Solution to Exercise 1: Matrix Computation

Given the $n \times k$ matrix **A** and the $k \times n$ matrix **B**:

1. Use an example to show that **AB** ≠ **BA** even if $n = k$.

Solution:

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then we have $\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and $\mathbf{BA} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Obviously we have **AB** ≠ **BA**.

2. When $n \neq k$, do we have *trace*(**AB**) = *trace*(**BA**)? Prove your conclusion.

Solution:

by definition

$$\begin{aligned} trace(AB) &= (AB)_{11} + (AB)_{22} + \cdots + (AB)_{nn} \\ &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1k}b_{k1} \\ &+ a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2k}b_{k2} \\ &+ \vdots \\ &+ a_{n1}b_{1n} + a_{n2}b_{2n} + \cdots + a_{nk}b_{kn} \end{aligned}$$

if you view the sum according to the columns, then you see that it is the $trace(BA)$. therefore,

$$trace(AB) = trace(BA).$$

3. What is the relationship between eigenvalues of **AB** and eigenvalues of **BA**? What is the relationship between eigenvectors of **AB** and eigenvectors of **BA**?

Solution:

Suppose that λ and x are the eigenvalue and eigenvector of matrix (**AB**), then by definition we have (**AB**)x = λx. Then we have **B**(**AB**)x = **B**λx, which indicates that (**BA**)**B**x = λ**B**x, i.e., λ is also the eigenvalue of matrix (**BA**), and the corresponding eigenvector is **B**x.