COMP7180 Assignment 1

Note:

1. **Instruction of assignment submission:**
   a) **write all your answers in a Microsoft Word document.**
   b) **hand written answers or importing relevant pictures is not allowed, otherwise, corresponding problems will be given Zero Mark;**
   c) **name your document using the following format: "COMP7180_A1_StudentID_Surname_Givenname.doc"; and**
   d) **submit the document on Moodle.**
2. **The submission deadline is 5pm, Nov. 8, 2022.**
3. **This is an individual work. Plagiarism is strictly forbidden. Students who plagiarized and who were plagiarized will be given Zero Mark.**

Problem 1 (20 marks)

Suppose

$$f(x_1, x_2) = 10 - 2(x_2 - x_1^2)^2$$
$$S = \{(x_1, x_2) \mid -11 \le x_1 \le 1, -1 \le x_2 \le 1\}$$

Is $f(x_1, x_2)$ a convex function on S? Prove your conclusion.

$$\frac{\partial f}{\partial x_1} = 8x_1(x_2 - x_1^2), \quad \frac{\partial f}{\partial x_2} = -4(x_2 - x_1^2),$$
$$\frac{\partial^2 f}{\partial x_1^2} = 8(x_2 - 3x_1^2), \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 8x_1, \quad \frac{\partial^2 f}{\partial x_2^2} = -4,$$

Hesse matrix

$$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 8(x_2 - 3x_1^2) & 8x_1 \\ 8x_1 & -4 \end{bmatrix}.$$

It is not a semi-positive definite matrix on the set S. For example, the point (0, 1) is an obvious counterexample. Therefore, it is not a convex function on S.

Problem 2 (20 marks)

Use the definition of the eigenvalue/eigenvector to prove the following statements:

(a) **(10 marks)** Show that if 5 is an eigenvalue of an $n \times n$ matrix A, then 25 is an eigenvalue of A^2.

$$A^2 v = A \cdot Av = A \cdot 5v = 5 \cdot Av = 5 \cdot 5v = 25v$$

(b) **(10 marks)** Let A be an invertible matrix with an eigenvalue 3. Show that 1/3 is an eigenvalue of A^{-1}.

$$A^{-1}(3u) = A^{-1}(Au) = A^{-1}Au = Iu = \frac{1}{3} \cdot (3u)$$

Problem 3 (20 marks)

Consider the following design matrix, representing four sample points $X_i \in \mathbb{R}^2$.

$$X = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 5 & 4 \\ 1 & 0 \end{bmatrix}$$

We want to represent the data in only one dimension, so we turn to principal component analysis (PCA).

(a) **(10 marks)** Compute all the principal component directions of X, and state which one the PCA algorithm would choose if you request just one principal component.

We center X, yielding

$$\dot{X} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 2 \\ -2 & -2 \end{bmatrix}.$$

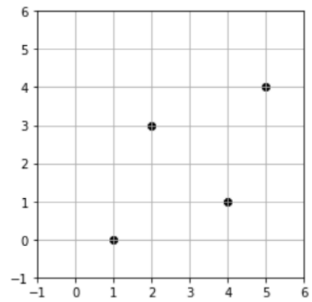
Then $\dot{X}^\top \dot{X} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$. (Divide by 4 if you want the sample covariance matrix. But we don't care about the magnitude.)

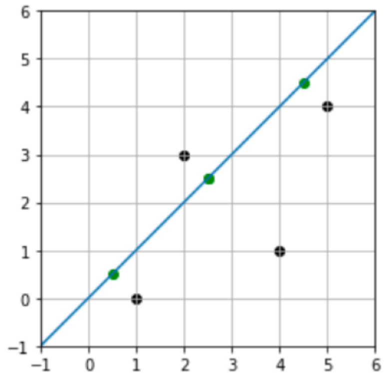
Its eigenvectors are $[1/\sqrt{2} \quad 1/\sqrt{2}]^\top$ with eigenvalue 16 and $[1/\sqrt{2} \quad -1/\sqrt{2}]^\top$ with eigenvalue 4. The former eigenvector is chosen.

(Negated versions of these vectors also get full points.)

(b) **(10 marks)** The plot below depicts the sample points from X. We want a one-dimensional representation of the data, so draw the principal component direction (as a line) and the projections of all four sample points onto the principal direction.

Label each projected point with its principal coordinate value. Give the principal coordinate values exactly.





The principal coordinates are $\frac{1}{\sqrt{2}}$, $\frac{5}{\sqrt{2}}$, $\frac{5}{\sqrt{2}}$, and $\frac{9}{\sqrt{2}}$. (Alternatively, all of these could be negative, but they all have to have the same sign.)

$$P_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$P_2 = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{5}{\sqrt{2}}$$

$$P_3 = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{5}{\sqrt{2}}$$

$$P_4 = \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{9}{\sqrt{2}}$$

Problem 4 (20 marks)

Determine if b is a linear combination of the vectors formed from the columns a_1, a_2, a_3 of the matrix A. Prove your conclusion.

$$\mathrm{A} = [a_1, a_2, a_3] = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix} \; and \; \mathrm{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

(A)

Suppose X=[x1,x2,x3], AX=b

x1-2x2-6x3=11;3x2+7x3=-5;x1-2x2+5x3=9

X1=245/33 , X2= -41/33 , X3= -2/11

So, b is a linear combination of the vectors [[245/33],[-41/33],[-2/11]] of matrix A

(B)

Elimination to A:

$[[1,-2,6],[0,3,7],[1,-2,5]] \Rightarrow [[1,-2,-6],[0,3,7],[0,0,11]]$

$$R(A) = 3$$

Therefore, b is a linear combination of a_1, a_2, a_3.

Problem 5 (20 marks)

Let $\mathrm{v}, \mathrm{w} \in \mathrm{V}$ for some vector space V. Show that $\{\mathrm{v}, \mathrm{w}\}$ are linearly independent if and only if $\{\mathrm{v}+\mathrm{w}, \mathrm{v}-\mathrm{w}\}$ are linearly independent.

If $\{u, v\}$ are linear independent and $c_1(v+w) + c_2(v-w) = 0$, then

$$(c_1 + c_2)v + (c_1 - c_2)w = 0\,.$$

Since $\{u, v\}$ are independent,

$$\begin{aligned} c_1 + c_2 &= 0\,, \\ c_1 - c_2 &= 0\,. \end{aligned}$$

This implies $c_1 = c_2 = 0$, so $\{v+w, v-w\}$ are linearly independent.
Now consider $\{v+w, v-w\}$ are linearly independent and $c_1 v + c_2 w = 0$. First we write u and v as a linear combination of $u+v$ and $u-v$,

$$\begin{aligned} u &= \frac{1}{2}\left((u+v)+(u-v)\right)\,, \\ v &= \frac{1}{2}\left((u+v)-(u-v)\right)\,. \end{aligned}$$

So

$$\begin{aligned} \frac{c_1}{2}\left((u+v)+(u-v)\right) + \frac{c_2}{2}\left((u+v)-(u-v)\right) = 0\,, \\ (c_1+c_2)(u+v) + (c_1-c_2)(u-v) = 0\,. \end{aligned}$$

Since $\{u+v, u-v\}$ are linearly independent, $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Thus $c_1 = c_2 = 0$ and $\{u, v\}$ are a linearly independent set.