MS Analysis Seminar 1:
Bump Functions and Partitions of Unity

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1 Introduction

The goal of this handout is to explore how different decompositions of sets can lead to interesting analytic and topologic structures, particularly centering around the idea of "locally finite" decompositions.
Dual to how the improper Riemann integral is defined by taking the limit of integrals on increasing intervals of $\mathbb{R}$, this technology proves helpful in the definition of the extended Riemann integral by breaking up functions into "increasing pieces."

2 Locally Finite Covers

We start with the following theorem, which provides us with an explicit construction for interesting covers of open sets.

Theorem 1. *Let $U = \cup_{m\in\mathbb{N}} U_m \subseteq \mathbb{R}^n$ all be open. Then there exists $\{R_i\}_{i\in\mathbb{N}}$ such that the following hold:*

- $U \subseteq \bigcup_{i\in\mathbb{N}} R_i^\circ$.
- *For all $i \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $R_i \subseteq U_m$.*
- *For all $x \in U$, there exists $\varepsilon > 0$ with $B(x, \varepsilon)$ intersecting only finitely many R_i.*

Proof. We start with the following lemma:

Lemma 1. *Suppose U is open. Then there exists $\{C_i\}_{i\in\mathbb{N}}$ such that $A = \bigcup_{i\in\mathbb{N}}$ and $C_i \subseteq C_{i+1}^\circ$ for all i.*

Proof. Set

$$C_i = \{x : d(x, U^c) \geq 1/i\} \bigcap B[0, i]$$

which is clearly compact as the intersection of a closed set and a compact set. Openness of U tells us that this is a cover of A, and to see the last condition, see

$$C_i \subseteq \{x : d(x, U^c) < 1/(i + 1)\} \bigcap B(0, i + 1) \subseteq C_{i+1}^\circ$$

as desired. □

Now let C_i be as above, and for convenience put $C_i = \varnothing$ for $i \leq 0$. Now for each $i \in \mathbb{N}$ put $D_i = C_i \setminus C_{i-1}^\circ$. Observe that D_i is bounded (as a subset of C_i) and closed (as $C_i \cap (C_{i+1}^\circ)^c$) and hence compact. Furthermore, D_i is completely disjoint from $C_{i-2} \subseteq C_{i-1}^\circ$.
Now for each $x \in D_i$ choose m with $x \in U_m$ and ε with $B(x, \varepsilon) \subseteq U_m \cap C_{i-2}^\circ$. Then choosing $R_x \subseteq B(x, \varepsilon)$ for every such x, we observe that $\{R_x^\circ\}_{x \in D_i}$ is an open cover of D_i so we can pass down to a finite $\{R_j^\circ\}_{j \in [n]}$ covering D_i. Now to find our desired $\{R_j\}_{j \in \mathbb{N}}$ take the union of this collection over all D_i.
Clearly the first two properties are satisfied, so it suffices to exhibit the last point. Let $x \in U$ be arbitrary. Then $x \in C_i^\circ$ for some i and we can choose $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq C_i^\circ$. Then C_i° intersects only the rectangles from $D_1, \cdots D_i$ which are finite, and we're done. □

3 Bump Functions

In order to better "support" (haha) the covers we developed in the previous part, we now quickly prove a couple results about the existence of C^∞ "indicator functions" that'll be useful in our partitions of unity.

Theorem 2. *Let $R = \prod_{i \in n} [a_i, b_i]$. Then there exists $I_R \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that $I_R(x) > 0$ for all $x \in R^\circ$ and 0 otherwise.*

Proof. We'll explicitly construct such a function in $\mathbb{R}$, then use some coordinate trickery to construct our desired function.

Claim.

$$g : \mathbb{R} \to \mathbb{R} \textit{ via } g(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & \textit{otherwise} \end{cases}$$

is C^∞.

Proof. We'll proceed by showing that for each $n \in \mathbb{N}$ that

$$g_n(x) = \begin{cases} e^{-1/x}/x^n & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is continuous at 0. To do so, first recall that that $\alpha < e^\alpha$ for all $\alpha \in \mathbb{R}$. Setting $\alpha = t/2n$, we find that

$$t/2n < e^{t/2n} \implies \frac{t^n}{e^t} < \frac{(2n)^n}{e^{t/2}}$$

which implies

$$e^{-1/x}/x^n < (2n)^n \cdot e^{-1/x}$$

and from this, the fact that each g_n is C^∞ clearly follows. □

Now observe that we can convolute g with itself to find a function $f = g(x) \cdot g(1 - x)$ that's positive on $(0, 1)$ and 0 everywhere else. Now we can let

$$I_R(x) = \prod_{i \in [n]} f\left(\frac{x_i - a_i}{b_i - a_i}\right)$$

and we're done. $\square$

To better support the content of the next section, for $f : \mathbb{R}^n \to \mathbb{R}$ we'll define the **support** S_f of f to be

$$S_f = \overline{\{f \neq 0\}}$$

4 Partitions of Unity

Now that all of our technology has been assembled, we'll finish with the following result:

Theorem 3. *Let $U = \bigcup_{k\in\mathbb{N}} U_k \subseteq \mathbb{R}^n$ all be open. There there exists $\{f_i\}_{i\in\mathbb{N}} \subseteq C^\infty(\mathbb{R}^n;\mathbb{R})$ such that:*

- *f_i is non-negative, and the support of each f_i is compact and contained in U.*
- *For all $x \in U$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon)$ intersects only finitely many S_{f_i}.*
- $\displaystyle\sum_{i\in\mathbb{N}} f_i = x \mapsto \begin{cases} 1 & x \in U \\ 0 & \textit{otherwise} \end{cases}$
- *For all $i \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that U_k contains the support of f_i.*

We call such a collection a partition of unity of class C^∞ with compact support dominated by $\{U_k\}_{k\in\mathbb{N}}$.

Proof. Let $\{R_i\}_{i\in\mathbb{N}}$ be as in the first theorem, and $\{I_i\}_{i\in\mathbb{N}}$ be the corresponding indicator functions from the second result. Let $\lambda = \sum_{i\in\mathbb{N}} I_i$, which, by our local finiteness condition, clearly converges. Now set $I_i = f_i/\lambda$ to produce the desired functions. $\square$