09-E-01

Arrangement

Duality: Transform

有天地，然后有万物；

有万物，然后有男女；

有男女，然后有夫妇；

有夫妇，然后有父子；

有父子，然后有君臣；

有君臣，然后有上下；

有上下，然后礼义有所错。

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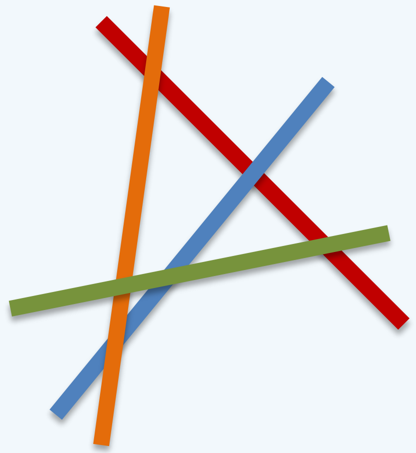
Configuration

- A finite set P of points in $\mathcal{E}^d$

  is called a configuration



- Dual transform:

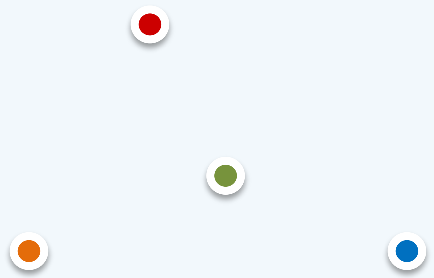
  a 1-to-1 correspondence between

  $\boxed{\text{points}}$ and $\boxed{\text{hyperplanes}}$ in $\mathcal{E}^d$

- Such a dual transform (if exists) will also be

  a 1-to-1 correspondence between

  $\boxed{\text{configurations}}$ and $\boxed{\text{arrangements}}$ in $\mathcal{E}^d$



Dual Transform $\mathcal{D}$

- For any point $p = (\ \pi_1,\ \dots,\ \pi_d\ )^T \in \mathcal{E}^d$, let

  Height : $d(p) = \pi_d$

  Projection : $\Pi(p) = (\ \pi_1,\ \dots,\ \pi_{d-1},\ 0\ )^T$

  Normal : $N(p) = \boxed{2}\Pi(p) - (\ 0,\ \dots,\ 0,\ \boxed{1}\ )^T = (\ \boxed{2}\pi_1,\ \dots,\ \boxed{2}\pi_{d-1},\ \boxed{-1}\ )^T$

- Note that $N(p) = N(q)$ iff p lies in the same vertical line with q

- Dual transform $\mathcal{D}$ between points and $\boxed{\text{non-vertical}}$ hyperplanes

  $\mathcal{D}(p)$ = hyperplane h = $\{x \in \mathcal{E}^d \mid N(p)^T x = d(p)\}$, and vice versa

  $\mathcal{D}(\mathcal{D}(h))$ = h for any $\boxed{\text{non-vertical}}$ h

- Duality: primal space ~ dual space

  configurations ~ arrangements (without vertical hyperplanes)

Geometric Explanation

❖ Unit paraboloid in $\mathcal{E}^d$:

$U = \{ x = (x_1, \ldots, x_d)^T \in \mathcal{E}^d \mid x^d = \sum_{i=1}^{d-1} x_i^2 \}$

//consider a tangent hyperplane h of U ...

❖ If $h \cap U = \{p = (\pi_1, \ldots, \pi_{d-1}, \sum_{i=1}^{d-1} \pi_i^2)^T \in U\}$, then the normal of h

1) is parallel to N(p), and

2) is parallel to any N(q) provided that $\prod(q) = \prod(p)$

❖ So geometrically

1) for a point $p \in U$, $\mathcal{D}(p)$ is just the tangent hyperplane of U at p

2) for a point q in the same vertical line with p,

$\mathcal{D}(q)$ is parallel to $\mathcal{D}(p)$

Geometric Explanation

- Now, we have built a 1-to-1 correspondence in $\mathcal{E}^d$

  1) between

     points and

     non-vertical hyperplanes

  2) between

     configurations and

     arrangements without vertical hyperplanes

- Why does $\mathcal{D}$ keep out of

  the vertical hyperplanes?