Question 1 (25 Marks)

A type of cake contains a prize voucher with probability 0.2. Whether or not a cake contains a prize voucher is independent of other cakes. A hungry child buys 5 cakes. Let X denote the number of prize vouchers that the child finds.

Answer the following questions.

1.1 What is the probability distribution P(X)? Please provide details of your computation.

(5 Marks)

$$P(X = x) = C_5^x \, 0.2^x 0.8^{5-x}, x = 0, \ldots, 5$$

1.2.1 What is the probability that the child finds no prize vouchers? Please provide details of your computation.

(5 Marks)

$$P(X = 0) = C_5^0 \, 0.2^0 0.8^5 = 0.32768$$

1.2.2 What is the expectation of the random variable X? Please provide details of your computation.

(5 Marks)

$$E(X) = \sum_{k=1}^{5} k \, C_5^k \, 0.2^k 0.8^{5-k} = 1$$

A second child keeps buying cakes until he finds a prize voucher. Let Y denote the number of cakes he buys.

1.3.1 What is the probability that the child buys more than 5 cakes? Please provide details of your computation.

(5 Marks)

$P(Y > 5) = 1 - \sum_{k=1}^{5} \mathrm{P(Y = k)} = 1 - 0.2 - 0.8 \times 0.2 - 0.8^2 \times 0.2 - 0.8^3 \times 0.2 - 0.8^4 \times 0.2 = 0.32768$

1.3.2 If each cake costs 5 dollars, what is the expected cost to this child? Please provide details of your computation.

(5 Marks)

$$E(Y) = \sum_{k=1}^{\infty} kP(Y = k) = \sum_{k=1}^{\infty} k0.2 * 0.8^{k-1} = 5$$

$$E(5Y) = 25$$

Question 2 (26 Marks)

Bayes' Theorem can be used to address many applications. Answer the following questions.

2.1 What is the Bayes' Theorem? Please describe it and prove it.

(2 Marks)

Bayes' Theorem is statistically stated as:
Given events A and B then $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Proof:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

2.2 In a university, admissions for the departments CS, MATH, IT are 75%, 95%, 70% respectively in the previous year. In the total of their output 10, 8, 5 percent are quick learners students. A student is taken at random from the department and is found to be quick learners. What are the probabilities that it was from the department CS, MATH, IT?

(12 Marks)

$P(A) = 0.75, P(B) = 0.95, P(C) = 0.7$
$P(X|A) = 0.1, P(X|B) = 0.08, P(X|C) = 0.05$
$P(X) = P(A)P(X|A) + P(B)P(X|B) + P(C)P(X|C) = 0.186$

$$P(A|X) = \frac{P(X|A)P(A)}{P(X)} = 25/62 = P(CS|\ quick\ leaners)$$
$$P(B|X) = \frac{P(X|B)P(B)}{P(X)} = 38/93 = P(MATH|\ quick\ leaners)$$
$$P(C|X) = \frac{P(X|C)P(C)}{P(X)} = 35/186 = P(IT|quick\ Learners)$$

2.3 The Research areas of faculty members in Department of mathematics in the university, Staff room I, II, III as follows:

- Room I: 1 Geometry, 2 Algebra and 1 Analysis
- Room II: 2 Geometry, 3 Algebra and 1 Analysis
- Room III: 1 Geometry, 2 Algebra and 2 Analysis

One room is chosen at random and two staff members selected, they happened to have the research area as Algebra and Analysis. What is probability that they come from room I, II, III?

(12 Marks)

$$P(I) = P(II) = P(III) = \frac{1}{3}$$

Let A_1, A_2 denote Algebra and Analysis

$$P(I|A_1A_2) = \frac{P(A_1A_2|I)P(I)}{P(A_1A_2)} = \frac{\frac{1}{3}\times\frac{1}{3}}{\frac{14}{45}} = \frac{5}{14}$$

$$P(II|A_1A_2) = \frac{P(A_1A_2|II)P(II)}{P(A_1A_2)} = \frac{\frac{1}{5}\times\frac{1}{3}}{\frac{14}{45}} = \frac{3}{14}$$

$$P(III|A_1A_2) = \frac{P(A_1A_2|III)P(II)}{P(A_1A_2)} = \frac{\frac{2}{5}\times\frac{1}{3}}{\frac{14}{45}} = \frac{3}{7}$$

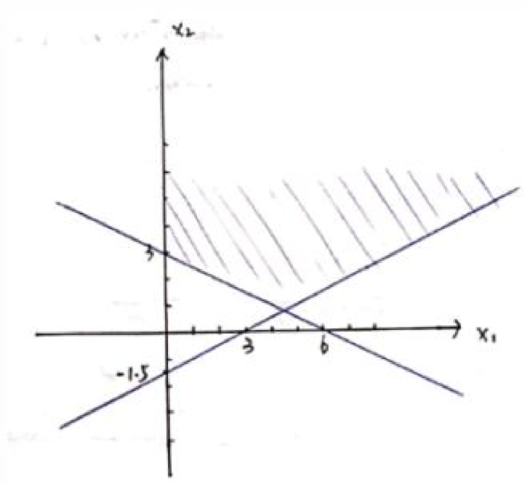
Question 3 (24 Marks)

Answer the following linear optimization issues. Please provide details of your computation.

3.1

Minimize x_1+3x_2
Subject to:
$x_1+2x_2 \geq 6$
$x_1-2x_2 \leq 3$
$x_1 \geq 0, x_2 \geq 0$

(12 Marks)



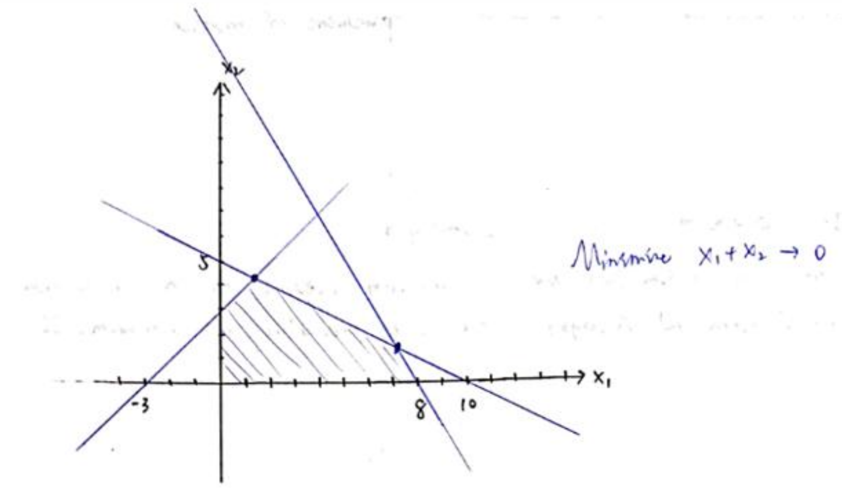
We need to compute $x_1 + 3x_2$ at points (0, 3) and (4.5, 0.75), then $x_1 + 3x_2$ are 9 and 6.75, respectively. So (4.5, 0.75) is the solution.

3.2

Minimize x_1+x_2
Subject to:

$$x_1+2x_2 \leq 10$$
$$2x_1 + x_2 \leq 16$$
$$-x_1+x_2 \leq 3$$
$$x_1 \geq 0, x_2 \geq 0$$

(12 Marks)



When $x_1 = x_2 = 0$, x_1+x_2 achieves 0, so (0, 0) is the solution.

Question 4 (25 Marks)

Consider the following convex optimization problem:

Minimize $\mathbf{x}^T\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$,

where $\mathbf{x} = (x_1, \dots, x_d)^T$ is a d-dimensional vector, $\mathbf{M}$ is a $d \times d$ dimensional matrix and $\mathbf{b} = (b_1, \dots, b_d)$ is a d-dimensional vector.

4.1.1 If

$$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -1 & 6 \end{pmatrix},$$

is Minimize $\mathbf{x}^T\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$ is a convex optimization problem? is Minimize $\mathbf{x}^T\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$ is a **strictly** convex optimization problem? Please provide details of your computation.

(5 Marks)

Firstly, the domain R^d is a convex set

$$x^{\top}Mx + bx \to \frac{\partial^2 f(x)}{\partial x \partial x} = M + M^{\top} = \begin{bmatrix} 2 & -2 \\ -2 & 12 \end{bmatrix}$$
$$(a,b)\begin{bmatrix} 2 & -2 \\ -2 & 12 \end{bmatrix}(a,b)^{\top} = 2(a^2 + 6b^2 - 2ab) = (a-b)^2 + 5b^2 \geq 0$$

Since Hessian Matrix is positive definite, $x^{\top}Mx + bx$ is a convex optimization problem and it is also a strictly convex problem

4.1.2 If

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is Minimize $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$ is a convex optimization problem? is Minimize $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$ is a **strictly** convex optimization problem? Please provide details of your computation.

(5 Marks)

Firstly, the domain R^d is a convex set

$$x^{\mathsf{T}}Mx + bx \rightarrow \frac{\partial^2 f(x)}{\partial x \partial x} = M + M^{\mathsf{T}} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(a,b,c)\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}(a,b,c)^{\mathsf{T}} = 2(a^2 + b^2 + c^2 - 2ab) = (a-b)^2 + c^2 \geq 0$$

Since Hessian Matrix is positive semi-definite, $x^{\mathsf{T}}Mx + bx$ is a convex optimization problem but it is not a strictly convex problem

4.1.2 If

$$\mathbf{M} = \begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix},$$

is Minimize $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$ is a convex optimization problem? is Minimize $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$ is a **strictly** convex optimization problem? Please provide details of your computation.

(5 Marks)

Firstly, the domain R^d is a convex set

$$x^{\mathsf{T}}Mx + bx \rightarrow \frac{\partial^2 f(x)}{\partial x \partial x} = M + M^{\mathsf{T}} = \begin{bmatrix} 2 & -10 \\ -10 & 2 \end{bmatrix}$$

$$(a,b)\begin{bmatrix} 2 & -10 \\ -10 & 2 \end{bmatrix}(a,b)^{\mathsf{T}} = 2(a^2 + b^2 - 10ab) = 2((a-b)^2 - 8ab)$$

Since Hessian Matrix is not positive semi-definite, $x^{\mathsf{T}}Mx + bx$ is not a convex optimization problem. So it is also not a strictly convex optimization.

4.2 If

$$\mathbf{M} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = (2, 1)$$

what is the solution of the convex optimization problem: Minimize $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x}+\mathbf{b}\mathbf{x}$. Please provide details of your computation.

(10 Marks)

$$\nabla f(x) = 2Mx + b^{\top} = 2\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution is $\nabla f(x) = 0$, i.e., $-\frac{M^{-1}B^{\top}}{2} =-\frac{1}{2}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ -\frac{2}{3} \end{bmatrix}$

END OF PAPER