Question 1 (30 Marks)

Answer the following questions about Python Programming.

1.1 What is the difference between lists and tuples in Python? (10 Marks)

Please check the slides for the answer.

1.2 What will be the output of the following Python code? (10 Marks)

```
print(123 + 456)
print('123' + '456')
```

The first line prints “579”.
The second line prints “123456”

1.3 What will be the output of the following Python code? (10 Marks)

```
a = [3, 5, 9 ,10, 2, 1]
a.sort()
print(a[5:2:-2])
```

[10, 5]

Question 2 (35 Marks)

Answer the following questions about basic matrix operations.

2.1 Find the values of c, d so that the linear combination $c\mathbf{v} + d\mathbf{w}$ produce vector $\mathbf{b}$:

$$\mathbf{v} = \begin{bmatrix}1\\3\end{bmatrix} \quad \mathbf{w} = \begin{bmatrix}3\\1\end{bmatrix} \quad \mathbf{b} = \begin{bmatrix}5\\-1\end{bmatrix}.$$

(10 marks)

$$c\begin{bmatrix}1\\3\end{bmatrix} + d\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix} \rightarrow \begin{cases} c + 3d = 5 \\ 3c + d = -1 \end{cases} \rightarrow \begin{cases} c = -1 \\ d = 2 \end{cases}$$

2.2 Find a unit vector $\mathbf{u}$ in the direction of $\boldsymbol{a} = \begin{bmatrix}8\\6\end{bmatrix}$. Choose a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{u}$ and a unit vector $\mathbf{v}$ that is in the reverse direction of $\mathbf{u}$.

(12 Marks)

$$\mathbf{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{1}{\sqrt{8^2 + 6^2}}\boldsymbol{a} = \frac{1}{10}\boldsymbol{a} = \begin{bmatrix}0.8\\0.6\end{bmatrix}$$

$$v = \begin{bmatrix}v_1\\v_2\end{bmatrix}, v^{\mathrm{T}}u = 0 \rightarrow \begin{cases} 0.8v_1 + 0.6v_2 = 0 \\ {v_1}^2 + {v_2}^2 = 1 \end{cases} \rightarrow v = \begin{bmatrix}-0.6\\0.8\end{bmatrix} \; or \; \begin{bmatrix}0.6\\-0.8\end{bmatrix}$$

2.3 Given $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, which of the following matrix operations are allowed, and what are the results?
AB, BA, ABB^T

(13 marks)

$$\mathbf{AB} = \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \end{bmatrix}$$

$$\mathbf{ABB^T} = \begin{bmatrix} 18 & 18 \\ 36 & 36 \end{bmatrix}$$

Question 3 (35 Marks)

Answer the following questions about linear equations system **Ax** = **b**.

3.1 Suppose you solve **Ax** = **b** for three special right side **b**:

$$\mathbf{Ax}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{Ax}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \text{ and } \mathbf{Ax}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

The three solutions are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(a) If a matrix **X** whose columns are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, what is the result of **AX**?

(10 marks)

$$\mathrm{AX} = \mathrm{A}[x_1 \quad x_2 \quad x_3] = [Ax_1 \quad Ax_2 \quad Ax_3] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) What is the inverse of **A**? Justify your answer.

(12 marks)

$$\mathbf{AX} = 3\mathbf{I} \rightarrow \mathbf{A}\left(\frac{1}{3}\right)\mathbf{X} = \mathbf{I} \rightarrow \boldsymbol{A^{-1}} = \frac{1}{3}\boldsymbol{X}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

3.2 Consider a linear equation system **Ax** = **b** where $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & a \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ b \end{bmatrix}$. Find the condition on *a*, *b* so that this system has a unique solution.

(13 marks)

To guarantee the unique solution, the determinant of **A** should not be zero. i.e., $\det(\mathbf{A}) \neq 0$. Thus, $1 \times \mathrm{a} - (-2) \times 4 \neq 0 \rightarrow \mathrm{a} \neq -8$.