Answer to COMP7180 Assignment 1

Problem 1
There are a lot of examples, such as Maximum Likelihood Estimation for Model Parameter Optimization and Naive Bayesian in Disease Diagnosis. In Maximum Likelihood Estimation, the probabilities are used to formulate the likelihood; while in Naive Bayesian, the joint probability distribution is constructed as the objective function to be maximized.

Problem 2
2.1
Since **event A** is "The outcome of the first throw is 1" and **event B** is "The summation of outcomes of three throws is no less than 10", then **events A and B** is "The outcome of the first throw is 1 and (at the same time) the summation of outcomes of three throws is no less than 10", which is equivalent to "The outcome of the first throw is 1 and the summation of outcomes of the second and the third throws is no less than 9". There are 10 situations that satisfy this condition: (outcome of the second throw, outcome of the third throw) = (3,6), (4,5), (4,6), (5,4), (5,5), (5,6), (6,3), (6,4), (6,5), (6,6). We also know that there are 6*6 = 36 situations of the second and the third throws. Therefore, P(B|A) = P(A,B)/P(A) = 10/36 = 5/18.

2.2
According to the definition, the independence of A and B requires that P(A,B) = P(A)*P(B) or P(B) = P(B|A) or P(A) = P(A|B). Let's see if one of these conditions can be satisfied in our case.

In our case, we know that event A is "At least one throw gives the outcome of 1". Therefore, the event ~A is "none of the throws gives the outcome of 1", and we have P(~A) = (5*5*5)/(6*6*6) = 125/216. So that P(A) = 1 - P(~A) = 91/216.
Moreover, we know that event B is "At least two throws have the same outcome". Then event ~B is "all throws have different outcomes", and we have P(~B) = (6*5*4)/(6*6*6) = 5/9. So that P(B) = 1 - P(~B) = 4/9.

For the events A and B, it means that "At least one throw gives the outcome of 1 AND at least two throws have the same outcome". There are three situations:
1. One 1 and two other same values: (1,2,2), (1,3,3), (1,4,4), (1,5,5), (1,6,6), (2,1,2), (3,1,3), (4,1,4), (5,1,5), (6,1,6), (2,2,1), (3,3,1), (4,4,1), (5,5,1), (6,6,1), totally 15 combinations;
2. Two 1s and one other value: (1,1,2), (1,1,3), (1,1,4), (1,1,5), (1,1,6), (1,2,1), (1,3,1), (1,4,1), (1,5,1), (1,6,1), (2,1,1), (3,1,1), (4,1,1), (5,1,1), (6,1,1), totally 15

combinations;
3. Three 1s: (1,1,1), totally 1 combination.
Therefore, P(A,B) = 31/216.

Obviously, P(A,B) $\neq$ P(A)*P(B). So A and B are not independent.

Problem 3 (20 Marks)
Let event E be “Alice wins a single game” and event F be “Bob wins a single game”, then we have P(E) = p and P(F) = 1-p.

Now let’s consider event A, i.e., “Alice finally wins”. It can be divided into two scenarios:
(1) Alice wins the first game and Alice finally wins;
(2) Alice loses the first game and Alice finally wins.

For scenario (1), the sequence starts with an “E”. Since Alice finally wins, the sequence ends with an “EEE”. But there could be zero, one, many or even infinite number of “EF” or “EEF” before the final “EEE” (note that “FF” is not allowed because this makes Bob finally wins). Therefore, in scenario (1), the sequences ensure that Alice finally wins could be
“EEE” or
“EF EEE” or “EEF EEE” or
“EF EF EEE” or “EF EEF EEE” or “EEF EF EEE” or “EEF EEF EEE” or
…

Since the result of each single game is independent to the results of other games, we have P(EEE) = p^3, P(EF) = $p(1-p)$, and P(EEF) = $p^2(1-p)$. Moreover, the event “EF” and “EEF” are mutually exclusive, so the total probability of scenario (1) is
$p^3[1 + (p(1-p)+p^2(1-p)) + (p(1-p)+p^2(1-p))^2 + (p(1-p)+p^2(1-p))^3 + (p(1-p)+p^2(1-p))^4 + \ldots] = p^3/(1-p+p^3)$.

For scenario (2), the sequence starts with an “F”, then it will exactly repeat scenario (1) since it cannot have “FF”.
So the total probability of scenario (2) is $(1-p)p^3/(1-p+p^3)$.

Therefore, P(A) = $p^3/(1-p+p^3) + (1-p)p^3/(1-p+p^3) = (2-p)p^3/(1-p+p^3)$.

Similarly, we can derive P(B), i.e., the probability that “Bob finally wins”. It can be divided into three scenarios:
(1) Bob wins the first game, and Bob finally wins;
(2) Bob loses the first game, wins the second game, and Bob finally wins;
(3) Bob loses the first two games, and Bob finally wins.

For scenario (1), the sequence starts with an “F”. Since Bob finally wins, the sequence ends with “FF”. But there could be zero, one, many or even infinite number of “FE” or “FEE” before the final “FF”. (note that “EEE” is not allowed because this makes Alice finally wins). Therefore, in scenario (1), the sequences ensure that Bob finally wins could be
“FF” or
“FE FF” or “FEE FF” or
“FE FE FF” or “FE FEE FF” or “FEE FE FF” or “FEE FEE FF” or
…

Since the result of each single game is independent to the results of other games, we have P(FF) =$(1-p)^2$, P(FE) = $(1-p)p$, and P(FEE) = $(1-p)p^2$. Moreover, the event “FE” and “FEE” are mutually exclusive, so the total probability of scenario (1) is
$(1-p)^2[1 + ((1-p)p+(1-p)p^2) + ((1-p)p+(1-p)p^2)^2 + ((1-p)p+(1-p)p^2)^3 + ((1-p)p+(1-p)p^2)^4 + \ldots] = (1-p)^2/(1-p+p^3)$.

For scenario (2), the sequence starts with an “E”, then the second is “F”, so from the “F” it will exactly repeat scenario (1). Therefore, the total probability of scenario (2) is $p(1-p)^2/(1-p+p^3)$.

For scenario (3), the sequence starts with “EE”, then the third is “F”, so from the “F” it will exactly repeat scenario (1). Therefore, the total probability of scenario (3) is $p^2(1-p)^2/(1-p+p^3)$.

Therefore, P(B) = $(1+p+p^2)(1-p)^2/(1-p+p^3)$.

Interestingly, we do have P(A) + P(B) = 1

Problem 4
4.1
According to the definition of expectation, we have
$E(x) = \sum_{x \in X} xp(x) = 0 * p(0) + 1 * p(1) = \mu$.
According to the property of variance, we have

$$Var(x) = E(x^2) - E(x)^2 = \left(0^2 * p(0) + 1^2 * p(1)\right) - \left(0 * p(0) + 1 * p(1)\right)^2$$
$$= \mu - \mu^2 = \mu(1-\mu)$$

4.2
Let $y_1 = \frac{x_1}{x_1+x_2+\cdots+x_n}$. Since $x_1, x_2, \ldots, x_n$ ($n > 1$) are independent and identically distributed random variables, we know that $y_1, y_2, \ldots, y_n$ ($n > 1$) are identically

distributed random variables. Therefore, we have $E(y_1) = E(y_2) = \cdots = E(y_n)$.
On the other hand, we know that $y_1 + y_2 + \cdots + y_n = 1$, so $E(y_1 + y_2 + \cdots + y_n) = E(y_1) + E(y_2) + \cdots + E(y_n) = 1$.

Therefore, $E(\frac{x_1}{x_1+x_2+\cdots+x_n}) = E(y_1) = \frac{1}{n}$.

Problem 5

Obviously, we have $E(X) = E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$, where X_i is a binary variable indicating whether the *i*-th group will be a "pair group" (1 indicates yes and 0 indicates no). For the *i*-th group, the probability that it forms a "pair group" is $\frac{n}{2n*(2n-1)/2} = \frac{1}{2n-1}$, where the numerator "$n$" indicates the number of ways to form a pair while the denominator "$2n * (2n-1)/2$" indicates the number of ways to select 2 shoes from $2n$ shoes. Then we have $E(X) = E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n) = n * \frac{1}{2n-1} = \frac{n}{2n-1}$.