Minimum Message Length and Kolmogorov Complexity

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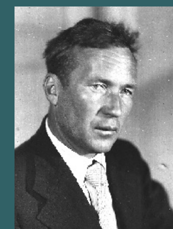
Presentation by Alkis Gkotovos

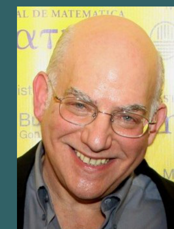
Overview

<u>Kolmogorov complexity</u>

Quantify complexity of binary strings via Turing Machines (early '60s)



A. Kolmogorov



G. Chaitin

P. Martin-Löf

?

<u>Universal induction</u>

Define algorithmic probability via Turing Machines and use it for induction (early '60s)

R. Solomonoff

<u>MML/MDL</u>

Infer a hypothesis about the data via two-part coding (late '60s and '70s)

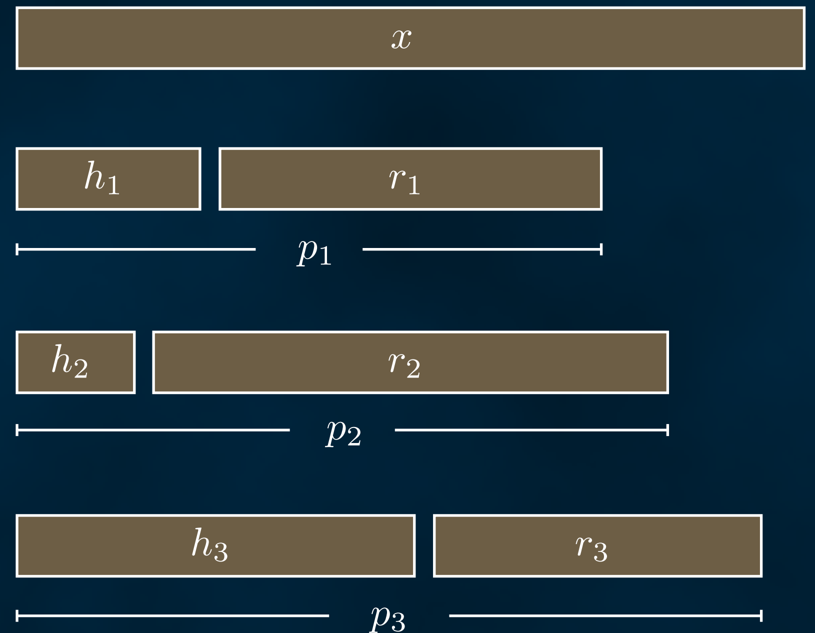
C. Wallace

J. Rissanen

Minimum Message/Description Length

Data string

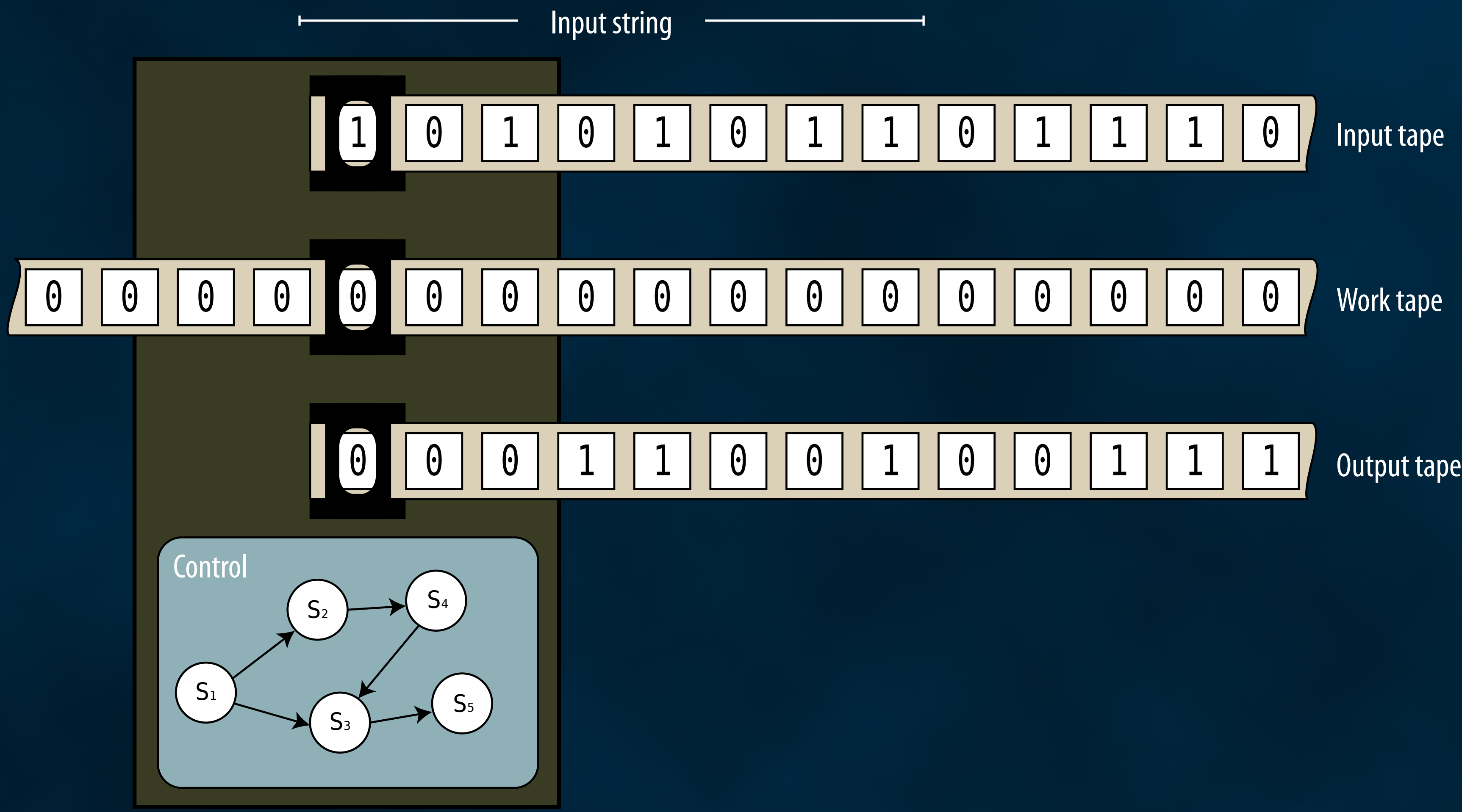
Encode x using a *two-part* scheme



Pick the hypothesis that results in the minimum encoding length

$$l(p_i) = l(h_i) + l(r_i) = -\log_2(p_H(h_i)) - \log_2(p_X(x \mid h_i))$$

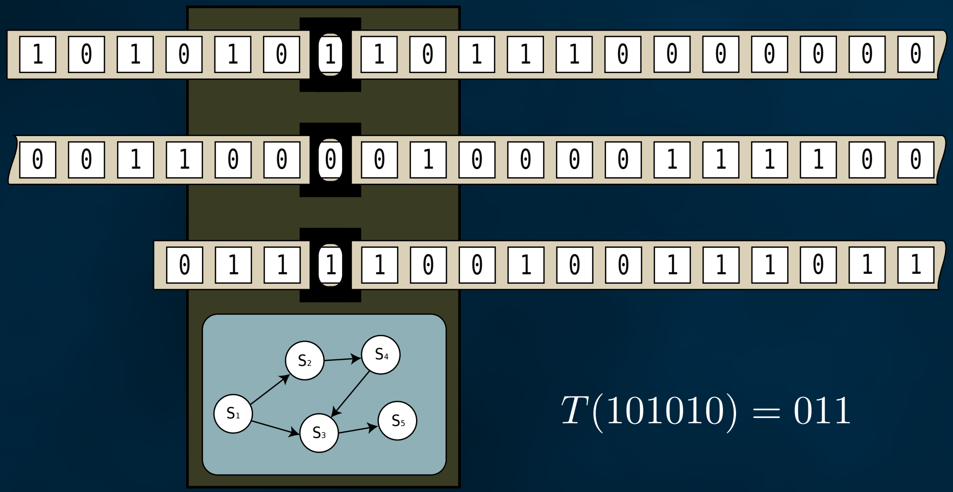
Introduction



Prefix TM

$T(p) = x$ if

- T reads all of the input p
- T writes x to the output
- T halts without reading/writing anything else



$T(101010) = 011$

T is a decoder of a prefix code (why prefix?)

Universal (prefix) TM = TM that can emulate any other (prefix) TM, e.g. $T(\langle i, p \rangle) = T_i(p)$

Introduction

(Prefix) Kolmogorov complexity of x = The length of the shortest input string required to produce x

$$K_T(x) = \min\{l(p) \mid T(p) = x\}$$

Definition dependent on T! Does this make any sense?

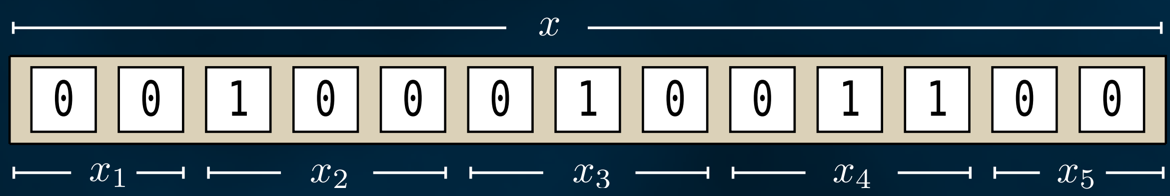
- Invariance theorem: Choose a *universal* prefix TM (among a special class) $\longrightarrow$ as good as any TM (up to a constant)
- Intuition: You can do (almost) as good as any TM by writing a "compiler" for it
- Caveat 1: Kolmogorov complexity is not computable (approximate, e.g. via compressor)
- Caveat 2: Constants can be large (more on this later)

Can define a "probability" measure from Kolmogorov complexity (hereafter: "probability" $\longleftrightarrow$ semimeasure)

$$P_T(x) = 2^{-K_T(x)} \quad \text{with} \quad \sum_x P_T(x) \leq 1 \quad \text{(why?)}$$

Data & Hypotheses

Data string x is a representation of observational data from a real world phenomenon



$$L = \{00, 100, 010, 011\}$$

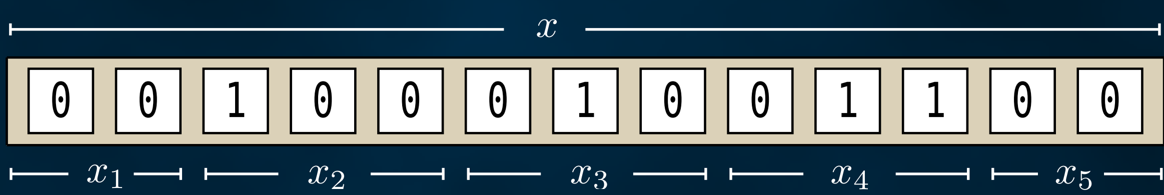
- "Sentences" $x_i \in L$, where L is a prefix-free set (data "language")
- Distinct sentences represent distinct real-world facts
- Sentences are conditionally independent given full knowledge of the phenomenon
- Strings are invariant to sentence permutation

Data & Hypotheses

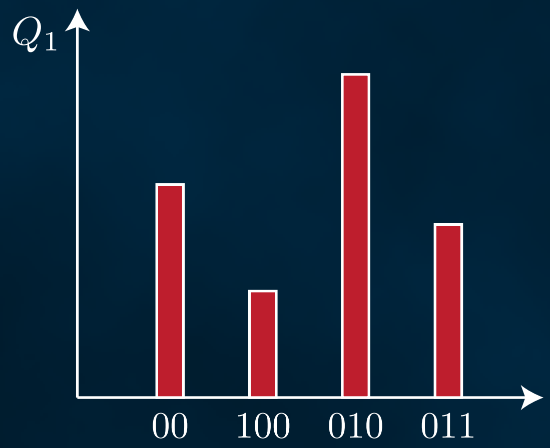
Hypothesis Q is a (computable) probability distribution over L

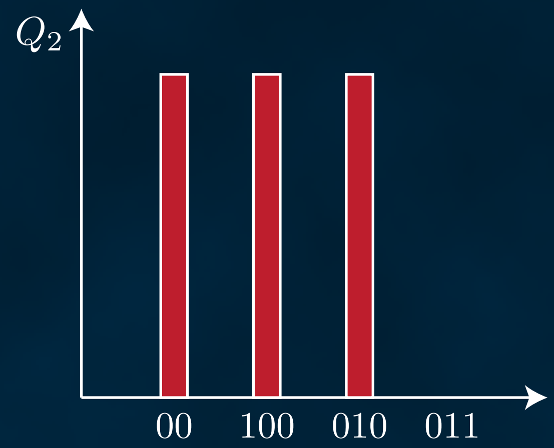
Conditional independence of sentences implies

$$x = x_1 \ldots x_n \Rightarrow Q(x) = Q(x_1) \times \ldots \times Q(x_n)$$



$$L = \{00, 100, 010, 011\}$$





Two-part encoding

How do we acquire a hypothesis-based encoding of data in the Algorithmic Complexity framework?

Idea

- Use conditional Kolmogorov complexity

$$K_T(x \mid y) = \min\{l(p) \mid T(\langle y, p\rangle) = x\}$$

  and interpret y as hypothesis and x as data

- Corresponding conditional algorithmic probability

$$P_T(x \mid y) = 2^{-K_T(x|y)}$$

Problem

Probability can never be 0, i.e. Popper-falsification not possible, because

$$K(x \mid y) < K(x) + O(1) \Rightarrow P_K(x \mid y) > P_K(x) + O(1)$$

Why? Hypothesis y acts as "extra info", instead of assertively

Proposal

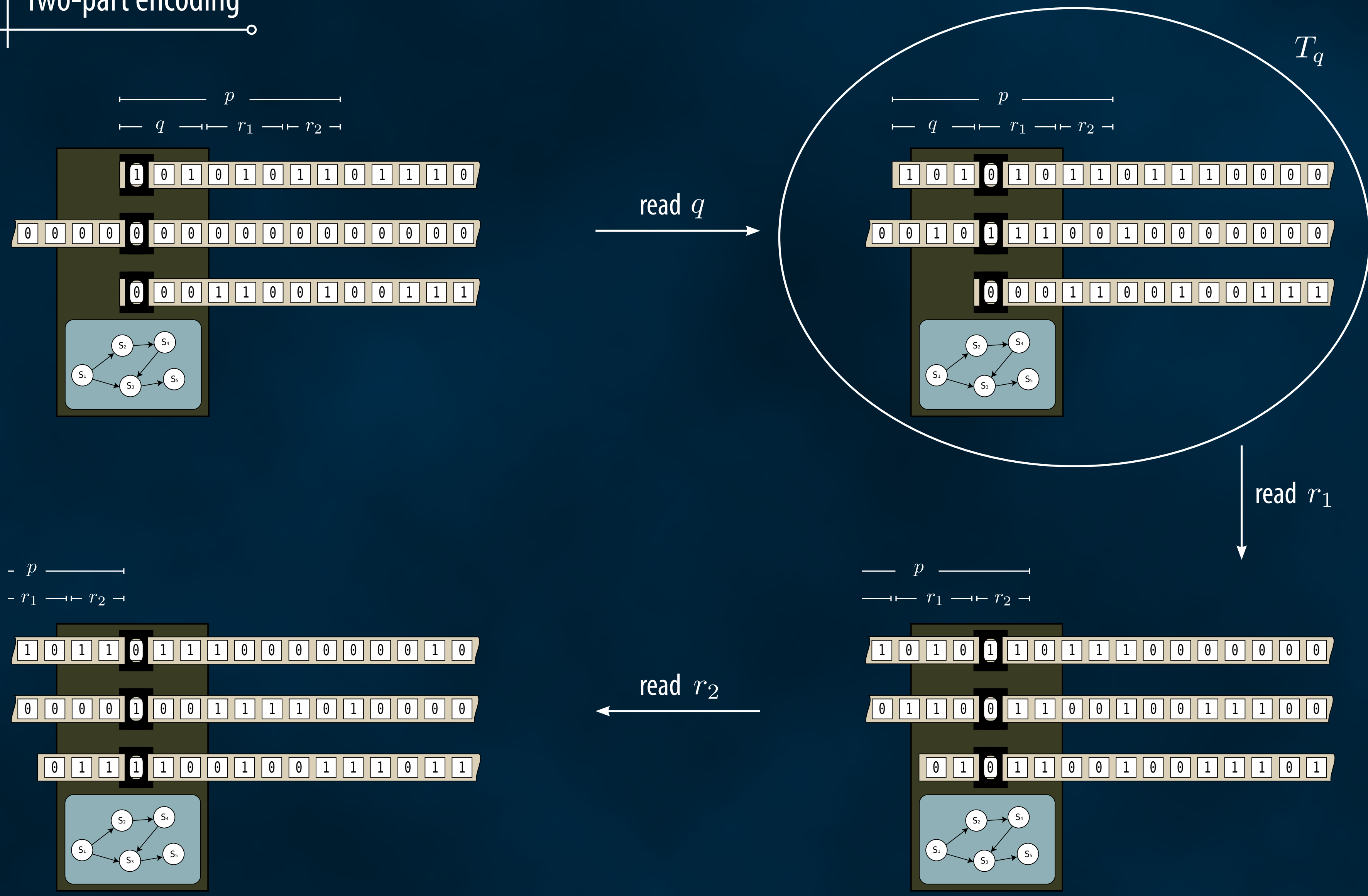
- Have hypothesis be a prefix of input string p
- Force intended two-part encoding by imposing conditions on p

Two-part encoding

Input p is an acceptable MML message encoding data string x, if

1)	$T(p) = x$	p encodes x
2)	$l(p) < l(x)$	some compression is achieved
3)	$p = qr$	two-part encoding
4)	$T(q) = \epsilon$	hypothesis q is does not determine data
5)	$T_q(rs) = xT_q(s)$	reading r does not alter the state of T
6)	$l(r) < K_T(x)$	hypothesis q is "significant"
7)	$x = x_1 \dots x_n \Rightarrow \begin{cases} r = r_1 \dots r_n \\ T_q(r_i) = x_i, \; i = 1 \dots n \end{cases}$	conditionally independent sentences
8)	$\begin{array}{l} x' = x^{(1)}x^{(2)} \\ j' = j^{(1)}j^{(2)} \end{array} \Rightarrow \begin{array}{l} T_q(j^{(1)}) = x^{(1)}, \; j^{(1)} < K_T(x^{(1)}) \\ T_q(j^{(2)}) = x^{(2)}, \; j^{(2)} < K_T(x^{(2)}) \end{array}$	hypothesis q is "general"
9)	No prefix of q satisfies all the above conditions	all of q is required

Two-part encoding



Two-part encoding

- The division of p into q and r is unique

- In what way exactly does hypothesis string q affect T?

Remember $T \xrightarrow{q} T_q$

T_q is a decoder of "second parts"

$$T_q : S \rightarrow W$$

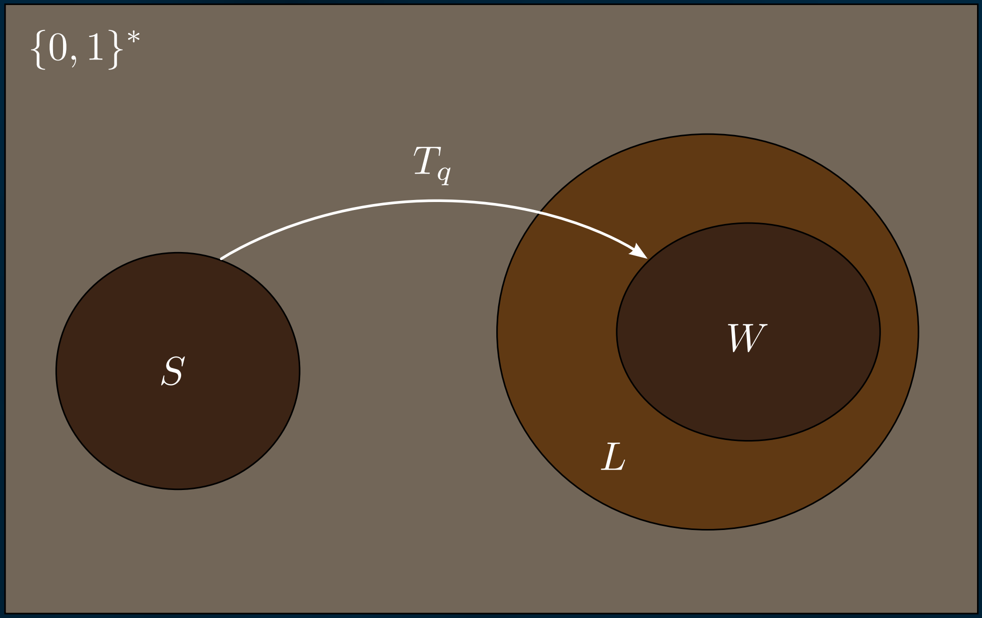
Code words

$$S = \{r_i \in \{0,1\}^* \mid T_q(r_i) \in L\}$$

Subset of L that is coded

$$W = \{x_i \in L \mid \exists r_i \in S : T_q(r_i) = x_i\}$$

In fact, T_q decodes a prefix code (why?)



Two-part encoding

- What is the hypothesis (probability distribution) Q implied by hypothesis string q?

$$Q(x_i) = \begin{cases} 2^{-l(p)} & \text{, if } p \text{ is a shortest codeword for sentence } x_i \in L \\ 0 & \text{, if there is no codeword for sentence } x_i \in L \end{cases}$$

Because of prefix code

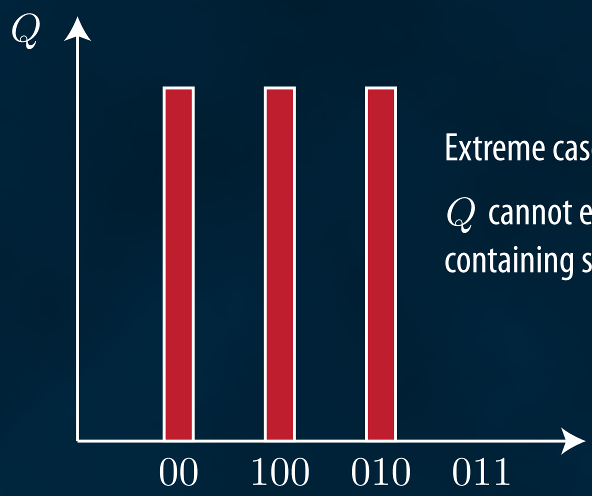
$$\sum_{x_i \in L} Q(x_i) = \sum_{x_i \in W} 2^{-l(p)} \overset{\text{Kraft}}{\leq} 1$$

- In this setting, hypotheses are falsifiable:

2) $l(p) < l(x) \Rightarrow l(r) < l(x)$

If Q assigns low probability (eq. high codeword length) to a sentence x_i, then adding enough such sentences to the data string will violate the above condition and falsify the hypothesis

Can Q assign lower codeword length to every sentence? (L is a complete prefix code for "data facts")



Extreme case:

Q cannot encode *any* data string containing sentence 011.

Two-part encoding

- What do we "pay" for enforcing a two-part encoding scheme?

Shortest acceptable MML input string: $M_T(x)$
Shortest unconstrained string: $K_T(x)$

with $M_T(x) \ ? \ K_T(x)$

$$\begin{aligned} M_T(x) - K_T(x) &= l(q) + l(r) - K_T(x) \\ &= K_T(Q) - \log_2(Q(x)) - K_T(x) \\ &= -\log_2\left(\frac{P_T(Q)Q(x)}{P_T(x)}\right) \\ &\approx -\log_2(\Pr(Q \mid x)) \end{aligned}$$

$$\boxed{P_T(x) = 2^{-K_T(x)}}$$

Finding the shortest MML string is like MAP, where $P_T(Q)$ plays the role of the prior

The log posterior odds ratio of two hypotheses is

$$\log_2\left(\frac{\Pr(Q_1 \mid x)}{\Pr(Q_2 \mid x)}\right) = l(p_1) - l(p_2)$$

where p_1 and p_2 are shortest input strings for their respective hypotheses