COMP7180: Quantitative Methods for DAAI

(Credits from Prof. Andrew Ng)

(Credits from HKBU)

Course Instructors: Dr. Yang Liu and Dr. Bo Han

Teaching Assistant: Mr. Minghao Li

About Me

- Dr. Bo Han
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Timetable

- Time of our classes
  - 6 weeks from Oct 25 to Nov 29
  - Regular Time: 18:30-21:30 PM (Thu)

- Classroom of our class
  - Lectures: OEE1017 (Thu)

Course Contents

- Conditional Probability and Independence (Week 7)
- Discrete and Continuous Random Variables (Week 8)
- Maximum Likelihood Estimation (Week 9)
- Mathematical Optimization (Week 10)
- Convex and Non-Convex Optimization (Week 11)
- Quiz and Explanation (Week 12)

Course Mode

- Instructor: 3-hour book knowledge
- Office-hour: 0.5~1-hour office hour
- 1 Assignment + 1 Quiz
- Final exam (50% my part)

Learning Outcomes

- COMP7180: To learn the various **quantitative methods** (i.e., **mathematical experience**) necessary for data analytics and artificial intelligence (DAAI).
- Knowledge:
  - Explain the essential concepts in probability and statistics for DAAI
  - Understand the essential concepts in optimization for DAAI
- Professional Skill:
  - Determine suitable quantitative methods for effective data analytics
  - Apply suitable quantitative methods for real-world problem solving

- Compared to COMP7250: To introduce the fundamentals, models and techniques commonly found in machine learning. To gain some **hands-on experience** on developing machine learning solutions.

Assessment Methods

- Continuous Assessment (40%)
  - Assignments and Quizzes
- Examination (60%)
- Important Notices
  - Plagiarism: **Students who plagiarized and who were plagiarized will be given 0 mark**.
  - Final Exam: In order to pass this course, students should attain at least **30% of the final examination mark**.
  - A Cumulative **GPA at least 2.50** for graduation.

Why Probability

- Probability theory is a mathematical framework for representing uncertain statements.

- The laws of probability tell us how AI systems should reason, so we design our algorithms to compute or approximate various expressions derived using probability theory.

- We can use probability and statistics to theoretically analyze the behavior of proposed AI systems.

Why Probability

- Machine learning must always deal with uncertain quantities and sometimes stochastic (nondeterministic) quantities. Uncertainty and stochasticity can arise from many sources.

- (1) Inherent stochasticity in the system being modeled. (2) Incomplete observability. (3) Incomplete modelling.

Why Probability: An Example

- Suppose you are trying to determine if a patient has inhalational anthrax (吸入性炭疽病). You observe the following symptoms:

A. The patient has a cough；

B. The patient has a fever；

C. The patient has difficulty in breathing.

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Why Probability: An Example

- You would like to determine how likely the patient is infected with inhalational anthrax given that the patient has a cough, a fever, and difficulty in breathing;

- We are not 100% certain that the patient has anthrax because of these symptoms. We are dealing with uncertainty!

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Why Probability: An Example

- Now suppose you order an x-ray and observe that the patient has a wide mediastinum ((胸腔)纵隔);

- Your belief the probability that the patient is infected with inhalational anthrax is now much higher.

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Why Probability: An Example

- In the previous slides, what you observed affected your belief that the patient is infected with anthrax;

- This is called reasoning with uncertainty;

- Wouldn’t it be nice if we had some methodology for reasoning with uncertainty? In fact, we do ! ☺

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What is Probability

- A probability can be regard as a function to estimate the value of every event.

As a function, we should have a domain (定义域). What is the domain?

Given a sample space S: set of all possible outcomes of an experiment.

The domain consists of some subsets of S.

The element E in the domain is called event.

What is Probability

Example: Toss a coin (1 time). Then, the outcome is H or T, where H is the head of a coin and T is the tail of a coin.

Then S= { H, T};

The domain is { {H,T}, {H}, {T}, Ø}.

{H,T}, {H}, {T}, Ø are called events.

What is Probability

The domain should satisfy some speial properties:

- S and Ø should be event;
- If E is an event, then E^{C} is an event (E^{C} = S-E);
- If E and F are both events, then E∩F is an event, that is event E and event F occur at the same time;
- If E and F are both events, then EUF is an event, that is event E occur or event F occur .

What is Probability

Example: Throw a coin (1 time). Then, the outcome is H or T, where H is the head of a coin and T is the tail of a coin.

Then S= { H, T}; The domain is { {H,T}, {H}, {T}, Ø}. {H,T}, {H}, {T}, Ø are called events.

- S and Ø should be event;
- S^C = Ø; $\{H\}^C$ = {T}; $\{T\}^C$={H}; $Ø^C$= S;
- S∩ Ø=Ø; S∩{H}={H}; S∩{T}={T}; {H}∩{T}=Ø;
- {H}∪{T}=S; {H}∪ Ø={H}; {T}∪ Ø={T}.

What is Probability

As a function, we should have a range (值域). What is the range?

Given an event E, a probability maps E into [0,1], that is $0 \leq P(E) \leq 1$.

If P(E)=0, then this event E will not occur.

If P(E)=1, then this event E occurs without uncertainty.

What is Probability

Example. Toss a coin (1 time). There are outcomes: H and T, where H is the head of a coin and T is the tail of a coin.

S= { H, T}; The domain is { {H,T}, {H}, {T}, Ø}.

{H,T}, {H}, {T}, Ø are called events.

P({H,T}) =1; P({H}) = 0.5; P({T}) =0.5; P(Ø)=0.

What is Probability

Probability is a special function, which should satisy some properities:

- P(S)=1; P(Ø)=0; 0$\leq$P(E)≤ 1;

- If event E belongs to event F, then P(E)$\leq$P(F);

- Given an event E, then P(E^{C}) = 1-P(E);

- Given events E and F, then P(E∪F) = P(E)+P(F)-P(E∩F).

What is Probability

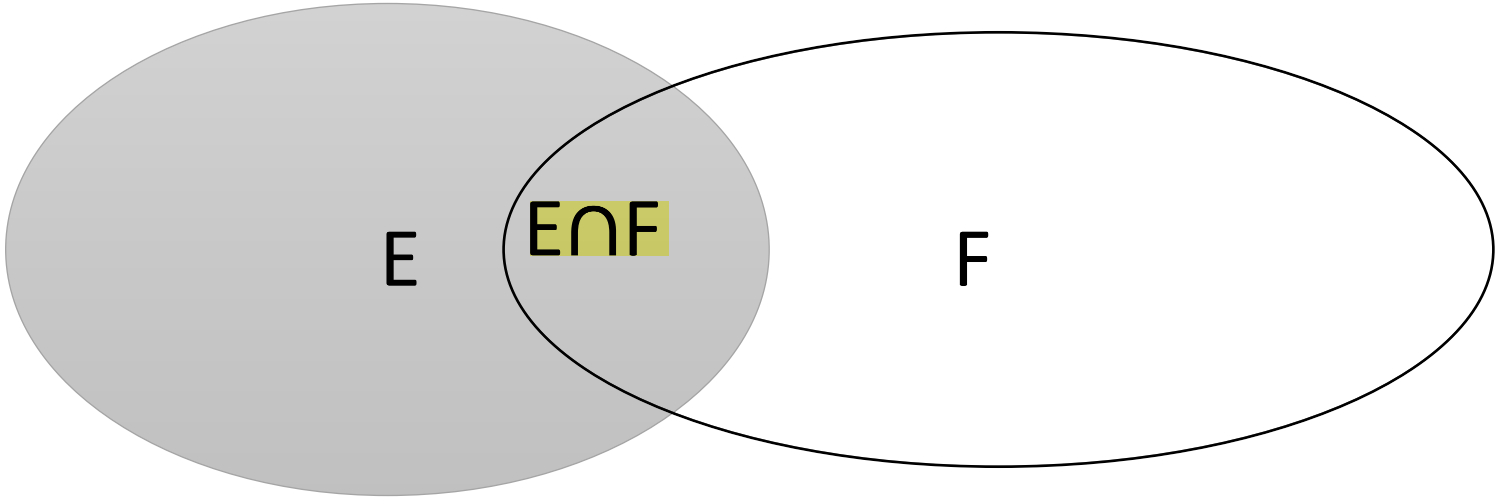
Example. Toss a coin (1 time). There are outcomes: H and T, where H is the head of a coin and T is the tail of a coin.

$P(\{H,T\}) = 1$; $P(\{H\}) = 0.5$; $P(\{T\}) = 0.5$; $P(\emptyset)=0$.

- $P(\{H,T\})=1$; $P(\emptyset)=0$;
- $P(\{H\})= 1-P(\{T\})$ and $P(\{H,T\}) = 1-P(\emptyset)$;
- $P(\{H\}\cup\{T\}) = P(\{H\})+P(\{T\})-P(\emptyset)$.

What is Probability

- How to understand $P(E \cup F) = P(E) + P(F) - P(E \cap F)$?



Random Variables

- Generally, it is very complex to represent an event;
- To deal with more complex events, researchers have developed random variables (随机变量).
- Example. Throw a coin (1 time). In the sample space S={ H, T}, we design a function X: S→{1,-1} such that X(H)=1 and X(T)=-1. Then X is

  a random variable.

  Moreover, P(X=1) = P({H}) = 0.5 and P(X=-1) = P({T})=0.5.

What are Random Variables

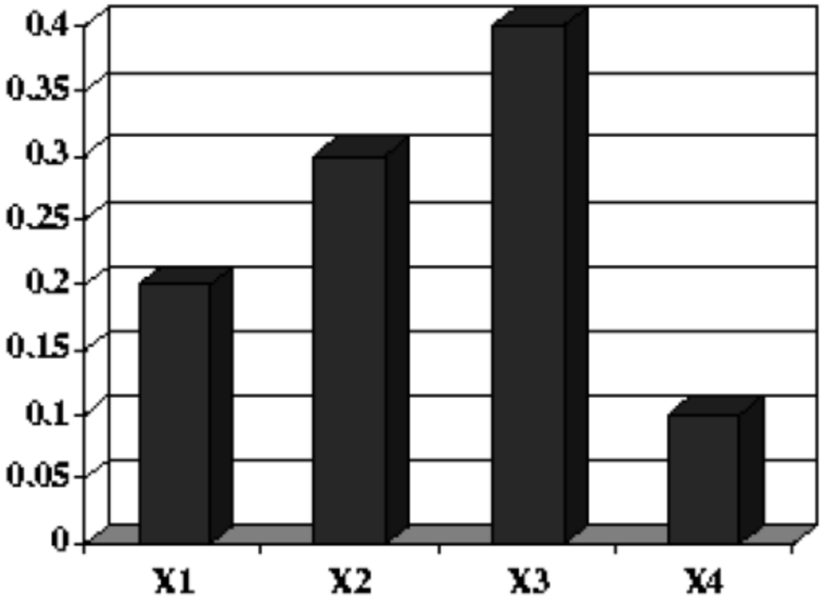
- A random variable is a variable that can take on different values randomly. We typically denote the random variable itself with an uppercase letter in plain typeface, and the values it can take on with lowercase letters.

- For vector-valued variables, we would write the random variable as X and one of its values as x.

- Random variables may be discrete or continuous. A discrete random variable is one that has a finite or countably infinite number of states. A continuous random variable is associated with a real value.

Probability Distributions

- A probability distribution is a description of how likely a random variable or set of random variables is to take on each of its possible states. The way we describe probability distributions depends on whether the variables are discrete or continuous.



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Discrete Variables and PMF

- A probability distribution over discrete variables may be described using a probability mass function (PMF, 概率质量函数)

- The probability mass function maps from a state of a random variable to the probability of that random variable taking on that state.

- $0 \leq \mathrm{P}(\mathrm{X} = \mathrm{x}) \leq 1$

- $\sum_{\mathrm{x} \in \mathrm{X}} \mathrm{P}(\mathrm{X} = \mathrm{x}) = 1$. We refer to this property as being normalized

Discrete Variables and PMF

A random variable X can be regarded a function:

the domain is the sample space S; but the range is discrete value:

- The range could be finite: $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}$;
- The range could be countably infinite: $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}, \ldots$

Discrete Variables and PMF: Examples

Discrete Random Variable with finite range:

Toss a coin (1 time).

In the sample space S={H, T}, we design
a random variable X: S→{1,-1} such that
X(H)=1 and X(T)=-1. Then X is a random variable with finite range.

The probability is P(X=1)=P(X=-1)= 0.5.

Discrete Variables and PMF: Examples

Discrete Random Variable with infinite range:

Toss a coin (countably infinity times).

We design a random variable X: X = n means that
the first head appears after throwing n times.

Then X is a random variable with countably infinite range.

The probability is P(X=n) = 0.5^{n}.

Discrete Variables and PMF: Examples

- Discrete uniform distribution (均匀分布) is one of the most important discrete distributions
- It is a finite discrete distribution
- Assume that the range is $\mathrm{x}_1,\mathrm{x}_2 \ldots \mathrm{x}_\mathrm{n}$, then
- $\mathrm{P}(\mathrm{X}=\mathrm{x}_\mathrm{i})=\frac{1}{n}; \quad \sum_i \mathrm{P}(\mathrm{X}=\mathrm{x}_\mathrm{i})=\sum_i \frac{1}{n}=\frac{n}{n}=1$

Continuous Variables and PDF

- A continuous variable X is a function;

- Range is not discrete and take values in real number;

- There is a probability density function (概率密度函数) $p_X(x)$ such that

1) $p_X(x) \geq 0$;

2) P(a≤X≤b) = $\int_a^b p_X(x)\, dx$;

3) $\int_{-\infty}^{+\infty} p_X(x)\, dx$=1.

Continuous Variables and PDF

- In principle variables such as height, weight, and temperature are continuous, in practice the limitations of our measuring instruments restrict us to a discrete (though sometimes very finely subdivided) world.

- However, continuous models often approximate real-world situations very well, and continuous mathematics (calculus) is frequently easier to work with than mathematics of discrete variables and distributions.

Continuous Variables and PDF: Example

- The weight of a certain animal like a dog.

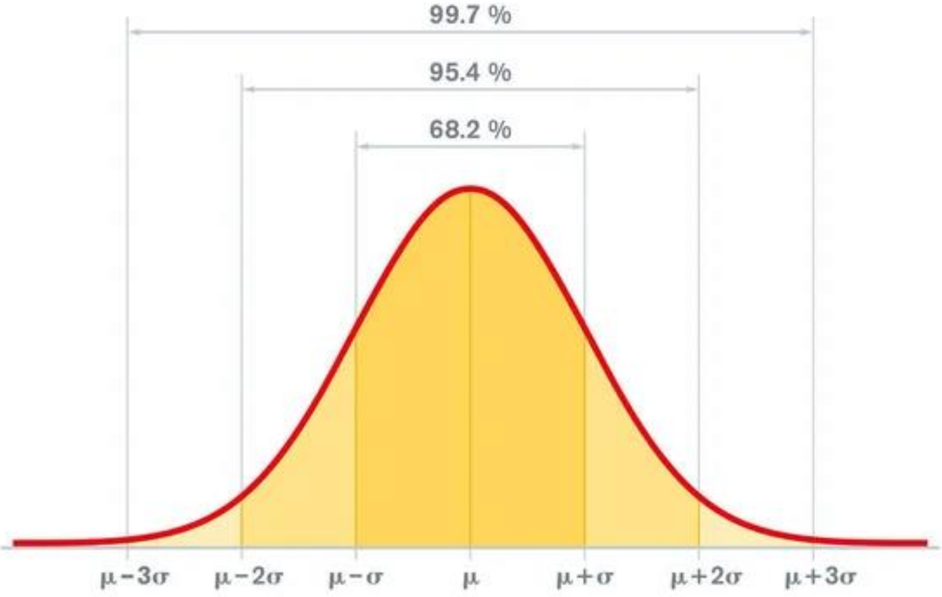
This is a continuous random variable because it can take on an infinite number of values. For example, a dog might weigh 30.333 pounds, 50.340999 pounds, 60.5 pounds, etc.

What is the distribution of dog’s weight?

Continuous Variables and PDF

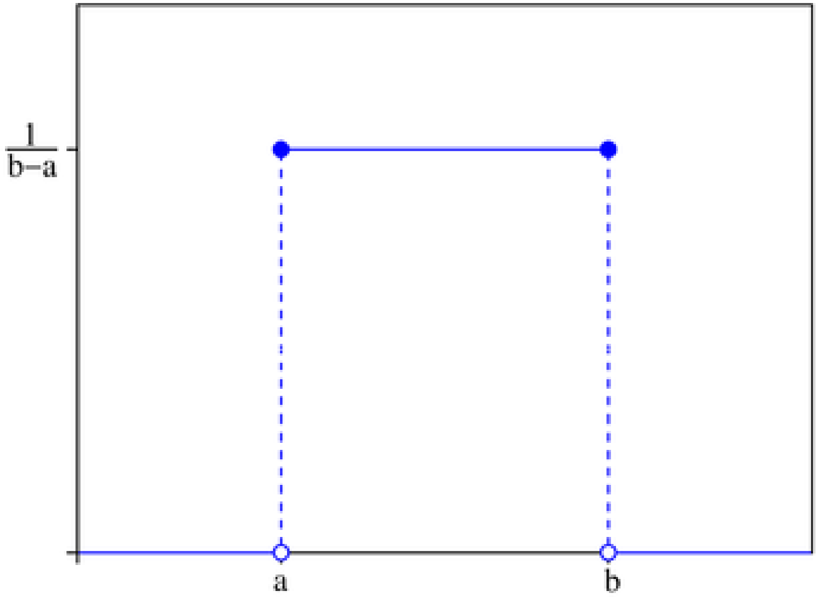
It is similar to a gaussian distribution. The probability density funtion is

$$\sqrt{\frac{1}{2\pi\sigma^2}}\exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$$



Continuous Variables and PDF

- Continuous uniform distribution is one of the most important continuous distributions.
- The probability density function of continuous unfiorm distribution can be written as $p(\mathrm{x}; a, b) = \frac{1}{b-a}$



Exercises

Classify each random variable as either discrete or continuous

1. The number of applicants for a job.
2. The time between customers entering a checkout lane at a retail store.
3. The temperature of a cup of coffee served at a restaurant.
4. The air pressure of a tire on an automobile.
5. The number of students who actually register for classes at a university next semester.

Exercises

Classify each random variable as either discrete or continuous

1. The number of applicants for a job. Discrete
2. The time between customers entering a checkout lane at a retail store. Continuous
3. The temperature of a cup of coffee served at a restaurant. Continuous
4. The air pressure of a tire on an automobile.Continuous
5. The number of students who actually register for classes at a university next semester. Discrete

Exercise

Determine whether or not the table is a valid probability distribution of a discrete random variable. Explain fully.

- X = -2, 0, 2, 4. P(X=-2)=0.2, P(X=0) = 0.3, P(X=2) = 0.3, P(X=4)=0.2.

- X = 0, 1, -1. P(X=0) = 0.3, P(X=1) = -0.0001, P(X=-1) = 0.7001.

- X= 1, 2, 3. P(X=1) = 0.2, P(X=2) = 0.2. P(X=3)= 0.2.

- X= 1, 2, 3, 4. P(X=1) = 0.2, P(X=2) = 0.2. P(X=3)= 0.2 P(X=4)= 0.5.

Exercise

Determine whether or not the table is a valid probability distribution of a discrete random variable. Explain fully.

- X = -2, 0, 2, 4. P(X=-2)=0.2, P(X=0) = 0.3, P(X=2) = 0.3, P(X=4)=0.2.

  Yes.

- X = 0, 1, -1.   P(X=0) = 0.3, P(X=1) = -0.0001, P(X=-1) = 0.7001.

  No. Because P(X=1)<0.

- X= 1, 2, 3.   P(X=1) = 0.2, P(X=2) = 0.2. P(X=3)= 0.2.

  No. Because P(X=1)+P(X=2)+P(X=3)=0.6<1.

- X= 1, 2, 3, 4.   P(X=1) = 0.2, P(X=2) = 0.2. P(X=3)= 0.2 P(X=4)= 0.5.

  No. Because P(X=1)+P(X=2)+P(X=3)+P(X=4)=1.1>1.

Exercise

A discrete random variable X has the following probability distribution:

X = 1, 3, 4, 70, 80, 90.

P(X=1) = 0.1, P(X=3)= 0.2, P(X=4) = 0.1, P(X=70)= 0.3, P(X=80) = 0.2.

What is P(X=90)?

What is P(X<70)?

What is P(X$\geq$70, X<90)?

Exercise

A discrete random variable X has the following probability distribution:

X = 1, 3, 4, 70, 80, 90.

P(X=1) = 0.1, P(X=3)= 0.2, P(X=4) = 0.1, P(X=70)= 0.3, P(X=80) = 0.2.

What is P(X=90)?

P(X=90) = 1-P(X=1)-P(X=3)-P(X=4)-P(X=70)-P(X=80)=0.1

What is P(X<70)? P(X<70)=P(X=1)+P(X=3)+P(X=4)=0.4

What is P(90>X≥70)?

P(90>X≥70)=P(X=70)+P(X=80)=0.3+0.2=0.5

Joint Distribution

- In some practice case, we need to consider multiple randon variables

For example, it is clear that the weight is related to the height. So we are interested in knowing the joint distribution related to dog's weight and height.

- If random variables X, Y are discrete random variables, then the joint distribution of X and Y is

$$P(X=x, Y=y)$$

Joint Distribution

- If random variables X, Y are continuous random variables, then the joint distribution of X and Y is

$$P(a1 \leq X \leq b1,\ a2 \leq Y \leq b2)$$

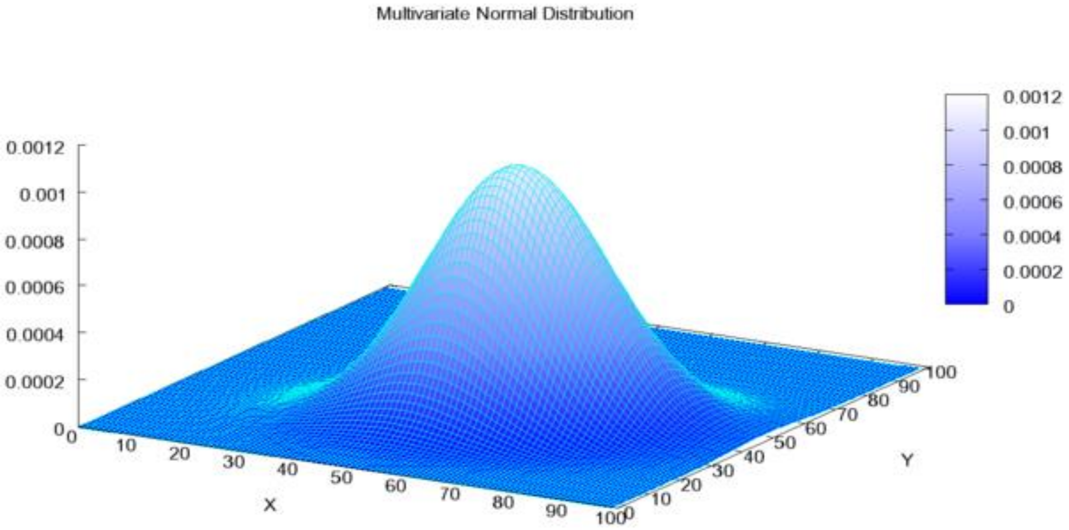
In fact, when X, Y are continuous random variables, there exists a probability density function for the joint distribution:

$$P(a1 \leq X \leq b1,\ a2 \leq Y \leq b2) = \int_{a1}^{b1} \int_{a2}^{b2} p_{XY}(x,y)\,dxdy,$$

where $p_{XY}(x,y)$ is the probability density function.

Joint Distribution

- If X represnets the weight of dog and Y represents the height of dog, then the joint distribuion P(X,Y) is similar to a two-dimensional Gaussian distribution.



Joint Probability

- If random variable X is continuous random variable, and Y is discrete random variable, then the joint distribution of X and Y is

$$P(a1 \leq X \leq b1,\ Y=\mathrm{x}_2)$$

In fact, when X, is continuous random variable, there exists a probability density function for the joint distribution:

$$P(a1 \leq X \leq b1,\ Y=y)=\int_{a1}^{b1} p_{XY}(x,y)dxdy,$$

where p_{XY}(x,y) is continuous with respect to x, but discrete with respect to y.

Marginal Probability

- Using joint distrbution, we can construct marginal distribution:
- If random variables X, Y are discrete random variables, then the marginal distributions are

$$P(X=x) = \sum_{y} P(X = x, Y = y)$$

$$P(X=y) = \sum_{x} P(X = x, Y = y)$$

Marginal Probability

- If random variables X and Y are continuous random variables, then the marginal distribution with respect to X is

$$\text{P(a} \leq \mathrm{X} \leq \mathrm{b}) = \int_a^b \int_{-\infty}^{+\infty} p_{XY}\,(x, y)\mathrm{dx}dy$$

and the density function of X is

$$p(\mathrm{x}) = \int_{-\infty}^{+\infty} p_{XY}\,(x, y)\mathrm{dx}dy$$

Similarly, we can obtain the marginal distribution with respect to Y and the density function of Y.

Marginal Probability

- If random variable X is continuous random variable, and Y is a discrete random variable, then the marginal distribution with respect to X is

$$\mathrm{P}(\mathrm{a} \leq X \leq \mathrm{b}) = \sum_y \int_a^b p_{XY}(x, y)dy$$

and the density function of X is

$$p(\mathrm{x}) = \sum_y p_{XY}(x, y)$$

The marginal distribution with respect to Y is

$$\mathrm{P(Y=y)} = \int_a^b p_{XY}(x, y)d\mathrm{x}$$

Joint Probability and Marginal Probability : An Example

- Joint probabilities can be between any number of variables
- For each combination of variables, we need to say how probable that combination is
- The probabilities of these combinations need to sum to 1
- Once you have the joint probability distribution, you can calculate any probability involving X, Y, and Z

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Joint Probability and Marginal Probability: An Example

- P(X=1, Y =1) = P(X=1, Y = 1, Z = 1) + P(X=1, Y = 1, Z = 0) = 0.2

- P(X=1, Y = 0) = P(X=1, Y = 0, Z = 1) + P(X=1, Y = 0, Z= 0) = 0.4

- P(X=1) = P(X=1, Y = 1) + P(X=1, Y = 0) =0.6

X	Y	Z	P(X,Y,Z)
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

- Try
  - P(Y= 1)

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Joint Probability and Marginal Probability: An Example

- Solution:

  $P(Y=1) = P(X=1,Y=1,Z=1)+$

  $P(X=0,Y=1,Z=1)+$

  $P(X=1,Y=1,Z=0)+$

  $P(X=0,Y=1,Z=0)$

  $= 0.15+0.05+0.05+0.05$

  $= 0.3$

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Conditional Probability

- Given an envent E and an event F,

the definition of condition probability is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

- P(X=1 | Y=1) = P(X = 1, Y = 1) / P(Y = 1)

= 0.2/0.3= 2/3

X	Y	Z	P(X,Y,Z)
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Conditional Distribution

- Conditional distributions seek to answer the question, what is the probability distribution over Y , when we know that X must take on a certain value x.

If X and Y are discrete random variables, then

$$P(Y = y|X{=}x){=}\ P(X{=}x,Y{=}y)/P(X{=}x)$$

Condition Distribution

$P(X=1 \mid Y=1) = P(X = 1, Y = 1) / P(Y = 1) = 0.2/0.3= 2/3$

$P(X=0 \mid Y=1) = P(X = 0, Y = 1) / P(Y = 1) = 0.1/0.3= 1/3$

$P(Y=1 \mid X=1) = P(X = 1, Y = 1) / P(X = 1) = 0.2/0.6= 1/3$

$P(Y=0 \mid X=1) = P(X = 1, Y = 0) / P(X = 1) = 0.4/0.6= 2/3$

$P(Z=1 \mid X=1) = P(X = 1, Z = 1) / P(X = 1) = 0.25/0.6= 5/12$

$P(Z=0 \mid X=1) = P(X = 1, Z = 0) / P(X = 1) = 0.35/0.6= 7/12$

Try:

$P(Z=0|X=1, Y=1)?$
$P(Y=0|X=1, Z=1)?$

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Condition Distribution

- Solution

$$P(X=1,Y=1) = P(X=1,Y=1,Z=1)+P(X=1,Y=1,Z=0)$$
$$= 0.15+0.05=0.2$$

$$P(X=1,Z=1) = P(X=1,Y=1,Z=1)+P(X=1,Y=0,Z=1)$$
$$= 0.15+0.1=0.25$$

$$P(Z=0|X=1,Y=1) = P(X=1,Y=1,Z=0)/P(X=1,Y=1)$$
$$= 0.05/0.2=1/4$$

$$P(Y=0|X=1,Z=1) = P(X=1,Y=0,Z=1)/P(X=1,Z=1)$$
$$= 0.1/0.25= 2/5$$

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Conditional Distribution

- If X and Y are continuous random variables, then

  $$\mathrm{P}(\mathrm{a2} \leq \mathrm{Y} \leq \mathrm{b2} | \, \mathrm{a1} \leq \mathrm{X} \leq \mathrm{b2}) = \int_{a1}^{b1} \int_{a2}^{b2} p_{\mathrm{XY}}(\mathrm{x,y})\mathrm{dxdy} / \int_{a1}^{b1} p_{\mathrm{X}}(\mathrm{x})\mathrm{dx},$$

  so as a1 and b2 get close to x, we obtain that

If $p_{\mathrm{X}}(x) > 0$, then

$$\mathrm{P}(\mathrm{a2} \leq \mathrm{Y} \leq \mathrm{b2} | \, \mathrm{X} = \mathrm{x}) = \int_{a2}^{b2} p_{\mathrm{XY}}(\mathrm{x,y})\mathrm{dxdy} / p_{\mathrm{X}}(\mathrm{x})$$

So if $p_{\mathrm{X}}(x) > 0$, then P(Y|X=x) is a continuous distribution with probability density function

$$p_{\mathrm{Y|X}}(\mathrm{y|x}) = p_{\mathrm{XY}}(\mathrm{x,y}) / p_{\mathrm{X}}(\mathrm{x})$$

Chain Rule of Conditional Probabilities

- Given n events E_1, E_2,…,E_n

  P($E_1 \cap E_2$) = P($E_2 | E_1$) P(E_1);

  P($E_1 \cap E_2 \cap E_3$) = P($E_3 \cap E_2 | E_1$)P(E_1)=P($E_3 | E_2 \cap E_1$)P($E_2 | E_1$)P(E_1)

  ………..

  P($E_1 \cap E_2$….$\cap$ E_n) = $\mathrm{P}(E_1) \prod_{i=2}^{n} \mathrm{P}(E_i | E_1 \cap E_2 \cap \cdots \cap E_{i-1})$

Above equation is called *chain rule*.

Chain Rule of Conditional Probabilities

- Chain rule with respect to random variables

Given random variables $X^1, \ldots, X^n$

$$P(X^1, \ldots, X^n) = P(X^1) \prod_{i=2}^{n} P(X^i | X^1, \ldots, X^{i-1})$$

P(X=1, Y=1, Z=1) =0.15;

P(X=1) = 0.6;

P(Y=1|X=1)= 1/3;

By chain rule, we obtain that

P(Z=1|X=1, Y=1)

= P(X=1, Y=1, Z=1)/(P(Y=1|X=1)P(X=1))

= 0.15/0.2 = 3/4

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Chain Rule of Conditional Probabilities

$P(X=0, Y=1, Z=1) = 0.05;$

$P(X=0) = 0.4;$

$P(Y=1|X=0) = 0.1/0.4 = 1/4;$

By chain rule, we obtain that

$$
\begin{aligned}
&\quad P(Z=1|X=0, Y=1) \\
&= P(X=0, Y=1, Z=1)/(P(Y=1|X=0)P(X=0)) \\
&= 0.05/0.1 = 1/2
\end{aligned}
$$

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Chain Rule of Conditional Probabilities

Exercise: Please use china rule to compute

P(Z=1|X=0,Y=0)

X	Y	Z	P(X,Y,Z)
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Chain Rule of Conditional Probabilities

- Solution

P(X=0, Y=0, Z=1) =0.2;

P(X=0) = 0.4;

P(Y=0|X=0)= (0.1+0.3)/0.4=3/4;

By chain rule, we obtain that

P(Z=1|X=0, Y=0)

= P(X=0, Y=0, Z=1)/(P(Y=0|X=0)P(X=0))

= 0.2/0.3 = 2/3

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Independence and Conditional Independence

- Two events E and F are independent if P(E∩F)=P(E)P(F)

- How is independence useful?

Suppose you have *n* coin flips, and you want to calculate the joint distribution P(E1∩E2∩...∩En). If the coin flips are not independent, you need 2^n values in the table.

If the coin flips are independent, then

P(E1∩E2∩...∩En) =P(E1)P(E2)...P(En).

Each P(Ei) table has 2 entries and there are n of them for a total of 2n values

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Independence and Conditional Independence

- If events E and F are independent, then

$$P(E|F) = P(E\cap F)/P(F)=P(E)P(F)/P(F)=P(E).$$

So

$$P(E|F) = P(E)$$

We can also obtain that

$$P(F|E) = P(F)$$

Independence and Conditional Independence

- Two random variables X and Y are independent if their probability distribution can be expressed as a product of two factors:

$$P(X, Y) = P(X)P(Y)$$

If X and Y are both discrete random variables, then X and Y are independent, if for any x, y

$$P(X=x, Y=y) = P(X=x)P(Y=y)$$

We can also obtain that $P(X=x|Y=y) = P(X=x)$ and $P(Y=y|X=x)=P(Y=y)$.

Independence and Conditional Independence

If X and Y are continuous random variables, then X and Y are independent, if for any x, y,

$$p_{\mathrm{XY}}(x,y)=p_{\mathrm{X}}(x)p_{\mathrm{Y}}(y)$$

where p_{XY}(x,y) is the density function of the joint distribution, p_{X}(x) is the density function with respect to random variable X, and p_{Y}(x) is the density function with respect to random variable Y.

Independence: An Example

- Throw a coin twice. The second time won't be affected by the first time.

- If we set X1 =1 if the outcome is head at the first time; X1=-1 if the outcome is tail at the first time.

- If we set X2 =1 if the outcome is head at the second time; X2=-1 if the outcome is tail at the second time.

- X1 and X2 are independent.

Exercise

In this table, please answer

1. Are X and Y independent?

2. Are Y and Z independent?

3. Are X and Z independent?

X	Y	Z	P(X,Y,Z)
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Exercise

1. P(X=1) = 0.6, P(Y=1)=0.3

   P(X=1,Y=1) = 0.2

   P(X=1,Y=1)>P(X=1)P(Y=1).

   So X and Y are not independent.

2. P(Z=1) = 0.5, P(Y=1,Z=1)=0.2

   P(Z=1)P(Y=1)<P(Y=1,Z=1).

   So Y and Z are not independent.

3. P(X=1, Z=1)=0.25

   P(Z=1)P(X=1)>P(X=1,Z=1)

   So X and Z are not independent.

X	**Y**	**Z**	**P(X,Y,Z)**
0	0	0	0.1
0	0	1	0.2
0	1	0	0.05
0	1	1	0.05
1	0	0	0.3
1	0	1	0.1
1	1	0	0.05
1	1	1	0.15

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Exercise

In this table, please answer

Are X and Y independent?

X	Y	P(X,Y)
0	0	0.25
0	1	0.25
1	0	0.25
1	1	0.25

- Solution:

$P(X=1)=P(Y=1)=P(X=0)=P(Y=0)=0.5$

$P(X=1,Y=1)=P(X=1,Y=0)=P(X=0,Y=1)=P(X=0,Y=0)=0.25$

So it is easy to check that for any $i=1$ or 2 and $j=1$ or 2

$P(X=i,Y=j)=P(X=i)P(Y=j)=0.25$

So X and Y are independent.

Exercise

In this table, please answer

1. Are X and Y independent?

2. Are Y and Z independent?

X	Y	Z	P(X,Y,Z)
0	0	0	0.05
0	0	1	0.2
0	1	0	0.1
0	1	1	0.15
1	0	0	0.1
1	0	1	0.15
1	1	0	0.05
1	1	1	0.2

Exercise

- Solution

1. Let i=1 or 2 and j=1 or 2
P(X=i)P(Y=j)=P(X=i,Y=j)=0.25
So X and Y are independent.

2. P(Y=1)=0.5, P(Z=1)=0.7,
P(Y=1,Z=1)=0.35=P(Y=1)P(Z=1)
P(Y=1,Z=0)=0.15=P(Y=1)P(Z=0)
P(Y=0,Z=1)=0.35=P(Y=0)P(Z=1)
P(Y=0,Z=0)=0.15=P(Y=0)P(Z=0)
So Y and Z are independent.

X	Y	Z	P(X,Y,Z)
0	0	0	0.05
0	0	1	0.2
0	1	0	0.1
0	1	1	0.15
1	0	0	0.1
1	0	1	0.15
1	1	0	0.05
1	1	1	0.2

Independence and Conditional Independence

Given three events A, B and C, we say A and B are independent with respect to C, if

$$P(A \cap B|C) = P(A|C)P(B|C)$$

It is easy to check that if A and B are independent with respect to C, then

$$P(A|B,C) = P(A|C)$$

$$P(B|A,C) = P(B|C)$$

Independence and Conditional Independence

1. $P(A \cap B|C) = P(A|C)P(B|C)$
2. $P(A|B,C) = P(A|C)$
3. $P(B|A,C) = P(B|C)$

How to derive the first equation from the second or third one?

Independence and Conditional Independence

How to derive

$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$ from $P(A \mid B, C) = P(A \mid C)$ or $P(B \mid A, C) = P(B \mid C)$?

$$P(A \mid B, C) = P(A \mid C)$$

$$P(A \cap B \cap C) / P(B \cap C) = P(A \cap C) / P(C)$$

$$P(A \cap B \cap C) / P(C) = (P(A \cap C) / P(C)) * (P(B \cap C) / P(C))$$

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

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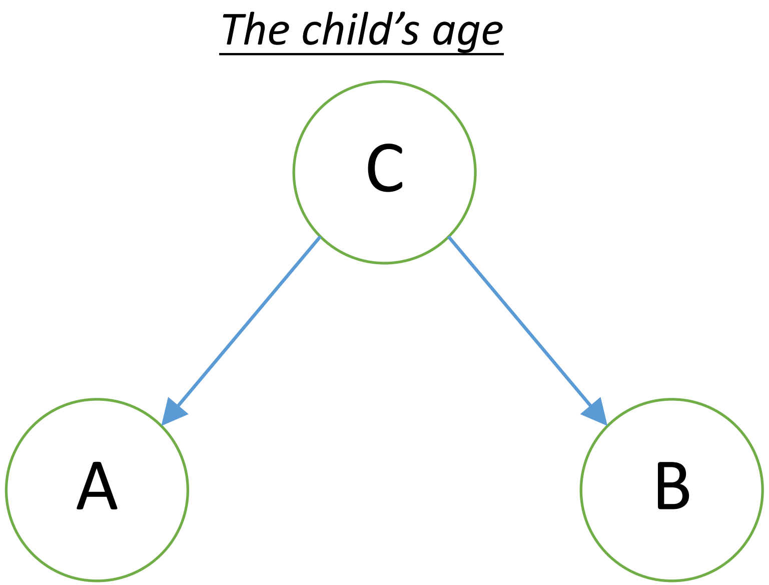
Conditional Independence: An Example



A is the *height of a child* and
B is the *number of words that the child knows*.
It seems when A is high, B is high too.

There is, however, a single piece of information that will make A and B completely independent. What would that be?
The child's age!

The height and the # of words known by the kid are NOT independent, but they are conditionally independent if you provide the kid's age.

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Conditional Independence: An Example

- A box contains two coins: a regular coin and one fake two-headed coin (P(H)=1). I choose a coin at random and toss it twice. Define the following events.

A = First coin toss results in an H.

B = Second coin toss results in an H.

C = Coin 1 (regular) has been selected.

Find $P(A|C)$,$P(B|C)$,$P(A\cap B|C)$,$P(A)$,$P(B)$, and $P(A\cap B)$.

Show that A and B are NOT independent, but they are conditionally independent given C.

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Conditional Independence: An Example

A = First coin toss results in an H.

B = Second coin toss results in an H.

C = Coin 1 (regular) has been selected.

We have $P(A|C)=P(B|C)=1/2$. Also, given that Coin 1 is selected, we have $P(A,B|C)=(1/2)*(1/2)=1/4$. So A and B are conditionally independent on C!

To find $P(A)$,$P(B)$, and $P(A\cap B)$, we use the law of total probability:

$P(A) =P(A|C)P(C)+P(A|C^c)P(C^c) =(1/2)*(1/2)+1*(1/2) =3/4$.

Similarly, $P(B)=3/4$. For $P(A\cap B)$, we have

$P(A\cap B)= P(A\cap B|C)P(C)+P(A\cap B|C^c)P(C^c) = P(A|C)P(B|C)P(C) + P(A|C^c)P(B|C^c)P(C^c) = (1/2)*(1/2)*(1/2)+1*1*(1/2)=5/8$.

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Conditional Independence: An Example

A = First coin toss results in an H.

B = Second coin toss results in an H.

C = Coin 1 (regular) has been selected.

As we see, $P(A,B)=5/8 \neq P(A)*P(B) = 9/16$, which means that A and B are not independent. We can justify this intuitively:

If we know A has occurred (i.e., the first coin toss has resulted in heads), we would guess that it is more likely that we have chosen Coin 2 than Coin 1. This in turn increases the conditional probability that B occurs. This suggests that A and B are not independent. On the other hand, given C (Coin 1 is selected), A and B are independent.

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Conditional Independence: An Example

One important lesson:

<u>Conditional independence neither implies (nor is it implied by) independence</u><u>.</u>

Thus, we can have two events that are conditionally independent but they are not unconditionally independent (such as A and B above). **Also, we can have two events that are independent but not conditionally independent, given an event C**.

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Conditional Independence: An Example

Alarm System Example

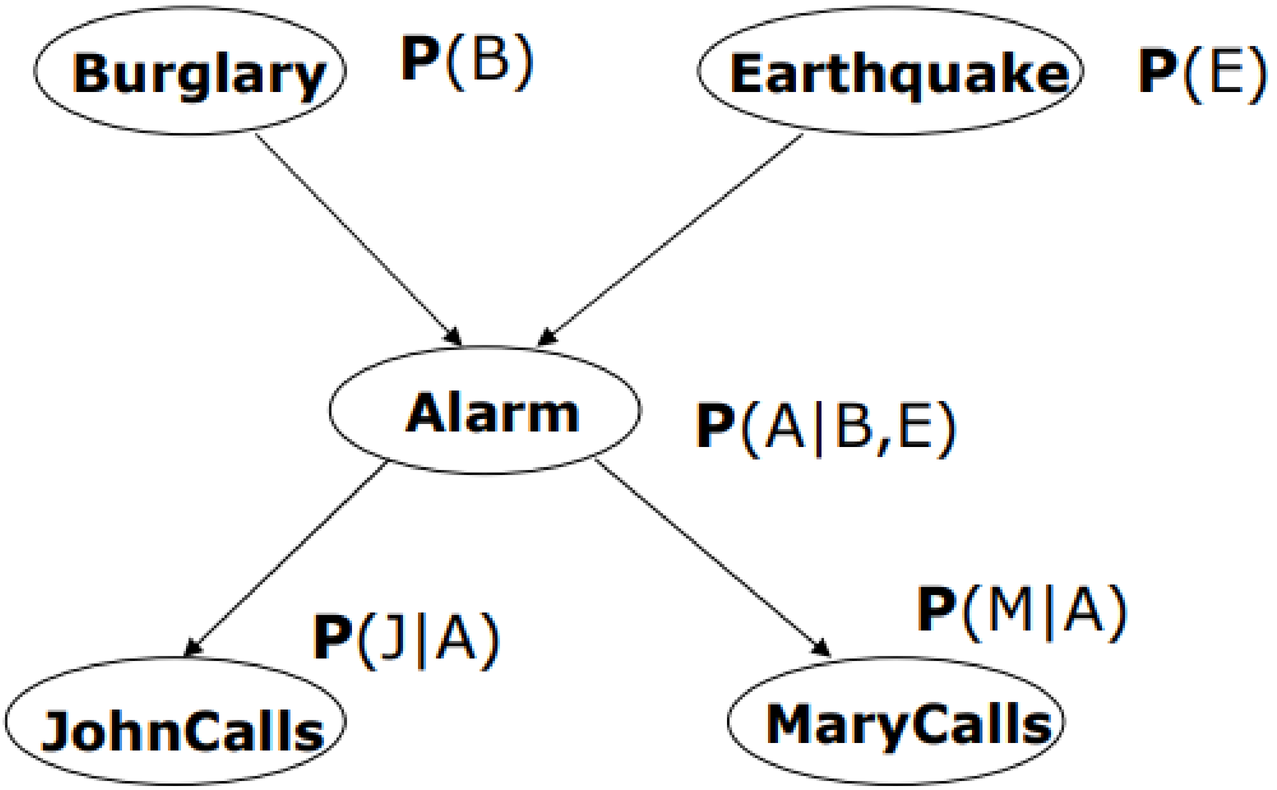
- Assume your house has an **alarm system** against **burglary**. You live in the seismically active area and the alarm system can get occasionally set off by an **earthquake**. You have two neighbors, **Mary** and **John**, who do not know each other. If they hear the alarm they might call you, but this is not guaranteed.

- Now we represent the probability distribution of events:
  - Burglary, Earthquake, Alarm, Mary calls, and John calls

Credits from Credit: CS 2740 Knowledge Representation, M. Hauskrecht

Conditional Independence: An Example

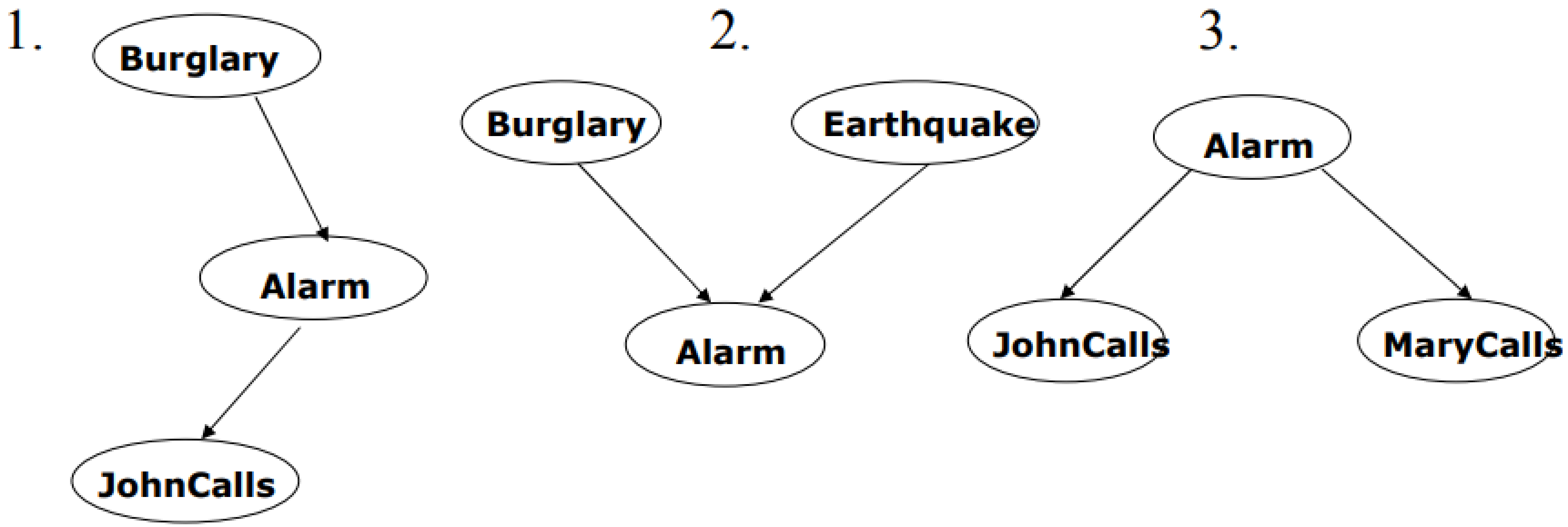
Alarm System Example



Credits from Credit: CS 2740 Knowledge Representation, M. Hauskrecht

Conditional Independence: An Example

Alarm System Example: 3 basic structures



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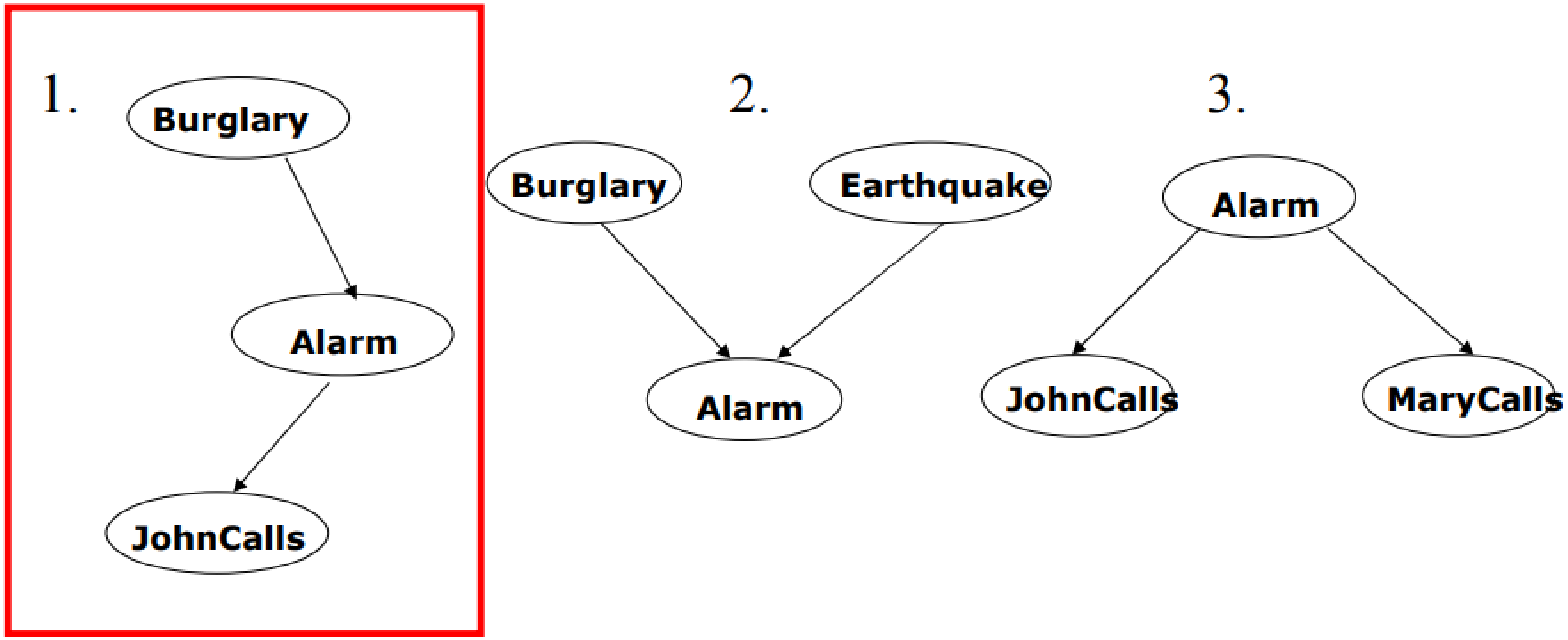
Conditional Independence: An Example

Alarm System Example: 3 basic structures



1. JohnCalls **is independent** of Burglary given Alarm

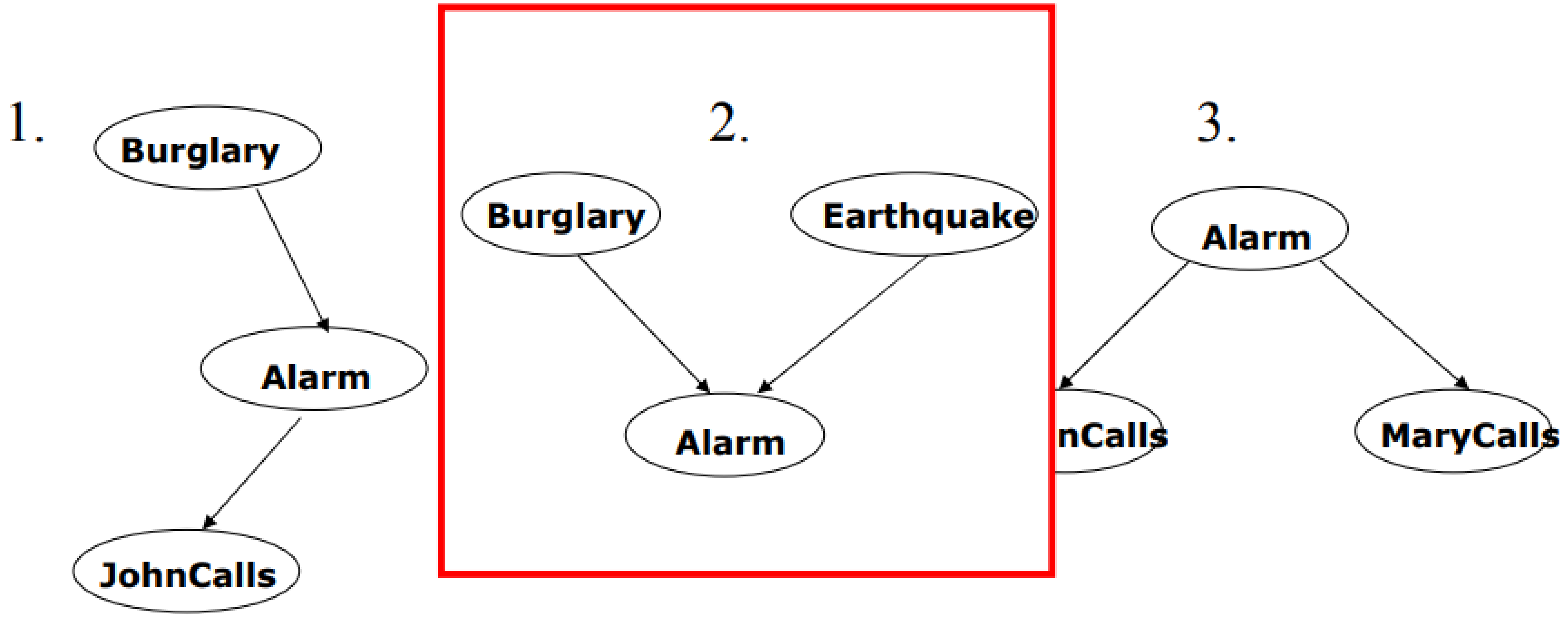
$$P(J \mid A, B) = P(J \mid A)$$

$$P(J, B \mid A) = P(J \mid A)P(B \mid A)$$

Credits from Credit: CS 2740 Knowledge Representation, M. Hauskrecht

Conditional Independence: An Example

Alarm System Example: 3 basic structures



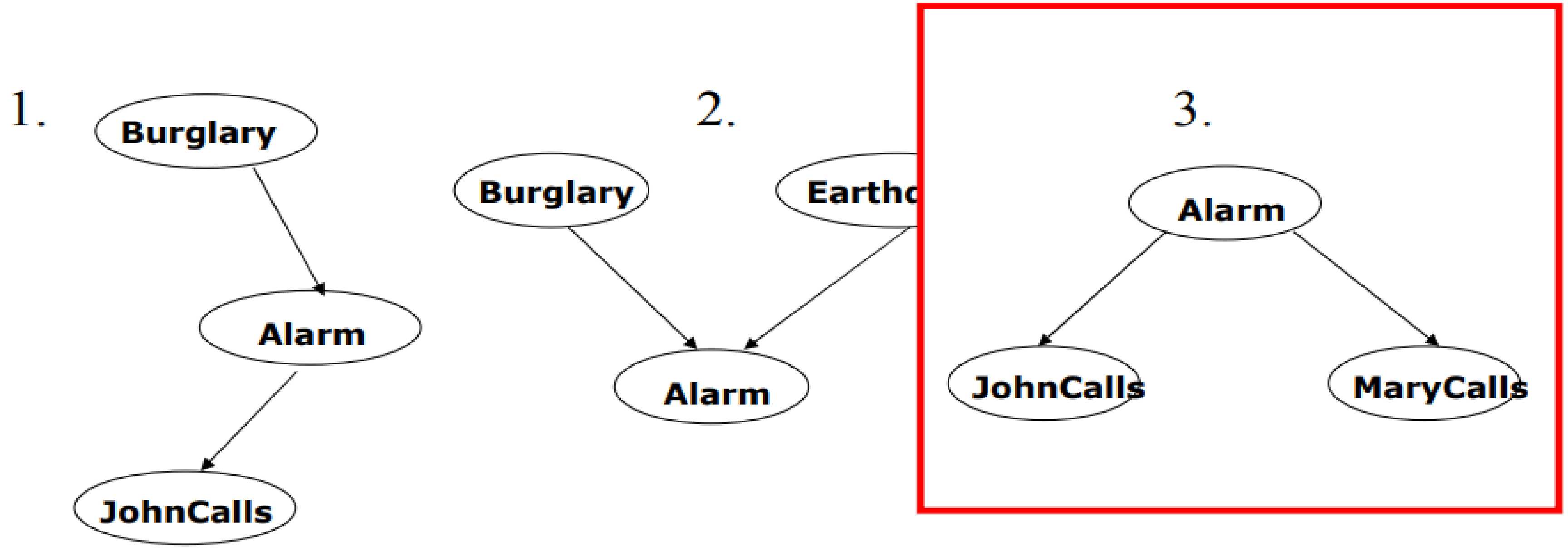
2. Burglary **is independent** of Earthquake (not knowing Alarm)
   Burglary and Earthquake **become dependent** given Alarm !!

$$P(B,E) = P(B)P(E)$$

Credits from Credit: CS 2740 Knowledge Representation, M. Hauskrecht

Conditional Independence: An Example

Alarm System Example: 3 basic structures



3. MaryCalls **is independent** of JohnCalls given Alarm

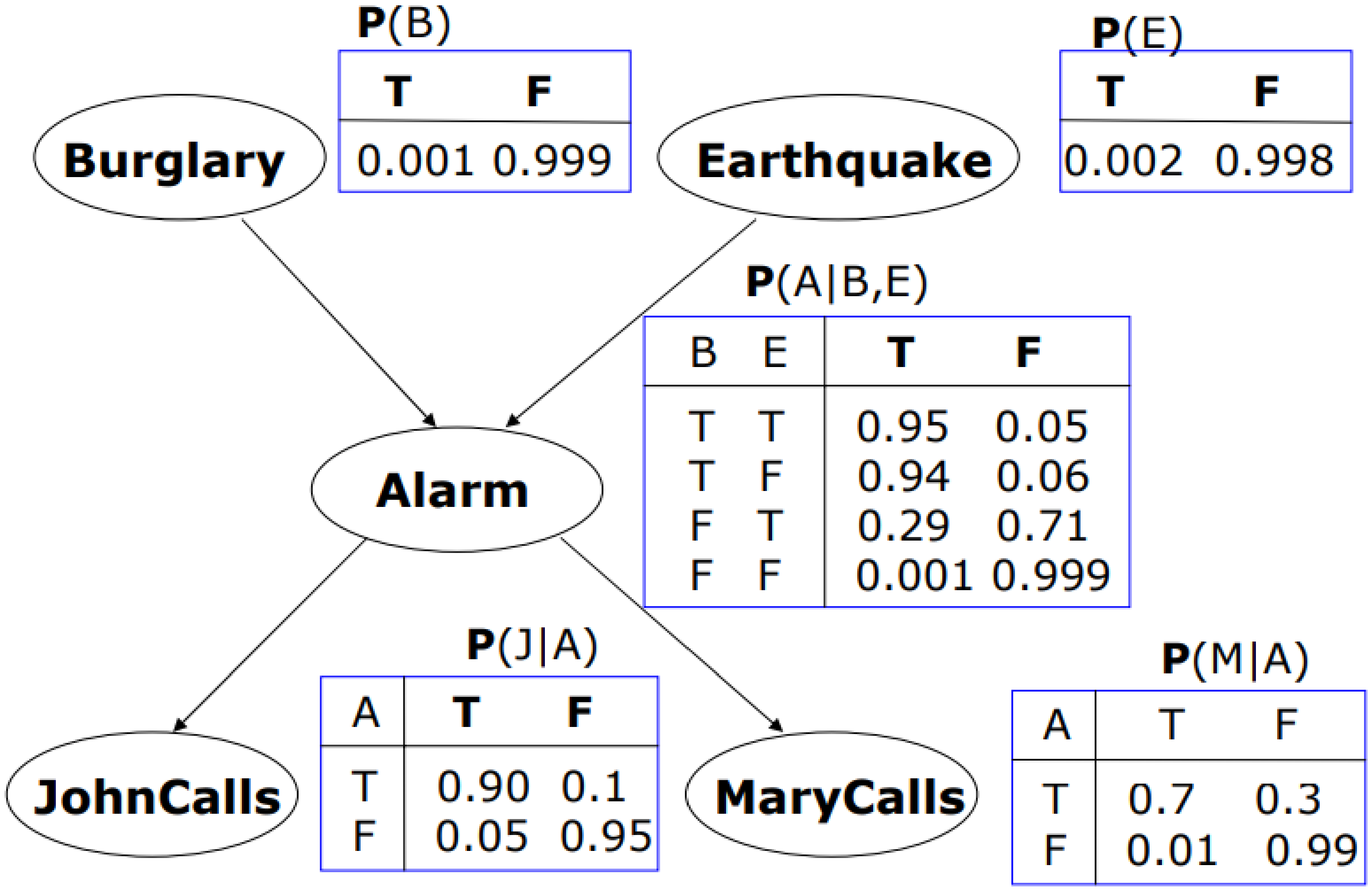
$$P(J \mid A, M) = P(J \mid A)$$

$$P(J, M \mid A) = P(J \mid A) P(M \mid A)$$

Credits from Credit: CS 2740 Knowledge Representation, M. Hauskrecht

Conditional Independence: An Example

Alarm System Example: 3 basic structures



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

Credits from Credit: CS 2740 Knowledge Representation, M. Hauskrecht

Conditional Independence to Random Variables

Two random variables X and Y are conditionally independent given a random variable Z if the conditional probability distribution

$$P(X,Y|Z) = P(X|Z)P(Y|Z)$$

When X, Y and Z are discrete variable variables, if X and Y are conditionally independent given a random variable Z: for any x, y, z

$$P(X=x,Y=y|Z=z) = P(X=x|Z=z)P(Y=y|Z=z)$$

Conditional Independence to Random Variables

When X, Y and Z are continouos variable variables, if X and Y are conditionally independent given a random variable Z: for any x, y, z,

then the probability density functions satisfy that

$$p_{\mathrm{XY|Z}}(\mathrm{x},\mathrm{y}|\mathrm{z}) = p_{\mathrm{X|Z}}(\mathrm{x}|\mathrm{z})p_{\mathrm{Y|Z}}(\mathrm{y}|\mathrm{z})$$

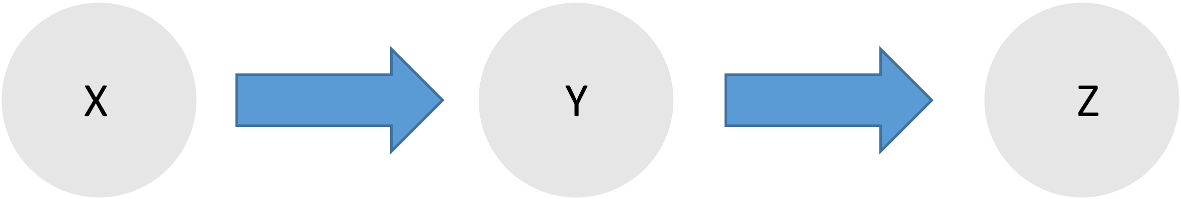
So

$$\frac{p_{\mathrm{XYZ}}(\mathrm{x,y,z})}{p_{\mathrm{Z}}(\mathrm{z})} = \frac{p_{\mathrm{XZ}}(\mathrm{x,z})}{p_{\mathrm{Z}}(\mathrm{z})}\frac{p_{\mathrm{YZ}}(\mathrm{y,z})}{p_{\mathrm{Z}}(\mathrm{z})}$$

$$p_{\mathrm{XYZ}}(\mathrm{x},\mathrm{y},\mathrm{z}) = p_{\mathrm{XZ}}(\mathrm{x},\mathrm{z})p_{\mathrm{YZ}}(\mathrm{y},\mathrm{z})$$

Exercise

- Given random variables X, Y, and Z



This graph is associated with $P(X,Y,Z) = P(Z|Y)P(Y|X)P(X)$

To show that X and Z are conditional independent given Y

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Solution: By chain rule, $P(X, Y, Z)=P(X)P(Y|X)P(Z|X,Y)$.

Because $P(X, Y, Z) = P(X)P(Y|X)P(Z|Y)$.

We obtain that $P(Z|X,Y)=P(Z|Y)$. So $P(Z|X,Y)P(X|Y) = P(X|Y)P(Z|Y)$.

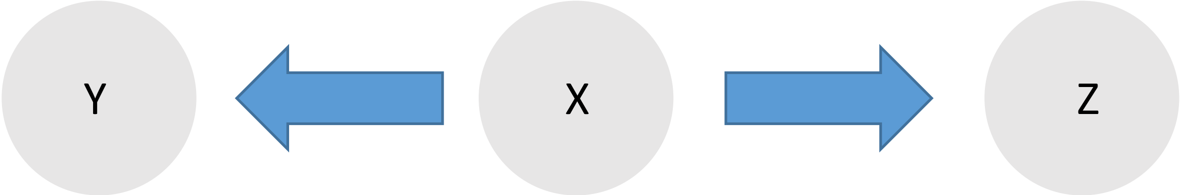
Note that $P(Z|X,Y)P(X|Y)=P(X,Z|Y)$, so $P(X,Z|Y)=P(X|Y)P(Z|Y)$.

Answer: X and Z are independent given Y.

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Given random variables X, Y, and Z



This graph is associated with P(X,Y, Z) = P(Y|X)P(Z|X)P(X)

To show that Y and Z are conditional independent given X

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Solution: By chain rule, P(X, Y, Z)=P(X)P(Y|X)P(Z|X,Y).

Because P(X, Y, Z) = P(X)P(Y|X)P(Z|X).

We obtain that P(Z|X,Y)=P(Z|X). So P(Z|X,Y)P(Y|X) = P(Z|X)P(Y|X).

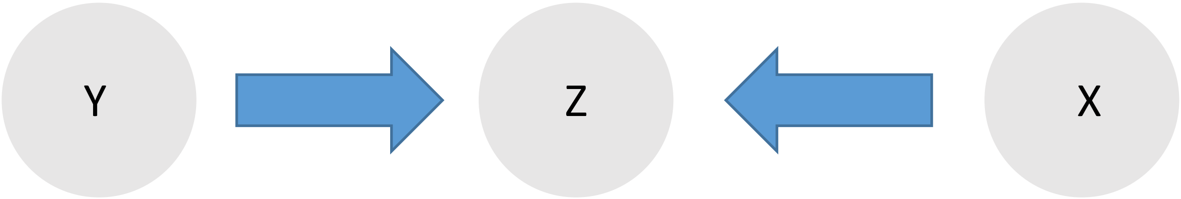
Note that P(Z|X,Y)P(Y|X)=P(Y,Z|X), so P(Y,Z|X)=P(Z|X)P(Y|X).

Answer: Y and Z are independent given X.

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Given random variables X, Y, and Z



This graph is associated with P(X,Y,Z) = P(Z|X,Y)P(X)P(Y)

To show that Y and X are independent.

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Solution: By chain rule, $P(X, Y, Z)=P(X)P(Y|X)P(Z|X,Y)$.

Because $P(X, Y, Z) = P(Z|X,Y)P(X)P(Y)$.

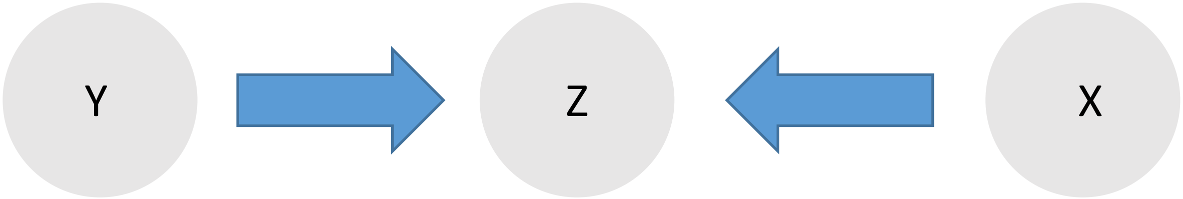
We obtain that $P(Y|X)=P(Y)$. So $P(X,Y)=P(Y|X)P(X)=P(Y)P(X)$.

Answer: X and Y are independent.

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Given random variables X, Y, and Z



This graph is associated with $P(X,Y,Z) = P(Z|X,Y)P(X)P(Y)$

Are Y and X independent given Z?

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Solution: There exists case that X and Y are not independet given Z.

Case. Suppose that Z indicates whether our lawn is wet one morning; X and Y are two explanations for it being wet: either it rained (indicated by X), or the sprinkler turned on (indicated by Y).

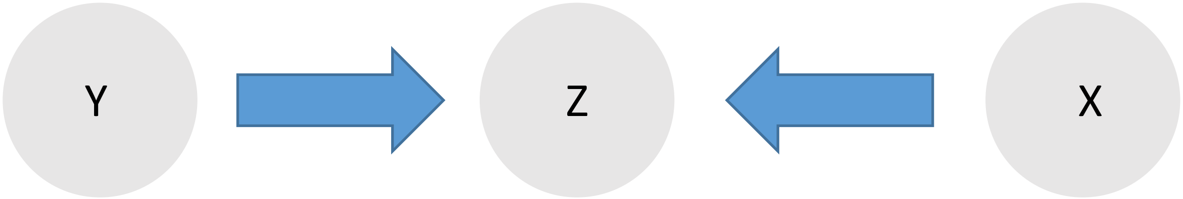
If we know that the grass is wet (Z is true) and the sprinkler didn’t go on (Y is false), then the probability that X is true must be one, because that is the only other possible explanation.

Hence, in this case, X and Y are not independent given Z.

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Exercise

- Given random variables X, Y, and Z



X and Y are independent. But there exists case that X and Y are not independent given Z.

This exercise indicates that
It is possible that X and Y are not independent given some other random variables, even if X and Y are independent.

Credits from Prof. Schmidt-Thieme, Randolf Scholz

Bayes’ Rule

- Bayes' theorem is an important tool in statistics and machine learning:

Given events A and B, then $\mathrm{P}(\mathrm{A}|\mathrm{B}) = \frac{\mathrm{P}(\mathrm{B}|\mathrm{A})\mathrm{P}(\mathrm{A})}{\mathrm{P}(\mathrm{B})}$.

Proof:

$$\mathrm{P}(\mathrm{A}|\mathrm{B}) \;=\; \mathrm{P}(\mathrm{A} \cap \mathrm{B})/\mathrm{P}(\mathrm{B}) \;=\; (\mathrm{P}(\mathrm{B}|\mathrm{A})\mathrm{P}(\mathrm{A}))/\mathrm{P}(\mathrm{B}).$$

Bayes’ Rule

- How to use Baye’s Rule?

Example: A bag I contains 4 white and 6 black balls while another bag II contains 4 white and 3 black balls.

One ball is drawn at random from one of the bags, and it is found to be black.

Find the probability that it was drawn from bag I.

Bayes' Rule

Soultion: Let E1 be the event of choosing bag I, E2 the event of choosing bag II, and A be the event of drawing a black ball.

Then, P(E1) = P(E2)= 0.5.

P(A|E1) = P(drawing a black ball from Bag I) = 6/10 = 3/5.

P(A|E2) = P(drawing a black ball from Bag II) = 3/7

P(A) = P(A|E1)P(E1)+P(A|E2)P(E2) = 18/35

By using Bayes' theorem, the probability of drawing a black ball from bag I out of two bags,

P(E1|A) = P(A|E1)P(E1)/P(A)=0.6*0.5/(18/35) = 7/12

Bayes’ Rule

- How to use Baye’s Rule?

Example: A man is known to speak the truth 2 out of 3 times. He throws a dice and reports that the number obtained is a four. Find the probability that the number obtained is four.

Bayes’ Rule

Soultion: Let A be the event that the man reports that number four is obtained. Let E1 be the event that four is obtained and E2 be its complementary event.

Then, P(E1) = Probability that four occurs = 1/6.

P(E2) = Probability that four does not occur = 1- P(E1) = 1 – (1/6) = 5/6.

P(A|E1)= Probability that man reports four and it is actually a four = 2/3

P(A|E2) = Probability that man reports four and it is not a four = 1/3.

So P(A) = P(A|E1)P(E1)+P(A|E2)P(E2) = 1/9+5/18 = 7/18.

By using Bayes’ theorem, probability that number obtained is actually a four, P(E1|A) = P(A|E1)P(E1)/P(A) = 2/18/(7/18) = 2/7.

Expectation (or Mean)

The expectation or mean of a random variable X *is denoted by* E[X] *and defined as:*

- For discrete variable,

$$\mathrm{E[X]} = \sum_x \mathrm{P}(X = x)x$$

- For continuous variable,

$$\mathrm{E[X]} = \int_{-\infty}^{+\infty} p_{\mathrm{X}}(\mathrm{x})\mathrm{x}\ \mathrm{dx}$$

In words, we are taking a weighted sum of the values that x can take on, where the weights are the probabilities of those respective values. The expected value has a physical interpretation as the “center of mass” of the distribution.

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Expectation (or Mean): An Example

- Assume that X is a discrete random variable with 4 possible values: x_1, x_2, x_3, x_4, and with the equal probability on these 4 values. What is the expectation of X?

Solution: According to the definition, we have

$$E[X] = \sum_x P(X = x)x$$

$$= \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4$$

$$= \frac{x_1+x_2+x_3+x_4}{4}$$

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Expectation (or Mean): An Example

- An exercise:

Let us consider an European roulette game that players can bet on any single number from 1, 2, …, 36. The number 0 is considered as winning for the casino. The probability of the appearance of any number from 0-36 is equal, i.e., 1/37.

Bob places 1 dollar on a specific number. If he wins, he can take the 1 dollar back and gains the extra 35 dollars from the casino; if he loses, he will lose the 1 dollar he bet on that number. ***Now, the question is: what is the expectation of the gain of Bob?***

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Expectation (or Mean): An Example

- An exercise:

Let us consider an European roulette game that players can bet on any single number from 1, 2, …, 36. The number 0 is considered as winning for the casino. The probability of the appearance of any number from 0-36 is equal, i.e., 1/37.

Bob places 1 dollar on a specific number. If he wins, he can take the 1 dollar back and gains the extra 35 dollars from the casino; if he loses, he will lose the 1 dollar he bet on that number. ***Now, the question is: what is the expectation of the gain of Bob?***

Solution:

$E[X] = (35)\times(1/37) +$

$(-1)\times(36/37)$

$= -1/37$

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Expectation of Functions

The expectation or mean of f(X) (a function of random variable X) is denoted by E[f(X)] and defined as:

For discrete variable,

$$\mathrm{E[f(X)]} = \sum_{x} \mathrm{P}(X = x)\mathrm{f}(x)$$

For continuous variable,

$$\mathrm{E[f(X)]} = \int_{-\infty}^{+\infty} p_{\mathrm{X}}(\mathrm{x})\mathrm{f}(\mathrm{x})\ \mathrm{dx}$$

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Properties of Expectation

For any two random variables X and Y, functions f and g, and any constants a, b ∈ R, the following equations hold:

- $E[a] = a,$ $E[f(a)] = f(a)$
- $E[X+Y] = E[X]+E[Y]$ $E[f(X)+g(Y)] = E[f(X)]+E[g(X)]$
- $E[aX] = aE[X]$ $E[af(X)] = aE[f(X)]$
- $E[aX+bY]=aE[X]+bE[Y]$ $E[af(X)+bg(Y)]=aE[g(X)]+bE[g(Y)]$

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Properties of Expectation

- For any two random variables X and Y and any constants a, b ∈ R, the following equations hold:

$$E[a] = a \tag{1}$$

Proof: From the definition of expectation, we have

$$E[a] = aP(X=a) = a \times 1 = a.$$

So, we have: the expectation of a constant is the constant itself.

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Properties of Expectation

- For any two random variables X and Y and any constants a, b ∈ R, the following equations hold:

$$E[aX] = aE[X] \qquad (2)$$

Proof: From the definition of expectation, we have: if X is disctete,

$$E[aX] = \sum_x P(X = x)ax = a\sum_x P(X = x)x = aE[X].$$

If X is continuous, the proof is similar.

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Properties of Expectation

- For any two random variables X and Y and any constants a, b ∈ R, the following equations hold:

$$E[X + Y] = E[X] + E[Y] \tag{3}$$

Proof: From the definition of expectation, we have: if X and Y are disctete,

$$E[X + Y] = \sum_x \sum_y (x + y)P(X = x, Y = y)$$

$$= \sum_x \sum_y xP(X = x, Y = y) + \sum_x \sum_y yP(X = x, Y = y)$$

$$= \sum_x x(\sum_y P(X = x, Y = y)) + \sum_y y(\sum_x P(X = x, Y = y)$$

$$= \sum_x xP(X = x) + \sum_y yP(Y = y) = E[X] + E[Y]$$

If X and Y are continuous, the proof is similar.

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Properties of Expectation

- For any two random variables X and Y and any constants a, b ∈ R, the following equations hold:

$$E[aX + bY] = E[aX] + E[bY] \tag{4}$$

Proof: From the definition of expectation, we have: if X and Y are disctete,

$$E[aX + bY] = E[aX] + E[bY]$$

$$= aE[X] + bE[Y]$$

If X and Y are continuous, the proof is similar.

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Properties of Expectation

- For any two random variables X and Y, and functios f and g and any constants a, b ∈ R, the following equations hold:

(5)

- $E[f(X)+g(Y)] = E[f(X)]+E[g(X)]$
- $E[af(X)] = aE[f(X)]$
- $E[af(X)+bg(Y)]=aE[g(X)]+bE[g(Y)]$

Proof: note that
f(X) and g(Y) are both random variables, therefore, using the properites (2), (3) and (4), we can prove the results.

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Expectation: Exercise

For any two random variables X and Y, if X and Y are independent, then the following equations hold:

$$E[XY]=E[X]E[Y]$$

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Expectation: Exercise

For any two random variables X and Y, if X and Y are independent, then the following equations hold:

$$E[XY]=E[X]E[Y]$$

Solution: According to the definition, we have

$$E[XY] = \sum_x \sum_y (xy) P(X = x, Y = y) .$$

Since X and Y are independent, we know that

$P(X = x, Y = y) = P(X = x) P(Y = y)$. Therefore, we have

$$E[XY] = \sum_x \sum_y (xy) P(X = x) P(Y = y)$$

$$= \left(\sum_x x P(X = x)\right) \left(\sum_y y P(Y = y)\right) = E[X]E[Y].$$

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Variance

- Expectation provides measure of the “center” of a distribution, but sometimes we are also interested in what the “spread” is about that center. Therefore, we define the variance Var(X) of a random variable X as follows:

$$\mathrm{Var}(X) = E[(X - E(X))^2]$$

- In words, this is the average squared deviation of the values of X from the mean of X.

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Variance of Functions

- Given a function f and random variable X, we define the variance Var(f(X)) as follows:

$$\mathrm{Var}(f(X)) = E[(f(X) - E(f(X)))^2]$$

Properties of Variance

For any random variable X and any a, b ∈ R, the following equations hold:

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

$$\mathrm{Var}(X) = E[X^2] – E[X]^2$$

Properties of Variance

For any random variable X and any a, b ∈ R, the following equations hold:

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

Proof: From the definition of variance, we have

$\mathrm{Var}(X) = E[(X - E(X))^2] = E[X^2 - 2XE(X) + E(X)^2]$

$= E[X^2] - E[2XE(X)] + E[E(X)^2].$

Note that E[X] and $E[X]^2$ are constants, so that

$\mathrm{Var}(X) = E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - E[X]^2.$

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Properties of Variance

For any random variable X and any a, b ∈ R, the following equations hold:

$$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$$

Proof: From $\mathrm{Var}(X) = E[X^2] - E[X]^2$, we have

$$\begin{aligned}\mathrm{Var}(aX+b) &= E[(aX+b)^2] - E[aX+b]^2 \\ &= E[a^2X^2 + 2abX + b^2] - (aE[X]+b)^2 \\ &= (a^2E[X^2] + 2abE[X] + b^2) - (a^2E[X]^2 + 2abE[X] + b^2) \\ &= a^2E[X^2] - a^2E[X]^2 = a^2\,(\,E[X^2] - E[X]^2\,) = a^2\mathrm{Var}(X)\end{aligned}$$

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Properties of Variance

If two random variables X and Y are independent, then

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Proof: Using the linearity of the expectation and the identity E(XY) = E(X)E(Y), which holds by the independence of X and Y, we can write

$\mathrm{Var}(X+Y) = E[(X+Y)^2] – E[X+Y]^2$

$= E[X^2+2XY+Y^2] – (E[X]^2+2E[X]E[Y]+E[Y]^2)$

$= (E[X^2] – E[X]^2) + (E[2XY] – 2E[X]E[Y]) + (E[Y^2] – E[Y]^2)$

$= \quad \mathrm{Var}(X) \quad + \quad 0 \quad + \quad \mathrm{Var}(Y)$

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Exercises

X has values 0, 1, 2 with P(X=0)=0.3; P(X=1)=0.3; P(X=2)=0.4

Y has values 0, 1, 3 with P(Y=0)=0.3; P(Y=1)=0.2; P(Y=3)=0.5

Please answer:

1. Var(X) and Var(2X+1000000)

2. Var(Y) and Var(3Y+10000000000)

3. If X and Y are independent, what is Var(2X+3Y+10000000)?

Exercises

X has values 0, 1, 2 with P(X=0)=0.3; P(X=1)=0.3; P(X=2)=0.4
Y has values 0, 1, 3 with P(Y=0)=0.3; P(Y=1)=0.2; P(Y=3)=0.5

- Solution:

1. E[X]=0*0.3+1*0.3+2*0.4=1.1, E[X^2]=0*0*0.3+1*1*0.3+2*2*0.3=1.5

Var(X) = E[X^2]-E[X]=1.5-1.1=0.4, Var(2X+1000000)=2*2*Var(X)=1.6

2. E[Y] = 0*0.3+1*0.2+3*0.5=1.7, E[Y^2]=0+1*1*0.2+3*3*0.5=4.7

Var(Y) = E[Y^2]-E[Y]= 3.0, Var(3Y+10000000000)=3*3*Var(Y)=27

3.Because X and Y are independent: Var(2X+3Y+10000000)=
Var(2X+3Y)=Var(2X)+Var(3Y)=4Var(X)+9Var(Y)=1.6+27=28.6

Covariance

- The covariance gives some sense of how much two values are linearly related to each other, as well as the scale of these variables

The covariance of two random variables X and Y is denoted by Cov(X, Y) and defined as

$$\mathrm{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] .$$

Covariance of functions: given functions f and g, then

$$\mathrm{Cov}(f(X), g(Y)) = E[(f(X) - E(f(X)))(g(Y) - E(g(Y)))] .$$

Properties of Covariance

It is clear that

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X);$$

$$\mathrm{Cov}(f(X),f(X)) = \mathrm{Var}(f(X)).$$

Properties of Covariance

If two random variables X and Y are independent, then

$$\mathrm{Cov}(X, Y) = 0 .$$

Proof: From the definition of covariance, we have

$\mathrm{Cov}(X, Y) = E[(X – E(X))(Y – E(Y))]$

$= E[XY – E(X)Y – XE(Y) + E(X)E(Y)]$

$= E[XY] – E[E(X)Y] – E[XE(Y)] + E[E(X)E(Y)]$

$= E[XY] – E(X)E(Y) – E(X)E(Y) + E(X)E(Y)= E[XY] – E(X)E(Y) = 0.$

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Covariance: Exercise

Please compute:

Cov(X,Y)

X	**Y**	**P(X,Y)**
0	0	0.25
0	1	0.25
1	0	0.25
1	1	0.25

Covariance: Exercise

Please compute:

Cov(X,Y)

X	Y	P(X,Y)
0	0	0.25
0	1	0.25
1	0	0.25
1	1	0.25

Solution:

Because X and Y are independent, so

Cov(X,Y)=0

Covariance matrix

If we have multiple random variables:

X =(X1, X2,...,Xn),

then we can define the convariance matrix

Cov(**X**) is a n*x*n matrix and the ij-th element of the matrix is

Cov(Xi,Xj)

The covariance between two variables is positive when they tend to move in the same direction and negative if they tend to move in opposite directions.

Covariance matrix

Let **X** = (X,Y)

Please compute: Cov(**X**)

X	**Y**	**P(X,Y)**
0	0	0.25
0	1	0.25
1	0	0.25
1	1	0.25

Covariance matrix

X	**Y**	**P(X,Y)**
0	0	0.25
0	2	0.25
2	0	0.25
2	2	0.25

Let **X** = (X,Y)

Please compute: Cov(**X**)

Because X and Y are independent ,

Cov(X,Y) = 0.

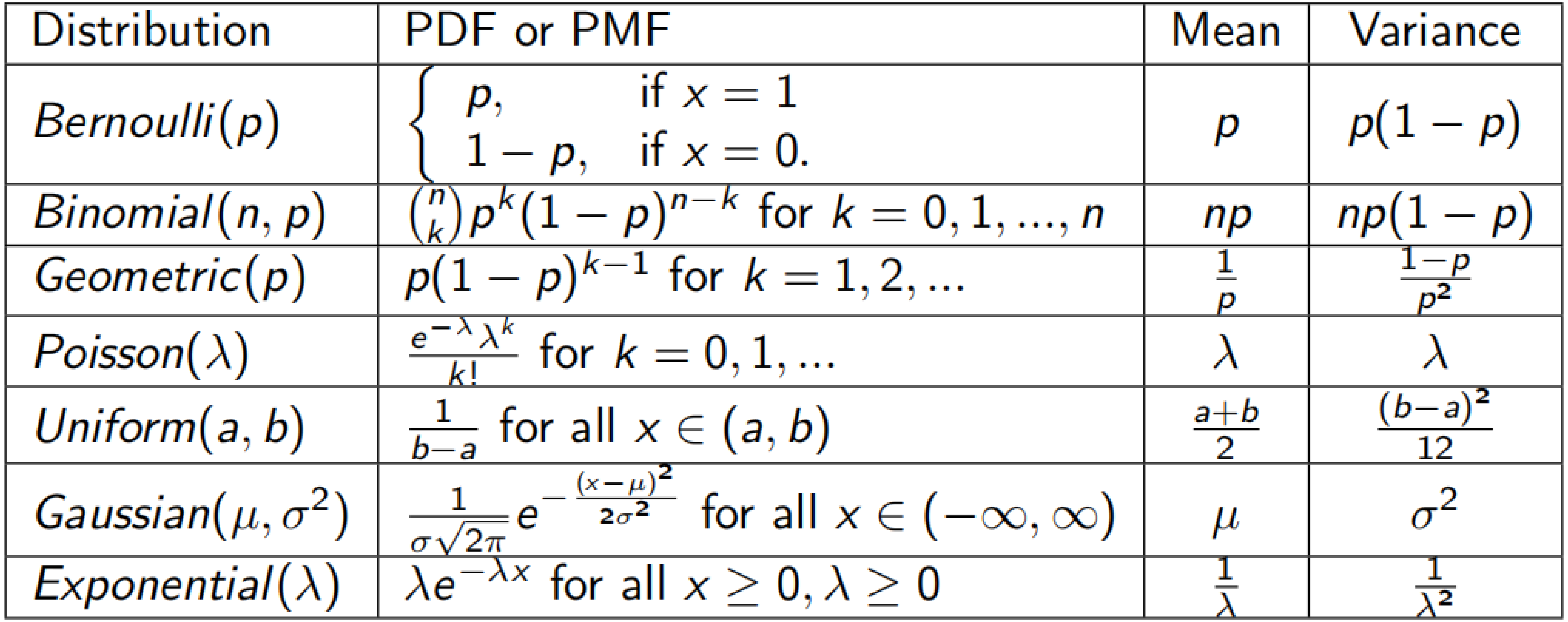
Cov(X,X) = Var(X) = E[X^2]-E[X]= 2*2*0.5 - 2*0.5 =1

Cov(Y, Y) = Var(Y) = E[Y^2]-E[Y]= 2*2*0.5 - 2*0.5 =1

So

$$\mathrm{Cov}(\mathbf{X}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Common Probability Distribution



Distribution	PDF or PMF	Mean	Variance
$Bernoulli(p)$	$\begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0. \end{cases}$	p	$p(1 - p)$
$Binomial(n, p)$	$\binom{n}{k} p^k (1 - p)^{n-k}$ for $k = 0, 1, ..., n$	np	$np(1 - p)$
$Geometric(p)$	$p(1 - p)^{k-1}$ for $k = 1, 2, ...$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$Poisson(\lambda)$	$\frac{e^{-\lambda} \lambda^k}{k!}$ for $k = 0, 1, ...$	λ	λ
$Uniform(a, b)$	$\frac{1}{b-a}$ for all $x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$Gaussian(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for all $x \in (-\infty, \infty)$	μ	σ^2
$Exponential(\lambda)$	$\lambda e^{-\lambda x}$ for all $x \geq 0, \lambda \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

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Bernoulli Distribution

Bernoulli(p)

P(X=1) = p
P(X=0) = 1-p

Less formally, it can be thought of as a model for the set of possible outcomes of any single experiment that asks a yes–no question.

For example, tossing a coin (only one time). If the possibility to appear the head is p, then fliping a coin is a Bernoulli Distribution with parameter p.

It is easy to check that
E[X] = p, Var(X)=p(1-p)

Credits from Wikipedia

Binomial Distribution

Binomial(p,n)

It is a generalization of the Bernoulli distribution to a distribution over integers. It can be used to describe the probability of observing m occurrences of X=1 in a set of n samples from a Bernoulli distribution where P(X=1)=p. The Binomial distribution is defined as

$$P(Y = m) = P(m; n, p) = \binom{n}{m} p^{m}(1 - p)^{n-m}$$

The expectation and variance of the binomial distribution are

$$E[Y] = np \text{ and } Var(Y) = np(1-p)$$

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Gaussian Distribution (Normal Distribution)

- **Gaussian Distribution (Normal Distribution)**: It is the most widely used model for the distribution of continuous variables. For a single variable X, the Gaussian distribution can be represented as follows:

$$N\left(\mu, \sigma^2\right).$$

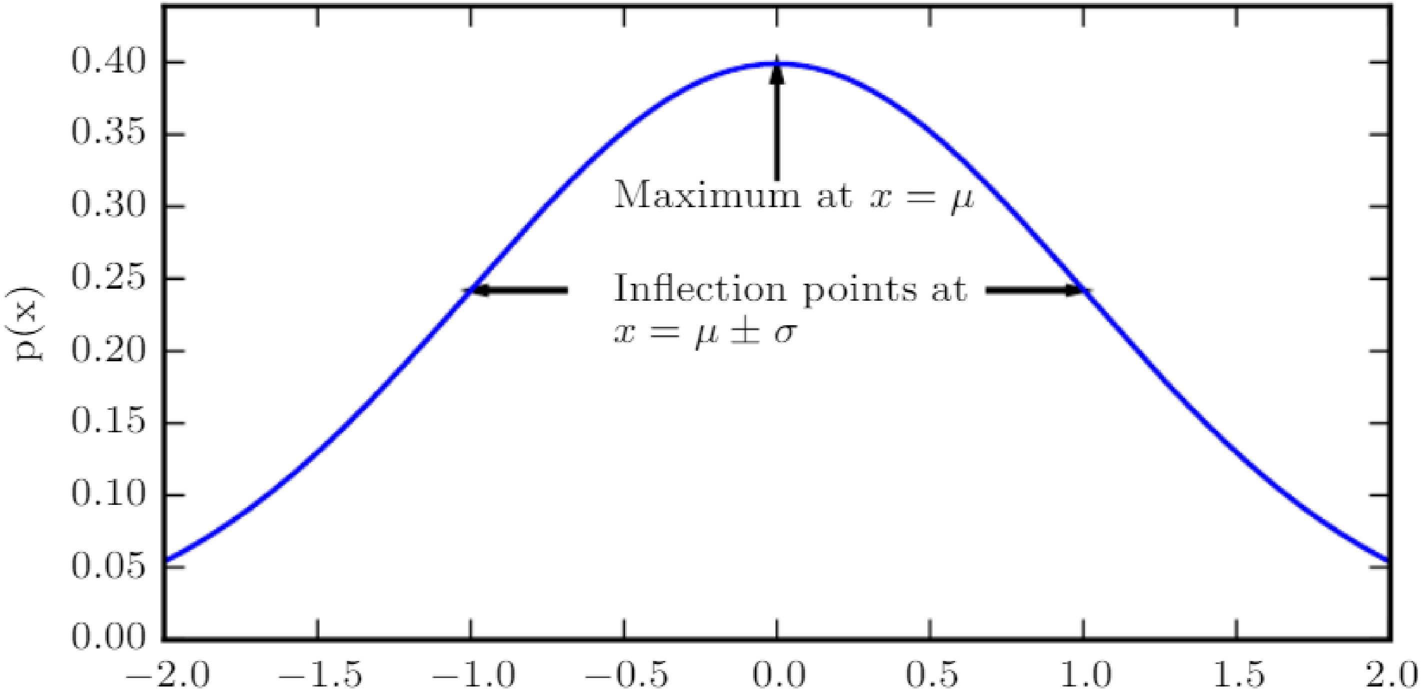
- It is a continuous distribution with the probability density function

$$\sqrt{\frac{1}{2\pi\sigma^2}}\exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$$

Gaussian Distribution (Normal Distribution)

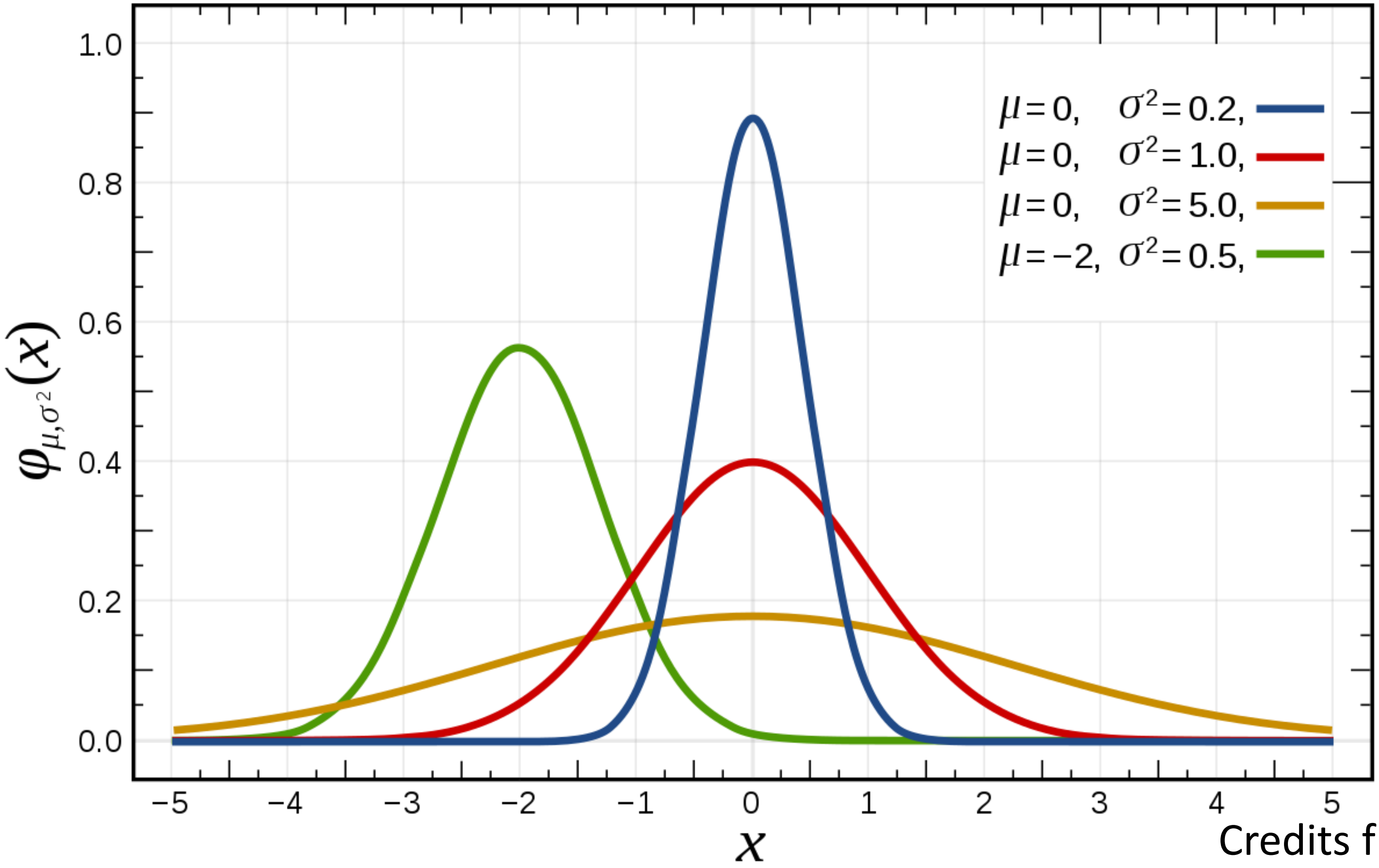
Gaussian distribution depends on two parameters μ, σ.

Then the expectation is μ and the variance is σ^2.



Gaussian Distribution (Normal Distribution)

- Gaussian distributions with different expectations and variances



Credits from Wikipedia

Exponential Distribution

- In the context of deep learning, we often want to have a probability distribution with a sharp point at $\mathrm{x} = 0$. To accomplish this, we can use the exponential distribution: the probability density function is

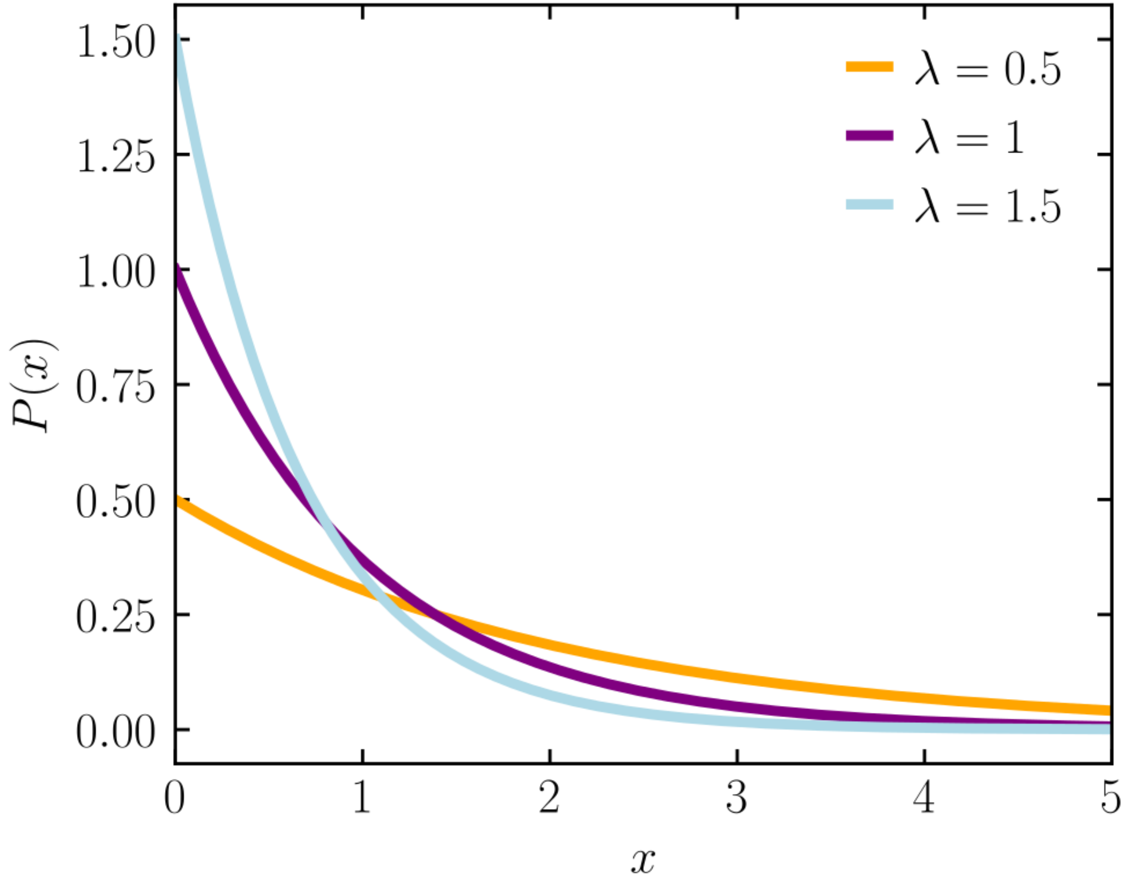
$$p(\mathrm{x}; \lambda) = \lambda \mathbf{1}_{\mathrm{x} \geq 0} \exp(-\lambda \mathrm{x})$$

- The exponential distribution is the probability distribution of the time between events in a Poisson point process, i.e., a process in which events occur continuously and independently at a constant average rate.

Credits from Wikipedia

Exponential Distribution

- Exponential distribution with different parameter λ.



Then the expectation is $\frac{1}{\lambda}$ and the variance is $\frac{1}{\lambda^2}$.

Credits from Wikipedia

Laplace Distribution

- A closely related probability distribution that allows us to place a sharp peak of probability mass at an arbitrary point μ is the Laplace distribution: the probability density function is

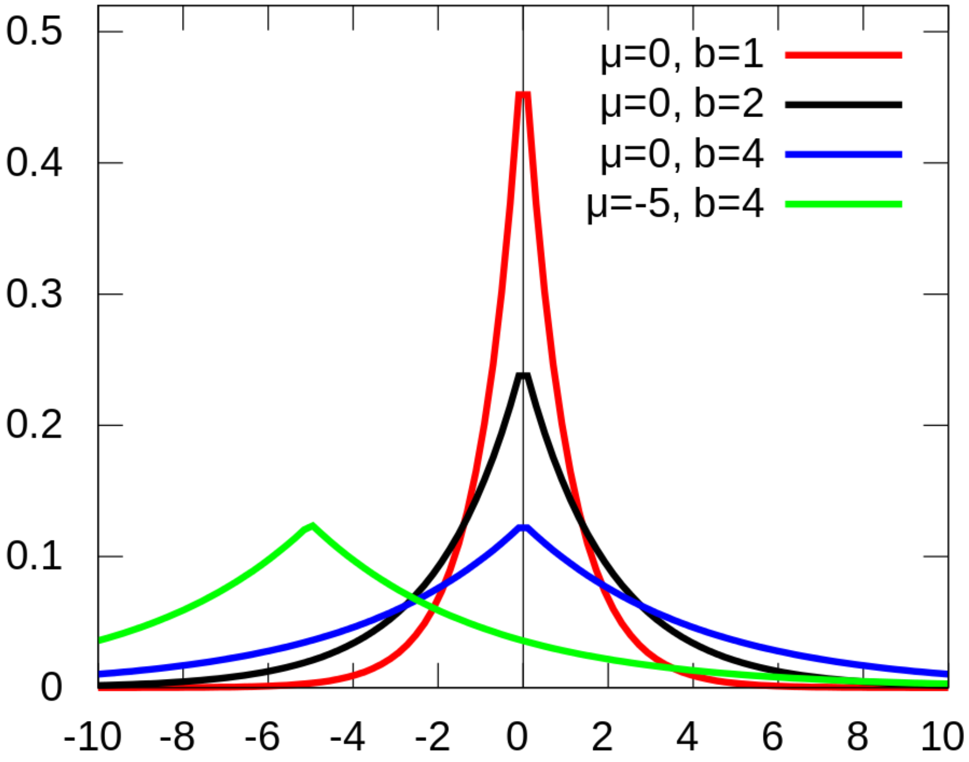
$$\mathrm{Laplace}(\mathrm{x};\mu,\mathrm{b}) = \frac{1}{2\mathrm{b}}\exp(-\frac{|\mathrm{x}-\mu|}{\mathrm{b}})$$

- Laplace distribution represents the distribution of differences between two independent variables having identical exponential distributions. It is also called double exponential distribution.

Credits from Wikipedia

Laplace Distribution

- Laplace distribution with different parameters b and μ .



Then the expectation is μ and the variance is $2\mathrm{b}^2$.

Credits from Wikipedia

Mixtures of Distributions

How to construct distributions by exisiting distributions?

Mixture of distributions is an important method.

- If we have n discrete distributions P1(x), P2(x),...,Pn(x), then we can construct a mixture distribution by weights w1,...,wn

$$P(x) = \sum_{i=1}^{n} wi * Pi(x) .$$

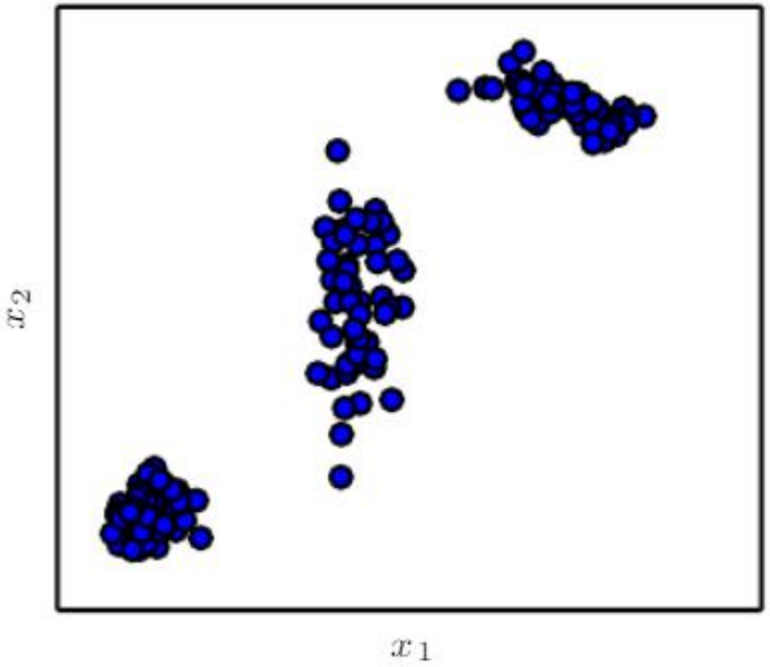
To ensure that P(x) is a distribution, wi should be non-negative and w1+w2+...+wn=1.

Mixtures of Distributions

If we have n continuous distributions with density function p1(x), p2(x),...,pn(x), then we can construct a mixture distribution by weights w1,...,wn: the mixture distribution's density function is

$$p(x) = \sum_{i=1}^{n} wi * pi(x) .$$

- Mixture distributions with three different Gaussian distributions



Change of Variables

How to construct distributions by exisiting distributions?

Contructing a function to deform the random variables is an important method.

- If X is a discrete random variable and g is a function, then

$$P(g(X) = y) = P(\ x: X \in g^{-1}(y))$$

Change of Variables

If X is a continuous random variable and g is a differentiable function, then

$$p_{\mathrm{X}}(\mathrm{x}) = p_{\mathrm{Y}}(\mathrm{g}(\mathrm{x}))\left|\frac{\partial \mathrm{g}(\mathrm{x})}{\partial \mathrm{x}}\right|$$

where $p_{\mathrm{X}}(\mathrm{x})$ is the probability density function for X and $p_{\mathrm{Y}}(\mathrm{g}(\mathrm{x}))$ is the probability density function for g(X).

Exercise

For a gaussian distribution with probability density function

$$\sqrt{\frac{1}{2\pi}}\exp(-\frac{1}{2}\mathrm{x}^2)$$

- If the function g(x) = 2x+1, then what is the density function after transforming by function g?

Exercise

Solution: $\frac{\partial g(x)}{\partial x}$ =2. By $p_X(x) = p_Y(g(x)) \left|\frac{\partial g(x)}{\partial x}\right|$, we know that

$$\sqrt{\frac{1}{2\pi}}\exp(-\frac{1}{2}x^2) = p_Y(2x+1) * 2$$

Let y=g(x)=2x+1.

Then x=0.5y-0.5, which implies that $\sqrt{\frac{1}{2\pi}}\exp(-\frac{1}{2}x^2) = p_Y(y) * 2$

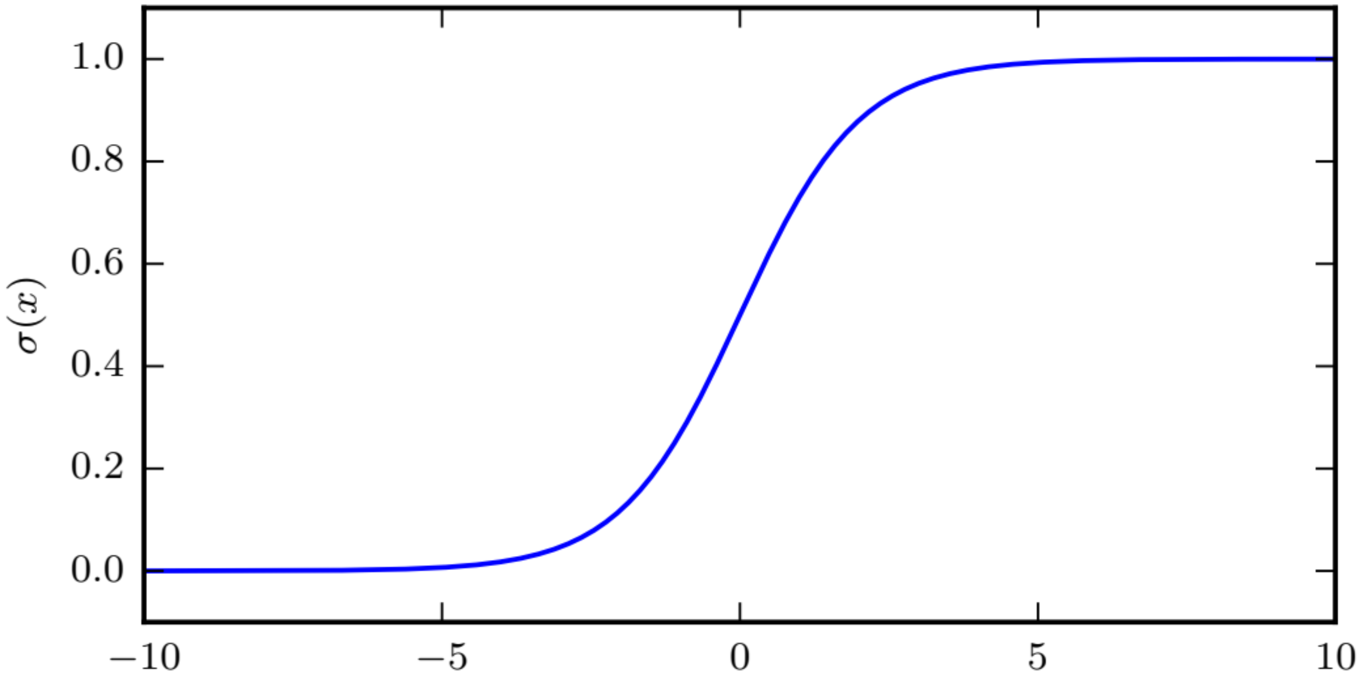
Replacing x by 0.5y-0.5, we have

$$p_Y(y) = \sqrt{\frac{1}{8\pi}}\exp(-\frac{1}{8}(y-1)^2)$$

Logistic Sigmoid

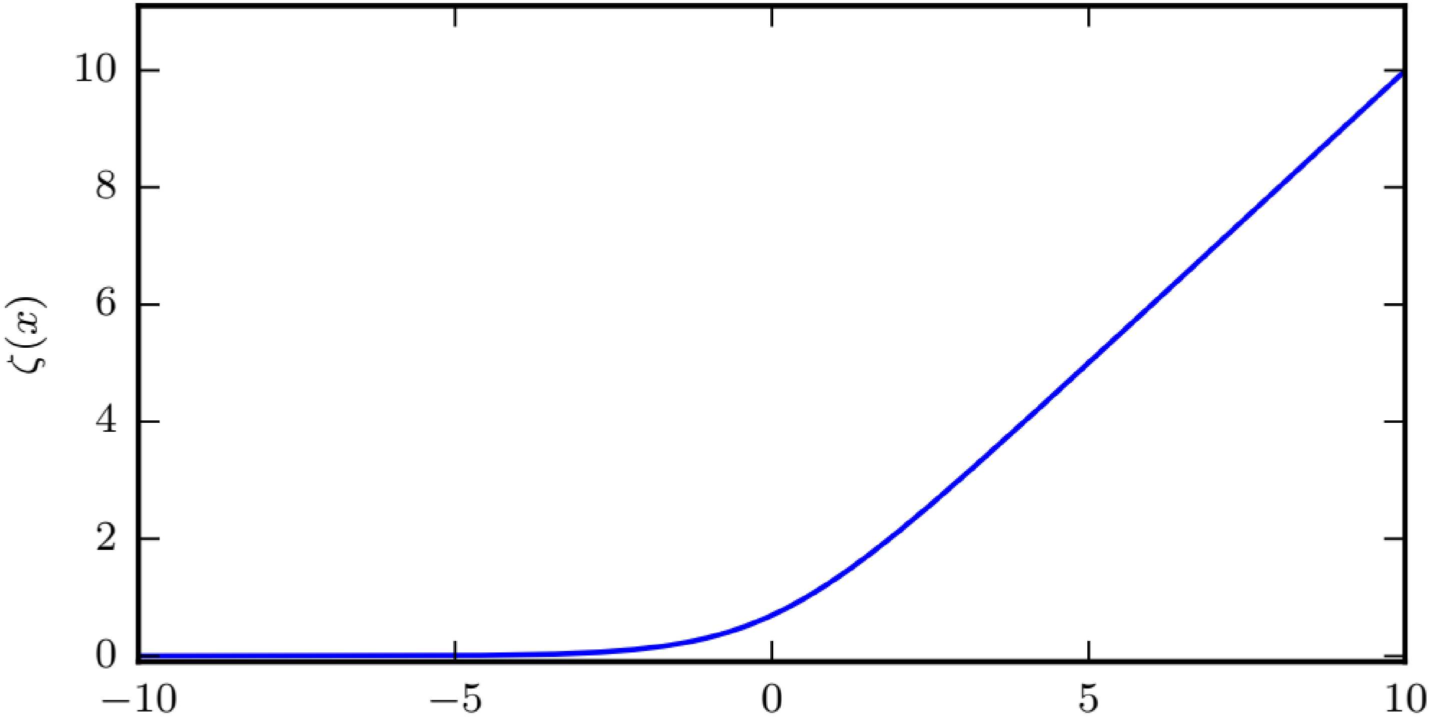
In deep learning, we construct many different functions to deform distributions.

- $\sigma(x) = \frac{1}{1+\exp(-x)}$



Softplus Function

- $\sigma(x) = \log(1 + \exp(x))$



Thank You!