Formulae's

Fourier Series:-

$$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n x \pi}{l}.$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n x\pi}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n x \pi}{l}\,dx$$

(ii) Half Range. $(0<x<l)$ extend to $(-l<x<l)$. $\boxed{\begin{matrix}\text{Sine} \\ \text{half} \\ \text{range}\end{matrix} = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx}$

(iii) Fourier Integral.

$$f(x) = \int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega.$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx.$$

(iv) Fourier Sine, Cosine Integral.

(v) Fourier Transform:-

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(x)\,e^{i\omega x}\,d\omega \quad \text{(Fourier inverse Transform)}$$

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int f(x)\,e^{-i\omega x}\,dx. \quad \text{(Fourier Transform)}$$

a) $z = ($, Differentiate

b) $f(x-y, x+y)$

$\downarrow u \quad \downarrow v$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\right.$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

Solving matrix.

(7) Solution of PDE by direct Integration.

Roots of A.E.	C.F.
(i) m_1, m_2, m_3, m_4	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots$
(ii) $m_1 = m_2, m_3$	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x}$
(iii) $a+ib, a-ib$ two roots	$e^{ax}(c_1 \cos bx + c_2 \sin bx)$

ex. $\frac{\partial^2 z}{\partial x^2} + z = 0$.

$(D^2+1) z = 0$

$D = \pm i$ $\quad a = 0$

$b = 1$

thus, $z = e^{0x}(c_1 \cos x + c_2 \sin x)$

$z = c_1 \cos x + c_2 \sin x$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Two combos.

$$\phi(c_1, c_2)$$

(9) Non-Linear PDE of 1st Order

Forms :

a) $f(p, q) = 0$ form :-

$$p = c_1 \qquad q = f(c_1)$$

$$\boxed{z = c_1 x + f(c_1) y + c_3}$$

b) $f(p, q, z)$ form :-

$$p = c_1 q \qquad q = g(z)$$

$$\boxed{dz = p\,dx + q\,dy}$$

c) $f(p, q, x, y)$ form :-

Separating p, x and q, y

we equal $\phi(p, x) = \psi(q, y) = k.$

Now putting back

$$\boxed{dz = p\,dx + q\,dy}$$

d) $f(p, q, x, y, z)$ special case :-

$$z = px + qy + f(p, q)$$

Then put $p = c_1$, $q = c_2$

$$\boxed{z = c_1 x + c_2 y + f(c_1, c_2)}$$

(10) Charpit's Method :-

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

1 equation

Then $dz = p\,dx + q\,dy$

(11) Homogeneous & Nonhomogeneous Linear Equations with constant coefficients :- $\phi(D, D')z = f(x,y)$

a) Complementary Function :-

> Roots of form $(a_iD + b_iD' + c_i)$

$$\boxed{CF = e^{-c_i/a_i\,x} f_1(b_ix - a_iy)}$$

> Roots of form $(b_iD' + c_i)$

$$\boxed{CF = e^{-c_i/b_i\,y} f_1(b_ix)}$$

> Irreducible. $\phi(D, D')$

Then $\phi(a_i, b_i) = 0$

$$\boxed{CF = \Sigma c_i e^{a_ix + b_iy}}$$

> Roots $(D - m_i)$ $CF = f(y + m_ix)$

b) Particular Integral :-

> In general, (Only when factors of form $(D - mD')$)

$$P.I. = \frac{1}{\phi(D, D')} f(x,y)$$

$$\triangleright \quad P.I. = \frac{1}{\phi(D, D')} e^{ax+by}$$

$$PI = \frac{1}{\phi(a, b)} e^{ax+by}$$

$$\triangleright \quad P.I. = \frac{1}{\phi(D, D')} f(x,y)$$

$$PI = \phi(D, D')^{-1} f(x,y)$$

$$\triangleright \quad P.I. = \frac{e^{ax+by}}{\phi(D, D')} f(x,y)$$

$$P.I. = e^{ax+by} \left(\frac{f(x,y)}{\phi(D+a, D'+b)} \right)$$

$$\triangleright \quad P.I. = \frac{\sin(ax+by)}{\phi(D^2 + D'^2 + DD')}$$

$$= \frac{\sin(ax+by)}{\phi(-a^2 - b^2 - ab)}$$

$$\triangleright \quad P.I. = \frac{\sin(ax+by)}{\phi(D' + 1)}$$

$$= \frac{(D'-1)}{(D'^2 - 1)} \sin(ax+by)$$

(12) Cumulative Distribution Functions :-

(i) $P(a \le x \le b) = F(b) - F(a)$

(ii) $F(x) = P(X \le x)$

(13) Probability Mass function :-

$p(x_i)$

$$F(x) = \sum p(x_i)$$

(14) Probability Density function :-

$f(x)$

$$F(x) = \int_{-\infty}^{\infty} f(x)\, dx.$$

(15) Expectation

> $E(X) = \sum x_i\, p(x_i)$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)$$

> $E(a) = a$

> $E(aX) = a\,E(X)$

> $E(X+Y) = E(X) + E(Y)$

> $E(g(x)) = \sum g(x)\, p(x)$

or

$$E(g(x)) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

(16) Variance

> $Var(X) = E((x-\mu)^2) = \sigma^2$

> $Var(X) = E(X^2) - (E(X))^2 = \sigma^2$

> $Var(aX) = a^2\, Var(X) = \sigma^2$

S.D. σ

$$S.D = \sigma = \sqrt{Var(X)}$$

(18) Moment Generating Function

$$MGX = E(e^{tX})$$

(19) Binomial Distribution

(i) $p(x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

(ii) Expectation $= np$.

(iii) Variance $= npq$

(iv) MGX $= (q + pe^{t})^{n}$.

(20) Poisson's Distribution

(λ : average)
x : random variable)

(i) $p(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

(ii) Expectation $= \lambda$

(iii) variance $= \lambda$.

(iv) MGX $=$

(21) Normal Distribution

(i) $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^{2}}$ $\qquad z = \dfrac{x-\mu}{\sigma}$

(ii) Cumulative Distribution $= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z} e^{-u^{2}/2}\, du.$

(iii) $\phi(z) = F(x) = \phi\left(\dfrac{x-\mu}{\sigma}\right)$

(iv) $\phi(\infty) = 1$

(v) $\phi(-\infty) = 0$ $\qquad$ (vi) $\phi(0) = 1/2$

(vii) Z (standardized Random Variable)

Mean = 0

Variance = 1

> $X^* = a + bX$

(i) $\mu^* = a + b\mu$ } property

(ii) $\sigma^{*2} = b^2\sigma^2$

(viii) $\phi(-z) = 1 - \phi(z)$

(22) Chebyshev's Inequality

$$\boxed{P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}}$$

> Markov's Inequality

$$P(X \geq a) \leq \frac{E(X)}{a}$$

(23) 2-D Random Variable's

(Joint Distribution)

(i) $P(X \leq x, Y \leq y) = F(x, y)$ Discrete (Cumulative)

✓(ii) $P(X = x_i) = \sum_j p(x_i, y_j)$ (Marginal) Discrete (Cumulative)

or

✓(iii) $P(Y = y_i) = \sum_i p(x_i\ y_j)$ (Marginal) Discrete (Cumulative)

(iv) $P(X \in A, Y \in B) = \iint_{A\,B} f(x,y)\,dx\,dy$ (Continuous Cumulative)

✓(v) $P(X \in A) = \int_A \int_{-\infty}^{\infty} f(x,y)\,dy\,dx.$ (Marginal) (Continuous Cumulative)

✓(vi) $P(Y \in B) = \int_B \int_{-\infty}^{\infty} f(x,y)\,dx\,dy$ (Marginal) (Continuous Cumulative)

~~Marginal Densities~~

$$\cancel{P_x(x) = \int f(x,y)\,dx}$$

(VII) $P(a < x \le b,\ Y \le y)$

$= F(b,y) - F(a,y)$

(VIII) $P(X \le x,\ c < y \le d)$

$= F(x,d) - F(x,c)$

(IX) $P(X \le x,\ Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$

(X) $P(a \le x \le b,\ c \le y \le d)$

$= F(b,d) + F(a,c)$

24 Marginal Density

(i) $f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

(ii) $f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

25 Conditional Density

$$P(X = x_i \mid Y = y_i) = \frac{P(X = x_i,\ Y = y_i)}{P(Y = y_i)} \quad \begin{matrix}\text{(Joint)} \\ \text{(Marginal)}\end{matrix}$$

26 Covariance

(i) $Cov(X,Y) = E(X,Y) - E(X)\cdot E(Y)$

(ii) $Cov(X,Y) = \frac{1}{n}\sum x_i y_i - \bar{x}\bar{y}$

(iii) $Cov(X,Y) = Cov(Y,X)$

(iv) $Cov(X+Z,Y) = Cov(X,Y) + Cov(Z,Y)$

(27) Correlation

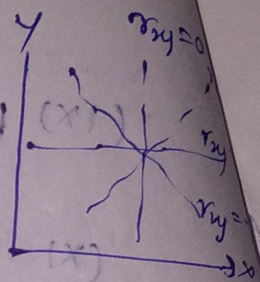
degree of strength of rel^n ship b/w (x, y)

(i) Correlation Coefficient

$$\boxed{r_{xy} = \frac{Cov(x, y)}{\sigma_x \sigma_y}}$$

σ_x : S.D of x

σ_y : S.D of y



(ii) $r_{xy} = r_{yx}$

(iii) $r_{xy} = 0$ (x, y unrelated

(iv) $-1 \leq r_{xy} \leq 1$

(v) Strongly related when $r_{xy} \to 1$

(28) Regression

(i) Regression line of 'y on x'

$$\boxed{(Y - \bar{Y}) = b_{yx} (X - \bar{X})}$$

where $b_{yx} = r_{xy} \frac{\sigma_y}{\sigma_x}$

(ii) Regression line of 'x on y'

$$\boxed{(X - \bar{X}) = b_{xy} = (X - \bar{X})}$$

where $b_{xy} = r_{xy} \frac{\sigma_x}{\sigma_y}$

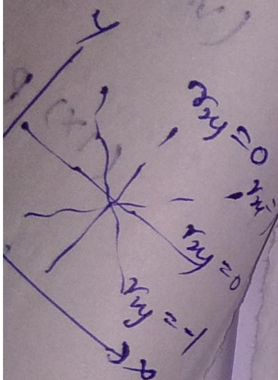
(iii) b_{xy} : regression coefficient of x on y

b_{yx} : regression coefficient of y on x.

$b_{xy} \cdot b_{yx} = (r_{xy})^2$.

v) Gradient of line 'y on x' = b_{yx}.

Gradient of line 'x on y' = $1/b_{xy}$



(i) **1D wave :-**

$$c^2 = \frac{T}{m}. \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution —

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4$$

(ii) **1D Heat :-**

$$c^2 = \frac{k}{\rho s}. \quad \text{diffusivity.}$$

$$\cancel{\frac{\partial^2 u}{}} \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution —

$$u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}.$$

(iii) **2D Heat :-**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Solution —

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

Covariance

$\text{Cov}(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$

$= E((X-\mu_x)(Y-\mu_y))$

(ii) $E(X\cdot Y) = E(X)\cdot E(Y)$ x, Y independent

$\text{Cov}(X,Y) = 0$

(iii) $\text{Cov}(X,Y) = \text{Cov}(Y,X)$

(iv) $\text{Cov}(X+Z, Y) = \text{Cov}(X,Y) + \text{Cov}(Z,Y)$

$$\boxed{\text{Cov}(x,y) = \frac{1}{n}\sum x_i y_i - \bar{x}\bar{y}}$$

Correlation :-

- degree of strength of rel^n ship b/w (x, y)

- Correlation coefficient, $\boxed{r_{xy} = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y}}$

  σ_x : SD of x

  σ_y : SD of y

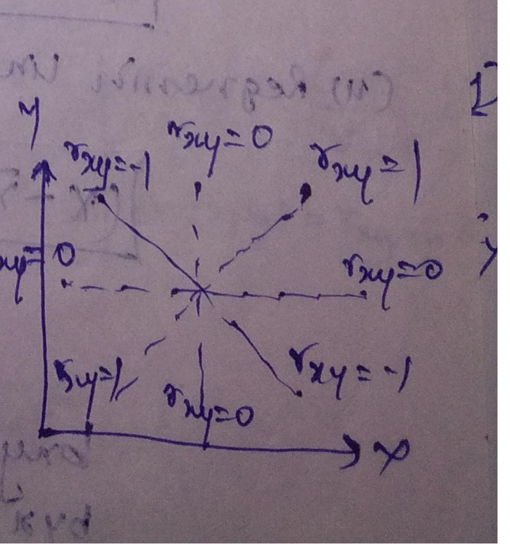
(I) $r_{xy} = r_{yx}$

(II) $r_{xy} = 0$ - unrelated. (x, y)

~~$r_{xy} \leq 1$~~

(III) $r_{xx} = 1$

(IV) $r_{x(-x)} = -1$



- estimation / prediction of the average value of one variable corresponding to a particular value of a number.

Regression line eq^n :-

(I) R.L. of y on x : (estimation of y for given x)
(II) R.L. of x on y (estimation of x for given y)

Theorem :-

Let (X, Y) be bivariate,
$(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$
Let $x_1, x_2, x_3, \ldots x_n$ has mean $\bar{x}$ and so σ_x
Let $y_1, y_2 \ldots y_n$ has mean $\bar{y}$ and so σ_y

(I) Regression line of 'y on x' is

[redacted] $\frac{\sigma_y}{\sigma_x} r_{xy}$ [redacted] $= b_{yx}$

$$\boxed{(y - \bar{y}) = b_{yx} (X - \bar{x})}$$

(II) Regression line of 'x on y' is

$$\boxed{(X - \bar{x}) = b_{xy} (Y - \bar{y})}$$

where $b_{xy} = \frac{\sigma_x}{\sigma_y} r_{xy}$

b_{xy} :- regression coefficient of y on x
b_{yx} :- regression coefficient of x on y

Proof :- Let the best fitted line be $y = a + bx$

normal equation

$$\Sigma y_i = an + b\Sigma x_i \quad -(i)$$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad -(ii)$$

dividing both eqn by n.

$$\frac{1}{n}\Sigma y_i = a + \frac{b}{n}\Sigma x_i$$

$$\Rightarrow \bar{y} = a + b\bar{x} \quad -(iii)$$

and

$$\frac{1}{n}\Sigma x_i y_i = \frac{a}{n}\Sigma x_i + \frac{b}{n}\Sigma x_i^2$$

$$\Rightarrow \frac{1}{n}\Sigma x_i y_i = a\bar{x} + \frac{b}{n}\Sigma x_i^2 \quad -(iv)$$

From (iii) and (iv)

$$\frac{1}{n}\Sigma x_i y_i = (\bar{y} - b\bar{x})\bar{x} + \frac{b}{n}\Sigma x_i^2$$

$$\frac{1}{n}\Sigma x_i y_i = \bar{x}\bar{y} - b\bar{x}^2 + \frac{b}{n}\Sigma x_i^2$$

$$\frac{1}{n}\Sigma x_i y_i - \bar{x}\bar{y} = b\left(\frac{1}{n}\Sigma x_i^2 - \bar{x}^2\right)$$

$$\mathrm{cov}(x,y) = b(\sigma_x^2)$$

$$b = \frac{\mathrm{cov}(x,y)}{\sigma_x^2}$$

$$r_{xy} = \frac{\mathrm{cov}(x,y)}{\sigma_x \sigma_y}$$

$$\mathrm{cov}(x,y) = r_{xy}\sigma_x\sigma_y$$

$$b = \frac{r_{xy}\sigma_x\sigma_y}{\sigma_x^2}$$

$$b_{yx} = r_{xy}\frac{\sigma_y}{\sigma_x} \quad -(v)$$

$\therefore$

$$a = \bar{y} - b\bar{x}$$

$$a = \bar{y} - r_{xy}\frac{\sigma_y}{\sigma_x}\bar{x} \quad -(vi)$$

$$y = a + bx$$

$$= \left(\bar{Y} - r_{xy}\frac{\sigma_Y}{\sigma_x}\bar{X}\right)$$

$$(Y - \bar{Y}) = b_{yx}(X - \bar{X})$$

Properties:

(i) r_{xy}, b_{xy}, b_{yx} are of same sign.

(ii) $\boxed{\underbrace{b_{xy} \cdot b_{yx}}_{\text{Same sign}} = (r_{xy})^2 \geq 0}$

(iii) Both the regression lines meet at $(\bar{x}, \bar{y})$

(iv) Gradient of regression line 'y on x' $= b_{yx}$

Gradient of regression line 'x on y' $= 1/b_{xy}$

Ques: For two variables x and y regression lines are.

$$x + 4y + 3 = 0 \quad \text{—(i)}$$
$$4x + 9y + 5 = 0 \quad \text{—(ii)}$$

(i) Identify which is 'y on x'

$$(y - \bar{y}) = b_{yx}(x - \bar{x})$$
$$y - \bar{y} = r_{xy}\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y = r_{xy}\frac{\sigma_y}{\sigma_x}x + \left(\bar{y} - r_{xy}\frac{\sigma_y}{\sigma_x}\bar{x}\right)$$

A/T,

Let $x + 4y + 3 = 0$

be the 'y on x' line.

Then, $b_{yx} = -1/4$

$$\frac{1}{b_{xy}} = -4/9.$$

$$b_{yx} \cdot b_{xy} = (r_{xy})^2.$$

$$(r_{xy})^2 = 9/16$$

$$r_{xy} = \pm 3/4 < 1$$

and $\boxed{-1 < r_{xy} < 1}$

Thus satisfies.

Now

Contradiction:–

Let $4x + 9y + 5 = 0$

be the 'y on x' line.

$$b_{yx} = -4/9$$

$$1/b_{xy} = -1/4.$$

$$b_{xy} \cdot b_{yx} = 16/9 = (r_{xy})^2 > 1$$

not possible,

hence

contradic.

Thus $x + 4y + 3 = 0$ is line 'y on x'

$4x + 9y + 5 = 0$ is line x on y.

Find the correlation coefficient and conclude their relationship.

x_i	-6	-4	-3	-1	1	2	4	7
y	-4	-3	-1	-1	0	2	3	4
	24	12	3	1	0	4	12	28

$= 84$

Sol:- $r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$

$\text{Cov}(x,y) = \frac{1}{n} \sum x_i y_i - \bar{x}$

$= \frac{1}{n} \sum x_i y_i - 0$

$= \frac{1}{\cancel{8}_2} \times \cancel{84}^{21} = 10.5$

$\boxed{\sigma_x^2 = \frac{1}{n} \sum x_i^2 - \bar{x}^2}$

$= \frac{1}{n} \sum x_i^2 - 0 = \sqrt{33/2}$

$\boxed{\sigma_y^2 = \frac{1}{n} \sum y_i^2 - \bar{y}}$

$= \sqrt{7}$

where $\sum y_i^2 = 7$

$\sum x_i^2 = 33/4$

$\sum x_i y_i = 84$.

therefore $r_{xy} = \frac{10.5}{\sqrt{33/2}\,\sqrt{7}} = 0.997$

→ strongly correlated