COMP 7180
Quantitative Methods for Data Analytics and Artificial Intelligence

Lecture 1 – Introduction to Linear Algebra: Vectors and Matrices

What is Linear Algebra

- **Linear**
  - Having to do with lines/planes/etc.
  - For example, x-y = 1, x+y+3z = 7, not *sin*; *log*; x^2, etc.

- **Algebra**
  - Solving equations involving numbers and symbols
  - From al-jebr (Arabic), meaning reunion of broken parts
  - Abu Ja'far Muhammad ibn Muso al-Khwarizmi, 9th century

What is Linear Algebra

- Linear algebra is the study of vectors, matrices, and linear functions/equations.

- Vectors (columns of numbers) and matrices (2D arrays of numbers) are the language of data.

- **Linear algebra is a branch of mathematics, but the truth of it is that linear algebra is the mathematics of data.**
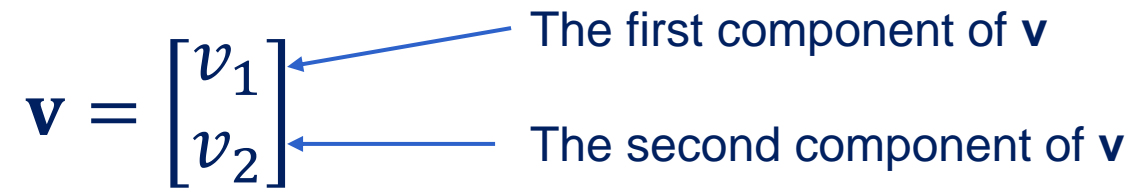
Vectors in two dimensions

- A two-dimensional vector **v**

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The first component of **v**

The second component of **v**

- We write **v** as a column. A single letter **v** (in boldface) to denote a vector.

Two basic operations in Linear Algebra

- Vector addition: add vectors

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \implies \mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

  – Subtraction follows the same ideas

$$\mathbf{v} - \mathbf{w} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \end{bmatrix}$$

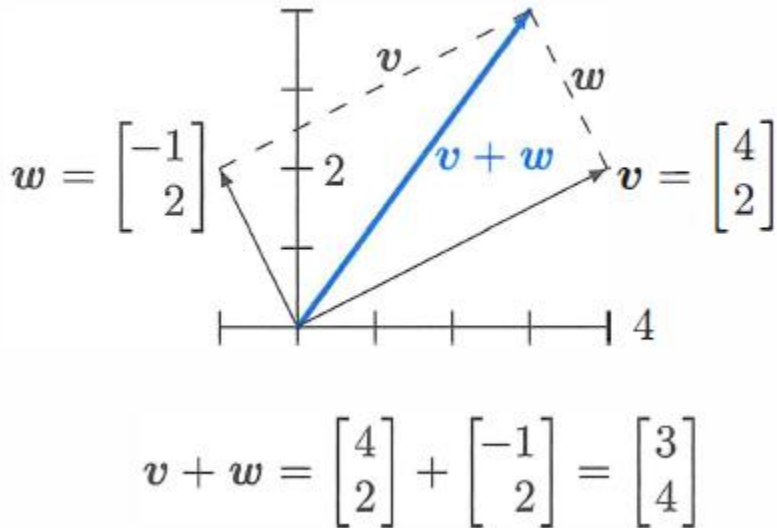- Scalar multiplication: vector can be multiplied by any number (scalar) c

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

Two basic operations in Linear Algebra

- Linear algebra is built on these two operations
  - Adding vectors: $\mathbf{v} + \mathbf{w}$
  - Multiplying by scalars: $c\mathbf{v}$

- Linear combination: combine addition with scalar multiplication
  - $c\mathbf{v} + d\mathbf{w}$: multiply $\mathbf{v}$ by c and multiply $\mathbf{w}$ by d, then add together
  - $c\mathbf{v} + d\mathbf{w}$ is a **"linear combination"** of vector $\mathbf{v}$ and $\mathbf{w}$

Visualize vector addition/subtraction

Vector Addition

Vector Subtraction

$$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$v - w$$

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$-w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- A vector can be represented by an arrow from the "**origin**" (0, 0)

- $v + w$: at the end of v, place the start of w. Then the third side is $v + w$
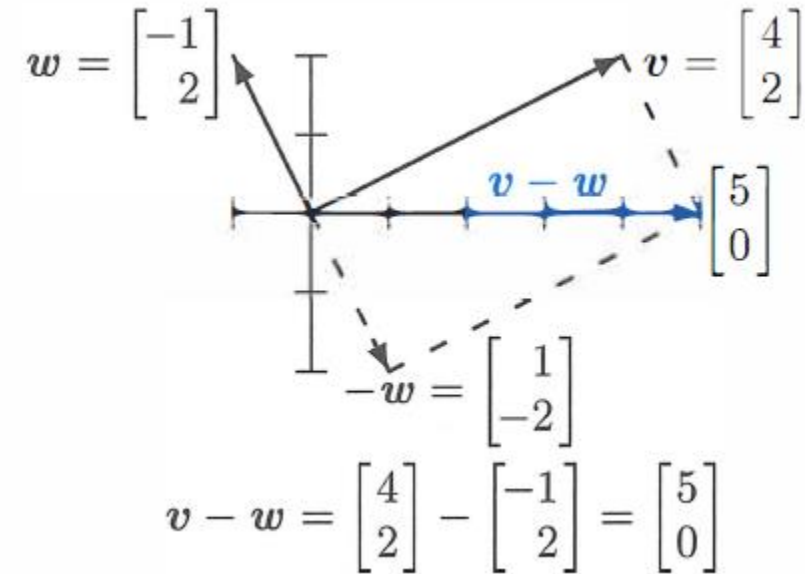
Visualize vector-scalar multiplication

- $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- $2\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

- If the scalar is positive, vector-scalar multiplication only changes the length (magnitude) of the vector, it does not change the direction of the vector.

- If the scalar is negative, it both changes the length and reverses the direction of the vector.
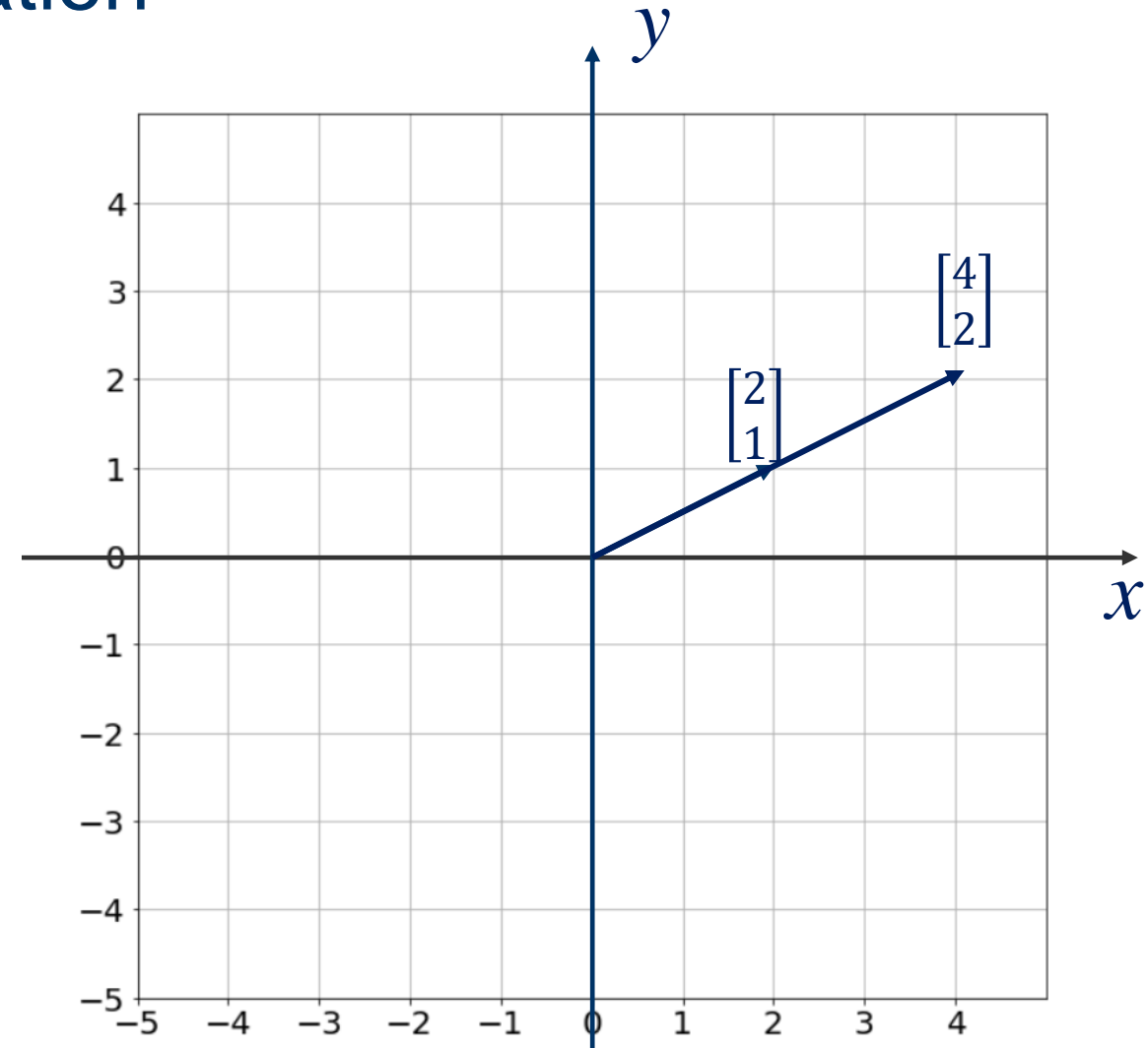
Vectors in Three Dimensions

- A vector with two components corresponds to a point in the xy plane.
  - The components of **v** are the coordinates of the point: $x = v_1$, $y = v_2$.

- Now we allow vectors to have three components, then the xy plane is replaced by three-dimensional xyz space.
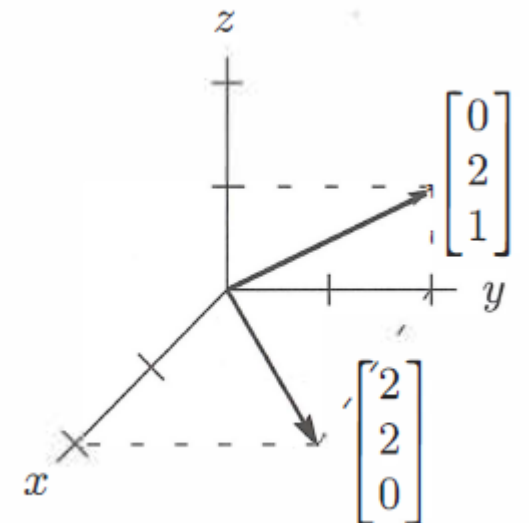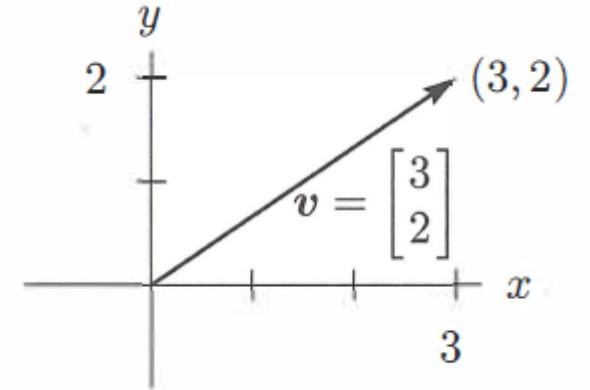
Vectors in Three Dimensions

- In three dimensions, **v + w** is still found by element-wise addition (the same as in two dimensions).
  - E.g.,

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{v} + \mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

- Then we can see how to add vectors in 4 or 5 or *n* dimensions. When **w** starts at the end of **v**, the third side is **v + w**.
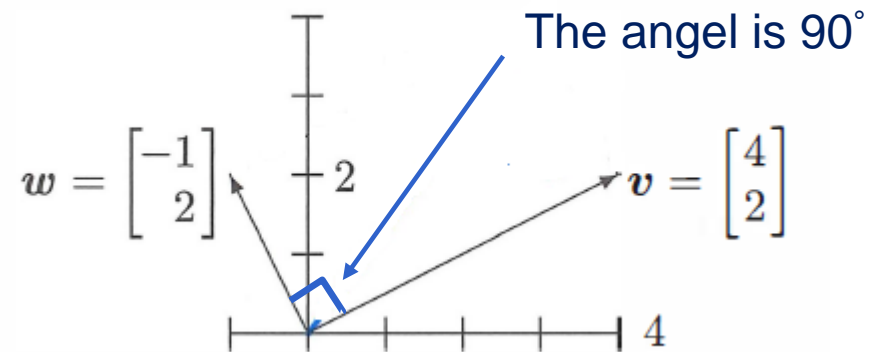
Dot product

- The dot product (or inner product) of $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is a scalar

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

- If the dot product of two vectors is 0, it means that ***these two vectors are perpendicular (orthogonal)***.

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 4 * (-1) + 2 * 2 = 0$$

The angel is 90°

$$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

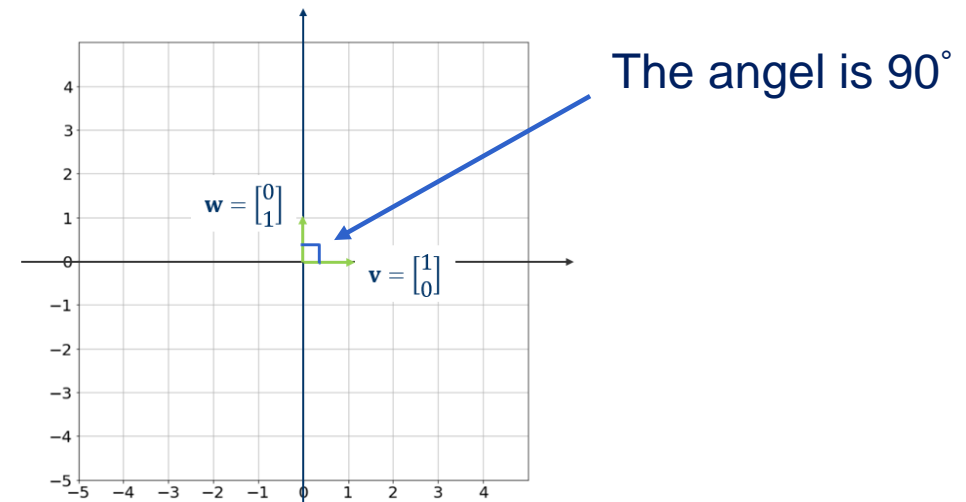$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Dot product

- The dot product (or inner product) of $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is a scalar

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

- If the dot product of two vectors is 0, it means that **these two vectors are perpendicular (orthogonal)**.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 * 0 + 0 * 1 = 0$$

The angel is 90°

Dot product example

- We have three goods to buy or sell. Their prices are (p_1, p_2, p_3) for each unit – this is the "price vector" $\mathbf{p}$. The quantities that we buy or sell are (q_1, q_2, q_3) – positive when we sell, negative when we buy. Selling q_1 units at the price p_1 brings in $q_1 p_1$. The total income (quantities $\mathbf{q}$ times prices $\mathbf{p}$) is the dot product $\mathbf{q} \cdot \mathbf{p}$ in three dimensions:

$$\textbf{Income} = \mathbf{q} \cdot \mathbf{p} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = q_1 p_1 + q_2 p_2 + q_3 p_3$$

  - Total sales equal to total purchases if $\mathbf{q} \cdot \mathbf{p} = 0$, which means that $\mathbf{p}$ is perpendicular to $\mathbf{q}$ in three dimensional space.
  - A supermarket with thousands of goods goes quickly into high dimensions.

Length of a vector

- An important case of dot product is the dot product with itself.
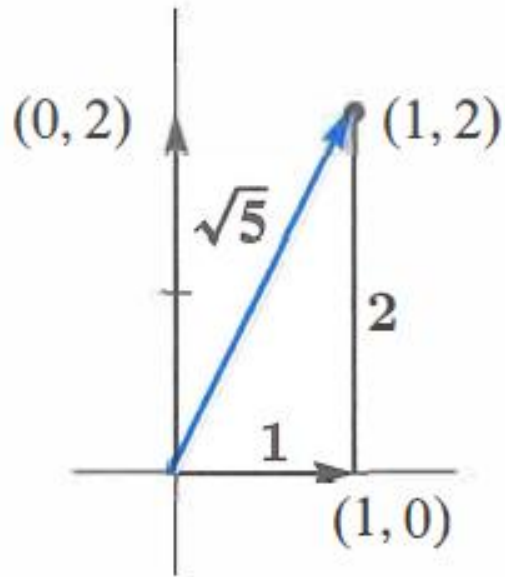 - E.g., $\mathbf{v} = (1, 2, 3)$

$$\mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14$$

- The dot product $\mathbf{v} \cdot \mathbf{v}$ gives the length of $\mathbf{v}$ squared: $\|\mathbf{v}\|^2$

- The length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is the square root of $\mathbf{v} \cdot \mathbf{v}$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_d^2}$$
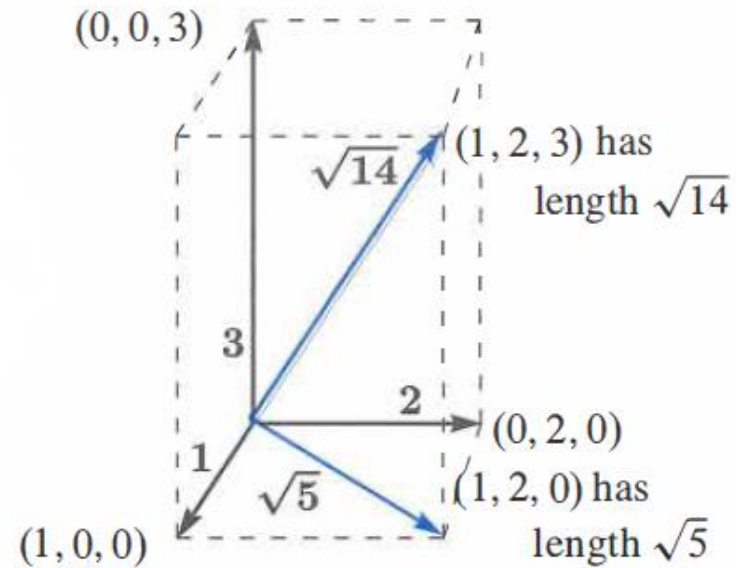
Length of a vector

- $\|\mathbf{v}\|$ is the length of the arrow that represents the vector.

Two-dimensional vector
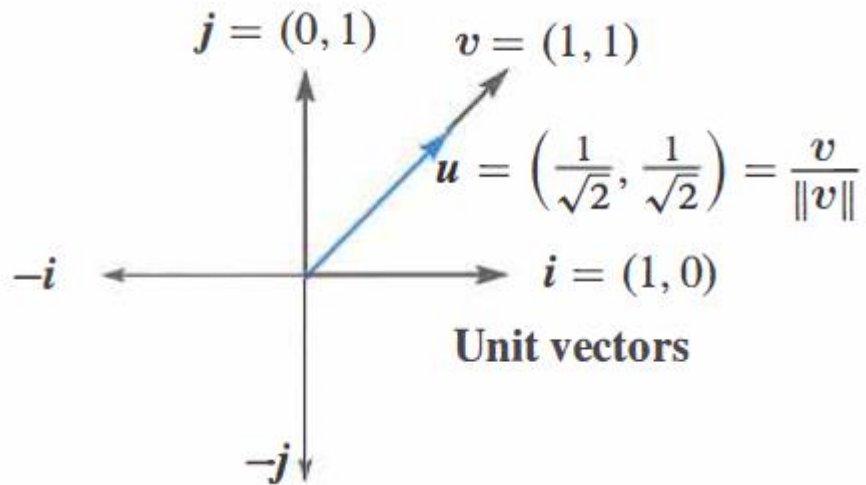
$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

Three-dimensional vector

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Unit Vector

- The word "unit" is always indicating that some measurement equals "one"

- A unit vector $\mathbf{u}$ is a vector whose length equal one: $\mathbf{u} \cdot \mathbf{u} = 1$.

- How to get a unit vector?
    - For any nonzero vector $\mathbf{v}$, we can obtain its unit vector by dividing it by its length $\|\mathbf{v}\|$

- $\boldsymbol{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector in the same direction as $\mathbf{v}$.

Unit Vector

$j = (0, 1)$ $v = (1, 1)$

$u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{v}{\|v\|}$

$-i$ $i = (1, 0)$

Unit vectors

$-j$

j $u = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$

1

$\sin \theta$

θ

$\cos \theta$ i

Unit circle

The unit vector **u** at angle 45° is obtained by dividing **v** = (1,1) by its length $\|v\| = \sqrt{2}$.
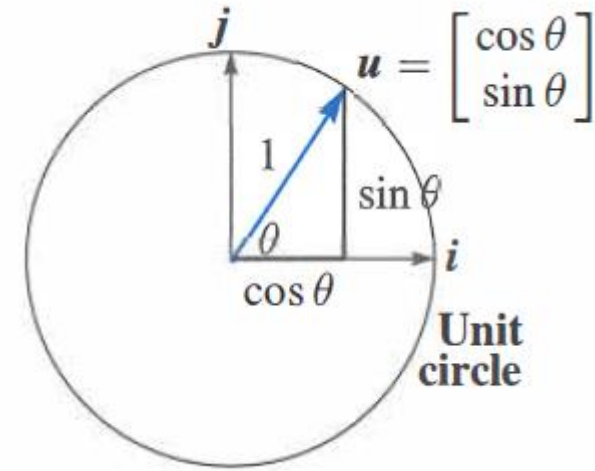
The **u** = (cosθ, sin θ) is a unit vector at angle θ.

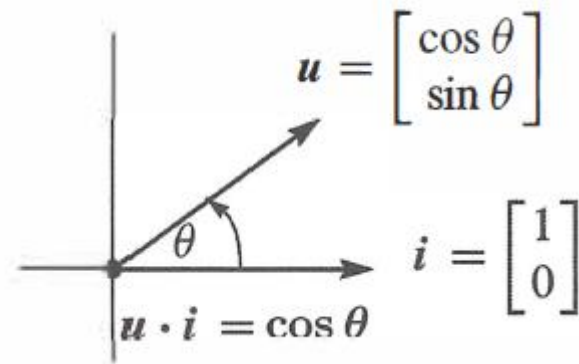$(\cos\theta)^2 + (\sin\theta)^2 = 1$

The Angle Between Two Vectors

- The dot product $\mathbf{v} \cdot \mathbf{w}$ is zero when the angle between these two vectors $\mathbf{v}$ and $\mathbf{w}$ is $\theta = 90°$. $\mathbf{v}$ is perpendicular to $\mathbf{w}$.

- Zero vector $\mathbf{v} = \mathbf{0}$ is perpendicular to every vector $\mathbf{w}$ because $\mathbf{0} \cdot \mathbf{v}$ is always zero.

- How about $\mathbf{v} \cdot \mathbf{w}$ is not zero?
  - The sign of $\mathbf{v} \cdot \mathbf{w}$ tells whether they are below or above a right angle.
  - The angle is less than 90° when $\mathbf{v} \cdot \mathbf{w}$ is positive
  - The angle is above 90° when $\mathbf{v} \cdot \mathbf{w}$ is negative
 - The borderline is where vectors are perpendicular to $\mathbf{v}$.

$v \cdot w = 0$

$v \cdot w > 0$

$v \cdot w < 0$

v

angle with v
less than 90°
in this half-plane

angle with v
greater than 90°
in this half-plane
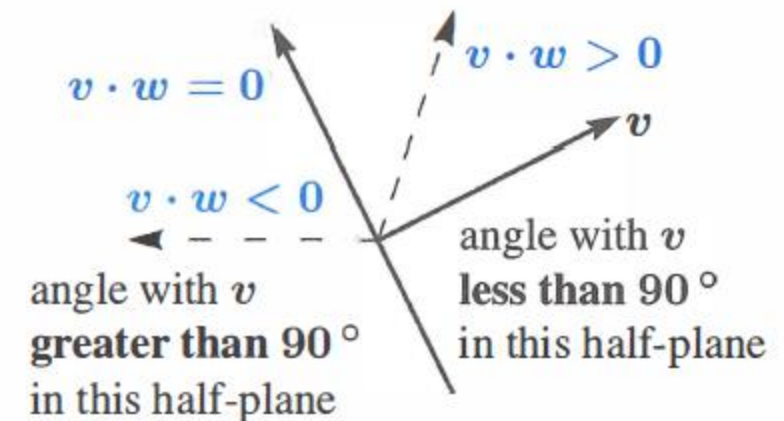
The Angle Between Two Vectors

- The dot product reveals the exact angle θ.

- For two unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, the dot product $\mathbf{u}_1 \cdot \mathbf{u}_2$ is the cosine of θ. This remains true in d dimensions.

$$\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u} \cdot i = \cos\theta$$

$$\mathbf{u} \cdot i = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cos\theta$$

$$\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = \mathbf{u}_1$$

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \mathbf{u}_2$$

$$\theta = \beta - \alpha$$
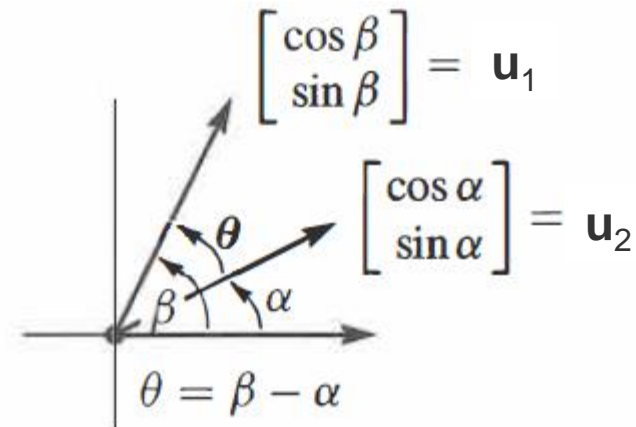
$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

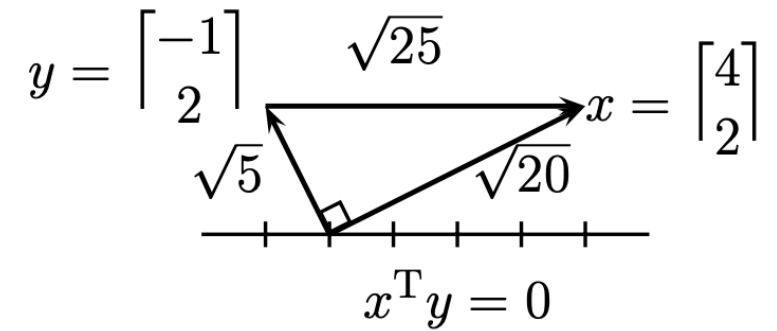$$= \cos(\beta - \alpha) = \cos\theta$$

The Angle Between Two Vectors

- How about the exact angle between two non-unit vectors **v** and **w**?

  - Divide the non-unit vectors by their length to get unit vectors $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\frac{\mathbf{w}}{\|\mathbf{w}\|}$

  - Then the dot product of unit vectors $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\frac{\mathbf{w}}{\|\mathbf{w}\|}$ gives $\cos\theta$

- **Cosine Formula**

  If **v** and **w** are nonzero vectors then $\frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} = \cos\theta$

The Angle Between Two Vectors – Another Point of View

- ▪ What is perpendicularity (orthogonality)?
  - – Two vectors are perpendicular/orthogonal provided they form a **right triangle**.

Sides of a right triangle $\qquad \|x\|^2 + \|y\|^2 = \|x - y\|^2.$

$$y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \sqrt{25}$$

$$x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\sqrt{5} \qquad \sqrt{20}$$

$$x^{\mathrm{T}} y = 0$$

  - – Applying the length formula, this test for orthogonality in $\mathbf{R}^n$ becomes:

$$(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2) = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2.$$

$$= (x_1^2 + \cdots + x_n^2) - 2(x_1 y_1 + \cdots + x_n y_n) + (y_1^2 + \cdots + y_n^2).$$

Orthogonal vectors $\qquad x^{\mathrm{T}} y = x_1 y_1 + \cdots + x_n y_n = 0.$

Matrix

Definition: An $m \times n$ matrix, A, is a rectangular array of elements

$$A = \left(a_{ij} \right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

m = # of rows

n = # of columns

dimensions = $m \times n$

Matrix Operations

- Matrix addition
  - Matrix can be added if their shapes are the same
  - Matrix addition is like vector addition: element-wise addition

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix}$$

- Matrix multiplied by a scalar c
  - Matrix can be multiplied by a scalar
  - Each entry in the matrix will be multiplied by the scalar

$$2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}$$

Matrix Operations: Matrix Multiplication

- When can we multiply matrix **A** by matrix **B**?

  - To multiply matrix **A** by matrix **B**, number of columns in **A** must equal to the number of rows in **B**.

$$\mathbf{A}_{m*n}\mathbf{B}_{n*p} = \mathbf{C}_{m*p}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

What is matrix product **AB**?

- The first way: the dot product way (the usual way to multiply matrices by hand) .

- The product **AB** is filled with dot products: take the dot product of each row of **A** with each column of **B**

- The ij-th (i.e., the i-th row and the j-th column) entry in matrix product **AB** is the dot product between i-th row of **A** and j-th column of **B**.

$$
\begin{bmatrix} & * & & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ & * & & & \\ & * & & & \end{bmatrix}
\begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix}
=
\begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}
$$

A is 4 by 5 $\qquad\qquad\qquad$ B is 5 by 6 $\qquad\qquad\qquad$ AB is $(4 \times 5)(5 \times 6) = 4$ by 6

What is matrix product **AB**?

- E.g.1,

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1*2+1*3+0*0 & 1*2+1*4+0*0 & 1*0+1*1+0*0 \\ 2*2+(-1)*3+0*0 & 2*2+(-1)*4+0*0 & 2*0+(-1)*1+0*0 \\ 0*2+0*3+1*0 & 0*2+0*4+1*0 & 0*0+0*1+1*0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- E.g. 2, a column times row $\mathbf{A}_{m*1}\mathbf{B}_{1*p} = \mathbf{C}_{m*p}$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0*1 & 0*2 & 0*3 \\ 1*1 & 1*2 & 1*3 \\ 2*1 & 2*2 & 2*3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

  - A column times a row is an "outer" product. The result is a matrix.
  - A row times a column is an "inner" product, that is another name for dot product. The result is a scalar.

What is matrix product **AB**?

- The second way (column picture): Each column of **AB** is a linear combination of the columns of **A**.
  - Matrix **A** times every column of **B**.

$$\mathbf{A}_{m*n}\mathbf{B}_{n*p} = \mathbf{A}\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \mathbf{Ab}_3 & \cdots & \mathbf{Ab}_p \end{bmatrix}$$

  - E.g.,

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$0\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

What is matrix product **AB**?

- The third way (row picture): each row of **AB** is a linear combination of rows of **B**.
  - Every row of **A** times matrix **B**

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1[2 \quad 2 \quad 0] +$$
$$1[3 \quad 4 \quad 1] +$$
$$0[0 \quad 0 \quad 0] = [5 \quad 6 \quad 1]$$

$$2[2 \quad 2 \quad 0] +$$
$$-1[3 \quad 4 \quad 1] +$$
$$0[0 \quad 0 \quad 0] = [1 \quad 0 \quad -1]$$

$$0[2 \quad 2 \quad 0] +$$
$$0[3 \quad 4 \quad 1] +$$
$$1[0 \quad 0 \quad 0] = [0 \quad 0 \quad 0]$$

What is matrix product **AB**?

- The fourth way (columns multiply rows): multiply columns 1 to n of **A** by rows 1 to n of **B**, then add those matrices together.

$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1 & - \\ & \vdots & \\ - & b_n & - \end{bmatrix} = \begin{bmatrix} a_1 b_1 + \cdots + a_n b_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 1 \\ -3 & -4 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The laws for matrix operations

- Addition laws
  - $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutative law)
  - $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ (distributive law)
  - $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (associative law)

- Multiplication laws
  - $\mathbf{AB} \neq \mathbf{BA}$ (the commutative "law" is usually broken)
  - $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (distributive law from the left)
  - $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ (distributive law from the right)
  - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative law for $\mathbf{ABC}$)

Small Exercise

- Find an examples of 2 by 2 matrices E and F such that EF = 0, although no entries of E or F are zero.

Linear combination revisit

- Linear combination of vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- The linear combinations of these three vectors are $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$

$$x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = x_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

Linear combination revisit

- Linear combination of vectors

$$x_1 \mathbf{u} + x_2 \mathbf{v} + x_3 \mathbf{w} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

- Represent linear combination of vectors using a matrix
  - Form a matrix **A** where vectors **u**, **v**, **w** are the columns of **A**
  - The linear combination of **u**,**v**,**w** is that matrix **A** multiplies the vector **x** = (x_1, x_2, x_3)

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}$$

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$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

Two different viewpoints of matrix-vector multiplication

- The usual way to view matrix-vector multiplication (the way you may familiar with)
  - Multiplication a row at a time (**row picture**)

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1*x_1 + 0*x_2 + 0*x_3 \\ -1*x_1 + 1*x_2 + 0*x_3 \\ 0*x_1 - 1*x_2 + 1*x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

- The new way is to view **Ax** as a linear combination of the columns of **A** (**column picture**). Linear combinations are the key to linear algebra.

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

Linear Equations

- Matrix-vector Multiplication

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{b}$$

- Given any input $\mathbf{x} = (x_1, x_2, x_3)$, we can compute the output $\mathbf{b}$
- E.g.,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \qquad \mathbf{b} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

- New question: Given $\mathbf{A}$, and $\mathbf{b}$, find $\mathbf{x}$ that satisfy $\mathbf{A}\mathbf{x} = \mathbf{b}$.
  - A system with linear equations.

Application of Linear Equations

- Civil Engineering: How much traffic flows throu the four labeled segments w, x, y, and z?

- System of linear equations:

  $w + 120 = x + 250$

  $x + 120 = y + 70$

  $y + 530 = z + 390$

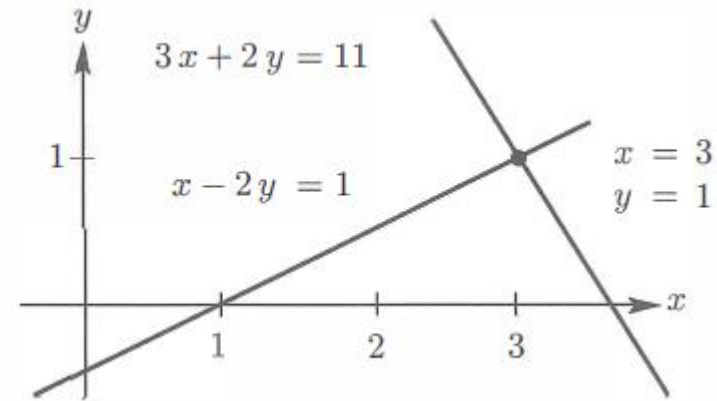 $z + 115 = w + 175$

Traffic flow (cars/hr)
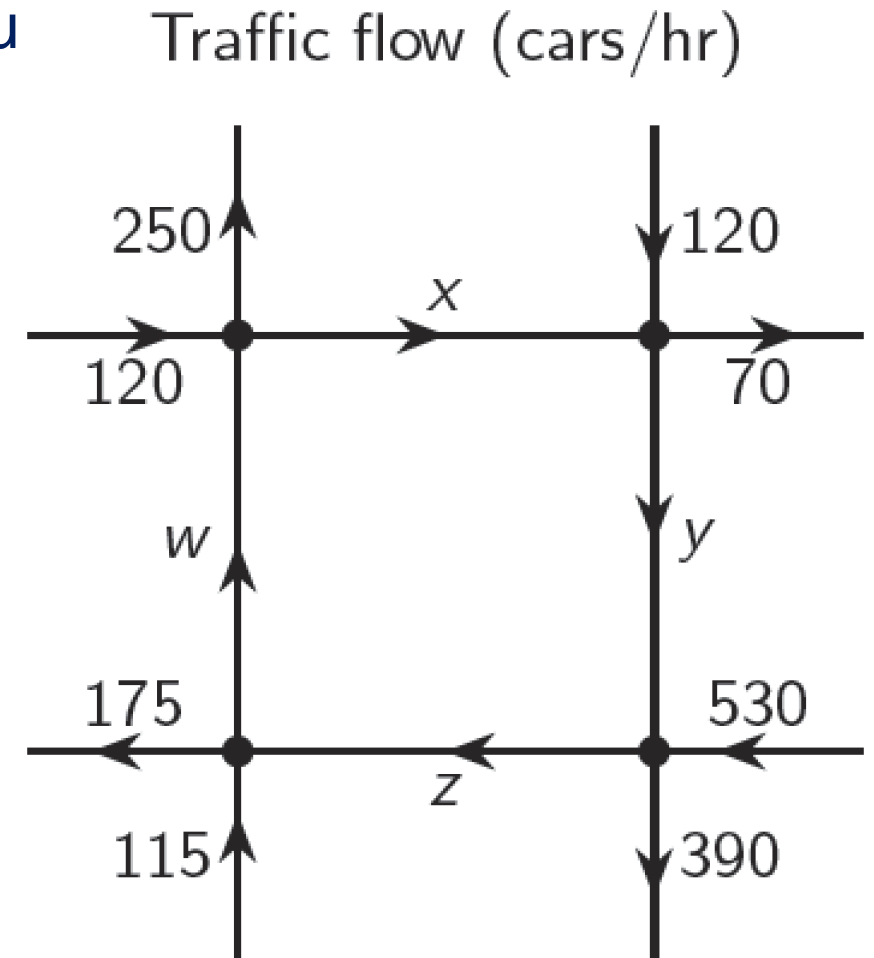
Solving Linear Equations

- Solving a system of linear equations is a central problem in linear algebra.

- A small linear system with two equations and two unknown variables.

$$x - 2y = 1$$
$$3x + 2y = 11$$

Row picture of the linear system

$$x - 2y = 1$$
$$3x + 2y = 11$$

- Row picture of this linear system
  - The first equation $x - 2y = 1$ corresponds to a line in the xy plane.
  - The second equation $3x + 2y = 11$ corresponds to another line in the xy plane.
  - The point (3, 1) where these two lines meet solves both equations.

Column Picture of the linear system

$$x - 2y = 1$$
$$3x + 2y = 11$$
$\longrightarrow$
$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

- This problem is to find the combination of two vector (1,3) and (-2, 2) on the left side that equals to vector (1, 11) on the right side.

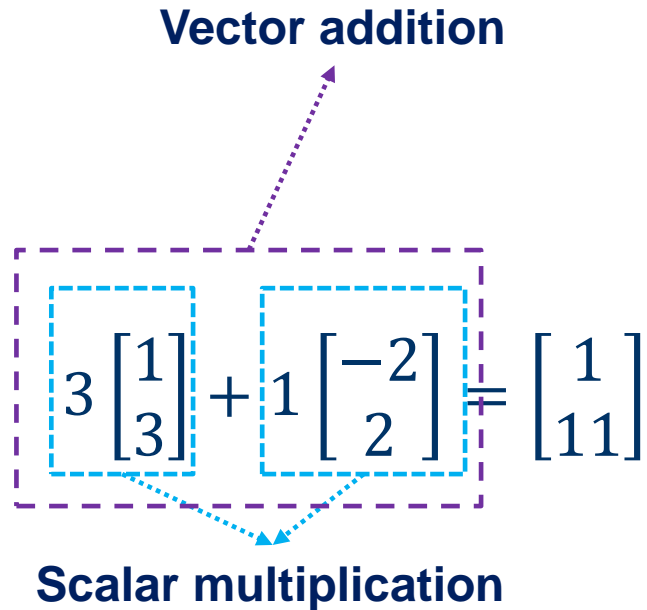- We know $x=3$ and $y=1$ (the same numbers as before) is the solution.

$$3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$
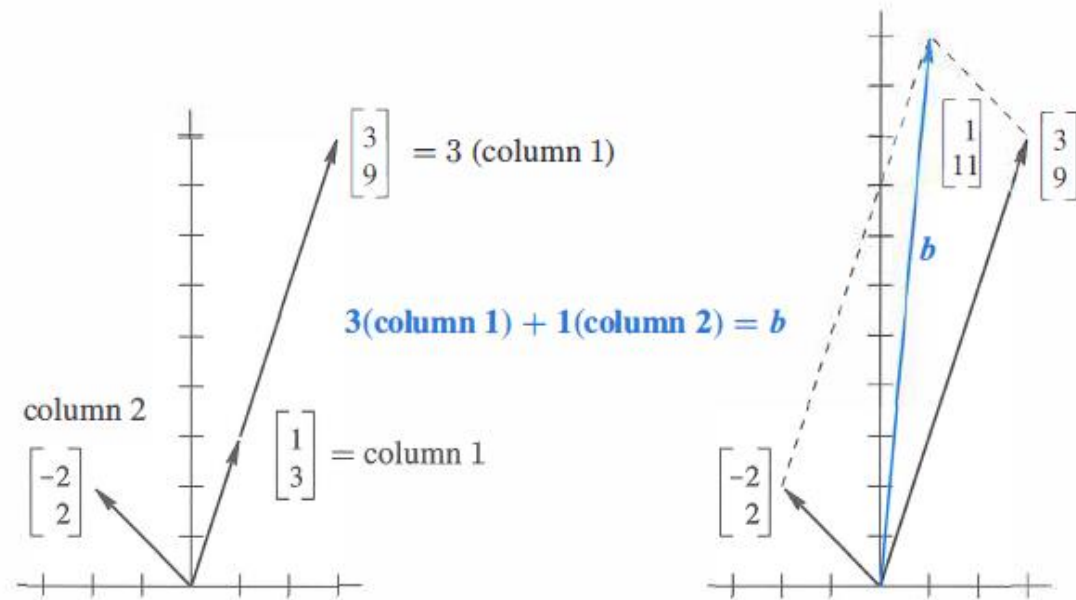
Column Picture of the linear system

Vector addition

Scalar multiplication

$$3\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

$\begin{bmatrix}3\\9\end{bmatrix} = 3$ (column 1)

$3(\text{column 1}) + 1(\text{column 2}) = b$

column 2

$\begin{bmatrix}-2\\2\end{bmatrix}$

$\begin{bmatrix}1\\3\end{bmatrix} = $ column 1

b

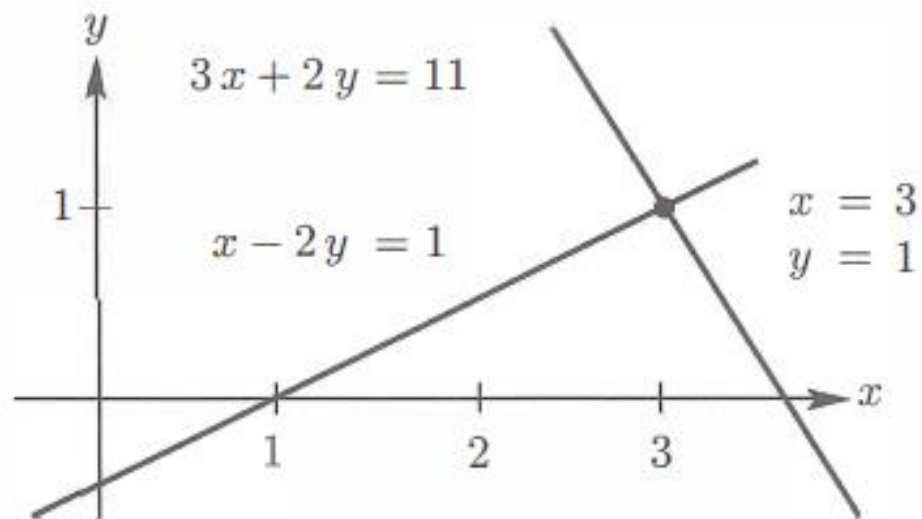$\begin{bmatrix}1\\11\end{bmatrix}$ $\begin{bmatrix}3\\9\end{bmatrix}$
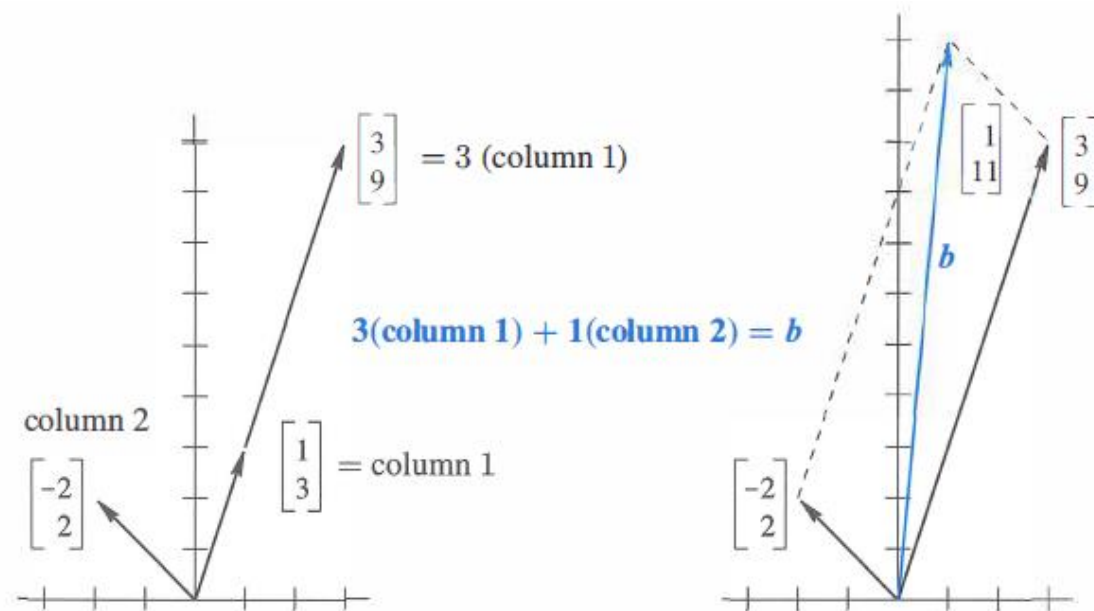
$\begin{bmatrix}-2\\2\end{bmatrix}$

- Linear combination consists of two basic operations: **scalar multiplication** and **vector addition**

Row picture vs. column picture

Row picture

$$3x + 2y = 11$$

$$x - 2y = 1$$

$$x = 3$$
$$y = 1$$

Column picture

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3 \text{ (column 1)}$$

$$\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

column 2

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \text{column 1}$$

$$3(\text{column 1}) + 1(\text{column 2}) = b$$

$$\begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$b$$

$$\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

The Matrix Form of Linear Equations

- We can represent the linear equations as a matrix problem **Ax** = **b**.

Ax = **b**

$$x - 2y = 1$$
$$3x + 2y = 11$$

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

- The bold capital letter **A** stands for the 2*2 coefficient matrix.
- The letter bold letter **b** denotes the column vector with two values 1, 11.
- The unknown variable **x** is also a column vector with two unknown values x an y.

How to systematically solve $\mathbf{Ax} = \mathbf{b}$?

- Suppose $\mathbf{A}$ is a square matrix (n unknown variables with n equations) and $\mathbf{A}$ is invertible. Then, the solution of $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ where $\mathbf{A}^{-1}$ is the inverse of matrix $\mathbf{A}$.

- Why?
    - Let discuss the following concepts before we talk about this solution.
        - Identity matrix
        - Inverse matrix

Identity matrix

- Let's see the following 2*2 matrix and 3*3 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- These two matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are a special type of matrix. They are **identity matrices**. Identity matrix is a square matrix that has 1s on the "main diagonal" and 0s everywhere else. Whatever vector this identity matrix multiplies, that vector is not changed.

- This is like multiply by 1, but for matrices and vectors.

Small Exercise

- If AB = I and BC = I, prove that A = C.

Inverse Matrix

- Suppose **A** is a square matrix. The inverse of A is a matrix B such that BA = I and AB = I. There is at most one such B, and it is denoted by A^{-1} (pronounced "A inverse"):

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

- Fundamental property is simple: If you multiply by A and then multiply by A^{-1}, you are back where you started.

Uniqueness of Inverse

- The matrix A cannot have two different inverses.

Property of Inverse

- The product AB of invertible matrices is inverted by $B^{-1}A^{-1}$.

- A similar rule holds with three or more matrices:

$$\textbf{Inverse of } ABC \qquad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

Small Exercise

- If A is invertible and AB = AC, prove that B = C.

Systematically solve a system of linear equations

- Solve $\mathbf{Ax} = \mathbf{b}$

- Suppose $\mathbf{A}$ is a square matrix and $\mathbf{A}$ is invertible.

$$\mathbf{Ax} = \mathbf{b} \quad \text{(multiply both sides by } \mathbf{A}^{-1})$$

$$\mathbf{A}^{-1}\,\mathbf{Ax} = \mathbf{A}^{-1}\,\mathbf{b} \quad (\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I})$$

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}$$

<u>Questions (We will answer them in the following lectures):</u>

1. Which square matrices are invertible?
2. What if A is not invertible?
3. What if A is not a square matrix?

The End