Periodic Function :-

→ $f(x+p) = f(x)$

$p \rightarrow$ period & $p > 0$ (positive)

Eg (i) $f(x) = x$
⇒ Not Periodic

(ii) $f(x) = \sin x$
⇒ Periodic with period
$p = 2\pi, 4\pi, 6\pi, \ldots$

(iii) $f(x) = \tan x$
⇒ Periodic with period
$p = \pi, 2\pi, 3\pi, \ldots$

(iv) $f(x) = 5\pi$
⇒ Periodic with period any real no., $R > 0$
⇒ Fundamental period cannot be defined

E.g (1) $f_1(x) = \sin x$, $p = 2\pi$

$f_2(x) = \cos 5x$, $p = \frac{2\pi}{5}$ ✓

$h(x) = a f_1(x) + b f_2(x)$

$= a\sin x + b\cos 5x$, $p = 2\pi$ (L.C.M of period of $f_1(x)$ & $f_2(x)$)

(2) $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots \infty$

$f(x) = a_0 + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$ → $p = 2\pi$ → Fourier Series ——— (1)

Orthogonal Property

$$\int_{-\pi}^{\pi} \sin nx . \sin mx \, dx.$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\sin nx \sin mx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n-m)x - \cos(n+m)x] \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(n-m)x}{(n-m)} - \frac{\sin(n+m)x}{(n+m)}\right]_{-\pi}^{\pi} \quad (m \neq n)$$

$$= \begin{cases} 0, & (n \neq m), \forall\, n, m \\ \pi, & (n = m) \end{cases}$$

& Integrating equ^n (i) w.r.t x from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos nx\,dx + \int_{-\pi}^{\pi} b_n \sin nx\,dx\right]$$

$$= 2\pi a^0 + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right]$$

$$= 2\pi a^0 + \sum_{n=1}^{\infty}\left[a_n \times 0 + b_n \times 0\right]$$

$$= 2\pi a^0$$

$$\Rightarrow \boxed{a^0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.} \quad \text{—(ii)}$$

<u>For finding a_n and b_n</u>

$a_n \Rightarrow$ First of all multiply $\cos mx$ & Integrating equ^n (i) w.r.t x from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} a_0 \cos mx\,dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos nx \cos mx\,dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx\,dx\right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0 + a_n \cdot \pi + 0$$

$$\Rightarrow \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx} \quad \text{—(iii)}$$

25.7.16

③ C.W

$b_n \Rightarrow$ First of all multiply $\sin mx$ & Integrating equ^n (i) w.r.t x from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} a_0 \sin mx\,dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos nx \sin mx\,dx + \int_{-\pi}^{\pi} b_n \sin nx \sin mx\,dx\right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin mx\,dx = 0 + 0 + b_n \cdot \pi$$

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx} \quad \text{—(iv)}$$

Equ^n (ii), (iii) & (iv) are known as Euler's Formula

1/8/16 C.W.

Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

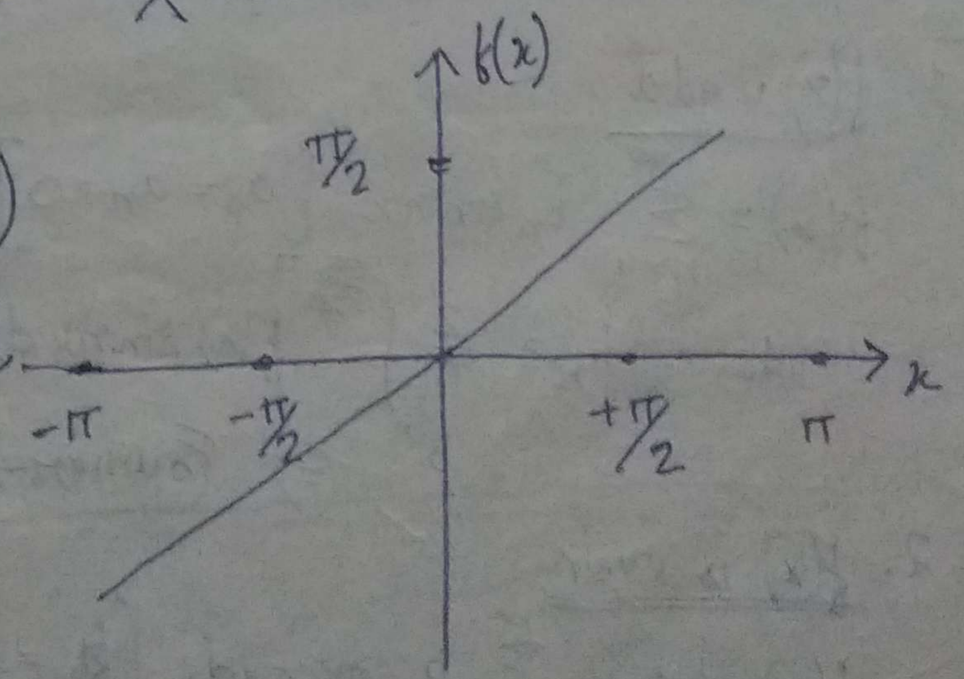
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Ex.1 Here, Given, $p = 2\pi$

$$\Rightarrow f(x+2\pi) = f(x)$$

Divide the interval into several parts,



$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ x, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

P.T.O

1.8.16

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2} 0\,dx + \int_{-\pi/2}^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} 0\,dx\right]$$

$$= \frac{1}{2\pi} \times \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{4}\right]$$

$$= 0.$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} 0 \cdot \cos nx\,dx + \int_{-\pi/2}^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} 0 \cdot \cos nx\,dx\right]$$

$$= \frac{1}{\pi} \times 0 = 0 \quad [\because x \text{ is odd, } \cos nx \text{ is even } \& \text{ odd} \times \text{even} = \text{odd}].$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx \quad [\because x \text{ is odd, } \sin nx \text{ is odd } \& \text{ odd} \times \text{odd} = \text{even}]$$

$$= \frac{1}{\pi} \times 2\int_0^{\pi/2} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x\int \sin nx\,dx - \int\left(\int \sin nx\,dx\right)dx\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}x\cos x + \frac{1}{n}\int \cos nx\,dx\right]_0^{\pi/2}$$

$$= \cancel{\frac{2}{\pi}\left[-\frac{\pi}{2} \times 0 + 0 + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]} = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]_0^{\pi/2}$$

$$= \cancel{\frac{2}{\pi}} \quad \frac{2}{n^2\pi}$$

$$\therefore f(x) = \frac{2}{\pi}\left[\frac{1}{2}\sin x + \left(\frac{\pi}{2^2}\right)\sin 2x - \frac{1}{2^2}\sin 3x - \frac{\pi}{2^3}\sin 4x + \dots\right]$$

Note:-

1. $f(x)$ is odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad [a_0 = a_n = 0]$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$ $\Rightarrow$ Fourier-Sine Series

2. $f(x)$ is even

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx\,dx \quad [b_n = 0]$$

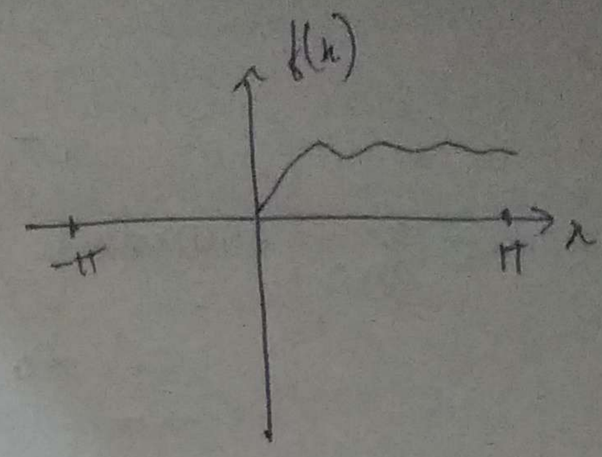
where $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$ $\Rightarrow$ Fourier-Cosine Series

$\pi/2$.

$\left(\frac{n\pi}{2}\right)$

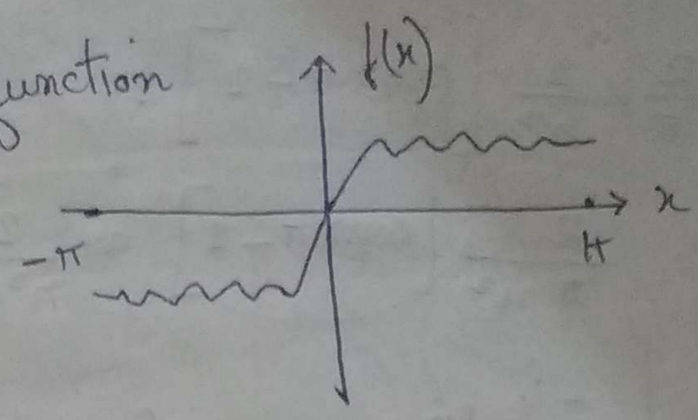
0

Half-Length Function



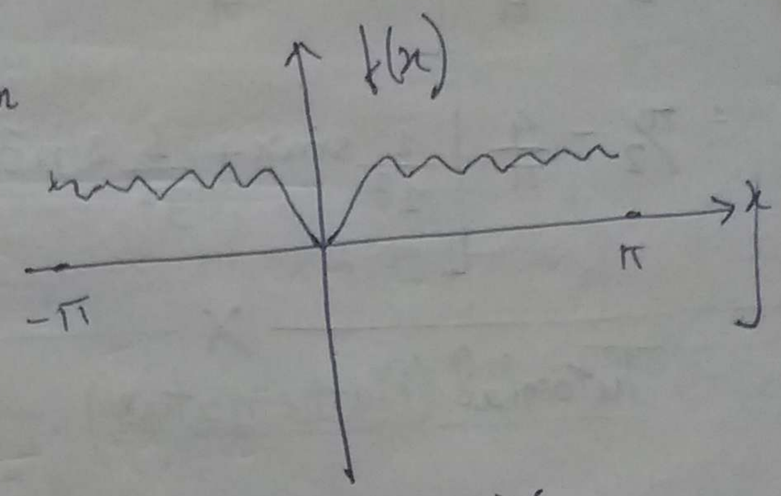
⇒ **f(x) is Odd :-**

(Full-length function is given by:-)



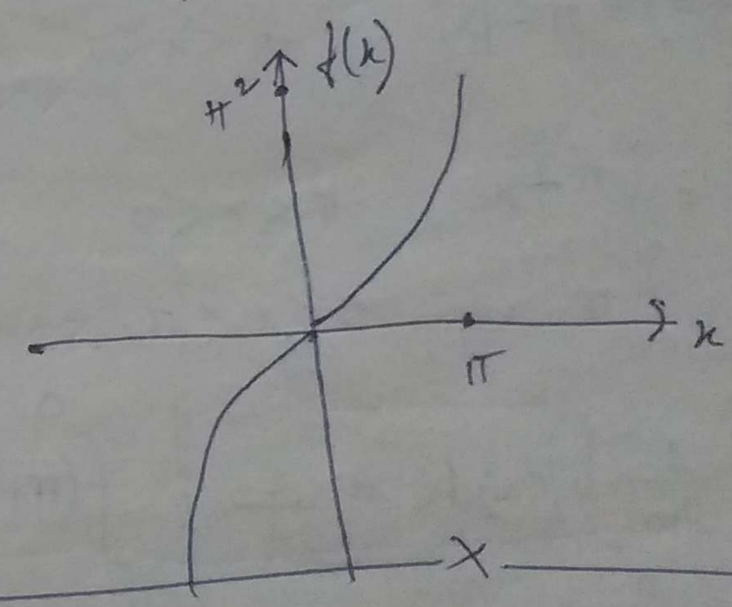
⇒ **f(x) is Even :-**

(Full length function is given by:-)



E.g. $f(x) = x^2 \quad , \quad 0 < x < \pi$.

$\quad\quad = -x^2 \quad , \quad -\pi < x < 0$



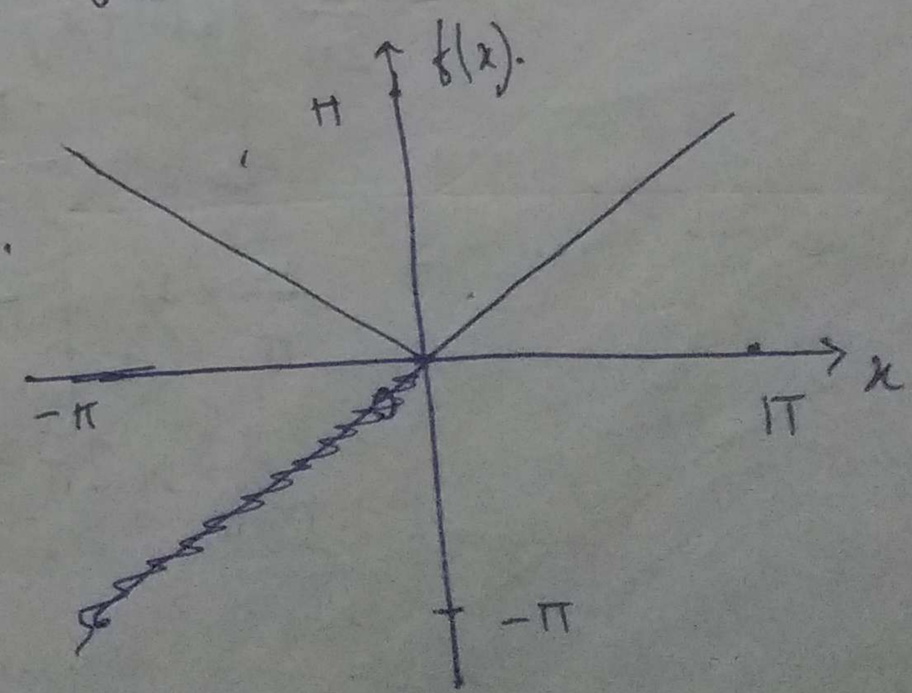
Q. Define Fourier Cosine series for $f(x) = x \quad , \quad 0 < x < \pi$

Solⁿ

$$f(x) = \begin{cases} x \ , & 0 < x < \pi \\ -x \ , & -\pi < x < 0. \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi} \times \pi^2$$

$$= \pi/2.$$



Sine Series

1.8.16

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\int\cos nx - \frac{1}{n}\int\sin nx\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\times\frac{1}{n^2}\left[\cos n\pi - 1\right].$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - 1\right] \text{ or } \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore f(x) = \frac{\pi}{2} + \left[\frac{-4}{\pi}\sin x - \frac{4}{9\pi}\sin 3x - \frac{4}{\pi\times 25}\sin 5x - \dots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\sin x + \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \dots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\sin x + \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x + \dots\right]$$

$\frac{2}{\pi\times 1}$

$\frac{2}{\pi\times 9}[-2]$

$\frac{2}{\pi\times 5^2}\times -2$

Tutorial (Mathematics)

- **Fourier Integral :— (Complex Form)**

$$f(x) = \int_0^\infty \left[\left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos\omega v \, dv \right) \cos\omega x + \left(\frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin\omega v \, dv \right) \sin\omega x \right] d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} \left[f(v) \left\{ \cos\omega v \cos\omega x + \sin\omega v \sin\omega x \right\} \right] dv \, d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v) \cos(x-v)\omega \, dv \, d\omega.$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \cos(x-v)\omega \, dv \, d\omega \quad \text{——— (i)}$$

$$\& \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \sin(x-v)\omega \, dv \, d\omega = 0 \quad \text{——— (ii)}$$

$$\Rightarrow \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) \sin(x-v)\omega \, dv \, d\omega = 0i = 0$$ [Adding $\{i\}$ on both sides of (ii)].

$$\therefore (i) + (ii) \Rightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{i(x-v)\omega} \, dv \, d\omega$$ [$e^{i\theta} = \cos\theta + i\sin\theta$

$$\Rightarrow \boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-iv\omega} \, dv \right] e^{ix\omega} \, d\omega.}$$

P.T.O

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$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \text{[Fourier Transform]}$$

or $F(\omega)$

$$F^{-1}[F(\omega)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \quad \text{[Inverse Fourier Transform]}$$

Property :-

(A) Linearly :-

$$F[a f(x) + b g(x)] = a F[f(x)] + b F[g(x)].$$

(B) Derivative :-

Let $f(x)$ is continuous on whole x-axis $(x \in R)$ & it is also absolutely integrable on whole x-axis & $f(x)$ goes to be zero (0) when $|x| \to \infty$ then

$$F[f'(x)] = i\omega F[f(x)].$$

Q. (1) $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ (2) $f(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0. \end{cases}$

(1). $$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{-1}{i\omega} \left[e^{-i\omega x}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{i}{\omega} \left[e^{-i\omega} - e^{i\omega}\right]$$

$$= \frac{i}{\omega\sqrt{2\pi}} \left[\cos\omega - i\sin\omega - \cos\omega - i\sin\omega\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$$

(2) $$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+i\omega)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{-1}{(a+i\omega)} \times \left[e^{-(a+i\omega)x}\right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{-1}{(a+i\omega)} \times -1 = \frac{1}{\sqrt{2\pi}} \times \frac{a-i\omega}{a^2+\omega^2} = \frac{1}{\sqrt{2\pi}} \frac{(a-i\omega)}{a^2+\omega^2} = \frac{1}{\sqrt{2\pi}} \frac{e^{(a-i\omega)}}{a^2+\omega^2}$$

10.2.16

(3) $f(x) = \begin{cases} e^{2ix}, & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$

Sol^n $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{(2i - i\omega)x} dx$

$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i(2-\omega)} \left[e^{(2i-i\omega)x} \right]_{-1}^{1}$

$= \frac{1}{\sqrt{2\pi}} \times \frac{-i}{(2-\omega)} \left[e^{(2i-i\omega)} - e^{(i\omega - 2i)} \right]$

$= \frac{1}{\sqrt{2\pi}} \times \frac{-i}{2-\omega} \left[e^{i(2-\omega)} - e^{-i(2-\omega)} \right]$

$= \frac{1}{\sqrt{2\pi}} \times \frac{-i}{(2-\omega)} \left\{ \cos(2-\omega) + i\sin(2-\omega) - \cos(2-\omega) + i\sin(2-\omega) \right\}$

$e^{3} - e^{2}$

$e^{2} - e^{-2}$

$= \frac{-i \times 2}{\sqrt{2\pi}} \left[i \sin(2-\omega) \right] \times \frac{1}{2-\omega}.$

$= \sqrt{\frac{2}{\pi}} \frac{\sin(2-\omega)}{(2-\omega)}.$

(4) $f(x) = \begin{cases} xe^{-x}, & -1<x<0 \\ 0, & \text{otherwise} \end{cases}$

[Ans. $F(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{1+i\omega e^{1+i\omega}}{(-\omega+i)^2} \right)$

(4). $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} xe^{-x} \cdot e^{-i\omega x} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{-(1+i\omega)x} dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{-x}{(1+i\omega)} e^{-(1+i\omega)x} + \frac{1}{(1+i\omega)} \int e^{-(1+i\omega)x} dx \right]_{-1}^{0}$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{(1+i\omega)} x e^{-(1+i\omega)x} - \frac{1}{(1+i\omega)^2} e^{-(1+i\omega)x} \right]_{-1}^{0}$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{(1+i\omega)} \left[1 e^{-(1+i\omega)} \right] - \frac{1}{(1+i\omega)^2} \left\{ 1 - e^{(1+i\omega)} \right\} \right]$

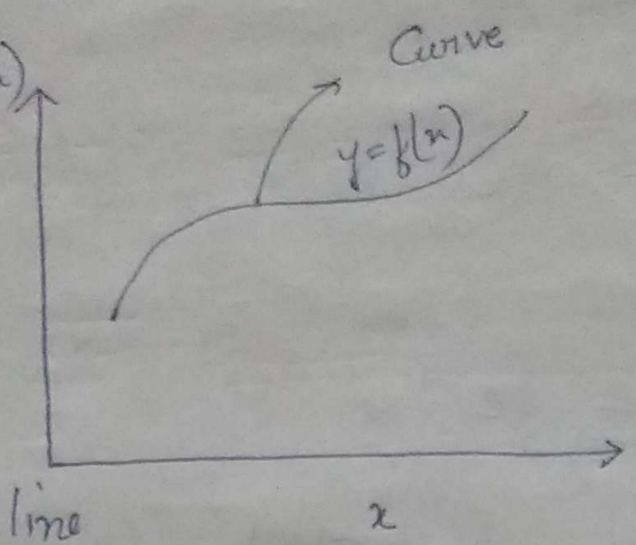
$= \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-(1+i\omega)}}{1+i\omega} - \frac{1}{(1+i\omega)^2} + \frac{e^{1+i\omega}}{(1+i\omega)^2} \right]$

X

17.8.16

Ch- Partial Differential Equation

• Single Independent Variable (x) — $f(x)$



$\frac{df}{dx}$ → Ordinary derivative

→ represents tangent line

dF → Total derivative

• Two Independent Variables (x, y) $z = f(x, y)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ → partial derivative

$$\therefore \boxed{dF = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy} \rightarrow \text{Total derivative}$$

E.g.

(i) $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 = 0$

Order → 2
Degree → 1.

(ii) $\left[\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right]^3 = \left[\left(-\left(\frac{\partial u}{\partial y}\right)^{+1/3}\right)\right]^3$

Order → 2
Degree → 3.

(iii) $x^{1/3} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = -\left(\frac{\partial u}{\partial y}\right)$

Order → 2
Degree → 1

(iv) $1/u + \frac{\partial u}{\partial x} = 0$ $\left[u \frac{\partial u}{\partial x} + 1 = 0\right]$

Order → 1
Degree → 1.

(v) $u^{1/3} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = -\left(\frac{\partial u}{\partial y}\right)$

Order → 2
Degree → 1.

Classification of Partial Differential Equation

(A) Linear :- E.g. $\frac{\partial u}{\partial x} + u = 0$. $\left[P = P(x, y),\ Q = Q(x, y),\ R = R(x, y, z)\right]$ linear

(B) Quasi-Linear :- Each term containing max^m 1 derivative term.

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E.g $u\frac{\partial u}{\partial x}+1=0$ $\left[P=P(x,y,z),\ Q=Q(x,y,z),\ R=R(x,y,z)\right]$

(c) Non-Linear :-

E.g $\left(\frac{\partial u}{\partial x}\right)^2+1=0$ $\left[P=P(x,y,z,p,q),\ Q=Q(x,y,z,p,q),\ R=R(x,y,z,p,q)\right]$

Notations used :- $\frac{\partial z}{\partial x}=p,\ \frac{\partial z}{\partial y}=q$ [General 1st Order P.D.E :- $Pp+Qq=R$]

OR

$Pp+Qq+Rz=S$

$R=R(x,y)$ $S=S(x,y)$.

1. $pz+qy=xy$ → Quasi-Linear (Non-Homogeneous)
2. $px+qy=pq$ → Non-Linear (Homogeneous)
3. $ax+by=pz$ → Quasi-Linear (Non-homogeneous)
4. $ax^3+bz^3=qy$ → Quasi-Linear (Non-homogeneous)

Formulation of PDE

(i) Elimination of arbitrary constants $\left(\frac{x^2}{a^2}+by=z^2\right)$.

(ii) " " " functions $\left(z=f(x)\right)$.

(i) Elimination of arbitrary constants

• Find out No. of unknowns & No. of independent variables [And check if No. of unknowns is greater than no. of independent variables]

E.g $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ —(i)

No. of unknown ~~variables~~ = 3

" " dependent = 1 [$x(y,z)$ or $y(x,z)$ or $z(x,y)$].

" " independent = 2

Note:- If No. of unknown > No. of independent variables (Order of PDE will be more than 1)

→ Partially diff. w.r.t to x,y,

$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$ —(ii) & $\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0$ —(iii).

$\frac{2}{a^2}+\frac{2}{c^2}\left[z\left(\frac{\partial^2 z}{\partial x^2}\right)+\left(\frac{\partial z}{\partial x}\right)^2\right]=0$ & $\frac{2}{b^2}+\frac{2}{c^2}\left[z\left(\frac{\partial^2 z}{\partial y^2}\right)+\left(\frac{\partial z}{\partial y}\right)^2\right]=0$

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$$\left(\frac{2}{c^2}\right)\left[-x\left\{z\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2\right\}+z\frac{\partial z}{\partial x}\right]=0$$

$$\Rightarrow -x\left(\frac{\partial z}{\partial x}\right)^2+z\left(\frac{\partial z}{\partial x}\right)-xz\frac{\partial^2 z}{\partial x^2}=0 \quad \left[\because \frac{2}{c^2}\neq 0\right].$$

E.g $y+\sqrt{x+a}+\sqrt{y-b}=0$ —— (i).

Diff. partially w.r.t x,

$$\frac{\partial y}{\partial x}+\frac{1}{2\sqrt{x+a}}=\cancel{0}+\frac{1}{\quad}$$

23.8.16.

Solution of PDE

(A) Direct Integration Method :— (Linear)

E.g (i) $\dfrac{\partial^3 z}{\partial x^2 \partial y}+\sin x\sin y=0$

Integrate w.r.t x, x, y,

$$\frac{\partial^2 z}{\partial x\partial y}-\cos x\sin y=C_1(y)$$

$$\frac{\partial z}{\partial y}-\sin x\sin y=x\cdot C_1(y)+C_2(y).$$

$$z+\sin x\cos y=x\int C_1(y)\,dy+\int C_2(y)\,dy+C_3(x).$$

$$\Rightarrow z=x\int C_1(y)\,dy+\int C_2(y)\,dy+C_3(x)-\sin x\cdot\cos y$$

(ii) $\dfrac{\partial^2 z}{\partial x^2}+z=0$, where $z=e^y$, $\dfrac{\partial z}{\partial x}=1$ when $x=0$.

Soln $A.E=m^2+1=0$

$\Rightarrow m=\pm i \quad (m=0\pm i)$

$z=e^{\alpha x}\left(C_1(y)\cos\beta x+C_2(y)\sin\beta x\right)$

$\Rightarrow z=C_1(y)\cos x+C_2(y)\sin x$ —— (i)

At $x=0$, $z=e^y$

$\Rightarrow e^y=C_1(y)\Rightarrow C_1(y)=e^y$

23.8.16

Now, $\frac{\partial z}{\partial x} = 1$, when $x = 0$

$\Rightarrow -\sin x\, C_1(y) + \cos x\, C_2(y) = 1$

$\Rightarrow \cos 0 \times C_2(y) = 1$

$\Rightarrow C_2(y) = 1.$

(i) $\Rightarrow$

$\therefore \boxed{z = e^y \cos x + 1.\sin x}$

(B) Lagrange's Method $(Pp + Qq = R)$ (Linear)

(Also applicable for Quasi-Linear)

E.g. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy.$

~~Integrate w.r.t x, y, z.~~

Assume $\phi(u, v) = 0$, where u, v are functions of x, y, z

Partially diff. w.r.t x, y. $[u(x, y, z(x,y))]$ & $[v(x, y, z(x,y))]$

$$\frac{\partial \phi}{\partial u}\cdot\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\cdot\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\cdot\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\cdot\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

For Non-trivial sol^n,

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot q \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + p\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} + q\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} + pq\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\right)$$
$$- \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} p + q\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} + pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow p\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}\right) + q\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right) = 0$$

or $\boxed{p.P + qQ = R}$

24.8.16

Let $u(x,y,z)=a$, $v(x,y,z)=b$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \text{—(i)}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad \text{—(ii)}$$

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

Solving (i) & (ii)

$$\therefore \frac{dx}{\left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}\right)} = \frac{dy}{\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}\right)} = \frac{dz}{\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)}$$

or $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$ Lagrange's Subsidiary Equation

---

E.g. (1) $\frac{y^2z}{x}p + xzq = y^2 \Rightarrow y^2zp + x^2zq = y^2x$

Solⁿ Lagrange's Subsidiary Equⁿ is —

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$$

$$\Rightarrow \frac{dx}{y^2z} = \frac{dy}{x^2z} \quad [z \neq 0]$$

$$\Rightarrow x^2dx = y^2dy$$

Integrating both sides,

$$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + C_1$$

$$\Rightarrow \boxed{x^3 - y^3 = C_1'} \quad [C_1' = 3C_1] \ \#$$

Again, $\frac{dx}{y^2z} = \frac{dz}{y^2x}$

$$\Rightarrow xdx = zdz$$

Integrating both sides,

$$\Rightarrow \boxed{x^2 - z^2 = C_2'}$$

$\therefore$ Solⁿ is,

$$\phi(u,v) = \phi(x^3-y^3, x^2-z^2) = 0 \ \#$$

X

Charpit's Method

$$F(x, y, z, p, q) = 0 \quad \text{——— (i).}$$

Charpit's subsidiary equ^n is —

$$\frac{dx}{-\left(\frac{\partial F}{\partial p}\right)} = \frac{dy}{-\left(\frac{\partial F}{\partial q}\right)} = \frac{dz}{-\left\{p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}\right\}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}}$$

Case – (I) – $F(p, q) = 0$ ——— (i)

$$\Rightarrow \boxed{p = c_1} \quad \text{——— (ii)}$$

$\therefore$ Putting (ii) in (i) $\Rightarrow$

$$\boxed{q = \phi(c_1)} \quad \text{——— (iii)}$$

$\therefore$ Sol^n is

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = c_1 dx + \phi(c_1) dy$$

$$\Rightarrow \boxed{z = c_1 x + \phi(c_1) y + c_2}$$

Q ① $\sqrt{p} + \sqrt{q} = 1 \Rightarrow q = (1 - \sqrt{p})^2$

Ans. Sol^n is,

$$z = c_1 x + \phi(c_1) y + c_2$$

where $\phi(c_1) = (1 - \sqrt{c_1})^2$

$\therefore$ Sol^n is

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = c_1 dx + (1 - \sqrt{c_1})^2 dy$$

$$\Rightarrow \boxed{z = c_1 x + (1 - \sqrt{c_1})^2 y + c_2}$$

② $pq = 1 \Rightarrow q = 1/p$

$\therefore$ Sol^n is, $\boxed{z = c_1 x + \frac{1}{c_1} y + c_2}$

Case (II) :– $F(z, p, q) = 0$ ——— (i)

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \boxed{\frac{p}{q} = c_1} \quad \text{——— (ii)}$$

$$\boxed{p = c_1 q}$$

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Solving eq^n (i) & (ii),

$$\phi = f(z), \quad q = g(z)$$

Q1. $p^2 z^2 + q^2 = 1$ —— (i).

$$\frac{p}{q} = c_1$$

$$\Rightarrow \boxed{p = c_1 q} \text{ —— (ii)}$$

From (i) & (ii),

$$c_1^2 q^2 z^2 + q^2 = 1$$

$$\Rightarrow q^2 = \frac{1}{(c_1^2 z^2 + 1)}$$

$$\Rightarrow q = \frac{1}{\sqrt{c_1^2 z^2 + 1}}$$

$$\therefore p = \frac{c_1}{\sqrt{c_1^2 z^2 + 1}}$$

$\therefore$ Sol^n is

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{c_1\,dx}{\sqrt{c_1^2 z^2 + 1}} + \frac{dy}{\sqrt{c_1^2 z^2 + 1}}$$

$$\Rightarrow \sqrt{c_1^2 z^2 + 1}\,dz = c_1\,dx + dy.$$

$$\Rightarrow c_1 \sqrt{z^2 + 1/c_1^2}\,dz = c_1\,dx + dy$$

$$\Rightarrow c_1 z \sqrt{1 + c_1^2 z^2} + \ln\left[c_1 z + \sqrt{1 + c_1^2 z^2}\right] = 2c_1(c_1 x + y + c_2)$$

Let $1 + c_1^2 z^2 = t$

$\Rightarrow c_1^2 \cdot 2z\,dz = dt$

$\Rightarrow dz = \frac{dt\, c_1}{2c_1^2 \sqrt{t-1}}$

$= \frac{dt}{2c_1}(t-1)^{-1/2}$

$\frac{1}{2c_1}\left[t(t-1)^{-1/2} dt\right]$

Q2 $p(1+q) = qz$ —— (i)

$$\boxed{p = c_1 q} \text{ — (ii)}$$

From (i) & (ii),

$$c_1 q(1+q) = qz$$

$$\Rightarrow c_1 q + c_1 q^2 = qz$$

$$\Rightarrow c_1 q^2 + q(c_1 - z) = 0.$$

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$$\Rightarrow q = \frac{3-c_1 \pm \sqrt{(c_1-3)^2}}{2c_1}$$

$$= \frac{3-c_1 \pm (c_1-3)}{2c_1}$$

$$= 0, \quad \frac{3-c_1-c_1+3}{2c_1}$$

$$= 0, \quad \frac{3-c_1}{c_1}$$

$$\therefore p = c_1 q = 3-c_1.$$

$\therefore$ Sol^n is,

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = (3-c_1)\,dx + \frac{(3-c_1)}{c_1}\,dy$$

$$\Rightarrow \frac{1}{3-c_1}\,dz = dx + \frac{dy}{c_1}$$

$$\Rightarrow \log_e|3-c_1| = x + \tfrac{1}{c_1}y + c_2$$

$$\Rightarrow 3-c_1 = e^{x+\frac{1}{c_1}y+c_2}$$

$$\Rightarrow 3 = e^{x+\frac{1}{c_1}y+c_2} + c_1.$$

Case III :- $F(x,y,p,q)=0$

or, $f(x,p) - g(y,q) = 0$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x}, \quad \frac{\partial F}{\partial y} = -\frac{\partial g}{\partial y}$$

$$\frac{dp}{\left(\frac{\partial f}{\partial x}\right)} = \frac{dq}{-\left(\frac{\partial g}{\partial y}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial g}{\partial q}}$$

$$\Rightarrow \boxed{p = \text{—}(x)}, \quad \boxed{q = \text{—}(y)}$$

Let,

or $f(x,p) = g(y,q) = a$

$\Rightarrow p = \phi(x,a), \quad q = \varphi(y,a)$

$$dz = p\,dx + q\,dy$$

$$z = \int \phi(x,a)\,dx + \int \varphi(y,a)\,dy + b.$$

Q. $p^2 y(1+x^2) = qx^2$

Sol^n :- $\cancel{p^2y + p^2x^2y = qx^2}$ $\quad \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a$

$\cancel{\Rightarrow q^2 + p^2 \quad p^2}$ $\quad \Rightarrow p^2 = \frac{x^2 a}{1+x^2} \Rightarrow \boxed{p = \sqrt{\frac{ax^2}{1+x^2}}}, \quad \boxed{q = ay}$

14.9.16.

$\therefore$ Sol^n is $z = \int \frac{\sqrt{a}\,x}{\sqrt{1+x^2}}\,dx + \int ay\,dy + b$

$\Rightarrow z = \frac{\sqrt{a}}{2}\sqrt{1+x^2} + \frac{a}{2}y^2 + b$

$\Rightarrow \boxed{z = \sqrt{a}.\sqrt{1+x^2} + ay^2 + b}$ #

Q2. $p.e^y = q.e^x$

$\Rightarrow pe^{-x} = qe^{-y} = a$

$\Rightarrow pe^{-x} = a$ & $qe^{-y} = a$

$\Rightarrow p = ae^x$, $q = ae^y$

$\therefore$ Sol^n is $z = \int ae^x dx + \int ae^y dy + b$

$\boxed{z = ae^x + ae^y + b}$

Case (IV) :- $F(x, y, z, p, q) = 0$ —— (i)

(or)

$z = px + qy + f(p, q)$ —— (ii)

$\Rightarrow \boxed{p = C_1}$, $\boxed{q = C_2}$ —— (iii) Clairaut's Form

Putting (iii) in (ii),

$\boxed{z = C_1x + C_2y + f(C_1, C_2)}$ #

Q1. $(p+q).(z - px - qy) = 1$. , $p = C_1, q = C_2$

$\Rightarrow z = px + qy + \frac{1}{p+q}$

Sol^n is, $\boxed{z = C_1x + C_2y + \frac{1}{C_1 + C_2}}$

Q2 $\sin^{-1}(px + qy - z) = pq$.

$\Rightarrow px + qy - z = \sin(pq)$

$\Rightarrow z = px + qy - \sin(pq)$. , $p = C_1, q = C_2$

$\therefore$ Sol^n is, $\boxed{z = C_1x + C_2y - \sin(C_1C_2)}$

$$\cancel{a_1 \partial^n u} \quad a_1 \frac{\partial^n u}{\partial x^n} + a_2 \frac{\partial^n u}{\partial x^{n-1} \partial y} + \cdots\cdots + a_n \frac{\partial^n u}{\partial y^n} + b_1 \frac{\partial^{n-1} u}{\partial x^{n-1}}$$

$$+ \cdots\cdots + b_{n-1} \frac{\partial^{n-1} u}{\partial y^{n-1}} + \cdots\cdots + c_0 u = f(x,y) \quad \text{——— (i).}$$

$$\frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D' \quad \text{(Operators)}$$

$$\text{or,} \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$$\left[\left(a_1 D^n + a_2 D^{n-1} D' + \cdots\cdots + a_n D'^n\right) + \left(b_1 D^{n-1} + b_2 D^{n-2} D' + \cdots + b_{n-1} D'^{n-1}\right) + \cdots\cdots c_0\right] u = f(x,y).$$

$$\Rightarrow F(D, D').u = f(x,y) \quad \text{——— (ii)}$$

[Homogeneous, when every term contain dependent ~~variabl~~ variable &

Non-homogeneous, when every term does not contain dependent variable]

When Homogeneous $\longrightarrow$ $u = u_{CF}$ (CF $\longrightarrow$ Complementary Function)

When Non-homogeneous $\longrightarrow$ $u = u_{CF} + u_{P.I}$ (P.I $\rightarrow$ Particular Integral)

<u>Theorem ①</u> :- If $u_{C.F}$ & $u_{P.I}$ are Complementary Function & Particular Integral of Linear Partial Differential Equ^n then, $u = u_{C.F} + u_{P.I}$ is the general sol^n of the given PDE

<u>Theorem ②</u> :- If $u_1, u_2, u_3 \cdots\cdots u_n$ are n independent sol^{n}s of given Linear PDE then $u = \sum_{i=1}^{n} c_i u_i$ is also a sol^n of that PDE.

Theorem ③ :- If the operator $F(D, D')$ is ~~reduced~~ reducible (factorizable to linear factors) i.e., if

$(a_i D + b_i D' + c_i)$ is a factor of $F(D, D')$ and

$\phi_i(\eta)$ is a function of η (arbitrary variable)

then sol^n is,

$$u_i = e^{\frac{-c_i}{a_i}x} \cdot \phi_i(b_i x - a_i y) \quad \text{where; } a_i \neq 0$$

For E.g ① $(D^2 - D'^2)u = 0$

$$\Rightarrow \underbrace{(D+D')}_{1}\underbrace{(D-D')}_{2}u = 0.$$

$\therefore$ Sol^n is,

$$u_1 = e^{-\frac{0.x}{1}} \cdot \phi_1(x-y) \quad \& \quad u_2 = e^{-\frac{0.x}{1}} \phi_2(x+y)$$

$$\Rightarrow u_1 = \phi_1(x-y). \qquad \Rightarrow u_2 = \phi_2(x-y)$$

$\therefore$ Sol^n is, ~~$u = c_1 u_1 + c_2 u_2 = \phi_1(x-y) + \phi_2(x+y)$~~

$\therefore$ Sol^n is, $u = c_1 u_1 + c_2 u_2$

$$= c_1 \phi_1(x+y) + c_2 \phi_2(x-y).$$

$$= \phi_1(x+y) + \phi_2(x-y).$$

For E.g ②

$$(D^2 + 2DD' + D'^2 - 2D - 2D')u = 0.$$

$$\Rightarrow \left((D+D')^2 - 2(D+D')\right)u = 0$$

$$\Rightarrow \left(D+D' - 2)(D+D')\right)u = 0.$$

$$u_1 = e^{-\frac{0}{1}x} \cdot \phi_1(x-y), \quad u_2 = e^{\frac{2}{1}x} \cdot \phi_2(x-y)$$

$$= \phi_1(x-y) \qquad\qquad = e^{2x}\phi_2(x-y).$$

$\therefore$ Sol^n is, $u = c_1 u_1 + c_2 u_2$

$$= c_1 \phi_1(x-y) + c_2 e^{2x} \phi_2(x-y)$$

$$= \phi_1(x-y) + e^{2x} \cdot \phi_2(x-y).$$

For E.g ③

$$(D^2+3DD'+2D'^2)u=0$$

$$\Rightarrow (D^2+2DD'+D'^2+DD'+D'^2)u=0$$

$$\Rightarrow ((D+D')^2+D'(D+D'))u=0$$

$$\Rightarrow (D+D')(D+2D')u=0.$$

Sol^n is, $u=\phi_1(x-y)+\phi_2(2x-y)$.

For E.g ④ $(D+D'-3)(D'+2)u=0$

Theorem ④ :- For $(b_iD'+c_i)$ form

If $(b_iD'+c_i)$ is a factor of $F(D,D')$ and $\phi_i(\eta)$ is an ~~ordinary~~ arbitrary function of η then,

$$\boxed{u_i=e^{-\frac{c_i}{b_i}y}\phi_i(b_ix)} \text{ where } \boxed{b_i\neq 0}$$

∴ Sol^n E.g ④ ~~u_i~~ Sol^n is, $u=e^{3x}\phi_1(x-y)+e^{-2y}\phi_2(x)$.

For E.g ⑤ $(D+2D'-3)^2(D+3).(2D'+7)u=0.$

$u=(c_1+c_2x)\phi_1(2x-y)$ —

Sol^n Sol^n is, $u=\cancel{(c_1+c_2x)\phi_1(x)+c_1\phi_1(x)+c_2y}$

$$u_1=e^{3x}\phi_1(2x-y).$$

$$u_2=\cancel{e^{-3x}\phi_1}\ xe^{3x}\phi_2(2x-y)$$

$$u_3=e^{-3x}\phi_3(-y)$$

$$u_4=e^{-7/2y}\phi_4(2x)$$

∴ Sol^n is, $u=e^{3x}\phi_1(2x-y)+xe^{3x}\phi_2(2x-y)$
$+e^{-3x}\phi_3(-y)+e^{-7/2y}\phi_4(2x)$.

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Linear PDE with constant coefficients

1. $(D^3 - 3D^2D' + 4D'^3)u = 0$

2. $(D^3 + DD'^2 - 10D'^3)u = 0.$

$D^2 \times 3DD' \times 4D'^2$

(94) c.w. 3.10.16

Solⁿ ② $(D-2D')(D^2+2DD'+5D'^2)u = 0$

c.w.

$x^3 + xy^2 - 10y^3 = 0$

x^3

3.10.16 **Second Order Linear PDE with constant coefficients (Higher)**

$$F(D, D')u = 0 \quad \text{———} \quad (i)$$

- When factor is Irreducible

Theorem ⑤ :- The solⁿ of $F(D,D')u = 0$ if $m(D,D')$ is an irreducible factor, Then the solⁿ is

$$u = \sum_{i=1}^{\infty} c_i e^{(a_i x + b_i y)} \quad \text{where } F(a_i, b_i) = 0$$

Ex. ① $(2D^2 - D'^2 + D)u = 0.$, $F(D,D') = 2D^2 - D'^2 + D.$

Solⁿ :- $u = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$

where $F(a_i, b_i) = 0$

$\Rightarrow 2a_i^2 - b_i^2 + a_i = 0$

$\therefore b_i = \pm\sqrt{2a_i^2 + a_i}$

General solⁿ is,

$$\therefore u = \sum_{i=1}^{\infty} c_i e^{a_i x \pm \sqrt{2a_i^2 + a_i}\, y}$$

→ **General Method** — $F(D,D')u = 0$ ——— (i)

(when every term has same order) (i.e when operator function is homogeneous)

Solⁿ :- Let, $u = \phi(y + mx)$ is a solⁿ of equⁿ (i):

Now, $F(D,D')u = 0$

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$$\Rightarrow \left(a_0 D^n + a_1 D^{n-1} D'^1 + a_2 D^{n-2} D'^2 + \dots + a_n D'^n\right) u = 0 \quad \text{—— (ii)}$$

Now, $u = \phi(y+mx)$

$\Rightarrow Du = m\phi'(y+mx)$

$\therefore D^n u = m^n \phi^{(n)}(y+mx)$

Again, $D'u = \phi'(y+mx)$

$D'^n u = \phi^{(n)}(y+mx)$ —— (iii)

~~Addin~~ Putting (iii) in (ii) $\Rightarrow$

$$\left[a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n\right] \phi^{(n)}(y+mx) = 0.$$

[Putting D by m & D' by 1]

A.E $\Rightarrow$ $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$ —— (iv).

Case (I) :- If m are the root of eq^n (iv) & all these roots are real/complex & distinct $(m_1, m_2, m_3, \dots, m_n)$.

$\therefore$ Sol^n is :- $\boxed{u = \sum_{i=1}^{n} C_i \cdot \phi_i(y+m_i x)}$ #.

Case (II) :- If roots are repeated (k times).

Sol^n is :- $\boxed{u = \sum_{i=1}^{n-k} C_i \phi_i(y+m_i x) + \sum_{j=1}^{k} \left(x^{j-1} \phi_j(y+m_j x)\right)}$

or $\boxed{u = \sum_{i=1}^{k} x^{i-1} \phi_i(y+m_i x) + \sum_{j=k+1}^{n} \phi_j(y+m_j x)}$

Q1 $\left(D^3 - 3D^2 D' + 4D'^3\right) u = 0.$ —— (i)

Sol^n :- $u = \phi(y+mx)$ is the sol^n of equ^n (i),

Now, $F(D, D') u = 0.$

$\Rightarrow \left(a_0 D^3 + a_1 D^{3-1} D'^1 + a_2 D^{3-2} D'^2\right) u = 0.$ —— (ii)

$\therefore a_0 = 1, \quad a_1 = -3, \quad a_2 = 4.$

$\therefore$ A.E = ~~$m^n - 3m^{n-1}$~~ $m^3 - 3m^2 + 4 = 0$

$\Rightarrow m(m^2 - 3m + 4) = 0$

$m = 0, \quad m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm i\sqrt{7}}{2}$

$\therefore$ Sol^n is, $u = \sum_{i=1}^{3} C_i \phi_i$

3.10.16. C.W. 24.10.

$$\Rightarrow m = -1, \quad m = 2, 2$$

$$\therefore u = \phi_1(y-x) + \phi_2(y-2x) + x\phi_3(y+2x) \quad \#$$

Sol

(2). $(D^3 + DD'^2 - 10D'^3)u = 0$

A.E $\rightarrow$ $m^3 + m^2 - 10 = 0$

$$(m-2)(m^2+2m+5) = 0$$

$$m = 2, \ -1 \pm 2i.$$

$$\therefore u = \phi_1(y+2x) + \phi_2(y-x+2ix) + \phi_3(y-x-2ix). \quad \#.$$

3.10.16. C.W

Non-Homogeneous Equations

Finding the P.I $\rightarrow$ $F(D, D')u = g(x,y)$.

P.I $\Rightarrow \dfrac{1}{F(D,D')} g(x,y)$

$$\Rightarrow u_{P.I} = \frac{1}{(D+m_1D')(D+m_2D')(D+m_3D')\cdots(D+m_nD')} \cdot g(x,y)$$

Take, $\dfrac{1}{(D+m_nD')} g(x,y) = z$

$$\Rightarrow Dz + m_n D'z = g(x,y)$$

$$\Rightarrow p + m_n q = g(x,y) \quad \left[p = \frac{\partial z}{\partial x} = Dz, \ q = \frac{\partial z}{\partial y} = D'z\right]$$

$\therefore$ Sol^n is $\dfrac{dx}{1} = \dfrac{dy}{m_n} = \dfrac{dz}{g(x,y)}$

$\Rightarrow y = m_n x + C$ & $\dfrac{dx}{1} = \dfrac{dz}{g(x, m_n x + C)}$

$$\Rightarrow g(x, m_n x + c)\,dx = dz$$

$$\Rightarrow z = \int g(x, m_n x + c)\,dx + C_2$$

Find the P.I :-

1. $(D^2+4DD'+4D'^2)u=0$. 2. $(D+D'+2)(D+3D'-2)u=0$.

Sol^n (1) $\Rightarrow (D^2+2DD'+4D'^2+4DD')u=0$

$\Rightarrow ((D+2D')^2+D'(D+2D'))=0$

$\Rightarrow (D+2D')(D'+1)=0$

$(D+2D')^2u=0$

$\therefore u_{C.F} = e^{-0x}\cdot\phi(2x-y)+x\phi_2(2x-y)$

$= \phi(2x-y)+x\phi_2(2x-y)$

Or

A.E $= m^2+4m+4=0$

(General Method) $\Rightarrow (m+2)^2=0$

$\Rightarrow m=-2,-2.$

Equal.

$\therefore$ Sol^n is, $u_{CF}=\phi_1(y-2x)+x\cdot\phi_2(y-2x)$.

2. $(D+D'+2)(D+3D'-2)u=0$.

Sol^n :- $u_{C.F} = e^{-2x}\cdot\phi(x-y)+e^{2x}\cdot\phi(3x-y)$.

General Method is not appicable as it is non-homogeneous

A.E

E.g $(D-D')u=e^{3x+y}$.

$\Rightarrow u_{P.I}=\dfrac{e^{3x+y}}{(D-D')}$, $u_{C.F}=\phi_1(x+y)$.

Take, $\dfrac{1}{D-D'}e^{3x+y}=z$.

$\Rightarrow p-q-e^{3x+y}=0$

$\therefore$ Sol^n is $\dfrac{dx}{1}=\dfrac{dy}{-1}=\dfrac{dz}{-e^{3x+y}}$

$\Rightarrow -x=y-c$

$\Rightarrow y+x=c$

$\Rightarrow y=c-x$

$\Rightarrow \dfrac{dx}{1}=\dfrac{dz}{-e^{3x+y}}$

$\Rightarrow \dfrac{dx}{1}=\dfrac{dz}{-e^{2x+c}}$

$\Rightarrow e^{2x+c}dx=+dz$

$\Rightarrow -e^{2x+c}+2z+c_2=0$

Important Formulas

1. $\dfrac{1}{F(D,D')} e^{ax+by} = \dfrac{1}{F(a,b)} e^{ax+by}, \; F(a,b) \neq 0$

2. $\dfrac{1}{F(D,D')} \left\{ e^{ax+by} \cdot f(x,y) \right\} = e^{ax+by} \cdot \dfrac{1}{F(D+a, D'+b)} f(x,y).$

3. $\dfrac{1}{F(D^2, DD', D'^2)} \begin{matrix} \sin(ax+by) \\ \text{or} \\ \cos(ax+by) \end{matrix} = \dfrac{1}{F(-a^2, -ab, -b^2)} \begin{matrix} \sin(ax+by) \\ \text{or} \\ \cos(ax+by) \end{matrix},$

$\quad , F(-a^2, -ab, -b^2) \neq 0.$

Ex $(D^2 + 6DD' + 9D'^2 + 4D + 12D')u = e^{x-3y}.$

Sol^n :- $\therefore u_{P.I} = \dfrac{1}{1 - 6.3 + 9.9 + 4 - 12.3} e^{x-3y}$

$= \dfrac{e^{x-3y}}{1 - 18 + 81 + 4 - 36}$

$86 - 54 = 32$

$\Rightarrow \left(D^2 + 2.D.3D' + 9D'^2 + 4(D+3D')\right)u = e^{x-3y}$

$\Rightarrow \left\{(D+3D')^2 + 4(D+3D')\right\}u = e^{x-3y}$

$\Rightarrow (D+3D')(D+3D'+4)u = e^{x-3y}.$

$\therefore u_{c.F} = \phi_1(3x-y) + e^{-4x}\phi_2(3x-y)$

$u_{P.I} = \dfrac{1}{(D+3D')(D+3D'+4)} e^{x-3y}$

$= \dfrac{1}{(1-9)(1-9+4)} e^{x-3y}$

$= \dfrac{1}{32} e^{x-3y}.$

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Application of Fourier Series

(A) Wave Equation $\left[\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \quad \nabla = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right]$ (Vector Notation)

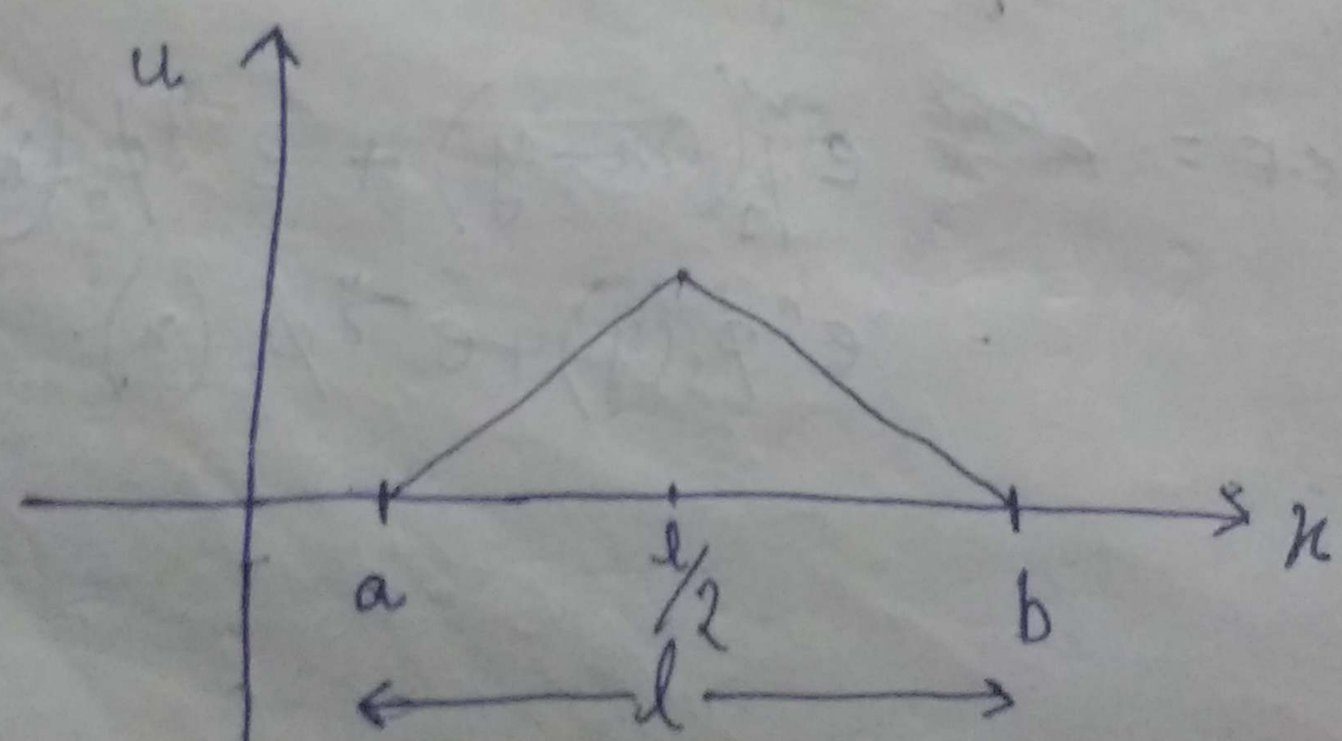
(B) Heat Equation

(C) Laplace Equation $\left[\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right]$ (Scalar Notation)

(A) Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \quad c^2 = \frac{T}{m}$$

Tension



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Initial condition at $t=0$, i.e ~~$u(x,y)$~~ $u(x,0) = f(x)$

$u_t(x,0) = 0$ [Initial velocity $= 0$]

where, $u_t = \frac{\partial u}{\partial t}$ (velocity)

Boundary conditions

$u(a,t) = 0$ & $u(b,t) = 0$

1) When $u(x=a, t) = ?$, $u(x=b, t) = ?$

$\longrightarrow$ Dirichlet's Condition

2) $u_x(x=a, t) = ?$, $u_x(x=b, t) = ?$

$\longrightarrow$ Neumann Boundary Condition

~~3) $c_1 u(x=a,t)$~~

3) $c_1 u(x=a, t) + c_2 u_x(x=a,t) = \ldots$,

$c_3 u(x=b, t) + c_4 \cdot u_x(x=b, t) = ?$

$\longrightarrow$ Mixed or Robin Boundary Condition

(B) Heat Equation :- $\frac{\partial u}{\partial t} = c^2 \nabla^2 u$

c^2 → Thermal Diffusivity

$c^2 = \frac{K}{\sigma \cdot \rho}$

K → Thermal conductivity

σ → Specific heat

ρ → Density

(C) Laplace Equation :- $\boxed{\nabla^2 u = 0}$

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Solution of Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

I.C's : $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ —— (ii)

B.C's : $u(x=a,t) = 0$, $u(x=b,t) = 0$ —— (iii)

Sol^n :-

Method used — Separation of Variable

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (i).}$$

Assume, $u(x,t) = X(x).T(t)$ —— (iv).

Substitute the value of $u(x,t)$ from (iv) to (i).

$$X(x).\ddot{T}(t) = c^2 T(t).X''(x)$$

$$\Rightarrow \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \underbrace{k\ (\text{say})}_{\text{constant}}$$

$$X''(x) - k.X(x) = 0 \quad \text{—— (v)}$$

$$\ddot{T}(t) - c^2.T(t) = 0 \quad \text{—— (vi)}$$

From equ^n — (v)

Case I :- When k is +ve, i.e $k = \lambda^2$

where λ is any real no.

$\therefore$ A.E $= m^2 - \lambda^2 = 0 \Rightarrow m = \pm\lambda$

$\therefore$ Sol^n is $X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

Similarly, $T(t) = C_3 e^{c\lambda t} + C_4 e^{-c\lambda t}$.

Case II :- When k is -ve, i.e $k = -\lambda^2$

$\therefore$ A.E $= m^2 + \lambda^2 = 0 \Rightarrow m = \pm i\lambda$

$\therefore$ Sol^n is $X(x) = C_1 \cos\lambda_n x + C_2 \sin\lambda_n x$

Similarly, $T(t) = C_3 \cos c\lambda_n t + C_4 \sin c\lambda_n t$

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Case III :- When k is zero, $k=0$

$$X(x) = C_1 x + C_2$$
$$T(t) = C_3 t + C_4.$$

From equ^n (iv),

1st sol^n :- $u(x,t) = (C_1 e^{\lambda x} + C_2 e^{-\lambda x}).(C_3 e^{c\lambda t} + C_4 e^{-c\lambda t})$

2nd sol^n :- $u(x,t) = (C_1 \cos\lambda x + C_2 \sin\lambda x).(C_3 \cos c\lambda t + C_4 \sin c\lambda t)$

3rd sol^n :- $u(x,t) = (C_1 x + C_2).(C_3 t + C_4).$

Now,

$$u(x,t) = X(x).T(t)$$
$$\therefore u(x,0) = X(x).T(0) = f(x) \text{ [Given]} \quad \text{——— (vii)}$$
$$u_t(x,0) = X(x).T'(0) = g(x) \quad \text{——— (viii)}$$
$$u(a,t) = X(a).T(t) = 0 \quad \text{or} \quad X(a) = 0 \; [\because T(t) \neq 0]$$
$$u(b,t) = X(b).T(t) = 0 \quad \text{or} \quad X(b) = 0 \; [\because T(t) \neq 0]$$

Case I :-

$$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$
$$\Rightarrow C_1 e^{\lambda a} + C_2 e^{-\lambda a} = 0 \;\&\; C_1 e^{\lambda b} + C_2 e^{-\lambda b} = 0.$$
$$\cancel{\Rightarrow C_1 e^{\lambda ab} + C_2 e^{-\lambda ab} = 0 \;\&\; C_1 e^{\lambda ab} + C_2 e^{-\lambda ab} = 0}$$
$$\therefore \cancel{C_1 e^{\lambda ab} + C_2 e^{-\lambda ab} + C_1 e^{\lambda ab}}$$
$$C_2\left[e^{\lambda(b-a)} - e^{-\lambda(b-a)}\right] = 0$$
$$\Rightarrow \boxed{C_2 = 0}, \boxed{C_1 = 0}$$

This is not an appropriate sol^n for that problem.

Case II :-

$$X(x) = C_1 x + C_2$$
$$\left.\begin{aligned} C_1 a + C_2 &= 0 \\ C_1 b + C_2 &= 0 \end{aligned}\right] \Rightarrow \boxed{C_1 = C_2 = 0}$$

This is not an appropriate sol^n for that problem.

Case III :- $X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$

$$\therefore \quad C_1 \cos \lambda a + C_2 \sin \lambda a = 0$$

$$C_1 \cos \lambda b + C_2 \sin \lambda b = 0$$

$$\Rightarrow \quad C_1 \sin \lambda b \cos \lambda a + C_2 \sin \lambda b \lambda a 0$$

$$\cancel{C_1 \sin \lambda b \cos \lambda} \; \cancel{C_1 \sin} \quad C_1 \sin \lambda a \cos \lambda b + C_2 \sin \lambda b \lambda a = 0$$

$$\therefore \quad C_1 \left(\sin \lambda a \cos \lambda a - \sin \lambda a \cos \lambda b \right) = 0$$

$$\Rightarrow \quad C_1 \sin \lambda a \left(\cos \lambda a - \cos \lambda b \right) = 0$$

$$\Rightarrow \quad \cos \lambda (a-b) = 0 = \cos (2n+1) \pi/2$$

$$\Rightarrow \quad \lambda (a-b) = (2n+1) \frac{\pi}{2} \quad , n = 0, 1, 2$$

$$\Rightarrow \quad \lambda_n = \frac{(2n+1)\pi}{2(a-b)}$$

or

$$C_2 \sin \lambda (b-a) = 0$$

$$\Rightarrow \quad \sin \lambda (b-a) = 0 = \sin n\pi$$

$$\Rightarrow \quad \boxed{\lambda_n = \frac{n\pi}{b-a}} \quad n = 1, 2, 3 \ldots$$

$\therefore$ From eq^n (iv), The complete solution is.

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_1 \cos \lambda_n x + C_2 \sin \lambda_n x \right) . \left(C_3 \cos \lambda_n t + C_4 \sin \lambda_n t \right)$$

From (ii) & on (vii)

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \left(C_1 \cos \lambda_n x + C_2 \sin \lambda_n x \right) . C_3 \quad \text{—— (ix)}$$

$$= \sum_{n=1}^{\infty} \left(C_1^* \cos \lambda_n x + C_2^* \sin \lambda_n x \right)$$

where, $C_1^* = C_1 . C_3$

$C_2^* = C_2 . C_3$

$$u_t(x=0) = g(x) = \sum_{n=1}^{\infty} \left(C_1 \cos \lambda x + C_2 \sin \lambda x \right) . C_4 \lambda_n$$

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$$\Rightarrow 0 = \sum_{n=1}^{\infty} \left(C_1 \cos \lambda_n x + C_2 \sin \lambda_n x \right) \cdot C_4 \cdot \lambda_n$$

$$\left[X(x) \cdot \dot{T}(0) = 0 \Rightarrow \boxed{\dot{T}(0) = 0} \right]$$

$$\Rightarrow T'(0) = 0.$$

$$T(t) = C_3 \cos \lambda_n t + C_4 \sin \lambda_n t$$

$$\dot{T}(0) = C_4 \cdot \lambda_n = 0$$

$$\Rightarrow \boxed{C_4 = 0}$$

From Fourier Series,

$$C_1^* = \frac{1}{2(b-a)} \int_a^b f(x) \cdot \cos \lambda_n x \, dx$$

$$C_2^* = \frac{1}{2(b-a)} \int_a^b f(x) \sin \lambda_n x \, dx$$

Maths Tutorial

8. $(D^2 - 4DD' + 4D'^2) = e^{2x+y}$.

Sol^n :- $m^2 - 4m + 4 = 0$

$\Rightarrow m = 2, 2$.

$\therefore u_{C.F} = \phi_1(y+2x) + x\phi_2(y+2x)$.

$$\therefore u_{P.I} = \frac{1}{(D-2D')^2} e^{2x+y}$$

$$= e^{2x+y} \cdot \frac{1}{(D+2-2D'-2)^2}\{1\}$$

$$= e^{2x+y} \cdot \frac{1}{(D-2D')^2}\{1\}$$

$$= e^{2x+y} \cdot \frac{1}{D^2\left(1-\frac{2D'}{D}\right)^2}\{1\}$$

$$= e^{2x+y} \cdot \frac{1}{D^2}\left\{\left(1-\frac{2D'}{D}\right)^{-2}\{1\}\right\}$$

$$= e^{2x+y} \cdot \frac{1}{D^2}\left\{\left(1+\frac{4D'}{D}\right)\{1\}\right\}$$

$$= e^{2x+y} \cdot \frac{1}{D^2}\{1\}$$

$$= \frac{x^2}{2} e^{2x+y}$$

$\therefore$ Sol^n is, $u = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{2x+y}$.

---

9. $(D+D'-1)(D+2D'-3)z = 2x+3y$

Sol^n :- $u_{C.F} = e^x \phi_1(x-y) + e^{3x}\phi_2(2x-y)$.

$$u_{P.I} = \frac{1}{3\left\{1-(D+D')\right\}\left\{1-\left(\frac{D+2D'}{3}\right)\right\}}(2x+3y)$$

$$= \frac{1}{3\left\{1-(D+D')\right\}}\left[\left\{1-\left(\frac{D+2D'}{3}\right)\right\}^{-1}(2x+3y)\right]$$

$$= \frac{1}{3\left\{1-(D+D')\right\}}\left[\left\{1+\left(\frac{D+2D'}{3}\right)\right\}(2x+3y)\right]$$

$$= \frac{1}{3\{1-(D+D')\}}\left[2x+3y+\frac{1}{3}(2+6)\right]$$

$$= \frac{1}{3}\{1-(D+D')\}^{-1}\left(2x+3y+\frac{8}{3}\right)$$

$$= \frac{1}{3}\left\{\left[1+(D+D')\right]\left(2x+3y+\frac{8}{3}\right)\right\}$$

$$= \frac{1}{3}\left\{2x+3y+\frac{8}{3}+2+3\right\}$$

$$= \frac{1}{3}\left\{2x+3y+\frac{23}{3}\right\}$$

$$= \frac{1}{9}(6x+9y+23).$$

$\frac{8}{3}+5 = \frac{23}{3}$

$\therefore$ Sol^n is, $u = e^x\phi_1(x-y)+e^{3x}\phi_2(2x-y)+\frac{1}{9}(6x+9y+23).$

Q. $(D^2-2DD'+D'^2)z = 12xy.$

Sol^n :- A.E $= m^2-2m+1=0$

$\Rightarrow m = 1,1$

$\therefore u_{C.F} = \phi_1(y+x)+x\phi_2(y+x).$

$$\therefore u_{P.I} = \frac{12}{(D-D')^2}xy$$

$$= \frac{12}{D^2\left(1-\frac{D'}{D}\right)^2}xy$$

$$= \frac{12}{D^2}\left\{\left(1-\frac{D'}{D}\right)^{-2}(xy)\right\}$$

$$= \frac{12}{D^2}\left\{\left(1+\frac{2D'}{D}\right)(xy)\right\}$$

$$= \frac{12}{D^2}\left\{xy+x^2\right\}$$

$$= 12\left\{\frac{x^3y}{6}+\frac{x^4}{12}\right\}$$

$$= 2x^3y+x^4$$

$\frac{2x^2}{2}$

$\frac{2}{D}(x)$

$= \frac{2x^2}{2}$

$\frac{1}{D}\frac{x^2y}{2}$

$= \frac{x^3y}{6}$

$\frac{x^3}{3}\ \frac{x^4}{12}$

$\therefore$ Sol^n is, $u = \phi_1(y+x)+x\phi_2(y+x)+2x^3y+x^4$

8. $r - 2s + t = \sin(2x+3y)$

$\Rightarrow (D^2 - 2DD' + D'^2)z = \sin(2x+3y)$

$\therefore$ A.E $= m^2 - 2m + 1 = 0$

$\Rightarrow m = 1, 1$

$\therefore u_{C.F} = \phi_1(y+x) + x\phi_2(y+x)$

$\therefore u_{P.I} = \dfrac{1}{D^2 - 2DD' + D'^2} \sin(2x+3y)$

$= \dfrac{1}{-4+12-9} \sin(2x+3y)$

$= -\sin(2x+3y)$

$\therefore$ Sol^n is, $u = \phi_1(y+x) + x\phi_2(y+x) - \sin(2x+3y)$

---

9. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$

Sol^n:- $\{(D+D')^2 - 2(D+D')\}z = \sin(x+2y)$

$\Rightarrow \{(D+D')(D+D'-2)\}z = \sin(x+2y)$

~~A.E~~ $u_{C.F} = e^{0x}\phi_1(x-y) + e^{2x}\phi_2(x-y)$

$= \phi_1(x-y) + e^{2x}\phi_2(x-y)$

$\therefore u_{P.I} = \dfrac{1}{D^2 + 2DD' + D'^2 - 2(D+D')} \sin(x+2y)$

$= \dfrac{1}{-1-4-4-2(D+D')} \sin(x+2y)$

$= \dfrac{-1}{2(D+D')+9} \sin(x+2y)$

$= \dfrac{-\{2(D+D')-9\}}{4D^2+4D'^2+8DD'-81} \sin(x+2y)$

$= \dfrac{9-2(D+D')}{4D^2+4D'^2+8DD'-81} \sin(x+2y)$

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$$= \frac{1}{-117}\left(9\sin(x+2y) - 2\left[3\cos(x+2y)\right]\right)$$

$$= \frac{6}{117}\cos(x+2y) - \frac{9}{117}\sin(x+2y)$$

$\therefore$ Sol^n is, $u = \phi_1(x-y) + e^{2x}\phi_2(x-y) + \frac{6}{117}\cos(x+2y) - \frac{9}{117}\sin(x+2y)$

Q $(D - D'^2)z = \cos(x-3y)$

Sol^n :- $F(D, D') = D - D'^2 = 0$ is not factorizable

$$\therefore u_{C.F} = \sum_{i=1}^{\infty} C_i e^{a_i x + b_i y}$$

where $F(a_i, b_i) = 0$

$\Rightarrow a_i - b_i^2 = 0$

$\Rightarrow b_i = \pm\sqrt{a_i}$

$$\therefore u_{C.F} = \sum_{i=1}^{\infty} C_i e^{a_i x \pm \sqrt{a_i} y}$$

$$\therefore u_{P.I} = \frac{1}{(D - D'^2)}\cos(x-3y)$$

$$= \frac{1}{D+9}\cos(x-3y)$$

$$= \frac{D-9}{D^2-81}\cos(x-3y)$$

$$= \frac{-1}{82}\left[-\sin(x-3y) - 9\cos(x-3y)\right]$$

$$= \frac{1}{82}\sin(x-3y) + \frac{9}{82}\cos(x-3y)$$

$\therefore$ Sol^n is, $u = \sum_{i=1}^{\infty} C_i e^{a_i x \pm \sqrt{a_i} y} + \frac{1}{82}\sin(x-3y) + \frac{9}{82}\cos(x-3y)$

Q. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— (i)

I.C's $u(x,0) = f(x)$ —— (ii) B.C's $u(0,t) = 0$ —— (iv)

$u_t(x,0) = 0$ —— (iii) $u(L,t) = 0$ —— (v)

$$u(x,t) = \sum_{n=1}^{\infty} C_3^* \cos\left(\frac{cn\pi}{L} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right).$$

where, $\frac{cn\pi}{L} C_3^* = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$ —— (vi), $T(t) = C_3 \cos c\lambda t + C_4 \sin c\lambda t$ —— (vii)

Also, ~~when~~ $u(x,t) = (C_1 \cos\lambda_n x + C_2 \sin\lambda_n x) \cdot (C_3 \cos c\lambda_n t + C_4 \sin c\lambda_n t)$; $n = 1, 2, \ldots$

when, $u(x,t) = X(x) \cdot T(t)$, substitute in (iv) & (v)

$0 \Rightarrow X(0) \cdot T(t)$

& $0 \Rightarrow X(L) \cdot T(t)$

$$\Rightarrow \begin{bmatrix} X(0) = 0 \\ \& \ X(L) = 0 \end{bmatrix} \text{ —— (viii)}.$$

From (vi) & (viii),

$$\boxed{C_1 = 0}$$

$\Rightarrow 0 = C_2 \sin\lambda L$

$\Rightarrow \sin\lambda L = 0 \Rightarrow \sin\lambda L = \sin n\pi.$

$\Rightarrow \lambda L = n\pi \Rightarrow \lambda = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

$$u_n(x,t) = C_2 \sin\lambda_n x \left(C_3 \cos c\lambda_n t + C_4 \sin c\lambda_n t\right)$$

$$= \sin\lambda_n x \left(C_3^* \cos c\lambda_n t + C_4^* \sin c\lambda_n t\right); \quad n = 1, 2, \ldots$$

$\therefore$ General sol^n,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left[\sin(\lambda_n x)\left(C_3^* \cos c\lambda_n t + C_4^* \sin c\lambda_n t\right)\right] \text{ —— (ix)}$$

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From (iii) & (ix)

$$\therefore u_t(x,t) = \sum_{n=1}^{\infty} \left[\sin(c\lambda_n x) \left[-C_3^* c\lambda_n \sin c\lambda_n t + C_4^* c\lambda_n \cos c\lambda_n t \right] \right]$$

$$\Rightarrow u_t(x,0) = 0 = \sum_{n=1}^{\infty} \left[\sin(c\lambda_n x) \times C_4^* c\lambda_n \right]$$

$$\Rightarrow \sum_{n=1}^{\infty} C_4^* c\lambda_n \sin(\lambda_n x) = 0$$

$$\Rightarrow C_4^* = 0$$

From (ix)

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[\left(C_3^* \cos c\lambda_n t \right) \left(\sin(\lambda_n x) \right) \right]$$

From (ii),

$$f(x) = \sum_{n=1}^{\infty} C_3^* \sin(\lambda_n x)$$

$$\therefore C_3^* = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x \, dx.$$

Q. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ —(i)

IC's: $u(x,0) = x$ —(ii)

$u_t(x,0) = 0$ —(iii)

B.C's: $u(0,t) = 0$ —(iv)

$u(1,t) = 0$ —(v).

$$\therefore C_3^* = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x \, dx$$

$$= \frac{2}{1} \int_0^1 x \sin\left(\frac{n\pi}{1} x\right) dx$$

$$= 2 \int_0^1 x \sin(n\pi x) \, dx$$

$$= 2 \left[-\frac{x}{n\pi} \cos(n\pi x) + \frac{1}{n\pi} \int \cos(n\pi x) \, dx \right]_0^1$$

$$= 2 \left[\frac{-1}{n\pi} \left(x \cos n\pi x \right)_0^1 + \frac{1}{(n\pi)^2} \sin(n\pi x) \Big|_0^1 \right]$$

$$= 2 \left[\frac{-1}{n\pi} (-1)^n + 0 \right] = \frac{-2}{n\pi} (-1)^n$$

$\therefore$ Sol^n is,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\left(\frac{-2(-1)^n}{n\pi} \cos \lambda_n t \right) \left(\sin(n\pi x) \right) \right]$$

$$= \sum_{n=1}^{\infty} \left[\left\{ \frac{(-2)(-1)^n}{n\pi} \cos n\pi t \right\} \sin(n\pi x) \right]$$

Solution of Heat Equation by Separation of Variable

Q. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ —— (i)

I.C's: $u(x,0) = f(x)$ —— (ii)

B.C's: $u(0,t) = 0$ —— (iii)

$u(L,t) = 0$ —— (iv)

Sol^n:- Assume, $u(x,t) = X(x) \cdot T(t)$ —— (v)

From (i) & (v),

$$X(x) \cdot \dot{T}(t) = c^2 X''(x) \cdot T(t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{c^2 T(t)} = k$$

$$\Rightarrow X''(x) - kX(x) = 0 \quad \text{—— (vi)}$$

& $\dot{T}(t) - kc^2 T(t) = 0$ —— (vii)

From (iii), (iv) & (v),

$$\left.\begin{aligned} 0 &= X(0) \cdot T(t) \\ 0 &= X(L) \cdot T(t) \end{aligned}\right] X(0) = 0,\ X(L) = 0 \quad \text{—— (viii)}$$

Case (III) :- When $k = -\lambda^2$

$$\therefore X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \quad \text{—— (ix)}$$

From (vii),

$$T(t) = C_3 e^{-\lambda^2 c^2 t} \quad \text{—— (x)}$$

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From (viii) & (ix),

$$0 = C_1$$

$$\& \quad 0 = C_1 \cos \lambda L + C_2 \sin \lambda L$$

$$\Rightarrow \quad 0 = C_2 \sin \lambda L$$

$$\lambda = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

$$\therefore \quad X(x) = C_2 \sin \lambda_n x \quad \text{——} (xi)$$

<u>From (v), (x) & (xi),</u>

$$u_n(x,t) = (C_2 \sin \lambda_n x) \cdot C_3 e^{-\lambda_n^2 c^2 t}, \quad n = 1, 2, 3$$

$\therefore$ General Sol^n,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} C_2^* \sin \lambda_n x \, e^{-\lambda_n^2 c^2 t} \quad \text{——} (xii).$$

From (ii),

$$\Rightarrow \quad u(x,0) = f(x) = \sum_{n=1}^{\infty} C_2^* \sin \lambda_n x$$

$$\therefore \quad C_2^* = \frac{2}{L} \int_0^L f(x) \sin \lambda_n x \, dx$$

$$\boxed{C_2^* = \frac{2}{1} \int_0^1 f(x) \sin \lambda_n x \, dx}$$

For $f(x) = \sin x$, $L = 1$.

$$\therefore \quad C_2^* = 2 \int_0^1 \sin x \sin \lambda_n x \, dx$$

$$= \int_0^1 \left\{ -\cos x (\lambda_n + 1) + \cos x (\lambda_n - 1) \right\} dx$$

$$= \frac{-1}{\lambda_n + 1} \sin(\lambda_n + 1) x \Big|_0^1 + \frac{1}{\lambda_n - 1} \sin(\lambda_n - 1) x \Big|_0^1$$

$$= \frac{-1}{\lambda_n + 1} \sin(n\pi + 1) + \frac{1}{\lambda_n - 1} \sin(n\pi - 1)$$

$$= \frac{1}{n\pi - 1} \sin(n\pi - 1) - \frac{1}{n\pi + 1} \sin(n\pi + 1)$$