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AN ANALYSIS OF SAND FILTRATION

Warren A. Hall*
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INTRODUCTION

Successful filters utilizing sand as a means of removing most of the suspended matter and bacteria from water supplies have been used at various times for more than forty centuries.(1) In spite of this, the nature of the process by which solids are removed by the filter has not been clearly defined. The history of sand filtration tells of a succession of filters, each more or less satisfactory, from which a set of design rules has been derived. While filters will continue to be improved by this trial and error process it has become apparent that a better knowledge of the fundamental mechanism of filtration must be obtained.

The American Society of Civil Engineers, through the Sanitary Engineering Division, instituted in 1935 an experimental study of the effects of differently composed filter sands on the operation of the rapid sand filter.(2) Baylis(3) has reported on increased infiltration rates made possible by improving the strength characteristics of the floe. The success of these experiments indicate that with a satisfactory theory of the mechanics of filtration for a basis, additional experimental advances may be possible. With this end in view an attempt is made in this paper to develop a relationship for the distribution of sediment in a filter in terms of space time coordinates and the physical properties of the components of the system.

DEVELOPMENT OF THE FUNDAMENTAL EQUATION

The limited data available on sediment distribution in the filter after a run appear to substantiate a postulation that only a portion of the suspended matter is removed at any given depth.(2) There is also a possibility that a "creep" or low velocity flow of the semi-fluid deposited sediment through the filter may occur. Both possibilities are accepted in the preliminary analysis. A second postulation requires that the reduction in concentration of suspended matter may be considered to be a continuous process. Utilizing the principle of mass conservation for one dimensional flow, an equation of continuity for the sediment will be derived. The weight rate of inflow of solids G_i to an

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* Asst. Prof. of Eng., Univ. of California, Los Angeles, Calif.

elemental volume Adz is the sum of the weight of sediment carried as suspension and that due to "creep" flow. Let C be the concentration of suspended matter, in weight per unit volume, v be the rate of water flow through the filter, distance per unit time, v_c be the creep flow rate, distance per unit time and γ_c be the specific weight of the creep flow sediment. Then the weight rate of inflow is

$$G_i = vCA\,dz + \gamma_c v_c A\,dz \tag{1}$$

The weight rate of outflow is

$$G_o = [vCA + \partial(vCA)/\partial z]dz + [\gamma_c v_c A + \partial(\gamma_c v_c A)/\partial z]\,dz \tag{2}$$

The rate of accumulation of sediment per unit total filter volume per unit time is equal to the net inflow over outflow divided by the element of volume. If W is the weight of sediment accumulated per unit filter volume at any point then:

$$\partial W/\partial t = -v\,\partial c/\partial z - \partial(\gamma_c v_c)/\partial z \tag{3}$$

This equation may be termed the fundamental differential equation of filtration. The second term on the right hand side is most probably a non linear function of the deposited sediment. Its analysis must await development of suitable methods to study flows of this type. Fortunately in many instances creep flow appears to be negligible, for example, during the initial portion of a run. This term will be omitted in the analysis to follow.

Immediately following the start of a filter run the filter may be considered to be a homogeneous isotropic porous medium. Thus the mechanism of separation, whatever its nature, should be the same throughout the filter. The distribution of suspended particles of a given size in a volume of the order of the dimensions of a pore is most probably uniform. From these conditions it follows that the rate of change of concentration of suspended matter of a given size is a function of the concentration of that size in suspension. Further, if each individual particle of the suspension behaves independently of the others, this function will be a simple proportionality. Designate the factor of proportionality by r, hereafter termed the rate factor. The rate factor is thus seen to vary with particle size but not with position in the filter so long as the conditions of homogeneity and isotropy are approximated. For any point in the filter at some instant of time

$$dC/dz = -rC \tag{4}$$

Separating the variables and integrating between the limits C_o and C, O and z:

$$C = C_o e^{-rz} \tag{5}$$

C_o represents the initial concentration of suspended matter in the water as it enters the filter and z is the depth below the surface of the filter. Taking the partial derivative of C with respect to z gives:

$$\partial C/\partial z = -rC_o e^{-rz} \tag{6}$$

Substituting equation 6 in equation (3) we have, neglecting the creep flow term:

$$dW/dt = vrC_o e^{-rz} \tag{7}$$

An equivalent result which will aid in interpreting the rate factor may be derived for a discontinuous filtration process, i.e., one in which the separation occurs only at certain points in the filter such as the front face of a sand grain or the narrow crevice at points of contact between sand grains. A new rate factor r' is defined to be the fraction of the concentration C removed in a distance equal to the center to center distance between sand grains. Designate this distance by h, then the relationship between r and r' may be expressed in terms of h.

$$r' = rh \tag{8}$$

In the first layer of sand the concentration will be reduced by an amount $r'C_o$. Thus the concentration leaving this layer and subsequent layers will be:

$$\begin{aligned} C_1 &= (1-r')C_o \\ C_2 &= (1-r')C_1 \\ C_i &= (1-r')C_{i-1} \end{aligned} \tag{9}$$

Substituting and summing to obtain the concentration after the ith layer gives:

$$C_i = C_o(1-r')^i \tag{10}$$

The rate of accumulation of sediment at any point in the filter is equal to the rate of reduction of concentration of suspension at that point. Hence

$$dW/dt = vr'C_o(1-r')^{z/h} \qquad z/h = 1, 2, 3, \cdots i \tag{11}$$

As h decreases relative to z this expression approaches the value given by equation (7).

Analysis of the Rate Factor

The development in the preceding section considers the continuity requirements but not the effect of the other physical properties of the system. These properties are embodied in the value of the rate factor. In order to estimate how these properties might affect the rate factor they will be reviewed very briefly.

The hydraulics of the flow through the sand is extremely complex. However, certain comments may be made. Experience with channel flow shows that in channels where the fluid is accelerating, velocity profiles tend to become more uniform and have less tendency to develop eddies. The opposite is true of a decelerating flow.(4) Hence we may presume that the flow in a pore approaching a constriction over the front face of a sand grain has a tendency toward more uniform velocity profiles, while the flow away from the constriction will tend to accentuate differences of the velocity profile and to form eddies when flow rates are high enough.

Most discussions of flow in porous media consider as a criterion for turbulence a Reynolds number of unity based on a means and diameter and a velocity equal to the discharge per total area of filter. Since many filters operate in or near the range $1 < N_R < 10$ this matter requires some discussion. The flow of a fluid about an isolated sphere may give some insight to the flow very close to the sand grain. At a Reynolds number of unity, based on the same data as used for the criterion in a porous medium, a separation of the laminar boundary layer occurs resulting in the formation of two small eddies behind the sphere. As the velocity is increased these eddies grow in size until they begin to be shed from the sphere. Subsequently the true turbulent boundary layer develops. Applying these facts to a porous medium is of course difficult. However, it is justifiable to expect that a similar progression of the nature of the flow in the boundary layer would occur although possibly at different values of N_R due to the presence of other sand grains and the effect of the non-spherical shape of the grains. Thus for porous media, the departure from a straight line of the plot of the log of the loss coefficient vs. the log of the Reynolds number which occurs at a Reynolds number of unity is probably due more to the formation of localized eddies behind the sand grains than to the incidence of self propagating turbulent (mixing) flow. These eddies, by dissipating energy and by increasing the actual velocity due to constriction of the areas, are probably responsible for the variation observed in the loss coefficient curves in the range $1 < N_R < 10$.

The floc in the water reaching the filter has passed through a coagulation, mixing and settling basin. Applying Stokes law to the settlement of particles, it is obvious that particles larger than the largest size which would settle in the time the water passes through the settling basin should not reach the filter. On the other hand if proper mixing and coagulation has occurred there should be few particles remaining in the near colloidal range. Thus particle sizes may be expected to lie in a relatively small range. The floc appears to possess cohesiveness or strength. It probably possesses adhesiveness with respect to the sand of the filter and to other particles.

The sand in the matrix of the filter may be considered to be spherical for most purposes of analysis with deviations in shape accounted for by correction factors. Although filters usually are not constructed with uniform sands, the results of the investigations of the American Society of Civil Engineers, Sanitary Engineering Division(2) indicate that uniform sands are as good or better than graded sands. Grading does not appear to be a necessary feature of a filter and sand size may be presumed constant for the theoretical filter.

The mechanism of filtration should be compatible with the foregoing remarks concerning velocity profiles, lack of fully developed turbulence, narrow range of particle sizes and floc properties. One possible mechanism to be considered postulates that an action occurs in the crevices adjacent to points of contact of the sand grains. In each of these angular areas normal to the flow there is a width of opening smaller than the geometric size of any given suspended particle. Whatever portion of the flow, however small, passes inside this point will be stripped of all particles of that size or larger. Thus the rate of removal at each sand layer for a given particle size is determined by the ratio of the flow passing within such boundaries to the total flow. A second compatible mechanism visualizes the small voids as settling basins in which gravitational sedimentation occurs assisted slightly by a dynamic effect due to the difference in densities of floc and water. It should be noted that if both straining and gravitational mechanisms are active, their results are not additive, since each removes only certain of those particles in close proximity to the surface of the sand grains.

The Interstitial Straining Theory

Figure 1 represents an idealized cross section through a constriction. If the area between the arc segments is presumed to be approximately the same as that between the chords, the area of the constriction with a dimension smaller than the particle size D, may be determined from geometric considerations to be approximately

$$A' = \left(3/2\sqrt{2}\right) D^{3/2} d^{1/2} \tag{12}$$

Where D is the diameter of the particle and d is the diameter of the sand grain, the total area of the constriction is:

$$A = 0.303\, d^2 \tag{13}$$

Introducing the constant c to include the effects of variation in velocity profile, non-symmetry of the particles and other grain arrangements than that used above, the rate factor for discontinuous separation is given approximately by the ratio of the two areas multiplied by the correction (experimental) constant.

$$r' = 35c\,(D/d)^{3/2} \tag{14}$$

Subsequent comparison of experimental data and the theoretical values indicates a value for c of 0.1 should be of the proper order of magnitude. Considering the effect of the converging channel on the velocity profile, this value appears quite reasonable.

The Gravitational Sedimentation Theory

Consider next the flow of a suspension of uniform size particles past an array of spheres. The direction of flow is parallel to the acceleration of gravity. If the particle had the same density as the water, the velocity and path of the particle would be identical to the velocity and a streamline of the fluid. Since the particle has a greater density than water, an unbalanced force will exist initially and the particle will accelerate relative to the fluid until a particle velocity relative to the fluid is reached for which the viscous drag is equal to the unbalanced gravitational force. A second unbalanced force exists due to the curvatures in the streamlines of the fluid in passing around a sphere. On the upstream face this acceleration tends to cast the particle nearer the sphere. The two unbalanced forces are directly proportional to the corresponding accelerations and the opposing viscous forces are proportional to the corresponding relative velocity components. Therefore, the ratio of the relative velocities is equal to the ratio of the accelerations. Since for the conditions encountered in sand filters the acceleration of gravity is very much greater than the acceleration due to curvatures, the relative displacement of a suspended particle due to flow curvatures may be neglected. The suspended particle may thus be considered to be moving downward everywhere in the filter with a velocity relative to the fluid determined by Stokes law. Let the specific weight of the particle be γ, the specific weight of the fluid to be γ_0 and the viscosity of the fluid be μ. The velocity of the particle relative to the fluid is:

$$V = 2/9\,(\gamma - \gamma_0)\,D^2/\mu \tag{15}$$

For convenience the coefficient of D^2 will be given by a single coefficient m in the subsequent analysis. Since V is the velocity of the particle relative to the fluid anywhere in the fluid it is also equal to the vertical velocity at which particles strike the boundaries of the fluid, i.e. the sand grains. Presume the particles do not reenter the fluid because of their adhesiveness to the sand. The rate at which particles are brought into the region h is vC and the rate at which they settle on the sand grains is VC. Hence the fraction removed in the region h is the ratio of the relative velocity V of the particles to the mean velocity of flow v with a correction constant as before caused by non-uniform velocity distribution in the pores and the ratio of solid area to void area at any crops section through the filter. The discontinuous rate factor for the gravitational theory is therefore:

$$r' = c' m\,D^2/v \tag{16}$$

Experimental data indicate a value for c' of 0.1, which appears to be reasonable.

Experimental Verification of the Theory

The theoretical relationships to be verified are:

(a) $$dW/dt = v r C_0 e^{-rz}$$

or (b) $$dW/dt = v r' C_0 (1 - r')^{z/h}$$

(c) $$r' = 35_c\,(D/d)^{3/2}$$ (straining theory)

and/or (d) $$r' = c' m\,D^2/v$$ (gravitational theory)

The relationship between r' and r is

(e) $$r' = rh$$

These theoretical relationships may be compared with data given by the Committee of the Sanitary Engineering Division on Filtering Materials(2) in figure 8 of their report. The results are given there in terms of "accumulated turbidity," that is, the sum of all material caught above a given depth. Since graphical differentiation of the data is somewhat inaccurate, the theoretical relationships will be integrated for comparison purposes. Integration of (a) with respect to time gives the total accumulation per unit volume at a point.

$$W = vrC_o t e^{-rz} \tag{17}$$

Integrating this expression with respect to z from O to z gives the total weight of accumulated turbidity T per unit filter area which lies above a depth z.

$$T = vC_o t\,(1 - e^{-rz}) \tag{18}$$

It may be noted that the coefficient qC_ot represents the total sediment to be removed during the time interval t, hence, the percentage of total sediment which is accumulated above a point z is

$$P = 1 - e^{-rz} \tag{19}$$

Substituting the discontinuous rate factor r', presuming $z \gg h$

$$P = 1 - e^{-r'z/h} \tag{20}$$

Although the data presented by the Committee(2) did not indicate particle size of floc entering the filter some information is given concerning the presedimentation basin from which approximate particle sizes were estimated as lying between 10 and 25 microns. Accordingly theoretical curves are plotted as solid lines in figure 2 for 10, 12, 15, 20 and 25 micron particle sizes for comparison purposes. The curves are plotted using rate factors derived for the gravitational straining theory. Quite by coincidence the rate factor for each of the two filtration mechanisms studies are approximately equal under the experimental conditions of sand size, particle size, and flow rates. Both theories thus seem to be acceptable. From the nature of the equations it would be expected that straining would predominate as the flow rate increases or the sand size decreases while the gravitational settling would predominate as the sand size increased or the flow rate decreases. It should also be noted that the two mechanisms are not additive in their effects since both remove particles nearest the sand surface.

The experimental data are replotted on figure 2 from the ASCE report in terms of percent of total accumulated sediment by presuming the maximum turbidity removed represents one hundred percent. There are three general groups of curves given in figure 8 of the report. Individual points from runs number 6 and 7 are replotted in figure 2 and are both seen to lie close to the theoretical curve for 15 microns. Runs number 2, 4, and 5 are also essentially congruent and lie close to the theoretical curve for 12 microns. Runs 1 and 3 are difficult to compare as it would appear that, if the filter had been deeper, a higher filtration efficiency would have been attained. If it is presumed that a deeper filter would have given approximately 50 ppm more total accumulated turbidity for run number 3, as is indicated by the shape of the other experimental curves, then run number 3 will lie very close to the curve for 10 microns. Run number 1 is not plotted since there is no indication of a break in the curve with which to estimate the probable initial total turbidity in the applied water.

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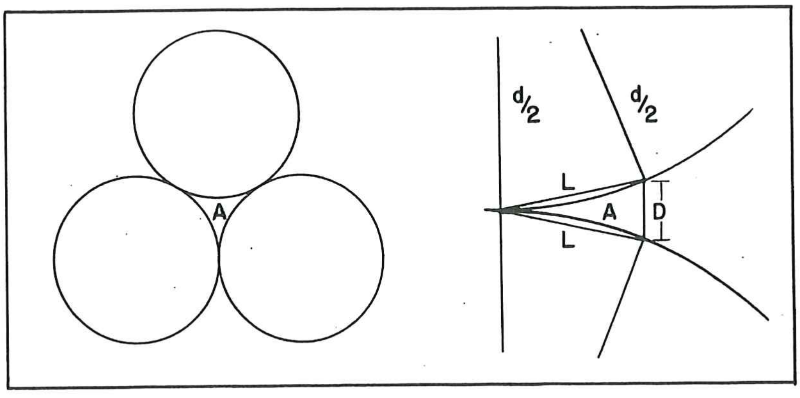

Fig. 1. Idealized Geometry of a Cross Section of a Pore Space.

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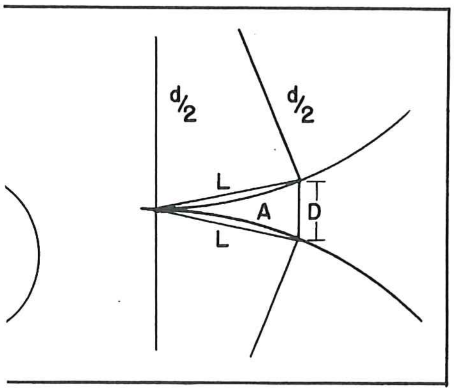

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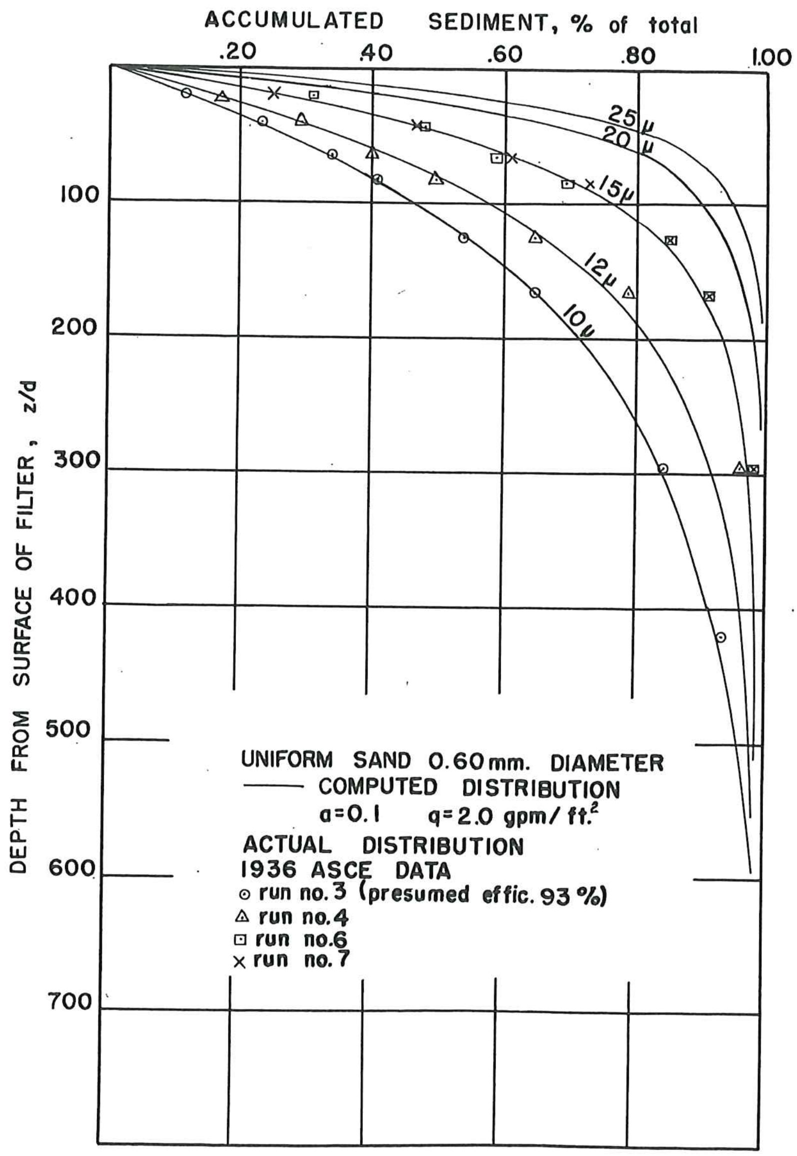

Fig. 2. Sediment Distribution as a Function of Depth of Filter. Solid Lines are Based on the Derived Theory. Individual Points are Based on Data Published by the A.S.C.E. Committee on Sanitary Engineering.(2)