10

Dissection

夥計說有大白饅頭、四喜肉、雞蛋、風肉。鴻漸主張切一碟

風肉夾了饅頭吃，李顧趙三人贊成，說是"本位文化三明治"，

要分付夥計下去準備。

- 錢鐘書，《圍城·第伍章》

Table of Contents

List of Figures

1 What You Will Learn in This Chapter

- Carathéodory's theorem
- Radon's partition/theorem
- Helly’s theorem
- Existence of a center point
- Ham-sandwich theorem
- Existence of a Ham-sandwich cut
- Minkowski's (first) theorem

2 Caratheodory's Theorem

2.1 C. Carathéodory

Fig 10-1 Constantin Carathéodory (1873/09/13 - 1950/02/02)

2.2 Caratheodory's Theorem

[Carathéodory, 1907][6]

Let $X \subseteq \mathcal{E}^d$.

Then each point of **conv**(X) is a convex combination of at most d+1 points of X.

Caratheodory's theorem is the fundamental dimensionality result in convexity and one of the cornerstones of combinatorial geometry.

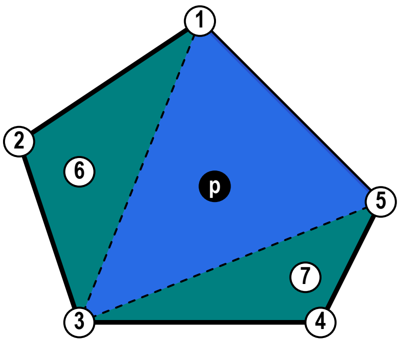

Fig 10-2 Each point p in the convex hull of finite points in the plane lies in the convex hull of no more than three of the points.
Note that the convex hull here is a triangle, or more generally, a planar simplex.

2.3 Separation Theorem

Let C, D $\subseteq \mathcal{E}^d$ be convex sets with $C \cap D = \varnothing$.

❶ Then there exists a hyperplane h such that C lies in one of the CLOSED halfspaces determined by h, and D lies in the opposite CLOSED halfspace.

In other words, there exist a unit vector $\alpha \in \mathcal{E}^d$ and a number $b \in \mathcal{R}$ such that for all $x \in C$ we have $\alpha^T x \geq b$, and for all $x \in D$ we have $\alpha^T x \leq b$.

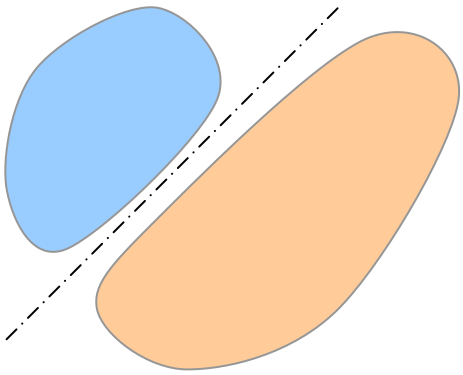

Fig 10-3 Disjoint convex sets might be separated by a hyperplane.

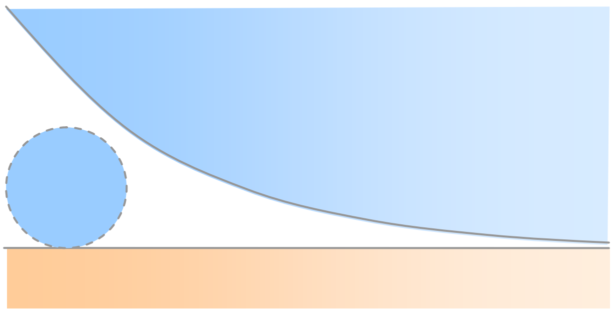

Fig 10-4 Disjoint convex sets are not necessarily strictly separable, even when both are CLOSED.

❷ If C is CLOSED and D is COMPACT, they can be separated strictly.
(in such a way that $C \cap h = D \cap h = \varnothing$)

3 Radon's Theorem

3.1 J. Radon

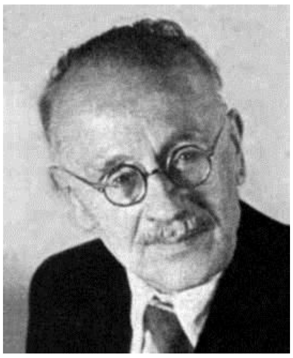

Fig 10-5 Johann Radon (1887/12/16 - 1956/05/25)

3.2 Radon Partition

Given P a family of sets in $\mathcal{E}^d$, if there are two disjoint, non-empty subfamilies P_1 and P_2 of P such that **conv**(P_1) $\cap$ **conv**(P_2) $\neq \varnothing$, then (P_1, P_2) is called a **Radon partition** of P.

$P_1 \cup P_2 \subseteq P \subseteq 2^{\mathcal{E}^d}$, $P_1 \cap P_2 = \varnothing$, **conv**($P_1$) $\cap$ **conv**(P_2) $\neq \varnothing$.

Fig 10-6 A family of no less than 4 points in the plane always admits a Radon partition.

3.3 Radon's Theorem

Radon gave a sufficient condition for the existence of Radon partition by a simple comparison between the cardinality of the family and the dimension of the space where the family lives.

[Radon, 1921][19]
Every family of $n \geq d+2$ sets in $\mathcal{E}^d$ admits a Radon partition.

- This theorem was employed (as a lemma) in Radon's first proof of Helly's Theorem.
- Radon himself never went back to the subject, and it is safe to say that he did not mean the lemma to be a theorem in its own right.
  Nevertheless, this theorem turns out to be extremely useful in combinatorial convexity theory.

3.4 Proof of Radon's Theorem

- It suffices to consider the families of singleton sets, i.e., the sets consisting of a single point.
  Consider such a family $\mathcal{F} = \{\{p_1\}, \ldots, \{p_n\}\}$.
  W.L.O.G., assume that all p_i's are distinct.
- $\because$ Any $n \geq d+2$ points in $\mathcal{E}^d$ must be affinely dependent.
  $\therefore$ $\exists\, \lambda_1, \ldots, \lambda_n \in R$, such that $\sum_{i=1}^{n}|\lambda_i| > 0$, $\sum_{i=1}^{n}\lambda_i = 0$, and $\sum_{i=1}^{n}\lambda_i p_i = \vec{o}$.
- Define
  ① $I^+ = \{i \mid \lambda_i \geq 0\}$ and
  ② $I^- = \{i \mid \lambda_i < 0\}$.
  Note that neither I^+ nor I^- is empty.
- Let $\lambda = \sum_{i\in I^+}\lambda_i = -\sum_{i\in I^-}\lambda_i > 0$
- Let
  ❶ $P^+ = \{p_i \mid i\in I^+\} = \{p_i \mid \lambda_i \geq 0\}$ and
  ❷ $P^- = \{p_i \mid i\in I^-\} = \{p_i \mid \lambda_i < 0\}$.
  Note that $P^+ \cup P^- = P$ and $P^+ \cap P^- = \varnothing$, i.e., (P^+, P^-) is a partition of P.
- Actually, (P^+, P^-) is a Radon partition of P.
  To prove it, it suffices to identify a point in the common intersection of **conv**(P^+) and **conv**(P^-).
- Consider the following "two" points
  ① $q^+ = \frac{1}{\lambda}\sum_{i\in I^+}\lambda_i p_i = \sum_{i\in I^+}\left(\frac{\lambda_i}{\lambda}\right)p_i \in$ **conv**(P^+), and
  ② $q^- = \frac{1}{-\lambda}\sum_{i\in I^-}\lambda_i p_i = \sum_{i\in I^-}\left(\frac{\lambda_i}{-\lambda}\right)p_i \in$ **conv**(P^-).
- We claim that q^+ and q^- are in fact a same point in $\mathcal{E}^d$.
  To see this, let's consider

$$
\begin{aligned}
& q^+ - q^- \\
&= \frac{1}{\lambda}\sum_{i\in I^+}\lambda_i p_i - \frac{1}{-\lambda}\sum_{i\in I^-}\lambda_i p_i \\
&= \frac{1}{\lambda}\sum_{i\in I^+}\lambda_i p_i + \frac{1}{\lambda}\sum_{i\in I^-}\lambda_i p_i \\
&= \frac{1}{\lambda}\sum_{i\in I}\lambda_i p_i \\
&= \vec{o}
\end{aligned}
$$

  $\therefore$ $q^+ = q^- \in$ **conv**$(P^+)\cap$**conv**$(P^-) \neq \varnothing$.
- [QED]

3.5 Minimal Radon Partition

A family of sets in $\mathcal{E}^d$ may have more than one Radon partition.

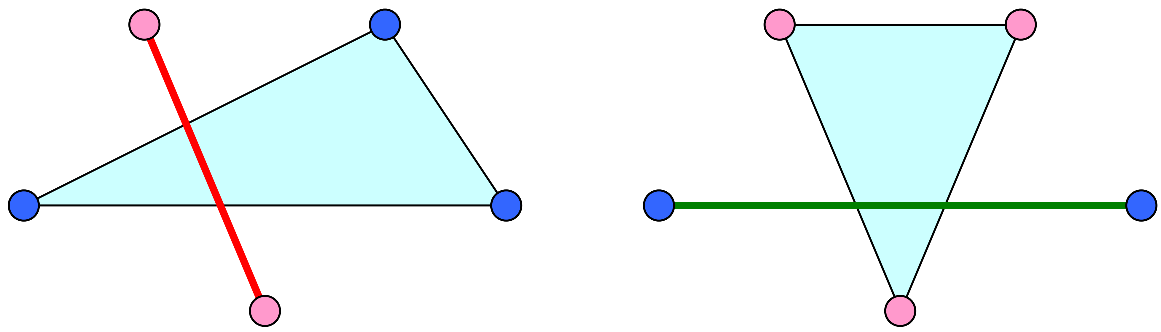

Fig 10-7 A family of 5 sets in the plane has ≥2 Radon partitions.

Let P be a family of sets in $\mathcal{E}^d$ with a Radon partition (P_1, P_2).

If for any $U \subset P$, $(P_1 \cap U, P_2 \cap U)$ is not a Radon partition of U, then
(P_1, P_2) is called a **minimal (or, primitive) Radon partition** of P.

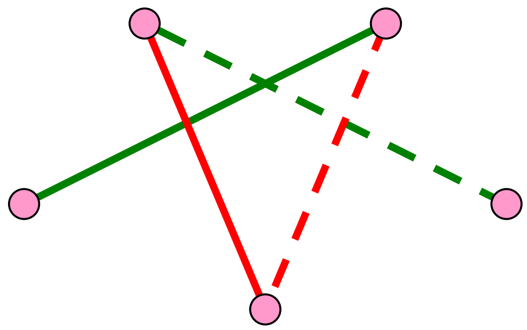

Fig 10-8 A family of 5 sets in the plane has ≥2 primitive Radon partitions.

3.6 Kirchberger's Theorem

[Kirchberger, 1903][14]

For any Radon partition (P_1, P_2) of a family P of $n \geq d+2$ sets in $\mathcal{E}^d$, there is a subfamily $U \subseteq P$ with $(\dim(P_1 \cup P_2)+2)$ sets such that $(P_1 \cap U, P_2 \cap U)$ is a Radon partition of U (and hence of P).

- Observe that $\dim(P_1 \cup P_2) + 2 \leq d + 2$.
- If fact, Kirchberger discussed only the intersection of the convex hulls of two families.
- Hare & Kenelly pointed out that the number of points of a primitive partition is determined by its affine dimension. More precisely,

[Hare & Kenelly, 1971][12]
Let P be a finite set of points.
A primitive Radon partition (Q_1, Q_2) of P contains $\dim(Q_1 \cup Q_2) + 2$ points.

I.e., a Radon partition (Q_1, Q_2) cannot be a primitive one if it contains more than $\dim(Q_1 \cup Q_2)+2$ points; but it can be reduced to a primitive one.

3.7 Tverberg's Theorem

[Tverberg, 1966][22]
❶ Every set of (m-1)(d+1)+1 points in $\mathcal{E}^d$ can be divided into m (pairwise disjoint) subsets whose convex hulls have a common point;
❷ the number (m-1)(d+1)+1 is the smallest which has the stated property.

Radon's Theorem is a special case of Tverberg's Theorem where m = 2.

The following is another special case when d = 2:

Every set of (3m-2) points in the plane can be divided into m subsets whose convex hulls have a common point.

For any integer $m \geq 2$, can you give a set of (3m-3) points in the plane which cannot be divided into m subsets whose convex hulls have a common point?

4 Helly's Theorem

4.1 E. Helly

Fig 10-9 Eduard Helly (1884/06/01 - 1943/11/28)

4.2 Helly's Theorem (Finite Version)

[Helly, 1923][11]
A family of finite convex sets admits a nonempty common intersection **iff**
each of its (d+1)-cardinality subfamilies does.

- This theorem was discovered by Helly in 1913 and communicated by him to Radon, who published a first proof in 1921[19].
- Helly proved this theorem during the WWI when he was jailed as a POW in Siberia for five years.
  The proof was published later in 1923[11], three years after his return to Vienna.
- The "only if" direction is trivial.
  To prove the theorem, it suffices to prove that

Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a family of $n \geq d+1$ convex sets in $\mathcal{E}^d$.
Their common intersection is non-empty if
the common intersection of any (d+1)-subfamily is non-empty.

4.3 Proof of Helly's Theorem (Finite Version)

- Induction on n.

➊ Inductive Base: when n = d+1, the statement is trivial.

❶ Inductive assumption : Helly's assertion holds for all $d+1 \le n < N$.

Let $\mathcal{S} = \{S_1, ..., S_N\}$ be a family of $N \ge d+2$ convex sets in $\mathcal{E}^d$, where any (d+1) sets have a common intersection.

Let $T_i = \bigcap_{k \ne i} S_k$, for $1 \le i \le N$.

By inductive assumption, we know that

For any $1 \le i \le N$,
① $T_i \ne \emptyset$, and
② T_i is convex.

Let $P = \{p_i \mid p_i$ is arbitrarily chosen from T_i, $1 \le i \le N\}$.

Observe that

① $p_k \in T_i$ whenever $k = i$, and
② $p_k \in S_i$ whenever $k \ne i$.

∵ **card**(P) = N ≥ d+2

∴ By Radon's Theorem,

P can be partitioned into P^+ and P^- such that
① $P^+ \ne \emptyset \ne P^-$,
② $P^+ \cup P^- = P$,
③ $P^+ \cap P^- = \emptyset$, and
④ $\exists$ point $q \in \mathbf{conv}(P^+) \cap \mathbf{conv}(P^-) \ne \emptyset$.

Assertion: $q \in \bigcap_{i=1}^{N} S_i$

To see this, it suffices to prove that

$\forall\ 1 \le i \le N,\ q \in S_i$

$\forall\ 1 \le i \le N$
∵ $P^+ \cap P^- = \emptyset$
∴ p_i **<u>cannot</u>** belong to both P^+ and P^-.
So we need to consider the following two cases:

❶ If $p_i \notin P^+$, then $\forall\ p_k \in P^+,\ p_k \ne p_i \Rightarrow k \ne i \Rightarrow p_k \in S_i$
∴ $P^+ \subseteq S_i$
∵ S_i is convex
∴ $q \in \mathbf{conv}(P^+) \subseteq S_i$.

❷ If $p_i \notin P^-$, then $\forall\ p_k \in P^-,\ p_k \ne p_i \Rightarrow k \ne i \Rightarrow p_k \in S_i$
∴ $P^- \subseteq S_i$

∵ S_i is convex

∴ $q \in \mathbf{conv}(P^-) \subseteq S_i$.

- In both cases, we can get $q \in S_i$.
- [QED]

4.4 Illustration

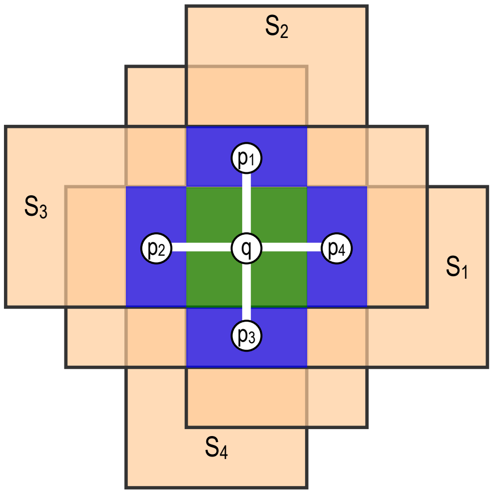

Fig 10-10 Helly's Theorem: four rectangles in the plane have a common intersection iff every three of them do.

4.5 Can the Theorem Hypothesis Be Weakened?

Helly's theorem does not hold for families with at least one **non-convex** set.

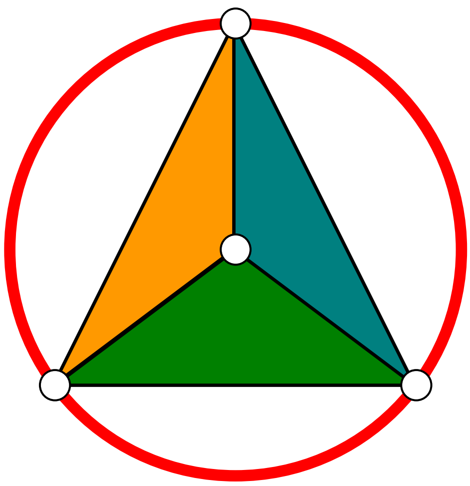

Fig 10-11 A family of four sets in the plane doesn't have Helly's property since one of them is not convex.

The finite version of Helly's theorem does not hold for families of **infinite non-compact convex** sets.

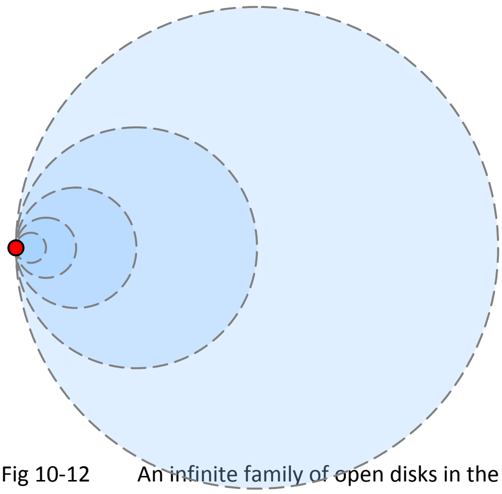

Fig 10-12 An infinite family of open disks in the plane is not necessarily to have Helly’s property.

- Construct an infinite family of closed (but unbounded) convex sets in the plane, where any 3 convex sets have a common intersection but all the sets don't. (Hint: an infinite family of "parallel" halfplanes)

4.6 Helly's Theorem (Infinite Version)

[Helly, 1923][11]
A family of infinite COMPACT convex sets admits a nonempty common intersection **iff** each of its (d+1)-cardinality subfamilies does.

- To get a proof, you can use the basic property of compactness to prove that

For any infinite family of COMPACT sets, if each of its finite subfamily has a non-empty common intersection, then the entire family has a non-empty common intersection.

5 Caratheodory = Radon = Helly

🖂 [Aleksandrov & Hopf, 1935][1]
[Danzer, Grunbaum & Klee, 1963][9]
➊ Caratheodory's theorem can be deduced from Radon's theorem;
➋ Caratheodory's theorem implies Radon's theorem.

🖂 [Eggleston, 1958][10]
➊ Caratheodory's theorem can be deduced from Helly's theorem;
➋ Caratheodory's theorem implies Helly's theorem.

6 Center Points

6.1 Center

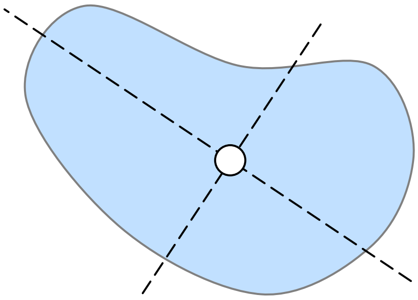

Fig 10-13 The center of a planar set is likely to lie in its "heart".

6.2 Center Points

Let P be a configuration of n points in $\mathcal{E}^d$.
A point x is called a **center point** of P if

for any halfspace K, $x \notin K$ **only if** **card**$(P \cap K) \le \frac{dn}{d+1}$.

Let x be a center point of P.

Then for any halfspace K, $x \notin K$ **only if** **card**$(P \cap K) \le \lfloor \frac{dn}{d+1} \rfloor$.

Given a configuration P of n points in $\mathcal{E}^d$, a point x is a center point of P if

neither of the two halfspaces defined by a hyperplane passing through x contains

$> \frac{dn}{d+1}$ (or, $> \lfloor \frac{dn}{d+1} \rfloor$)

points of P.

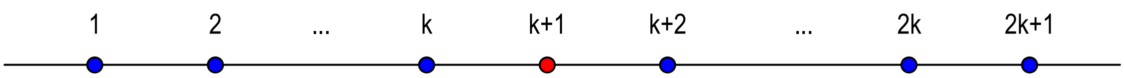

Fig 10-14 For a configuration of an odd number of points on a line ($\mathcal{E}^1$), the center point set consists of a singleton.

Fig 10-15 For a configuration of an eve[illegible] line ($\mathcal{E}^1$), the center point set is a closed line segment.

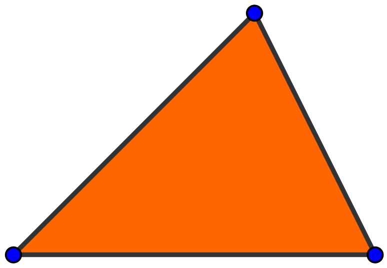

Fig 10-16 For 3 points in general position in the plane, the center point set is the close triangle of the three points ($\lfloor \frac{2*3}{2+1} \rfloor = 2$).

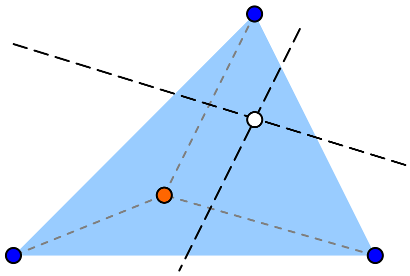

Fig 10-17 For 4 points in the plane whose convex hull is a triangle, the center point set is a singleton of the point inside the hull ($\lfloor \frac{2*4}{2+1} \rfloor = 2$).

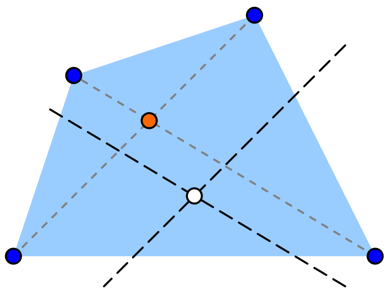

Fig 10-18 For 4 points in the plane whose convex hull is a quadrilateral, the center point set is a singleton of the intersection point of the two diagonals ($\lfloor \frac{2*4}{2+1} \rfloor = 2$).

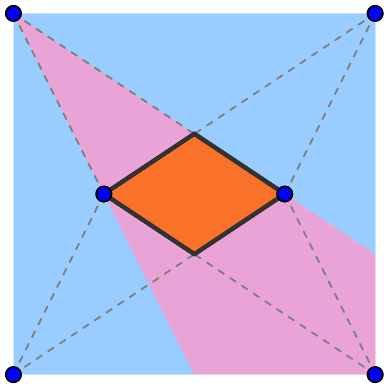

Fig 10-19 The center point set for a planar configuration of 6 points whose convex hull is a quadrilateral is also a convex quadrilateral ($\lfloor\frac{2*6}{2+1}\rfloor = 4$).

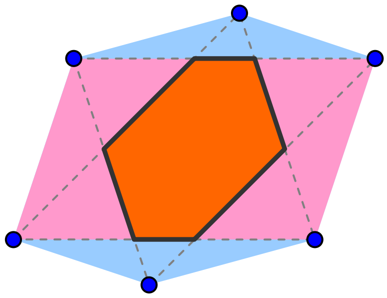

Fig 10-20 The center point set for a planar configuration of 6 points whose convex hull is a hexagon is also a hexagon ($\lfloor\frac{2*6}{2+1}\rfloor = 4$).

6.3 Conjectures

Note first that

A center point of a configuration **does not necessarily** come from this configuration.

Should a center point of a configuration necessarily lie inside the convex hull of the configuration?

Should the set of center points for a configuration necessarily be
❶ bounded?
❷ closed? and
❸ convex?

And of most importance,

Does each configuration permit at least one center point?

6.4 Why $\frac{dn}{d+1}$?

⌛ To define a center point, why not use a smaller number, say, $\frac{dn}{d+1}-1$?

■ Planar Configurations: ≥ 2n/3

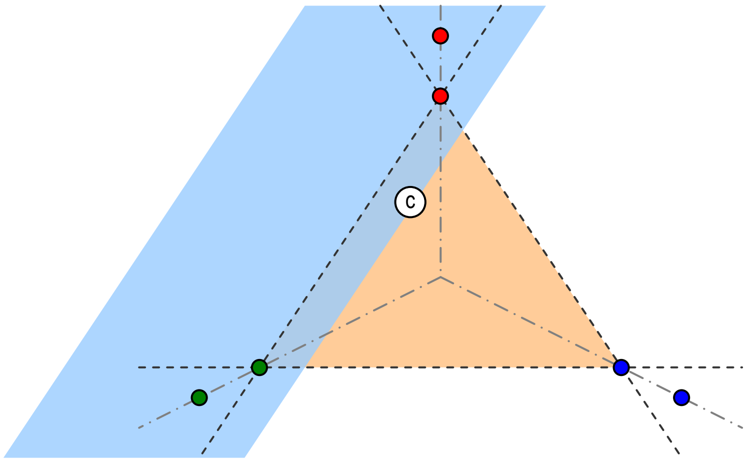

Fig 10-21 A planar configuration P of 6 points: for any point c, there is a halfspace with c on its boundary and containing 4 points of P.

■ $\geq \frac{dn}{d+1}$

For any positive integers d and n, $n \geq d+1$,

there is a point set $S \in \mathcal{E}^d$, **card**(S) = n, such that

for any point $x \in \mathcal{E}^d$, there is an (open) halfspace K, $x \notin K$, and **card**$(S \cap K) \geq \lfloor \frac{dn}{d+1} \rfloor$.

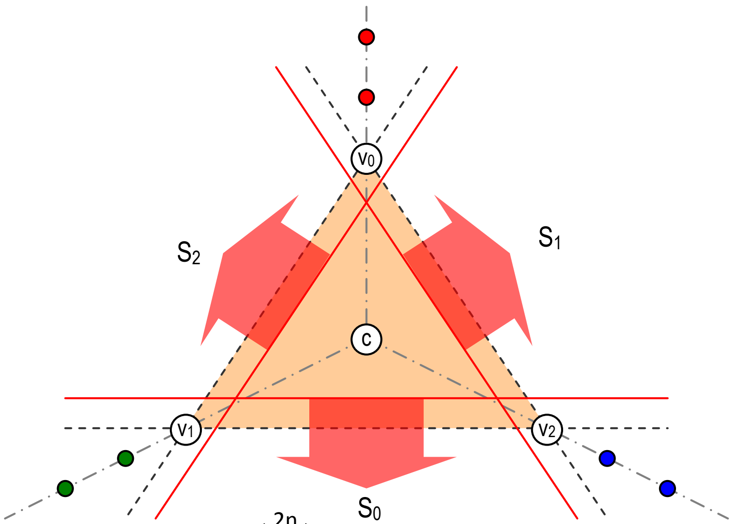

Fig 10-22 Necessity of $\lfloor \frac{2n}{3} \rfloor$ in the plane

[Proof]

[Simplex & rays]

⓿ Consider the d-dimensional Euclidean space $\mathcal{E}^d$.

❶ Choose a set of (d+1) affinely independent points, $P = \{v_0, v_1, \ldots, v_d\} \subset \mathcal{E}^d$.

(i.e., **conv**(P) is a simplex in $\mathcal{E}^d$)

❷ Choose arbitrarily a point $c \in$ **int**(**conv**(P)).

❸ Emanate (d+1) rays $\{r_0, r_1, \ldots, r_d\}$ from each point of P respectively, such that

the ray r_i is directed such that $r_i \subset$ line cv_i, but $c \notin r_i$.

[Placement of points]

For any positive integer $n \geq d+1$, let $j = n - (d+1) \times \lfloor \frac{n}{d+1} \rfloor$.

❶ Place arbitrarily $\lfloor \frac{n}{d+1} \rfloor$ distinct points on each ray r_i, for $j \leq i \leq d$.

❷ If $j > 0$, place arbitrarily $\lceil \frac{n}{d+1} \rceil$ points on each ray r_i, for $0 \leq i \leq j$.

Note that there are altogether $\lceil \frac{n}{d+1} \rceil \times j + \lfloor \frac{n}{d+1} \rfloor \times (d-j+1) = n$ points.

Consider the (d+1) hyperplanes $\{h_0, \ldots, h_d \mid h_i$ is defined by $P \setminus \{v_i\}, 0 \leq i \leq d\}$.

For each h_i,

❶ choose another hyperplane g_i parallel to h_i and lying between h_i and c; and

❷ let S_i be the open halfspace that takes g_i as its boundary and doesn't contain c.

Observe that

$$\bigcup_{i=0}^{d} \mathbf{compl}(S_i) = \mathcal{E}^d$$

I.e., the set of closed halfspaces $\{\mathbf{compl}(S_i) \mid 0 \leq i \leq d\}$ covers the whole space.

Therefore,

$\forall$ point $x \in \mathcal{E}^d$, $\exists\, 0 \le i \le d$, such that $x \in \mathrm{compl}(S_i)$ (i.e., $x \notin S_i$).

- Now, let's count the number of points of P contained in S_i.

  $\because$ $(\lfloor \frac{n}{d+1} \rfloor \le)$ $\mathbf{card}(P \cap \mathbf{compl}(S_i)) \le \lceil \frac{n}{d+1} \rceil$.

  $\therefore$ $\mathbf{card}(P \cap S_i) = \mathbf{card}(P) - \mathbf{card}(P \cap \mathbf{compl}(S_i)) \ge n - \lceil \frac{n}{d+1} \rceil = \lfloor \frac{nd}{d+1} \rfloor$.

- [QED]

Let d and n be two positive integers, $n \ge d+1$.

Then for any integer $D < \lfloor \frac{dn}{d+1} \rfloor$,

there is a point set $S \in \mathcal{E}^d$ of n points,

for which there doesn't exist a point $x \in \mathcal{E}^d$ such that

for any open halfspace K, $x \in \partial K$ only if $\mathbf{card}(S \cap K) \le D$.

- Now we understand that

If an integer D less than $\lfloor \frac{dn}{d+1} \rfloor$ is used in the definition of the center point, the **existence** of center point for all configurations will not be guaranteed.

6.5 Existence of Center Points

[Rado's Theorem, 1947][18]
Every finite set of points in d-dimensional Euclidean space admits one center point.

- In a dual setting, let L be the set of lines dual to the points in P, and let K_1, K_2 be the convex hulls of the $\lfloor n/3 \rfloor$- and $\lfloor 2n/3 \rfloor$- levels of the arrangement $\mathcal{A}(L)$, respectively.
- The dual of a center point of P is a line separating K_1 from K_2.
- This implies that the set of center points is a convex polygon with at most 2n edges.

7 Proof of Rado's Theorem - Using Helly's Theorem (Infinite Version)

7.1 Alternative Definition of Center Points

A point x is a center point of P **iff**

x **lies in** each (open) halfspace H which contains more than $\frac{dn}{d+1}$ points of P.

- In other words, **card**(P∩H) > $\frac{dn}{d+1}$ only if x∈H

7.2 Non-empty Intersection

- We claim that

The intersection of all halfspace H containing more than $\frac{dn}{d+1}$ points of P is non-empty.

- To prove it, we'd like to apply Helly's theorem.
- But we cannot proceed directly, since we have **infinite many** halfspaces and, even worse, they are **open and unbounded**.

7.3 Closeness

- Note first that P is a finite set.

For each ***open*** halfspace H containing more than $\frac{dn}{d+1}$ points of P,

let H' be a ***closed*** halfspace H' ⊂ H such that H ∩ P = H' ∩ P.

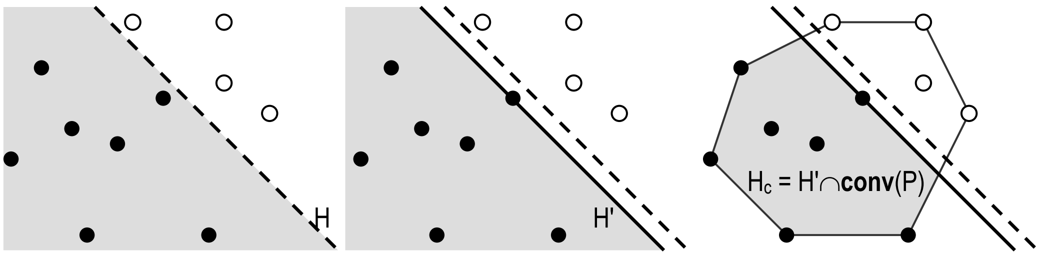

Fig 10-23 Constructing an infinite number of compact sets.

7.4 Boundedness

For each open halfspace H containing more than $\frac{dn}{d+1}$ points of P, let

$H_c = H' \cap \textbf{conv}(P)$,

where H' is defined as above.

Since H' is convex and **conv**(P) is both convex and compact, we have

Each H_c is convex and compact.

7.5 Using Helly's Theorem

For any configuration P in $\mathcal{E}^d$, let

$\mathcal{F} = \{H_c \mid H \text{ contains more than } \frac{dn}{d+1} \text{ points of P}\}$

Now we observe that

❶ $\mathcal{F}$ is an infinite family of convex and compact sets, and

❷ for each $H_c \in \mathcal{F}$, $H_c \cap P = H' \cap P = H \cap P$, i.e., $\textbf{card}(H_c \cap P) > \frac{dn}{d+1}$.

It is implied that

Any d+1 sets in $\mathcal{F}$ intersect in **at least one** point of P.

(Can you figure out the details?)

Now by the infinite version of Helly's Theorem, it follows that

All sets in $\mathcal{F}$ intersect in **at least one** point of P, which is a center point of P.

8 Proof of Rado's Theorem - Using Helly's Theorem (Finite Version)

- By an induction on d.

- Inductive base: $\mathcal{E}^1$, trivial.
- Inductive assumption: the assertion holds for $\mathcal{E}^k$, k < d.

- Consider P a finite set of points in $\mathcal{E}^d$.
- W.L.O.G., assume that P contains (d+1) affinely independent points.
  (Or, **dim**(**conv**(P)) < d and hence it could be reduced to a lower dimensional case.)
- Let M = $\{s_1, \ldots, s_m\}$ be the set of all maximal halfspaces.
  Observe that **card**(M) = m < +∞.
- ❶ If m ≤ d+1,

$$\mathbf{card}(P\cap\mathbf{compl}(\bigcup_{i=1}^{m} s_i)) = \mathbf{card}(P) - \mathbf{card}(P\cap(\bigcup_{i=1}^{m} s_i)) = \mathbf{card}(P) - \mathbf{card}(\bigcup_{i=1}^{m}(P\cap s_i))$$

$$\geq n - \sum_{i=1}^{m}\mathbf{card}(P\cap s_i)$$

$$\geq n - (d+1) \times (\lceil \frac{n}{d+1} \rceil - 1)$$

$$\geq n - (d+1) \times ((\frac{n-1}{d+1}+1)-1)$$

$$= 1$$

$$\therefore\ P\cap\mathbf{compl}(\bigcup_{i=1}^{m} s_i) \neq \varnothing \neq \mathbf{compl}(\bigcup_{i=1}^{m} s_i).$$

- ❷ If m > d+1
  ∀ (d+1) maximal halfspaces $s_{k_1}, \ldots, s_{k_{d+1}} \in M$,

$$\mathbf{card}(P\cap\mathbf{compl}(\bigcup_{i=1}^{d+1} s_{k_i})) = \mathbf{card}(P) - \mathbf{card}(P\cap\bigcup_{i=1}^{d+1} s_{k_i}) = \mathbf{card}(P) - \mathbf{card}(\bigcup_{i=1}^{d+1}(P\cap s_{k_i}))$$

$$\geq n - \sum_{i=1}^{d+1}\mathbf{card}(P\cap s_i)$$

$$\geq n - (d+1) \times (\lceil \frac{n}{d+1} \rceil - 1)$$

$$\geq n - (d+1) \times ((\frac{n-1}{d+1}+1)-1)$$

$$= 1$$

$$\therefore\ P\cap\mathbf{compl}(\bigcup_{i=1}^{d+1} s_{k_i}) \neq \varnothing \neq \mathbf{compl}(\bigcup_{i=1}^{d+1} s_{k_i}) = \bigcap_{i=1}^{d+1}\mathbf{compl}(s_{k_i}).$$

- Now consider all the sets in the family $\{$**compl**$(s_k) \mid 1 \leq k \leq m\}$.
  Observe that
  ① m < +∞,
  ② they are all convex (and closed), and
  ③ any (d+1) of them have a non-empty common intersection.

- By Helly's theorem, we know that $\bigcap_{i=1}^{m} \textbf{compl}(s_i) \neq \varnothing$.

  In other words, $\varnothing \neq \bigcap_{i=1}^{m} \textbf{compl}(s_i) = \textbf{compl}(\bigcup_{i=1}^{m} s_i)$.

- As a whole, $\varnothing \neq \textbf{compl}(\bigcup_{i=1}^{m} s_i)$ always holds.

  $\therefore$ $\exists$ a point $c \in \textbf{compl}(\bigcup_{i=1}^{m} s_i)$.

  $\therefore$ $c \notin \bigcup_{i=1}^{m} s_i$.

  In other words, c is not contained in any maximal space.

  $\therefore$ c is a center point of P.

- [QED]

9 Algorithms for Center Points

9.1 NP-Hardness

[Teng, 1991][21]
If d is not a constant, testing whether a point is a center point of a configuration in $\mathcal{E}^d$ is NP-complete.

9.2 Fixed-Dimensional Center Points

A center point of a configuration of n points in $\mathcal{E}^d$ can be computed by solving a set of $\Theta(n^d)$ linear inequalities, using linear programming.

9.3 Planar Center Points

[Cole, Sharir & Yap, 1987][8]
A center point of a configuration in $\mathcal{E}^2$ can be computed in $\mathcal{O}(n\log^5 n)$ time.

[Jadhav & Mukhopadhay, 1993][13]
A center point of a configuration in $\mathcal{E}^2$ can be computed in linear time.

9.4 Center Points in Space

[Cole, Sharir & Yap, 1987][8]
A center point of a configuration in $\mathcal{E}^3$ can be computed in $\mathcal{O}(n^2\log^7 n)$ time,

9.5 Approximate Center Points

- Clarkson et al., 1993[7]

10 Ham-Sandwich Cuts

10.1 Problem

Given a collection of objects, is there always a hyperplane bisecting them **simultaneously**?

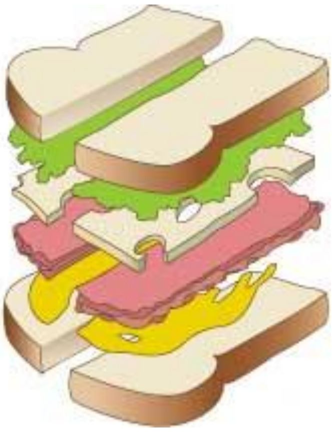

Fig 10-24 A ham-sandwich cut.

10.2 Multisets

A **multiset** is a collection of elements, each of which may have a finite multiplicity.

A multiset is also called a bag.

10.3 Bisectors

Let P be a multiset of n points in $\mathcal{E}^d$.

A hyperplane h is called a **bisector** of P, if neither of the two (open) halfspaces defined by h contains more than $\frac{n}{2}$ points of P.

In this case, we also say that h bisects P.

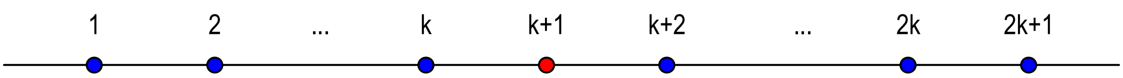

Fig 10-25 For a configuration of 2k+1 points on a line ($\mathcal{E}^1$), the bisector is the (k+1)-th point.

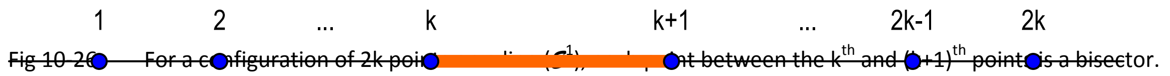

Fig 10-26 For a configuration of 2k poi[illegible] ($\mathcal{E}^1$), [illegible] nt between the k^{th} and (k+1)th point is a bisector.

■ Existence of Bisectors

Let P be a multiset of n points in $\mathcal{E}^d$, and let N be a normalized vector.
There exists a bisector of P taking N as its normal.

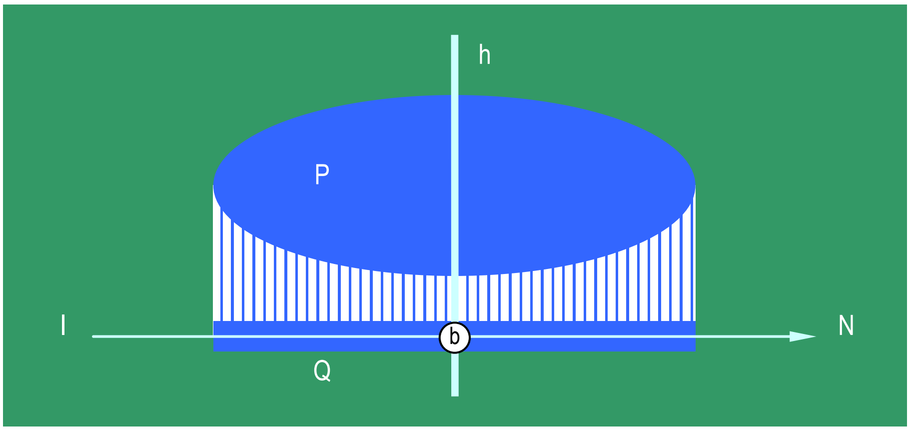

Fig 10-27 For a multiset P⊂$\mathcal{E}^d$, it has a bisector in any direction.

- [Proof]
- Let l be a line parallel to N.
- ∀ point q ∈ P, let l(q) be the orthogonal projection of q onto l.
  Define a multiset Q = l(P) = {l(p) | p∈P} ⊂ $\mathcal{E}^1$.
  Note that it is possible for more than one point of P to project onto a same point on l.
- Let point b ∈ l be a bisector of Q.
  Would such a bisector always exist?
  Define a hyperplane h ⊂ $\mathcal{E}^d$: {x∈$\mathcal{E}^d$ | $N^Tx = N^Tb$}.
  ① h takes N as its normal, and
  ② h passes through the point b.
- ∴ h is a bisector of P.
- [QED]
- This Lemma is different from the Mean-Value theorem since the function here is not continuous.

■ Complexity of Bisectors

Two bisectors are called **equivalent** if they define a same partition of P.

Every vertical line intersects the $\lfloor \frac{n}{2} \rfloor$-level and $\lceil \frac{n}{2} \rceil$-level of an arrangement of n non-vertical hyperplanes.

The number of non-equivalent bisectors of n points in $\mathcal{E}^d$ is $O(e^{(d)}_{\lfloor \frac{n}{2} \rfloor}(n))$.

10.4 Mass Distribution

ε-Ball

For any $p \in \mathcal{E}^d$, the set $b_\varepsilon(p) = \{x \mid d(p, x) < \varepsilon\}$ is called the **ε-ball** of point p.

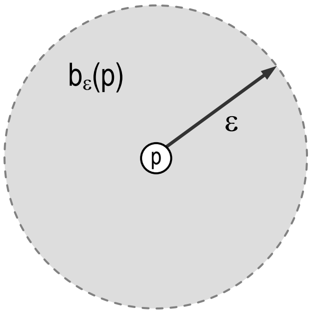

Fig 10-28 The ε-ball of a point p takes p as the center and ε as the radius. Note that each ε-ball is an open set.

Approximating Mass-Distribution

Let P be a finite multiset of points in $\mathcal{E}^d$.

Given any two positive reals ε and δ, the **approximating mass-distribution** $\mu_{\varepsilon,\delta}$ of P is defined as follows:

For any $x = (x_1, \ldots, x_d) \in \mathcal{E}^d$, $\mu_{\varepsilon,\delta}$ assigns to x the mass

$$\mu_{\varepsilon,\delta}(x) = i(x) + \delta e^{-(|x1|+\ldots+|xd|)},$$

where $i(x) = \textbf{card}\{p \in P \mid x \in b(p)\}$, i.e., the number of ε-balls containing x.

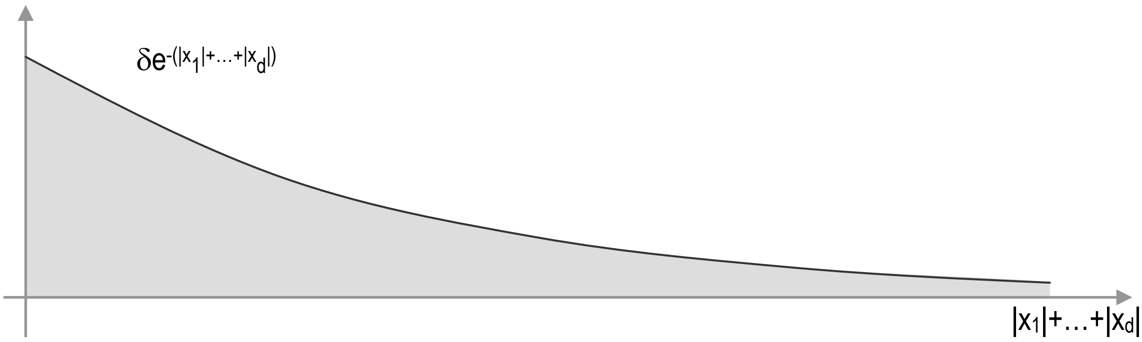

Fig 10-29 Approximating mass-distribution of a point set.

10.5 Measure of $\mu_{\varepsilon,\delta}$

Let P be a finite multiset of points in $\mathcal{E}^d$ with the approximating mass-distribution $\mu_{\varepsilon,\delta}$.

Then the measure of $\mu_{\varepsilon,\delta}$ on $\mathcal{E}^d$ is $M(\varepsilon, \delta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mu_{\varepsilon,\delta}(x)dx_1 \ldots dx_d$.

$M(\varepsilon, \delta) = \delta \cdot 2^d + n \cdot \varepsilon^d \cdot \mu_0$, where μ_0 is the measure of a unit ball.

[Proof]

Each ball $b_\varepsilon(p)$ contributes $\varepsilon^d \cdot \mu_0$ to $M(\varepsilon, \delta)$.

$\because\ 1 = \int_0^{\infty} \cdots \int_0^{\infty} e^{-(|x1|+\ldots+|xd|)} dx_1 \ldots dx_d$.

$\therefore\ M(\varepsilon,\delta) = \delta \cdot 2^d \cdot 1 + n \cdot \varepsilon^d \cdot \mu_0$.

[QED]

For a subset $S \subseteq \mathcal{E}^d$, the measure of $\mu_{\varepsilon,\delta}$ **restricted to S** is $M_S(\varepsilon, \delta) = \int_{x \in S} \mu_{\varepsilon,\delta}(x)dx$.

10.6 Bisector of $\mu_{\varepsilon,\delta}$

A hyperplane $g \subset \mathcal{E}^d$ is called a **bisector** of $\mu_{\varepsilon,\delta}$ if
the measure restricted to one side of g amounts to exactly one half of the total measure.

I.e., $M_{g+}(\varepsilon, \delta) = M_{g-}(\varepsilon, \delta) = \frac{1}{2} \cdot M(\varepsilon, \delta)$.

It can be proved that

If both ε and δ are sufficiently small, then
❶ $\mu_{\varepsilon,\delta}$ approaches P, and
❷ a bisector of $\mu_{\varepsilon,\delta}$ approaches a bisector of P.

10.7 Sufficiently Small

Given a family of sets in $\mathcal{E}^d$, a hyperplane is call a **transversal** hyperplane of the family if it intersects with all sets of the family.

Parameters ε and δ are called **sufficiently small** for a point set P if
❶ ε is small enough such that
any subfamily of $\{b(p) \mid p \in P\}$ has a transversal hyperplane only if
the centers of these balls lie in a same hyperplane;
❷ δ is small enough such that
$0 < \delta \times 2^d < \varepsilon^d \cdot \mu_0$.

❶ I.e., $\forall$ (d+1) points $p_{k1}, \ldots, p_{k(d+1)}$,
$\exists$ a hyperplane g, $g \cap b(p_{ki}) \neq \emptyset$, for $1 \leq i \leq d+1 \Leftrightarrow$ $\exists$ another hyperplane h, $\{p_{k1}, \ldots, p_{k(d+1)}\} \subset h$.

❷ I.e., the whole mass of exp(-|x|) contributes less than the mass of a single ball to the measure M(ε, δ).

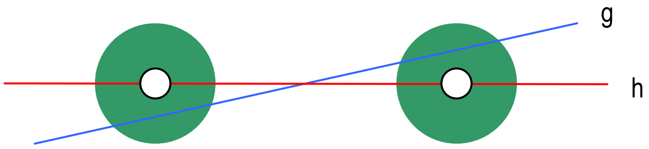

Fig 10-30 The centers of ≤d balls in $\mathcal{E}^d$ always share a common hyperplane.

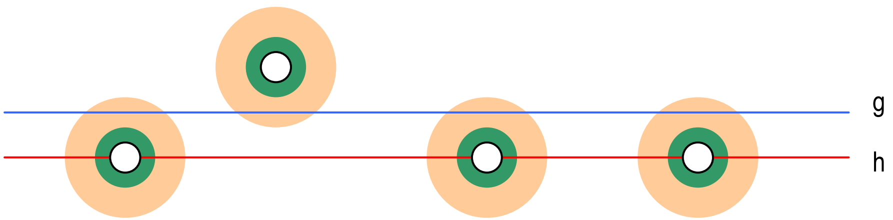

Fig 10-31 When ε and δ are sufficiently small, any (d+1) balls in $\mathcal{E}^d$ have a common transversal hyperplane if and only if the dimension of the affine hull of their centers is no more than (d-1).

Assume that both ε and δ are sufficiently small.
❶ It is still possible that the centers of all the n balls lie on a common hyperplane, and
❷ it is still possible that two balls intersect with each other. However,
❸ the number of intersecting balls could not be greater than d.
❹ For any n(≥d) balls intersecting a common hyperplane,
the hyperplane their centers belong to **must be unique**.
❺ For any n(<d) balls intersecting a common hyperplane,
the hyperplane their centers belong to is **not necessarily unique**.

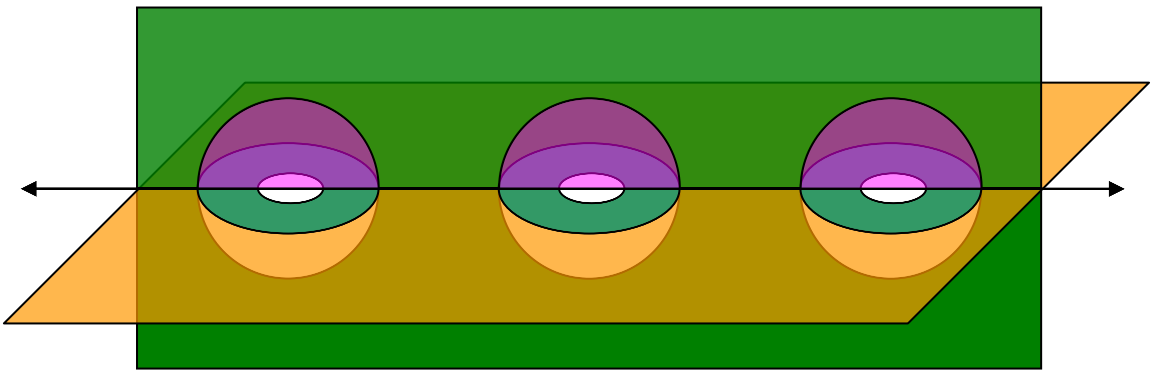

Fig 10-32 There are possibly multiple hyperplanes shared by n points.

❻ ε and δ are also sufficiently small for any subset of P.

10.8 Bisectors of P

Let P be a multiset of n points in $\mathcal{E}^d$ and let $\mu_{\varepsilon,\delta}$ be an approximating mass-distribution with ε and δ sufficiently small.

Then for any hyperplane g with open halfspaces g^+ and g^-,
there is a hyperplane h with open halfspaces h^+ and h^-, such that

❶ a point $p \in P$ belongs to h	if	the ball $b_\varepsilon(p)$ intersects g,
❷ a point $p \in P$ belongs to h^+	only if	the ball $b_\varepsilon(p)$ lies in g^+,
❸ a point $p \in P$ belongs to h^-	only if	the ball $b_\varepsilon(p)$ lies in g^-, and
❹ h bisects P	if	g bisects $\mu_{\varepsilon,\delta}$.

[Proof]

Consider an arbitrary hyperplane g.

❶ If $g \cap \bigcup_{p \in P} b_\varepsilon(p) = \varnothing$, then select h = g.

❷ Or, let $B_g = \{p \in P \mid b_\varepsilon(p) \cap g \neq \varnothing\} = \{p_1, \ldots, p_m\}$, $1 \le m \le n$. (Note that m is **<u>not necessarily</u>** less than d.)

$\because$ ε is sufficiently small.

$\therefore$ $\exists$ a hyperplane h, $B_g = \{p_1, \ldots, p_m\} \subset h$.

If there exists more than one such hyperplane, let h be the one lying closest to g.

(When will this happen? Is such a hyperplane well defined?)

Now consider an arbitrary point $p \in P$.

❶ If $b_\varepsilon(p) \cap g \neq \varnothing$, then $p \in B_g$, $p \in h$.

❷ If $p \in h^+$, we can show that $b_\varepsilon(p) \subset g^+$.

Otherwise,

① assume $b(p) \cap g \neq \varnothing$.

$\therefore$ $p \in B_g$. And by ❶, $p \in h$.

A contradiction.

② Assume $b_\varepsilon(p) \subset g^-$.

$\therefore$ $p \in b_\varepsilon(p) \subset g^-$.

$\therefore$ $p \in h^+ \cap g^-$.

$\therefore$ $\exists$ hyperplane g', $\forall$ $q \in \{p\} \cup B_g$, $b(q) \cap g' \neq \varnothing$. (Refer to the figure.)

$\because$ ε is sufficiently small.

$\therefore$ $\exists$ hyperplane h', $\{p\} \cup B \subset h'$ and more important, h' lies closer to g than h.

A contradiction with the minimum distance between g and h.

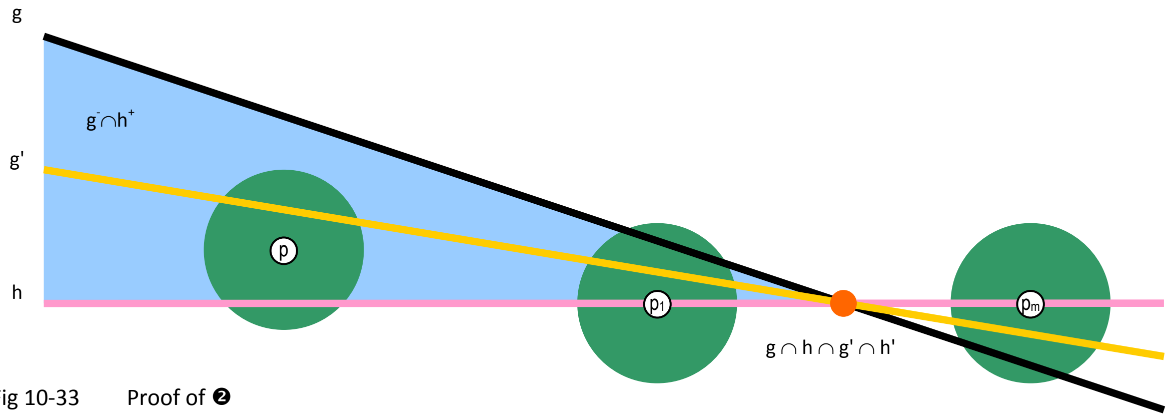

Fig 10-33 Proof of ❷

❸ If $b(p) \subset h^-$, we can show that $b_\varepsilon(p) \subset g^-$.
(Similar to ❷.)

❹ Suppose that g bisects $\mu_{\varepsilon,\delta}$.
Consider the semispace $P_h^+ = P \cap h^+$ and the pseudo-semispace $P_g^+ = \{p \in P \mid b(p) \subset g^+\}$.
by ❷, $\forall\ p \in P_h^+$, $b(p) \subset g^+$.
$\therefore\ \forall\ p \in P_h^+$, $p \in P_g^+$.
$\therefore\ P_h^+ \subseteq P_g^+$.
$\therefore$ **card**$(P_h^+) \le$ **card**(P_g^+)
$\because$ g bisects $\mu_{\varepsilon,\delta}$.
$\therefore\ \frac{1}{2}(n \times \varepsilon^d \times \mu_0 + \delta \times 2^d) = \int_{x \in g^+} \mu_{\varepsilon,\delta}(x)dx$.

Observe that
① $\int_{x \in g^+} \mu_{\varepsilon,\delta}(x)dx$ > **card**$(P_g^+) \times \varepsilon^d \times \mu_0$ ≥ **card**$(P_h^+) \times \varepsilon^d \times \mu_0$; On the other hand,
② $\frac{1}{2}(n \times \varepsilon^d \times \mu_0 + \delta \times 2^d)$ < $\frac{n+1}{2} \times \varepsilon^d \times \mu_0$.
$\therefore$ **card**$(P_h^+) < \frac{n+1}{2}$.
$\therefore$ **card**$(P_h^+) \le \frac{n}{2}$.
By the same reason, **card**$(P_h^-) \le \frac{n}{2}$.
As a conclusion, h bisects P.

[QED]

It is possible that there exists >1 such hyperplane h for a given hyperplane g.

10.9 Borsuk-Ulam Theorem

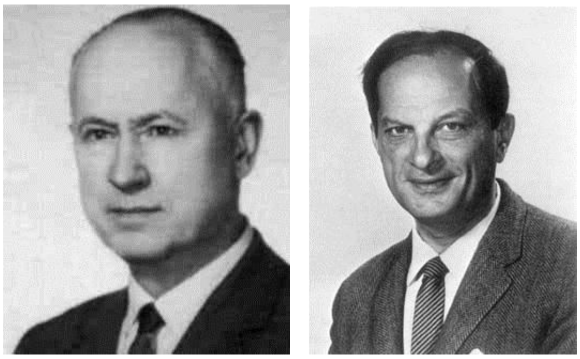

Fig 10-34 Karol Borsuk (1905/05/08 - 1982/01/24) and Stanislaw Marcin Ulam (1909/04/03 - 1984/05/13)

[Borsuk & Ulam, 1933][4]

❶ Let f be a continuous anti-symmetric function from $\mathcal{S}^{d-1}$ to $\mathcal{E}^k$, k < d.
Then f(x) = 0 for some $x \in \mathcal{S}^{d-1}$.

❷ Let f be a continuous function from $\mathcal{S}^{d-1}$ to $\mathcal{E}^{d-1}$.
Then f(x) = f(-x) for some antipodal pair x and -x on $\mathcal{S}^{d-1}$.

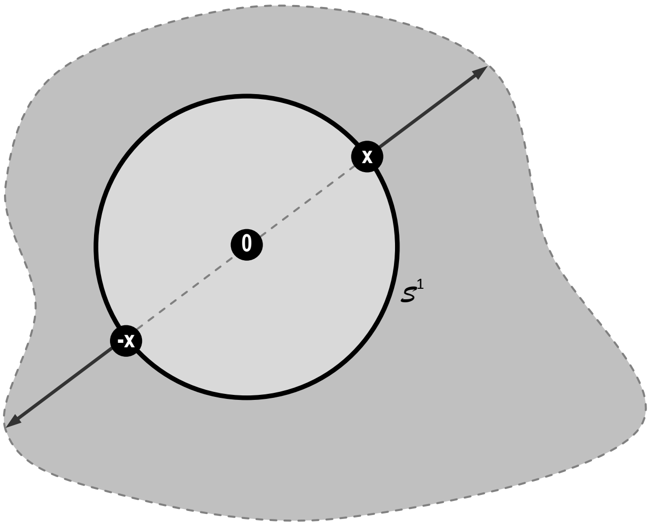

Fig 10-35 Borsuk-Ulam Theorem

10.10 Ham-Sandwich Theorem

[Ham-sandwich Theorem (discrete version)]
Let $P_1, \ldots, P_d$ be d finite sets of points in $\mathcal{E}^d$.
There exists a hyperplane h that simultaneously bisects $P_1, \ldots, P_d$.

- Note that $P_1, \ldots, P_d$ are not necessarily disjoint from each other.
- [Proof]

- Choose parameters ε and δ sufficiently small to approximate the multiset $P = \bigcup_{i=1}^{d} P_i$.
- Let μ_i be the mass-distribution that approximates P_i with parameter ε and δ, $1 \le i \le d$.
- We will construct a hyperplane g which bisects $\{\mu_1, \ldots, \mu_d\}$ simultaneously.
- Define a function g: $S^{d-1} \to$ {hyperplanes in $\mathcal{E}^d$} as following:

> $\forall$ point $y \in S^{d-1}$, let $g(y)$ = the hyperplane which
> ❶ takes y as its normal, and
> ❷ bisects μ_d

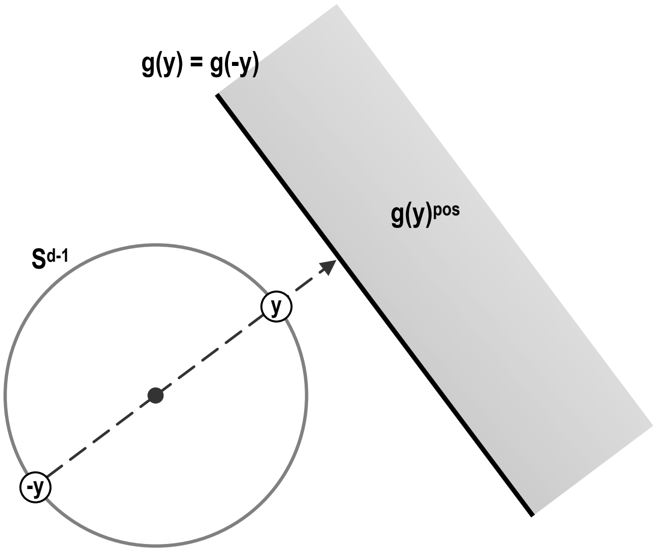

Fig 10-36 Existence of ham-sandwich cuts.

- Observe that

> ❶ g(y) is unique since $\mu_d > 0$ everywhere, and
> ❷ $g(y) = g(-y)$ for any y.

> Let $g(y)^{pos}$ be the open halfspace $\{z \in \mathcal{E}^d \mid y^T z > y^T z_0\}$, where z_0 is an arbitrary point on g(y).

- Observe that

> $\forall\, y \in S^{d-1}$
> ❶ $g(y)^{pos} \cap g(-y)^{pos} = \varnothing$, and
> ❷ $g(y)^{pos} \cup g(-y)^{pos} \cup g(y) = \mathcal{E}^d$.

- $\therefore\ \forall\, 1 \le i \le d$, $\displaystyle\int_{z \in g(y)^{pos}} u_i(z)dz + \int_{z \in g(-y)^{pos}} u_i(z)dz = \int_{z \in \mathcal{E}^d} u_i(z)dz$ = measure of μ_i.
- Define a function f: $S^{d-1} \to \mathcal{E}^{d-1}$ as following:

> $\forall$ point $y \in S^{d-1}$, let
>
> $$f(y) = \left< \int_{z \in g(y)^{pos}} u_1(z)dz,\ \int_{z \in g(y)^{pos}} u_2(z)dz,\ \ldots,\ \int_{z \in g(y)^{pos}} u_{d-1}(z)dz \right>.$$

- Observe that f is continuous.
- Now by Borsuk-Ulam theorem, $\exists\, x \in S^{d-1}$ such that f(x) = f(-x).

  I.e., $$\int_{z \in g(x)^{pos}} u_i(z)dz = \int_{z \in g(-x)^{pos}} u_i(z)dz = \frac{1}{2} \times \int_{z \in \mathcal{E}^d} u_i(z)dz = \frac{\text{measure of } \mu_i}{2}, \quad \forall\, 1 \le i \le d\text{-}1.$$
- ∴ The hyperplane g(x) bisects μ_i, $1 \le i \le d-1$.
- ∴ The hyperplane g(x) bisects μ_i, $1 \le i \le d$.
- This completes the proof.
- [QED]

10.11 Ham-Sandwich Algorithms

Planar Ham-sandwich

[Lo, Matousek & Steiger, 1994][16]
The Ham-Sandwich cut in the plane can be found in linear time.

11 Planar Equitable Cutting

11.1 Equitable g-Cutting

Let $m \geq 2$, $n \geq 2$ and g be positive integers.
Let R and B be two disjoint sets of points in the PLANE such that points of $R \cup B$ are in general position, **card**(R) = gn and **card**(B) = gm.
A partition of $R \cup B$ into g subsets $P_1, \ldots, P_g$ is called **an equitable g-cutting** if
① P_i and P_j are linearly separable for all $1 \leq i < j \leq g$, and
② **card**($P_i \cap R$) = n and **card**($P_i \cap B$) = m, for all $1 \leq i \leq g$?

11.2 Conjecture

[Kaneko & Kano, 1998][15]
Given two planar sets R and B of gn and gm points respectively, is there always an equitable g-cutting of $R \cup B$?

11.3 g = 2

When g = 2, this conjecture is equivalent to the planar Ham-Sandwich Theorem.

Kaneko & Kano's conjecture is true when g = 2.

11.4 n = 2

Kaneko and Kano managed to proved the case when n = 2.

[Kaneko & Kano, 1998][15]
Kaneko & Kano's conjecture is true when n=2.

11.5 Theorem

The complete conjecture was proven independently by Uehara et al. and Sakai:

✉ [Uehara et al., 1998][23] & [Sakai, 1998][20]

Kaneko and Kano's conjecture is true.

12 Planar Equitable Subdivision

12.1 Equitable Subdivision

Given gn red points and gm blue points in the plane in general position, a partition of the plane into g disjoint convex polygons is called an equitable subdivision if each convex polygon contains n red points and m blue points.

12.2 Conjecture

[Bespamyatnikh et al., 1999][3]
Given gn red points and gm blue points in the plane in general position, does there always exist an equitable subdivision of the plane?

12.3 $g = 2^k$

For $g=2^k$, $k>0$, an equitable subdivision can be found by applying the Ham Sandwich Theorem in a divide-conquer fashion.

12.4 Theorem

[Bespamyatnikh et al., 1999][3]
Given gn red points and gm blue points in the plane in general position, there exists **an equitable subdivision** of the plane into g disjoint convex polygons, each of which contains n red points and m blue points.

13 Planar Erasing Subdivision

13.1 Quarter-Cutting

For any set P of n points in $\mathcal{E}^2$, there are two lines that cut P into four wedges, each of which contains at most $\frac{n}{4}$ points of P.

- Choose the first line bisecting P into P_1 and P_2.
  The first line is not necessarily unique.
- By the Ham-sandwich Theorem, there exists a second line bisecting P_1 and P_2 simultaneously.

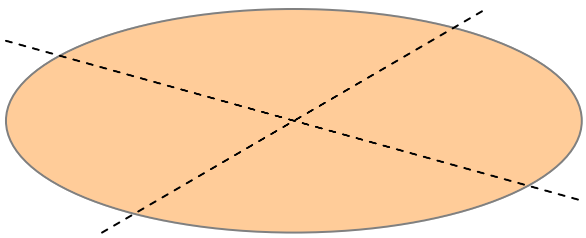

Fig 10-37 A quarter-cutting in the plane

13.2 Erasing Cell Complex

Given P a set of n points in $\mathcal{E}^d$, a cell complex C with convex cells is said to erase P if all points of P are contained in facets and lower-dimensional faces of C, and we call C an erasing cell complex

13.3 Erasing Subdivision

- An erasing cell complex in $\mathcal{E}^2$ is called an **erasing subdivision**.
- The maximal number of cells of C that intersect any one hyperplane, is called the stabbing number of C, denoted as s(C).

- Erasing cell complexes that having a small stabbing number are relevant in the design of data structure, e.g. the simplicial range queries

- The stabbing number of P is defined as
  $s(P) = \min \{s(\mathcal{D}) \mid \mathcal{D} \text{ is an erasing cell complex of } P\}$
- $s^{(d)}(n) = \max \{s(Q) \mid Q \text{ is a set of n points in } \mathcal{E}^d\}$

Every line intersects at most three of the open regions defined by any two lines in the plane.

13.4 Constructing Erasing Subdivisions

- Input:
  P a set of n points in $\mathcal{E}^2$
- Initialization:
  Find l_0 a bisector of P. (l_0 cuts $\mathcal{E}^2$ into regions $L(l_0)$ and $R(l_0)$)
  Initial C = { $l_0[L(l_0),R(l_0)]$ }
  Let ActiveEdge = { $l_0[L(l_0),R(l_0)]$ }
- Iteration:

```
While ActiveEdge ≠ ∅ do
    Choose e(L(e),R(e)) ∈ ActiveEdge
    ActiveEdge = ActiveEdge - { e(L(e), R(e)) }
    Unless P∩L(e) = P∩R(e) = ∅ do
        Find a line l that simultaneously bisects P∩L(e) and P∩R(e)
        Let edges f = l∩L(e) and g = l∩R(e)
        C = C + { f[L(f),R(f)], g[L(g),R(g)] }
        ActiveEdge = ActiveEdge + { f[L(f),R(f)], g[L(g),R(g)] }
    End Unless
End While
```

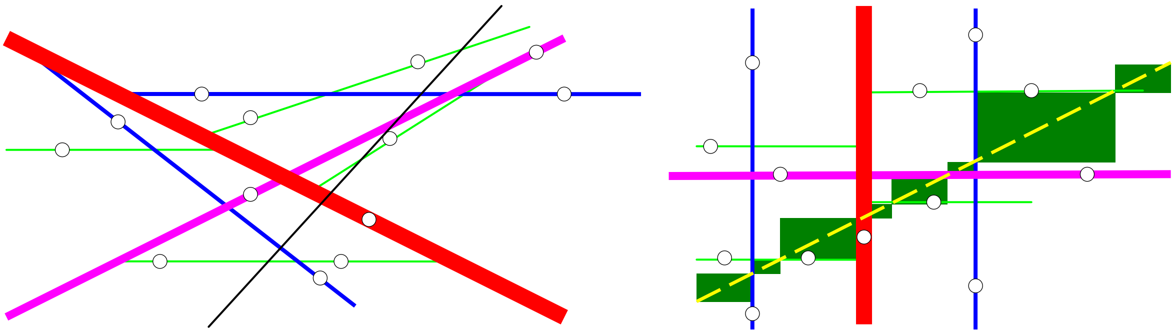

Fig 10-38 Erasing subdivision C for 13 points, with s(C)=8

13.5 Upper Bound

$s^{(2)}(n) = \mathcal{O}(n^{\alpha})$, where $\alpha = \log_2(\frac{1+\sqrt{5}}{2}) < 0.695$.

- [Proof]
- Observe that it can be the case when we need to bisect a pair of point sets where one of them is empty. This is simpler than cases when both sets are non-empty, since a bisection of the non-empty one will work.
- Notice that the algorithm above disactivates an edge immediately after finding an intersecting bisector.
  - This feature is necessary to obtain a reasonable stabbing number as will be shown in following.
- For a certain point set, this algorithm may produce different erasing subdivisions.

- Define $\bar{s}(n)=\max\{s(\mathcal{D}) \mid Q$ a set of n points in $\mathcal{E}^2$, $\mathcal{D}$ an erasing subdivision of Q that can be constructed by the algorithm above$\}$.
- $\therefore\ s^2(n) \le \bar{s}(n)$.

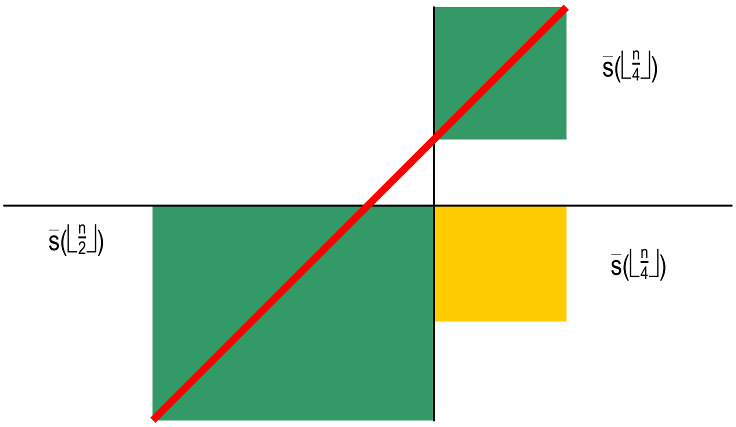

Fig 10-39 $\bar{s}(n) \le \bar{s}(\lfloor \frac{n}{2} \rfloor) + \bar{s}(\lfloor \frac{n}{4} \rfloor)$

- Observations:

① Any line intersects at most 3 of the 4 sectors defined by the first 2 lines used in any construction done by the algorithm above.
② The subdivision of each one of the sectors is an erasing subdivision of at most n/4 points, which is of the same kind as the entire subdivision.
③ The subdivisions of each half-plane defined by 1^{st} line is an erasing subdivision of at most n/2 points, which is of the same kind as the entire subdivision.

- $\therefore\ \bar{s}(n) \le \bar{s}(\lfloor \frac{n}{2} \rfloor) + \bar{s}(\lfloor \frac{n}{4} \rfloor)$
- $\because\ \bar{s}(0) \le 2$

$\therefore\ \bar{s}(n) \le a_{k+2}$, where a_k is the k^{th} Fibonacci numbers: $a_0=1$, $a_1=1$, and $a_k=a_{k-1}+a_{k-2}$ for $k\ge 2$

$\therefore\ a_k = \frac{1}{\sqrt{5}} \times [(\frac{1+\sqrt{5}}{2})^{k+1} - (\frac{1-\sqrt{5}}{2})^{k+1}]$

- $\because\ -1 = \frac{1-\sqrt{9}}{2} < \frac{1-\sqrt{5}}{2} < 0$

$\therefore\ |(\frac{1-\sqrt{5}}{2})^{k+1}| < 1$

$\therefore\ a_k < \frac{1}{\sqrt{5}} \times [(\frac{1+\sqrt{5}}{2})^{k+1} - (-1)] = \frac{1}{\sqrt{5}} \times [(\frac{1+\sqrt{5}}{2})^{k+1} + 1]]$

$$\begin{aligned}\therefore\ \bar{s}(n) &\le a_{\lfloor \log_2 n \rfloor + 2} \\ &\le \frac{1}{\sqrt{5}} \times [(\frac{1+\sqrt{5}}{2})^{\lfloor \log_2 n \rfloor + 3} + 1]] \\ &\le \frac{1}{\sqrt{5}} \times [(\frac{1+\sqrt{5}}{2})^{\log_2 n + 3} + 1]] \\ &= \mathcal{O}(n^{\log_2(\frac{1+\sqrt{5}}{2})})\end{aligned}$$

- [QED]

14 Planar Simultaneous Bisection

For h a line in the plane, define $h^+ = \{(x_1, x_2) \mid x_2>0\}$, and $h^- = \{(x_1, x_2) \mid x_2<0\}$.

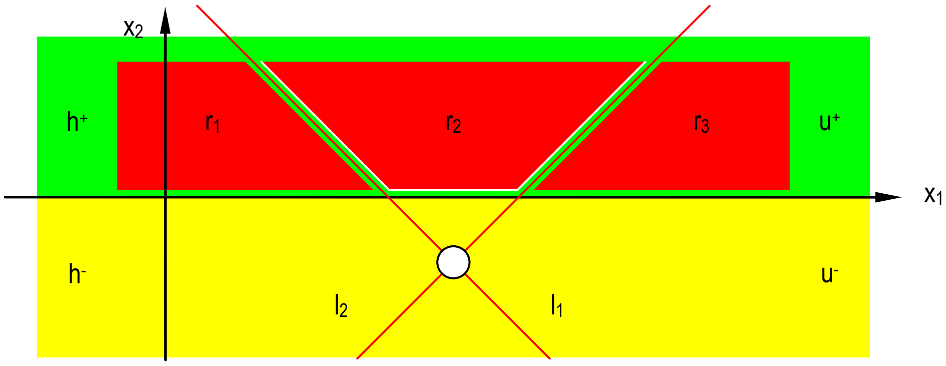

Fig 10-40 Uniqueness of 2-$\mathcal{D}$ Ham-sandwich cut under certain condition

Let u^+ and u^- be mass-distributions in $\mathcal{E}^2$ with bounded total measure, such that u^+ is positive in h^+ and vanishes in h^-, and the reverse is true for u^-. Then there is a unique line that bisects both u^+ and u^-.

- [Proof]
- Assume that there are 2 distinct simultaneous bisectors l_1 and l_2.
  l_1 and l_2 are impossible to be parallel.
  W.O.L.G., assume $l_1 \cap l_2 \notin h^+$
- l_1 and l_2 cut h^+ into 3 regions: r_1, r_2 and r_3, as shown in figure.
- Consider r_2
  ∵ Both l_1 and l_2 bisect h^+
  ∴ The measure of u^+ in $r_2 = 0$.
  ∴ A contradiction against that u^+ is positive in h^+.
- [QED]

15 3D Four-Section

- Given two sets P and Q of points in $\mathcal{E}^3$, **card**(P)=m, **card**(Q)=n.
  Two planes g and h are said to **four-sect** P and Q if each of the four open wedges defined by g and h contains at most m/4 points of P and at most n/4 points of Q.
- Two sets P and Q are called separable, if **conv**(P) $\cap$ **conv**(Q) = $\varnothing$.

Let P and Q be two separable sets of m and n points in $\mathcal{E}^3$.
Then there exist two planes that four-sect P and Q.

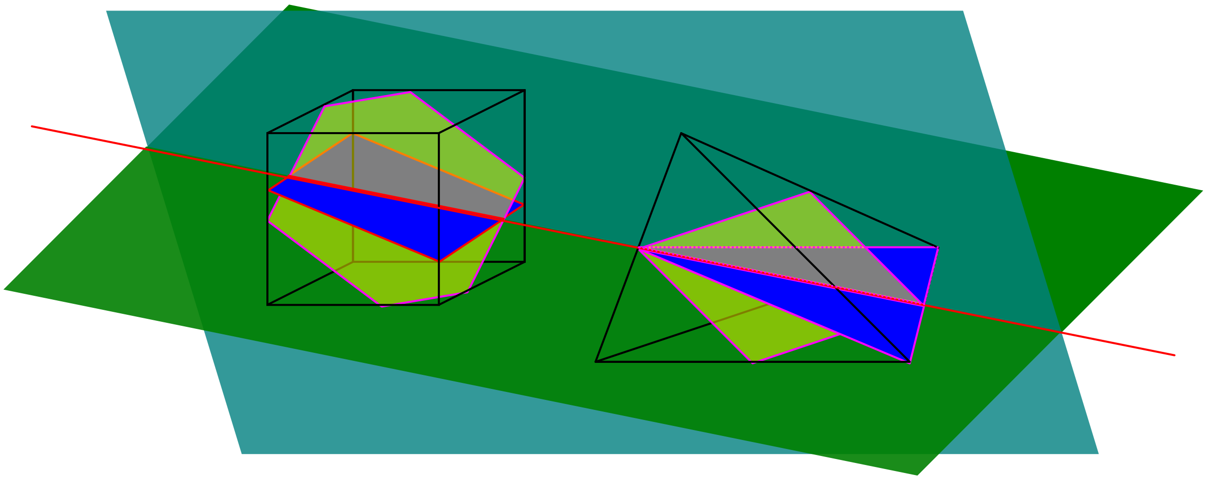

Fig 10-41 Four-section of two separable sets in $\mathcal{E}^3$

Let P and Q be two sets of m and n points in $\mathcal{E}^3$.
Then there exist two planes that four-sect P and Q.

16 Minkowski's First Theorem

✉ [Minkowski, 1891][17]

Let $C \subseteq \mathcal{E}^d$ be symmetric (around the origin), convex, bounded, and suppose that **vol**(C) > 2^d.

Then C contains at least one lattice point different from 0.

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