boxed{\text{affinely independent}}$

- To see this, first note that

  $p(u_0), p(u_1), \ldots, p(u_d)$ are affinely independent

  $\boxed{\text{iff}}$

- $$\begin{vmatrix} 1 & u_0 & u_0^2 & \cdots & u_0^d \\ 1 & u_1 & u_1^2 & \cdots & u_1^d \\ 1 & u_2 & u_2^2 & \cdots & u_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_d & u_d^2 & \cdots & u_d^d \end{vmatrix} \neq 0$$

Affine Independency of Points on γ

❖ $$\begin{vmatrix} 1 & u_0 & u_0^2 & \dots & u_0^d \\ 1 & u_1 & u_1^2 & \dots & u_1^d \\ 1 & u_2 & u_2^2 & \dots & u_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_d & u_d^2 & \dots & u_d^d \end{vmatrix} \neq 0$$

❖ Take the planar case as an example:

3 points are affinely independent iff

they define a triangle with a non-zero area

❖ The left side is the well-known $\boxed{\text{Van der Monde}}$ determinant

which is equal to $\boxed{\prod_{0 \le i < j \le d} u_j - u_i}$

❖ Since $\boxed{u_i \neq u_j}$, this determinant can't be $\boxed{\text{zero}}$ //QED

Simplicity of Cyclic Polytopes

- The affine independency of points on γ implies that

  $CP(u_0, u_1, \ldots, u_d)$ is a $\boxed{\text{simplicial}}$ d-polytope

  i.e., each facet of CP is a $\boxed{\text{(d–1)-simplex}}$,

  which consists of exactly d vertices

- This should be true. // Otherwise ...

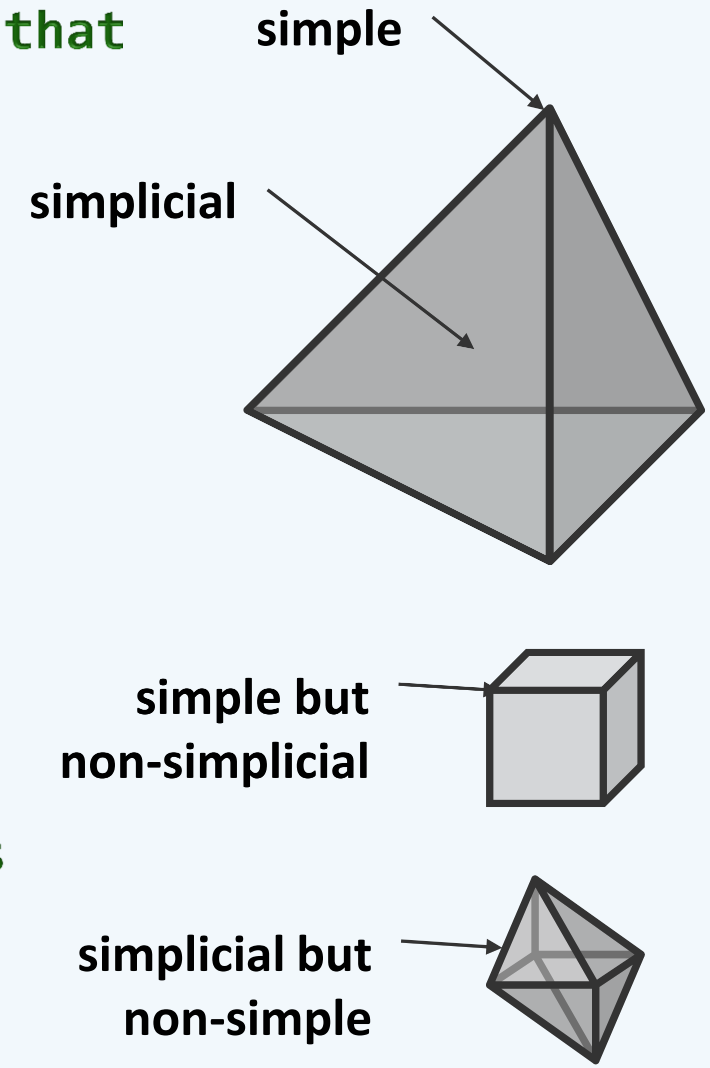
  there must be a facet which is not $\boxed{\text{simplicial}}$

  In other words,

  this facet consists of at least $\boxed{d + 1}$ vertices

  which are affinely dependent

  //A contradiction



Intersections between Hyperplanes and γ

- Claim#1: any hyperplane h $\boxed{\text{intersects}}$ the MC γ in $\boxed{\text{at most d}}$ points
- Every hyperplane h can be expressed by the equation

  $\boxed{\langle a, x \rangle = b}$, or $\boxed{a_1x_1 + a_2x_2 + \ldots + a_dx_d = b}$

- Each point of γ has the form $\boxed{(t, t^2, t^3, \ldots, t^d)}$, and

  if it lies in h, we get $\boxed{a_1t + a_2t^2 + a_3t^3 + \ldots + a_dt^d - b = 0}$

- This means that

  t is a $\boxed{\text{root}}$ of a nonzero polynomial of degree $\boxed{\text{at most d}}$, and hence

  the number of intersections is $\boxed{\text{at most d}}$

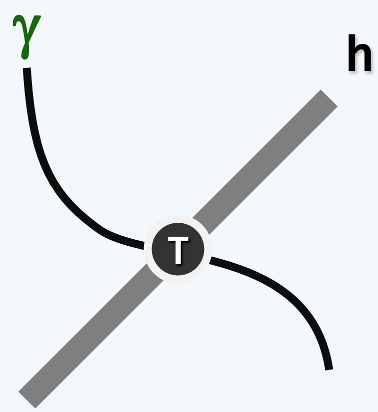
Intersections between Hyperplanes and γ

- Claim#2: if there are d intersections, then

  h $\boxed{\text{cannot}}$ be tangent to γ and thus,

  at each intersection,

  γ $\boxed{\text{passes}}$ from one side of h to the other

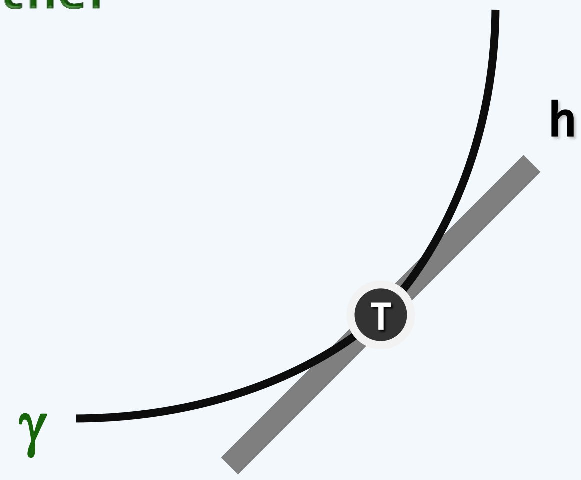


- If there are d distinct roots

  then they must be all simple, whereas

  if h is $\boxed{\text{tangent}}$ to γ

  then the touch point would imply a root of $\boxed{\text{multiplicity } \geq 2}$



d-Subsets of S

- How many facets does CP(S) have?
- We observe first that

  1) each facet of CP(S) is determined by a $\boxed{\text{d-subset}}$ of S, and

  2) different d-subsets determine $\boxed{\text{distinct hyperplanes}}$, hence $\boxed{\text{distinct facets}}$ (if they do define facets)
- As a set of cardinality n, S has $C(n, d) = O(n^d)$ d-subsets
- How many of these d-subsets, then, can define a facet for the CP(S)?

Evenness Criterion

- Denote the 2 halfspaces determined by h as $\boxed{h^+}$ and $\boxed{h^-}$ resp., where $\boxed{h^+}$ contains S
- A d-subset F of S defines a hyperplane h(F) which $\boxed{\text{supports}}$ a facet of CP(S)

  $\boxed{\text{iff}}$

  S\F (and hence the whole polytope) is $\boxed{\text{a subset of } h^+}$
- Now we can say that $\boxed{(1)}$ the d points of F split γ into $\boxed{d + 1}$ intervals
- And since only those $\boxed{\text{intervals in } h^+}$ can contain points of S and

  each point of F is a $\boxed{\text{crossing point}}$,

  we know that $\boxed{(2)}$ the $\boxed{d + 1}$ intervals alternate as in $\boxed{h^+}$ and $\boxed{h^-}$

Evenness Criterion

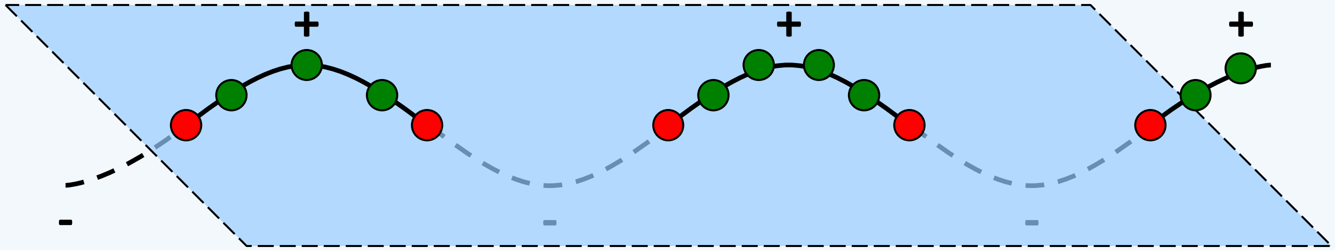
- These can be summarized as Gale's famous Evenness Criterion
- [Gale, 1967] B. Grünbaum, *Convex Polytopes*, Interscience, London, 1967

A d-subset $\boxed{F}$ of S defines a hyperplane $\boxed{h(F)}$ which supports a facet of CP

iff

between any 2 points in S\F,

there are an $\boxed{\text{even}}$ number of points in F along γ //Note that 0 is even



The hyperplane supporting a facet of CP splits the moment curve into (d+1) intervals, which alternate as in h^+ and h^-. Only those intervals in h^+ can contain points of S

Bead Pairs

- In the following,

  we split the analysis depending on the $\boxed{\text{parity}}$ of d

- But before that,

  we'd like to mention such a counting fact ...

- Given a string of n beads,

  the number of ways to choose k pairs of consecutive beads from the string

  is C(n-k, k)

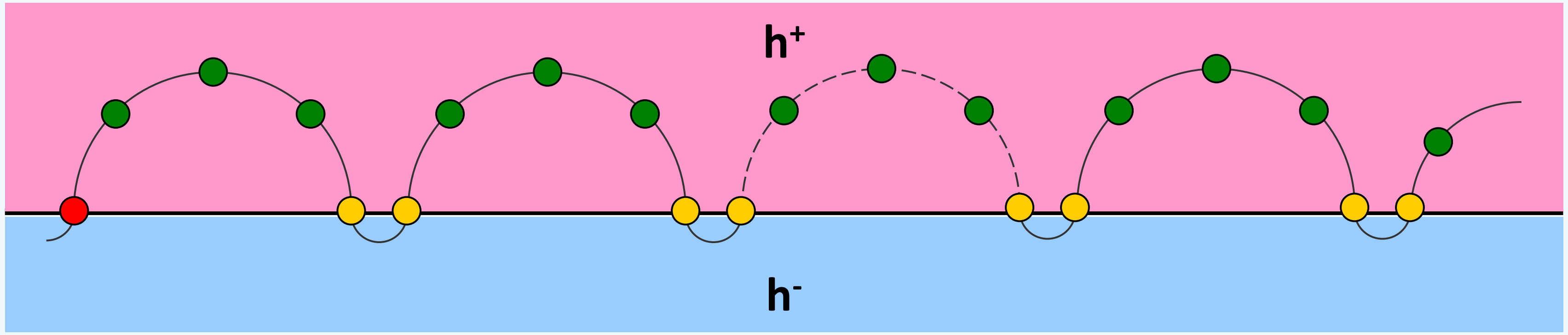
Bead Pairs: Odd Cases

- When d is $\boxed{\text{odd}}$

  1) there is a single crossing point in the first or last position, and

  2) all other crossing points appear in pairs

- How many valid configurations could be there?



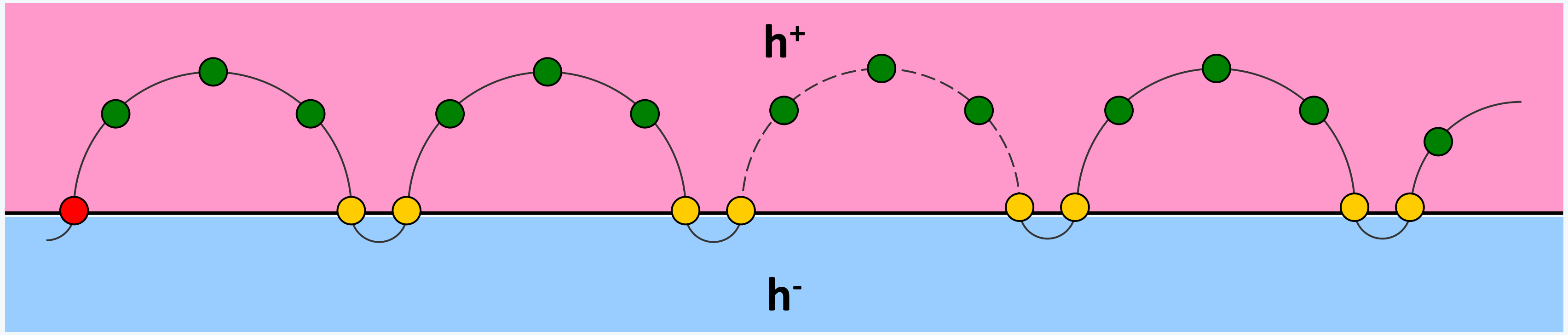
In odd dimensions, except for the first/last one (red), all the other (d-1) crossing points appear in consecutive pairs (yellow)

Bead Pairs: Odd Cases

- Once the single crossing point's position is determined, the number is equal #ways to choose $\lfloor d/2 \rfloor$ pairs of consecutive beads from a string of length $\boxed{d-1}$

  //So as a whole, we have ...

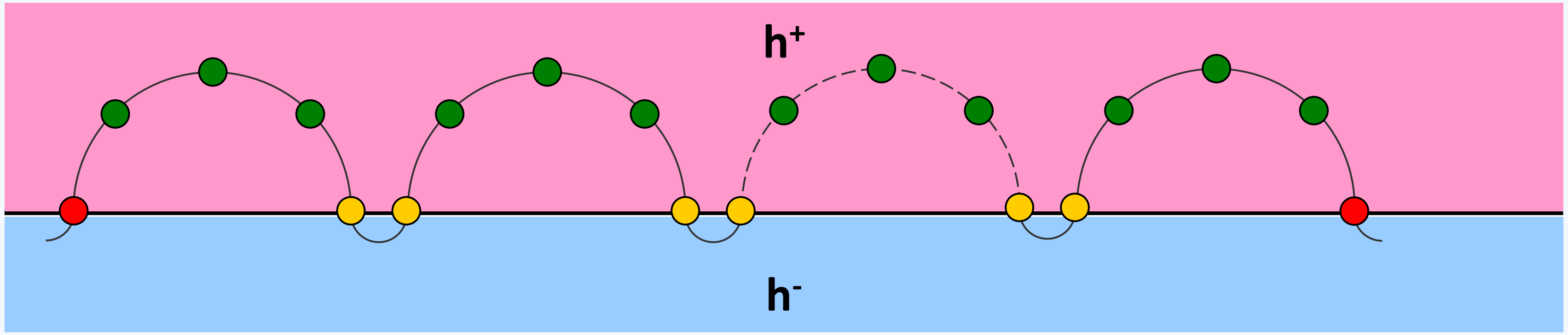
- In an odd-dimensional space, there are $\boxed{\Omega(\ 2 \times \boxed{C(\ n - \lfloor d/2 \rfloor - 1,\ \lfloor d/2 \rfloor\ )}\ )}$ d-subsets of S each of which determines a distinct facet of CP(S)



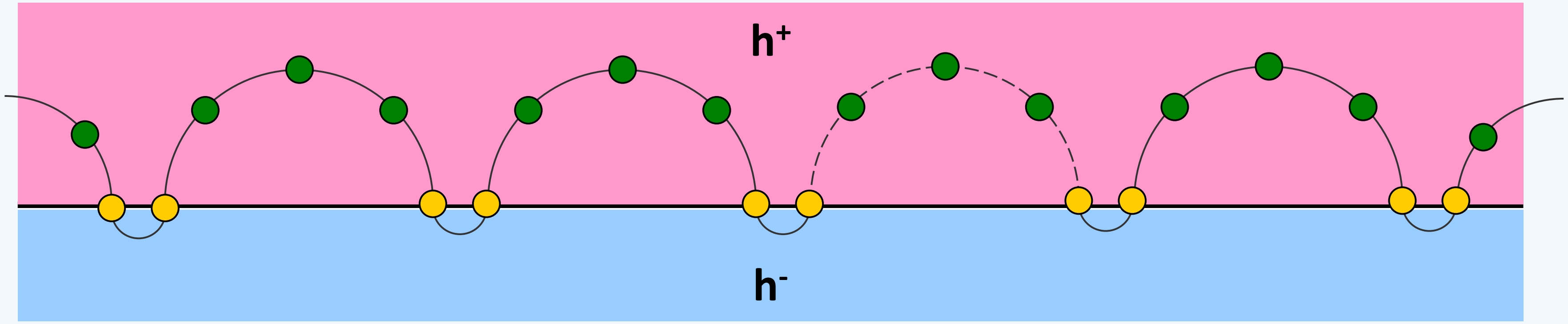
In odd dimensions, except for the first/last one (red), all the other (d-1) crossing points appear in consecutive pairs (yellow)

Bead Pairs: Even Cases

❖ When d is $\boxed{\text{even}}$, there could be two subcases



Subcase (a): except for the first and last one (red), all the other (d-2) crossing points appear in consecutive pairs (yellow)



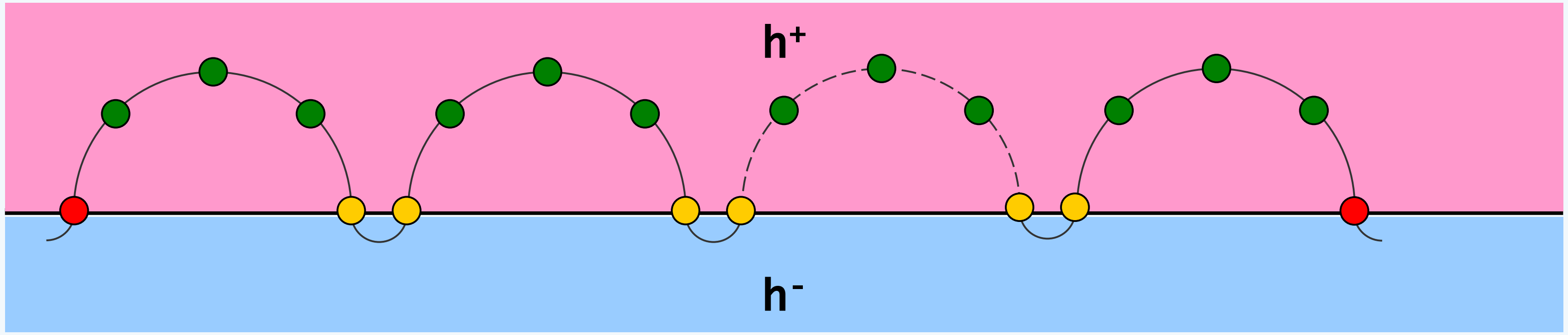
Subcase (b): all the d crossing points appear in consecutive pairs (yellow)

Bead Pairs: Even Cases: Subcase (a)

- When both the $\boxed{\text{first}}$ and $\boxed{\text{last}}$ crossing points are single

  while the other $\boxed{d - 2}$ appear in pairs, the number is

  $$C(\ (n - 2) - (\lfloor d/2 \rfloor - 1),\ \lfloor d/2 \rfloor - 1\ ) \quad = \quad C(\ n - \lfloor d/2 \rfloor - 1,\ \lfloor d/2 \rfloor - 1\ )$$



Bead Pairs: Even Cases: Subcase (b)

- When all the d crossing points appear in pairs, the number is

  $C(\ n - \lfloor d/2 \rfloor,\ \lfloor d/2 \rfloor\ )$

- As a whole, we have ...

- In an $\boxed{\text{even}}$-dimensional space, there are

  $\Omega(\ C(\ n - \lfloor d/2 \rfloor - 1,\ \lfloor d/2 \rfloor - 1\ ) + C(\ n - \lfloor d/2 \rfloor,\ \lfloor d/2 \rfloor\ )\ )$

  d-subsets of P each of which determines a distinct facet of CP(S)