Reference Solutions to Some Quiz Questions

- Suppose you solve **Ax** = **b** for three special right side vectors **b** (Here **A** is a 3*3 matrix, and **x** and **b** are 3*1 vectors):
- $$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \text{ and } \mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{A}\mathbf{x}_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$
- The three solutions are $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.
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- What is the inverse of **A**?

- Solution:
- $\mathbf{A}[\mathbf{x}_2\ \ \mathbf{x}_3\ \ \mathbf{x}_1] = [\mathbf{A}\mathbf{x}_2\ \ \mathbf{A}\mathbf{x}_3\ \ \mathbf{A}\mathbf{x}_1] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
- $\mathbf{A}\begin{bmatrix} 0 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ => $\mathbf{A}\begin{bmatrix} 0 & 1 & 1 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 1 & \frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ => $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ \frac{2}{3} & 0 & \frac{2}{3} \\ 1 & \frac{1}{3} & 1 \end{bmatrix}$

- Consider the linear equation system **Ax** = **b**, where **A** and **b** are given. If $\boldsymbol{\eta_1}$ and $\boldsymbol{\eta_2}$ are both the solutions, please judge if the following statements are correct or not.
- (a) $2\boldsymbol{\eta_1} - \boldsymbol{\eta_2}$ is also the solution of **Ax** = **b**.
- (b) $\frac{1}{2}\boldsymbol{\eta_1} - \frac{1}{2}\boldsymbol{\eta_2}$ is the solution of **Ax** = **0** (**0** is the zero vector).

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- Solution:
- (a) $\boldsymbol{\eta_1}$ and $\boldsymbol{\eta_2}$ are both the solutions, so we have $\mathbf{A}\boldsymbol{\eta_1}$ = **b** and $\mathbf{A}\boldsymbol{\eta_2}$ = **b**.
- Then we have $\mathbf{A}(2\boldsymbol{\eta_1} - \boldsymbol{\eta_2}) = 2\,\mathbf{A}\boldsymbol{\eta_1} - \mathbf{A}\boldsymbol{\eta_2} = 2\mathbf{b} - \mathbf{b} = \mathbf{b}$.
- So, yes, $2\boldsymbol{\eta_1} - \boldsymbol{\eta_2}$ is also the solution of **Ax** = **b.**

- (b) $\mathbf{A}(\frac{1}{2}\boldsymbol{\eta_1} - \frac{1}{2}\boldsymbol{\eta_2}) = \frac{1}{2}\mathbf{A}\boldsymbol{\eta_1} - \frac{1}{2}\mathbf{A}\boldsymbol{\eta_2} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{b} = \mathbf{0}$.
- So, yes, $\frac{1}{2}\boldsymbol{\eta_1} - \frac{1}{2}\boldsymbol{\eta_2}$ is the solution of **Ax** = **0**

- Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be two $n \times n$ projection matrices.
- (a) What are the eigenvalues of $\mathbf{P}_1$ and $\mathbf{P}_2$?
- (b) Do we have $\mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)^2 = (\mathbf{P}_1 - \mathbf{P}_2)^2\mathbf{P}_1$?

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- Solution: $\mathbf{P}_1$ and $\mathbf{P}_2$ are projection matrices, so we have $\mathbf{P}_1^2 = \mathbf{P}_1$ and $\mathbf{P}_2^2 = \mathbf{P}_2$.
- (a) Assume that $\mathbf{P}_1$ has the eigenvalues α, then $\mathbf{P}_1\mathbf{x} = \alpha\mathbf{x}$.
- For $\mathbf{P}_1^2$, we have $\mathbf{P}_1^2\mathbf{x} = \mathbf{P}_1\mathbf{P}_1\mathbf{x} = \mathbf{P}_1\alpha\mathbf{x} = \alpha\mathbf{P}_1\mathbf{x} = \alpha*\alpha\mathbf{x} = \alpha^2\mathbf{x}$
- Since $\mathbf{P}_1^2 = \mathbf{P}_1$, we have $\alpha^2 = \alpha \Rightarrow \alpha(\alpha - 1) = 0 \Rightarrow \alpha = 0 \; or \; \alpha = 1.$
- For $\mathbf{P}_2$, it is the same.

- (b)
- $\mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)^2 = \mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)(\mathbf{P}_1 - \mathbf{P}_2) = \mathbf{P}_1(\mathbf{P}_1^2 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_2^2) = \mathbf{P}_1^3 - \mathbf{P}_1^2\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_2^2$

$= \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1$

- $(\mathbf{P}_1 - \mathbf{P}_2)^2\mathbf{P}_1 = (\mathbf{P}_1 - \mathbf{P}_2)(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{P}_1 = (\mathbf{P}_1^2 - \mathbf{P}_1\mathbf{P}_2 - \mathbf{P}_2\mathbf{P}_1 + \mathbf{P}_2^2)\mathbf{P}_1 = \mathbf{P}_1^3 - \mathbf{P}_1\mathbf{P}_2\mathbf{P}_1 - \mathbf{P}_2\mathbf{P}_1^2 + \mathbf{P}_2^2\mathbf{P}_1$

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So, yes, we have $\mathbf{P}_1(\mathbf{P}_1 - \mathbf{P}_2)^2 = (\mathbf{P}_1 - \mathbf{P}_2)^2\mathbf{P}_1$

- Principal Component Analysis (PCA) and Linear Discriminant Analysis (LDA) are two representative methods for dimensionality reduction.
- In what situation, PCA and LDA will give the same projection result?
- In what situation, PCA and LDA will give totally different projection results?

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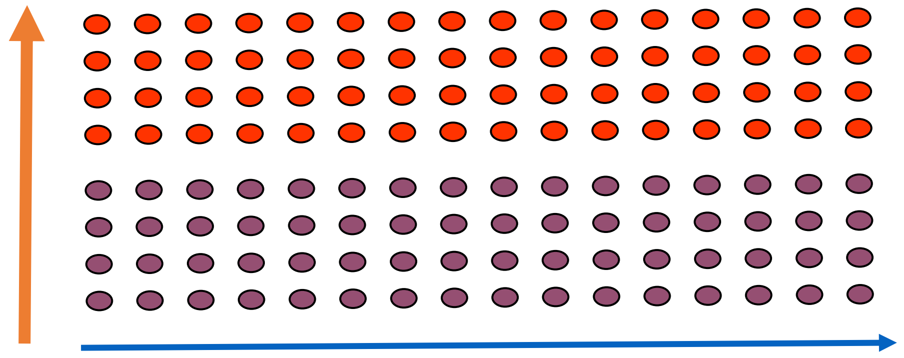
- Solution:



- Assume that $f(x) = x^2 + 2x + 1$, $g(x) = (x - 1)^4 - 10$, and $h(x) = e^x$.
- Let $F(x) = max\{f(x), g(x), h(x)\}$, where $max\{f(x), g(x), h(x)\}$ refers to the largest of the three values of $f(x)$, $g(x)$, and $h(x)$.
- Is $F(x)$ a convex function?

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- Is $F(x)$ a convex function?

- Solution:
- We can use the definition or the first order condition to prove that $f(x)$, $g(x)$, $h(x)$ are convex.
- For $F(x) = max\{f(x), g(x), h(x)\}$, we follow the definition of convexity:
- $F(\theta x + (1-\theta)y) = \ max\{f(\theta x + (1-\theta)y),\ \ g(\theta x + (1-\theta)y),\ \ h(\theta x + (1-\theta)y)\}$

$$\leq max\{\theta f(x) + (1-\theta)f(y),\ \theta g(x) + (1-\theta)g(y),\ \theta h(x) + (1-\theta)h(y)\}$$

$$\leq max\{\theta f(x),\ \theta g(x),\ \theta h(x)\} + max\{(1-\theta)f(y),\ \ (1-\theta)g(y),\ \ (1-\theta)h(y)\}$$

$$= \theta max\{f(x), g(x), h(x)\} + (1-\theta)max\{f(y), g(y), h(y)\} = \theta F(x) + (1-\theta)F(y)$$

So, yes, $F(x)$ a convex function.