I. For 3 variables x, y, z, they satisfy the equality $x + y + z = 0$. Calculate the angle between vector $v = (x, y, z)$ and vector $w = (z, x, y)$.

$$v \cdot w = xz + xy + yz = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$$

$$v \cdot w = 0 - \frac{1}{2}\|v\|\|w\|, \text{then } \cos\theta = -\frac{1}{2}, \theta = 120^o$$

II. Suppose $Q^T = Q^{-1}$.

(1) Show that the columns $q_1, \cdots q_n$ are unit vectors: $\|\boldsymbol{q_i}\|^2 = 1$.

(2) Show that every two columns of Q are perpendicular: $\boldsymbol{q_i^T q_j} = 0$.

(3) Find a 2 by 2 example (that $Q^T = Q^{-1}$) with first entry $q_{11} = \cos\theta$.

$Q^T = Q^{-1}$, so $Q^T Q = I$, as in $\begin{bmatrix} q_i^T \\ q_j^T \end{bmatrix} \begin{bmatrix} q_i & q_j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

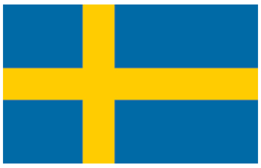
(1) The diagonal entries give $q_i^T q_i = 1$ and $q_j^T q_j = 1$: *unit vectors*

(2) The off-diagonal entry is $q_1^T q_2 = 0$, and in general $\boldsymbol{q_i^T q_j} = 0$

(3) The leading example for Q is the rotation matrix $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

III. These flags have rank 2. Find the singular value decomposition of

$\mathbf{A}_{\text{Sweden}}$, $\mathbf{A}_{\text{Finland}}$, $\mathbf{B}_{\text{Benin}}$.



Sweden

Finland

Benin

$$\mathbf{A}_{\text{Sweden}} = \mathbf{A}_{\text{Finland}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \qquad \mathbf{B}_{\text{Benin}} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

For A,

$$U \begin{matrix} 0.482 & 0.517 & 0.707 \\ 0.732 & -0.682 & 0.000 \\ 0.482 & 0.517 & -0.707 \end{matrix} \qquad S \begin{matrix} 5.402 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.907 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{matrix} \qquad V \begin{matrix} 0.449 & -0.362 & -0.816 & 0.000 \\ 0.628 & 0.778 & -0.000 & 0.000 \\ 0.449 & -0.362 & 0.408 & -0.707 \\ 0.449 & -0.362 & 0.408 & 0.707 \end{matrix}$$

For B,

U	0.566	0.824
	0.824	-0.566

S	5.285	0.000	0.000
	0.000	0.268	0.000

V	0.263	0.965	-0.000
	0.682	-0.186	-0.707
	0.682	-0.186	0.707

IV. Suppose A_0 is a 5 by 10 matrix with average grades for 5 courses over 10 years.

(1) How would you create the centered matrix A and the sample covariance matrix S?
(2) When you find the leading eigenvector of S, what does it tell you?

From each row of A_0, subtract the average of that row (the average grade for that course) from the 10 grades in that row. This produces the centered matrix A. Then the sample covariance matrix is $S = \frac{1}{9}AA^T$. The leading eigenvector of the 5 by 5 matrix S tells the weights on the 5 courses to produce the "eigencourse". This is the course whose grades have the most information (the greatest variance).

If a course gives everyone an A, the variance is zero and that course is not included in the eigencourse. We are looking for most information not best grade.