Solutions to Exercises 0.1 to 0.6 of Pedoe's "Geometry: A Comprehensive Course"

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0.1

Prove $\overline{z_1 + z_2 + ... + z_n} = \overline{z_1} + \overline{z_2} + ... + \overline{z_n}$

Base Case:

Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$

Expand $\overline{z_1 + z_2}$

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{(a_1, b_1) + (a_2, b_2)} \\
&= \overline{(a_1 + a_2, b_1 + b_2)} \\
&= (a_1 + a_2, -b_1 - b_2) \\
&= (a_1, -b_1) + (a_2, -b_2) \\
&= \overline{(a_1, b_1)} + \overline{(a_2, b_2)} \\
&= \overline{z_1} + \overline{z_2}
\end{aligned}$$

Inductive Case:

Assuming that for any k complex numbers, $\overline{z_1 + z_2 + ... + z_k} = \overline{z_1} + \overline{z_2} + ... + \overline{z_k}$ holds, we will show that $\overline{z_1 + z_2 + ... + z_k + z_{k+1}} = \overline{z_1} + \overline{z_2} + ... + \overline{z_k} + \overline{z_{k+1}}$ also holds.

Let $z_k' = z_k + z_{k+1}$

By the inductive hypothesis,

$\overline{z_1 + z_2 + ... + z_k'} = \overline{z_1} + \overline{z_2} + ... + \overline{z_k'}$

By the base case, $\overline{z_k'} = \overline{z_k + z_{k+1}} = \overline{z_k} + \overline{z_{k+1}}$

Therefore,

$\overline{z_1 + z_2 + ... + z_k + z_{k+1}} = \overline{z_1} + \overline{z_2} + ... + \overline{z_k} + \overline{z_{k+1}}$

QED

Prove $\overline{z_1 \cdot z_2 \cdot ... \cdot z_n} = \overline{z_1} \cdot \overline{z_2} \cdot ... \cdot \overline{z_n}$

Base Case:

Let $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$

Expand $\overline{z_1 \cdot z_2}$

$$\begin{aligned}\overline{z_1 \cdot z_2} &= \overline{(a_1, b_1) \cdot (a_2, b_2)} \\ &= \overline{(a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)} \\ &= (a_1a_2 - b_1b_2, -a_1b_2 - b_1a_2)\end{aligned}$$

Expand $\overline{z_1} \cdot \overline{z_2}$

$$\begin{aligned}\overline{z_1} \cdot \overline{z_2} &= (a_1, -b_1) \cdot (a_2, -b_2) \\ &= (a_1a_2 - b_1b_2, -a_1b_2 - b_1a_2)\end{aligned}$$

so

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

Inductive Case:

Assuming that for any k complex numbers, $\overline{z_1 \cdot z_2 \cdot ... \cdot z_k} = \overline{z_1} \cdot \overline{z_2} \cdot ... \cdot \overline{z_k}$ holds, we will show that $\overline{z_1 \cdot z_2 \cdot ... \cdot z_k \cdot z_{k+1}} = \overline{z_1} \cdot \overline{z_2} \cdot ... \cdot \overline{z_k} \cdot \overline{z_{k+1}}$ also holds.

Let $z_k' = z_k \cdot z_{k+1}$

By the inductive hypothesis,

$\overline{z_1 \cdot z_2 \cdot ... \cdot z_k'} = \overline{z_1} \cdot \overline{z_2} \cdot ... \cdot \overline{z_k'}$

From the base case, $\overline{z_k'} = \overline{z_k \cdot z_{k+1}} = \overline{z_k} \cdot \overline{z_{k+1}}$

Therefore,

$\overline{z_1 \cdot z_2 \cdot ... \cdot z_k \cdot z_{k+1}} = \overline{z_1} \cdot \overline{z_2} \cdot ... \cdot \overline{z_k} \cdot \overline{z_{k+1}}$

QED

0.2 Show $|z_1 + z_2 + ... + z_n| \leq |z_1| + |z_2| + ... + |z_n|$

We will show this by induction on the number of complex numbers involved.

Base Case:

$|z_1 + z_2| \leq |z_1| + |z_2|$ is true by the triangle inequality.

Inductive Case:

Assuming $|z_1 + z_2 + ... + z_k| \leq |z_1| + |z_2| + ... + |z_k|$ holds, we will show that $|z_1 + z_2 + ... + z_k + z_{k+1}| \leq |z_1| + |z_2| + ... + |z_k| + |z_{k+1}|$ also holds.

Let $|z_k'| = |z_k| + |z_{k+1}|$ then by the inductive hypothesis $|z_1 + z_2 + ... + z_k'| \leq |z_1| + |z_2| + ... + |z_k'|$. By the base case, $|z_k'| = |z_k + z_{k+1}| \leq |z_k| + |z_{k+1}|$ therefore $|z_1 + z_2 + ... + z_k + z_{k+1}| \leq |z_1| + |z_2| + ... + |z_k| + |z_{k+1}|$.

0.3 Prove $|\mathbf{z_1} + \mathbf{z_2}|^2 + |\mathbf{z_1}$ - $\mathbf{z_2}|^2 = 2(|\mathbf{z_1}|^2 + |\mathbf{z_2}|^2)$

LHS

$$\begin{aligned}
&|z_1 + z_2|^2 + |z_1 - z_2|^2 \\
&(z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \\
&2(|z_1|^2 + |z_2|^2)
\end{aligned}$$

which equals the RHS.

QED

The formula can be interpreted as two right angle triangles with the same hypotenuse. The RHS describes a right angle triangle that is always an isosceles triangle with sides $\sqrt{|z_1|^2 + |z_2|^2}$, $\sqrt{|z_1|^2 + |z_2|^2}$, $\sqrt{2(|z_1|^2 + |z_2|^2)}$ and the LHS describes a triangle with sides $|z_1 + z_2|$, $|z_1 - z_2|$, $\sqrt{|z_1 + z_2|^2 + |z_1 - z_2|^2}$.

0.4

Show $(z_1 - z_4) \cdot (z_2 - z_3) + (z_2 - z_4) \cdot (z_3 - z_1) + (z_3 - z_4) \cdot (z_1 - z_2) = 0$

Expanding this out and evaluating results in 0.

Show $|AD||BC| - |BD||CA| - |CD||AB| \leq 0$.

$|AD||BC| + |BD||CA| + |CD||AB|$ can be represented by $(z_1 - z_4) \cdot (z_2 - z_3) + (z_2 - z_4) \cdot (z_3 - z_1) + (z_3 - z_4) \cdot (z_1 - z_2)$ which we showed equals 0.

Since $|AD||BC| - |BD||CA| - |CD||AB| \leq |AD||BC| + |BD||CA| + |CD||AB| = 0$ and magnitudes are always non-negative, $|AD||BC| - |BD||CA| - |CD||AB| \leq 0$.

0.5 Express $(a_1^2 + b_1^2)(a_2^2 + b_2^2)$ as $a^2 + b^2$ using $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

$$\begin{aligned}
(a_1^2 + b_1^2)(a_2^2 + b_2^2) &= |z_1|^2 |z_2|^2 \\
&= |z_1||z_1||z_2||z_2| \\
&= |z_1 \cdot z_1 \cdot z_2 \cdot z_2| \\
&= |z_1 \cdot z_2|^2 \\
&= |a_1 a_2 + a_2 b_1 i + a_1 b_2 i - b_1 b_2|^2 \\
&= |(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i|^2 \\
&= (a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2
\end{aligned}$$

0.6

Show that the equation of a circle in the plane can be described by the equation $|z - z_0| = r^2$.

First, we will represent the x and y position of points on a circle by the real and imaginary parts of a complex number.

$$(x - x_0) + (y - y_0) i$$

Then we will constrain the equation so that the magnitude squared of the complex number equals r^2 the radius of the circle squared.

$$|(x - x_0) + (y - y_0) i|^2 = r^2$$

Let $z = x + yi$ and $z_0 = x_0 + y_0 i$

$$\begin{aligned} |z - z_0| &= r^2 \\ ((x - x_0) + (y - y_0)i)((x - x_0) - (y - y_0)i) &= r^2 \\ (x - x_0)^2 + (y - y_0)^2 &= r^2 \end{aligned}$$

which is the traditional equation of a circle in a plane.

Show that $|z|^2 - z\overline{z_0} - z_0\overline{z} + z_0\overline{z_0} - r^2 = 0$ is another valid equation describing a circle in a plane.

$$\begin{aligned} |z - z_0|^2 &= r^2 \\ (z - z_0)(\overline{z} - \overline{z_0}) &= r^2 \\ z\overline{z} - z\overline{z_0} - z_0\overline{z} + z_0\overline{z_0} &= r^2 \\ |z|^2 - z\overline{z_0} - z_0\overline{z} + z_0\overline{z_0} - r^2 &= 0 \end{aligned}$$