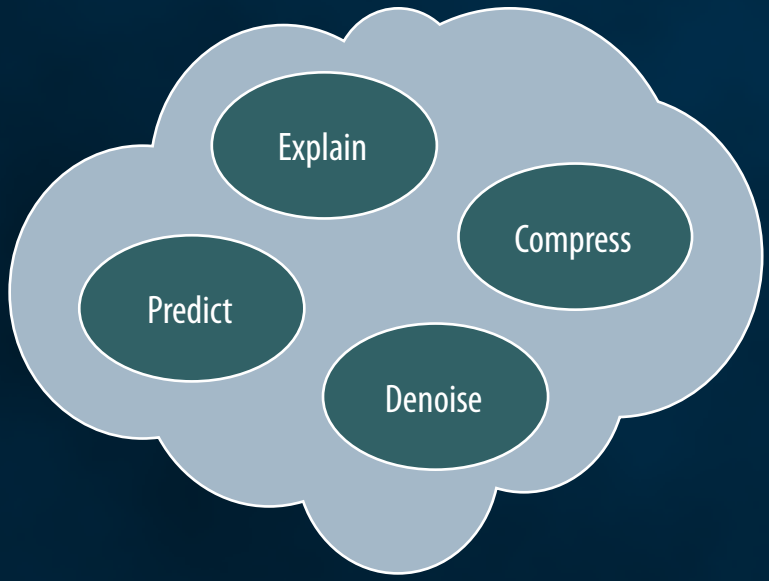
Minimum Message Length and Kolmogorov Complexity

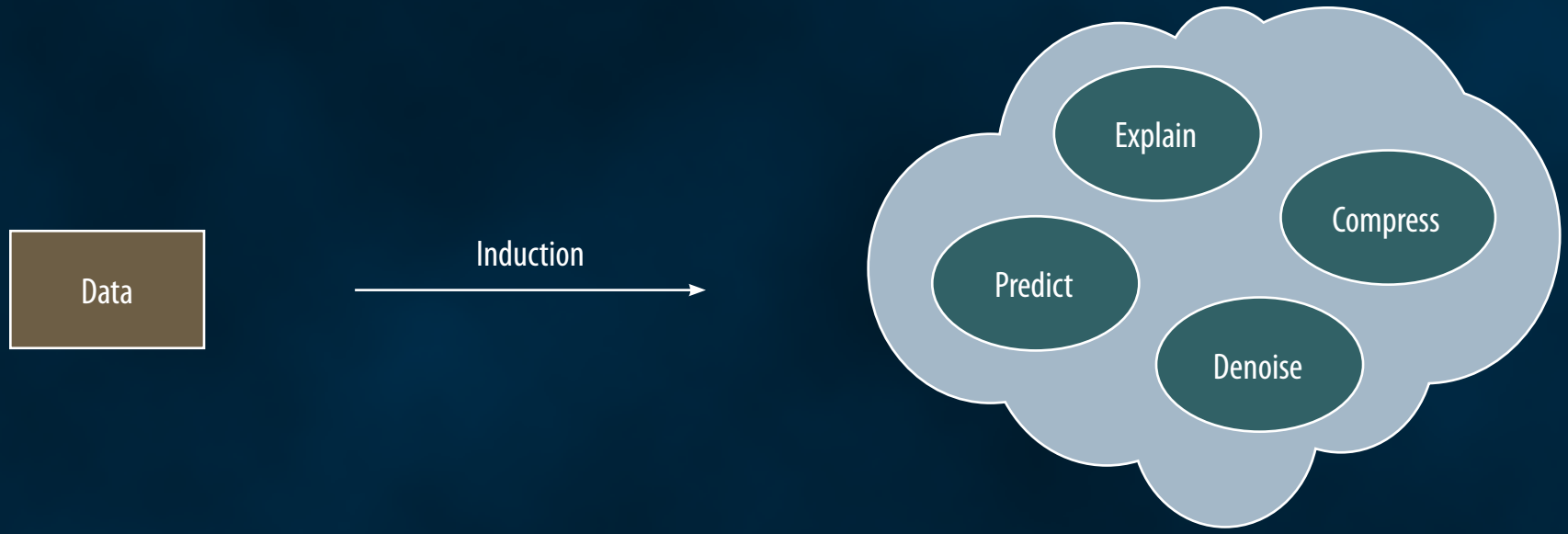
C. S. Wallace and D. L. Dowe

Presentation by Alkis Gkotovos

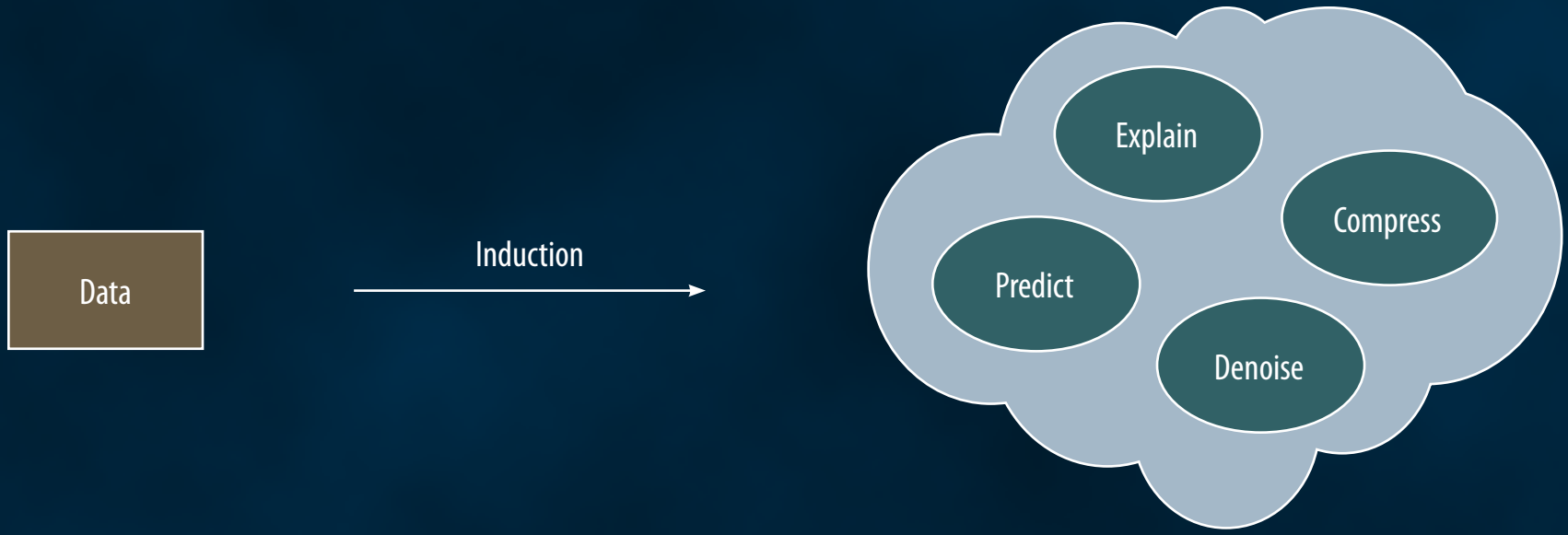
Data

Data




Data
Induction
Explain
Compress
Predict
Denoise

- Use some principle to guide inductive process, e.g. Occam's razor (bias towards "simplicity")



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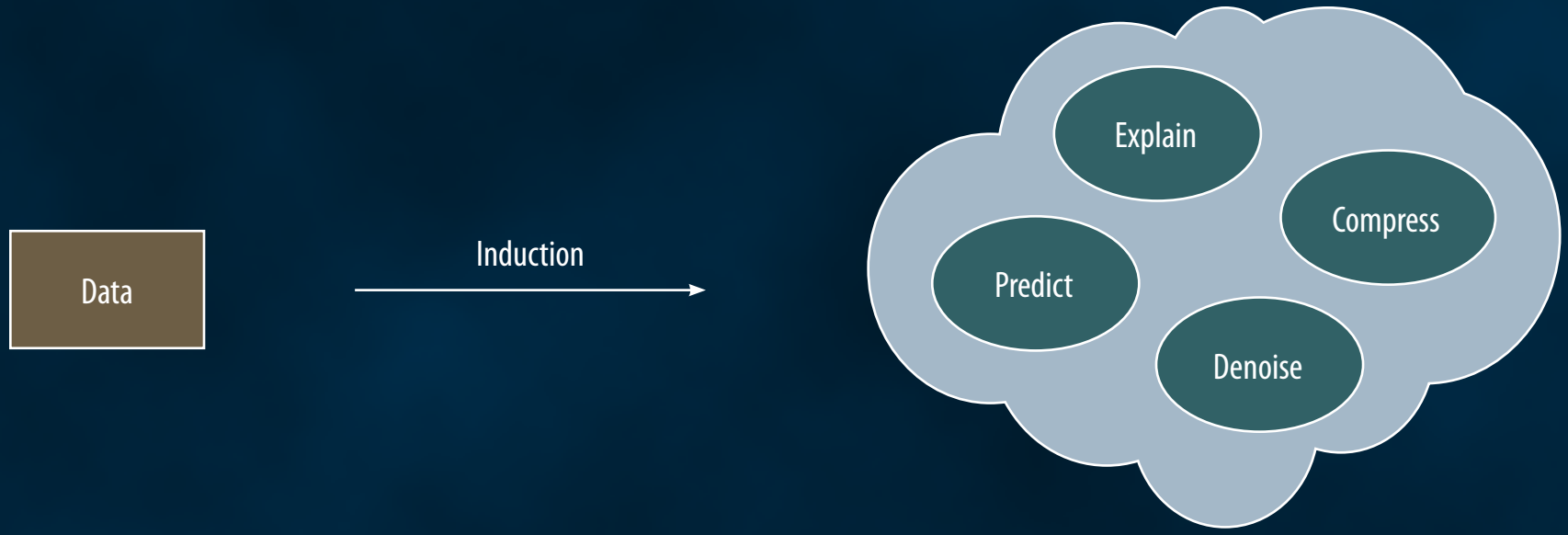

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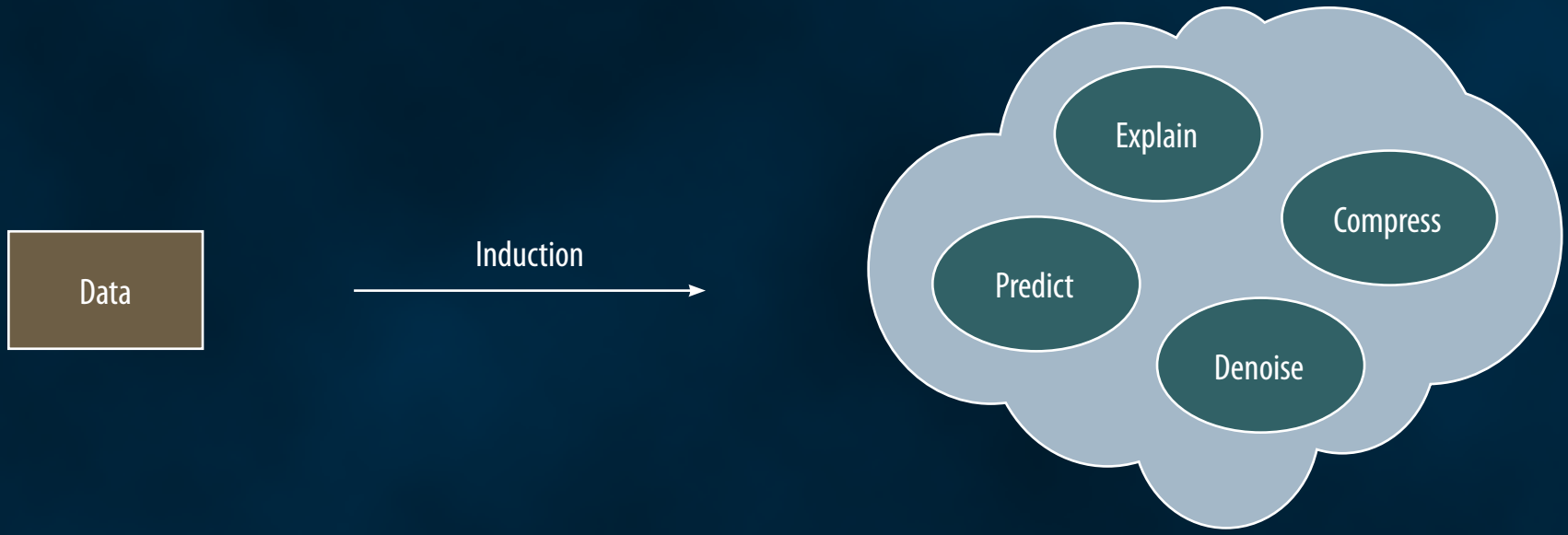
  - Information theory (e.g. MDL, MML)



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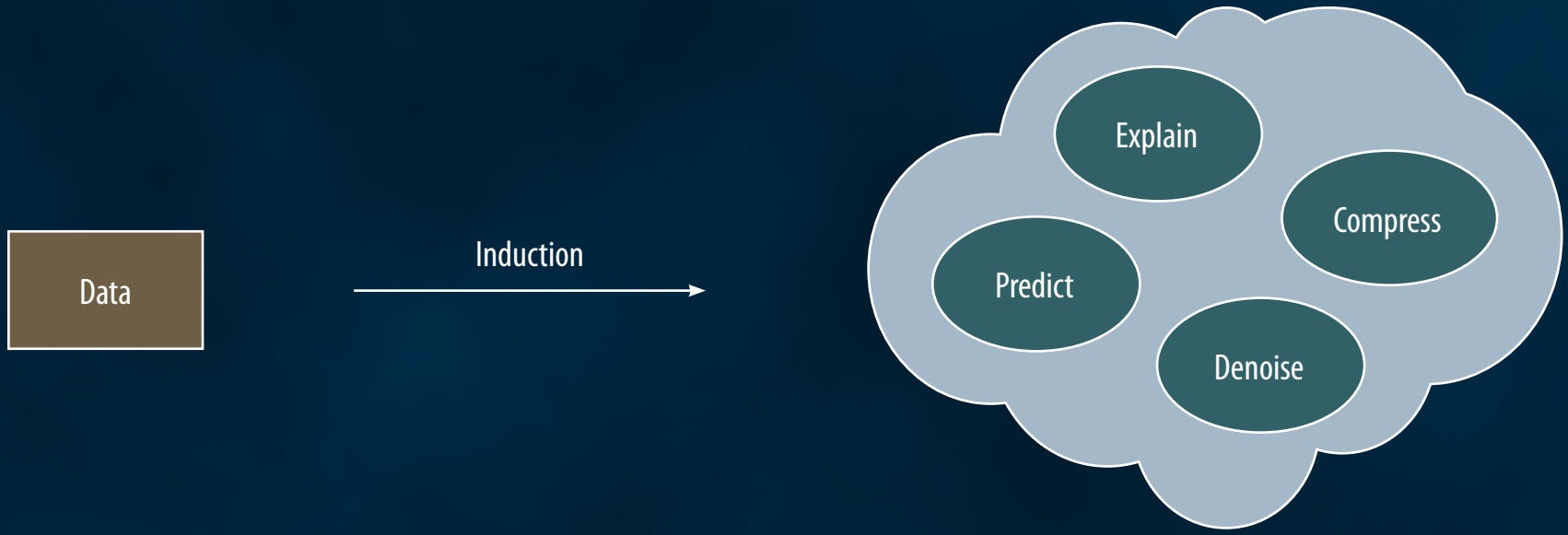

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  - Algorithmic complexity

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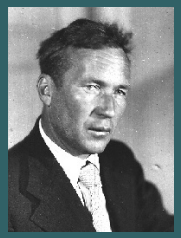
Kolmogorov complexity

Quantify complexity of binary strings via Turing Machines (early '60s)



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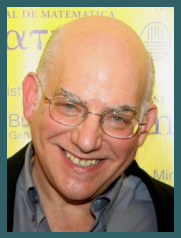

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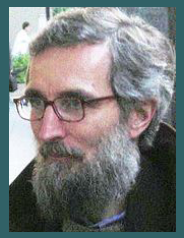

P. Martin-Löf

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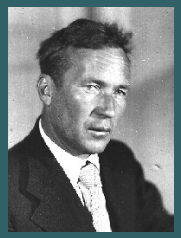
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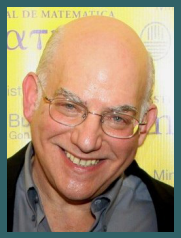
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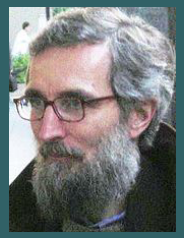
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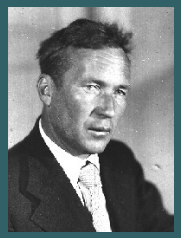
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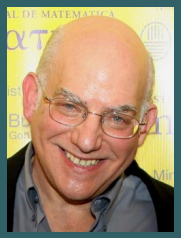
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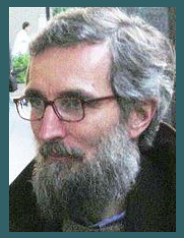
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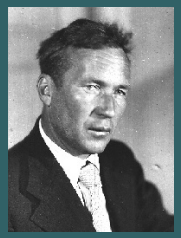
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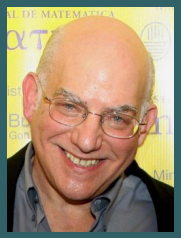
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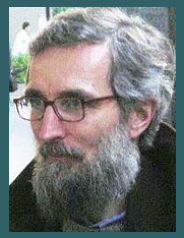
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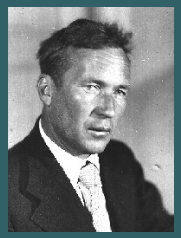
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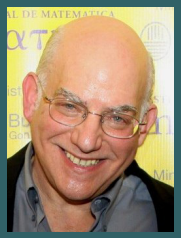
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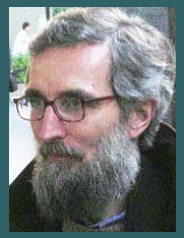
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Data string

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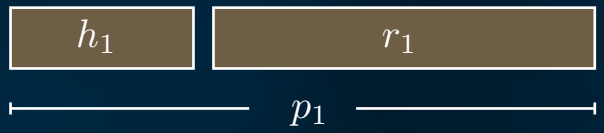
Encode x using a *two-part* scheme



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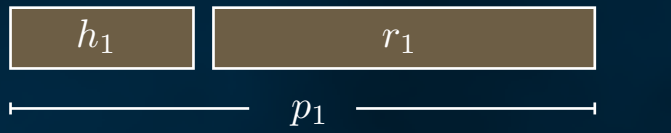

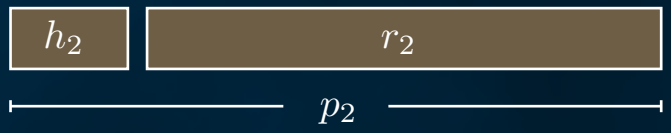

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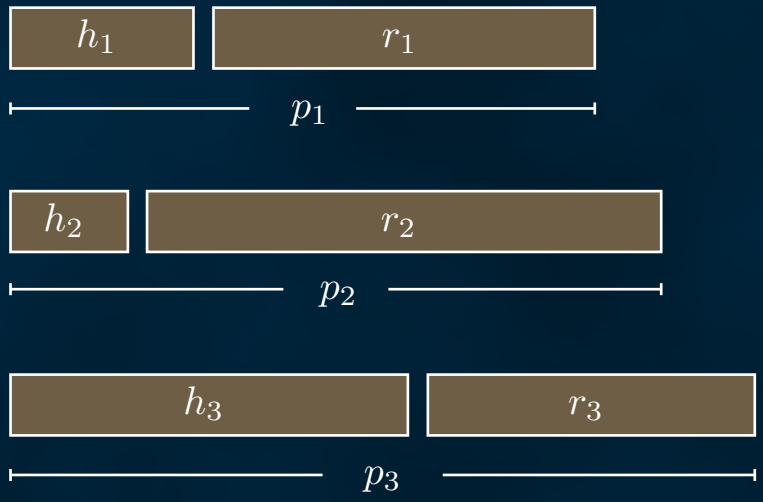

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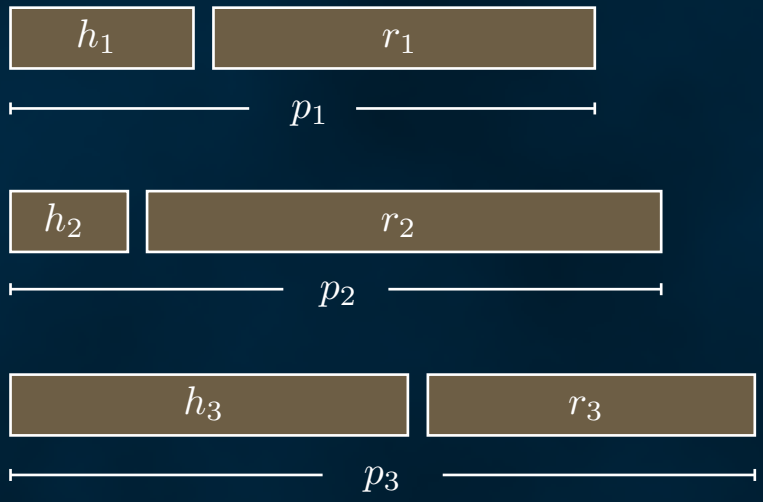
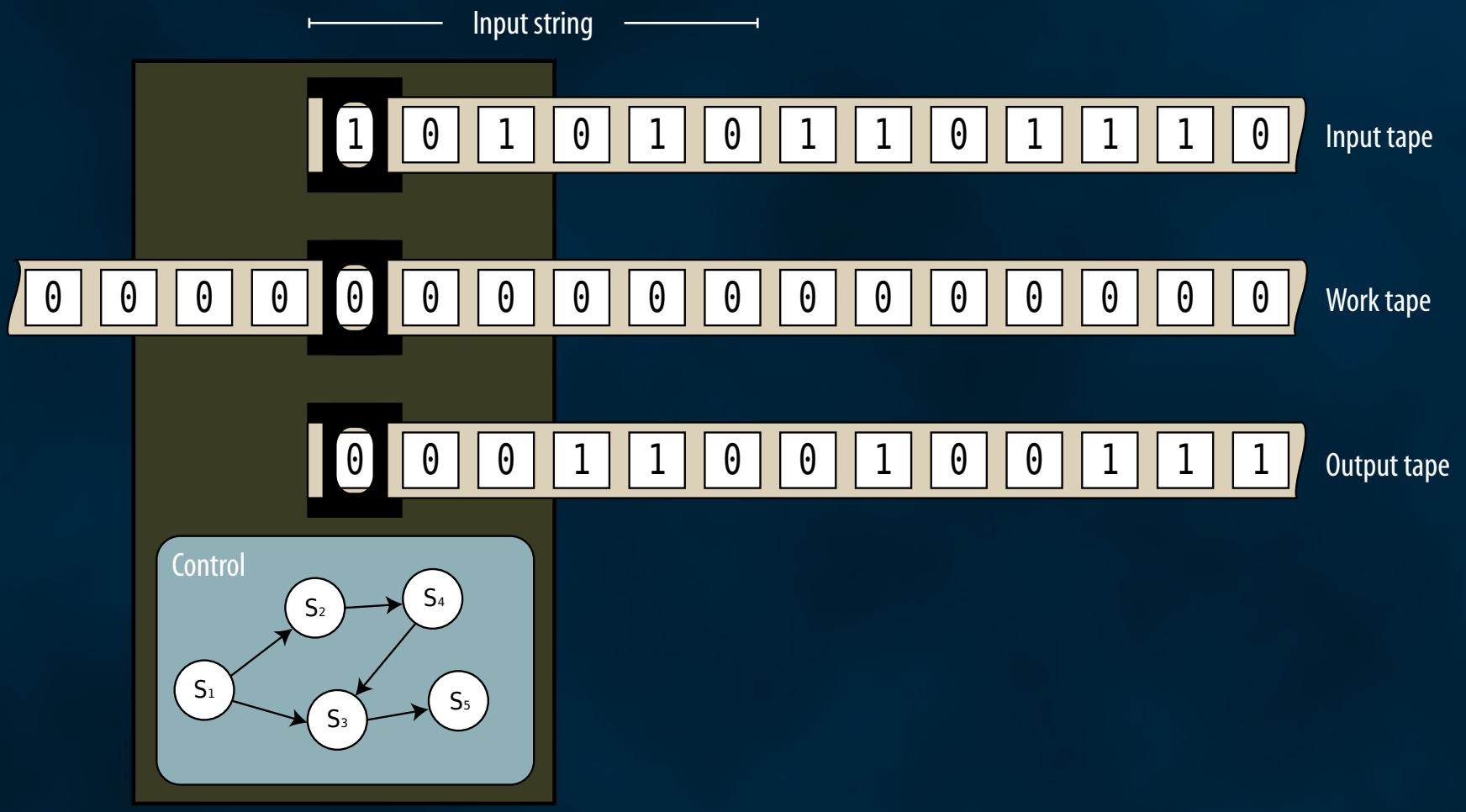
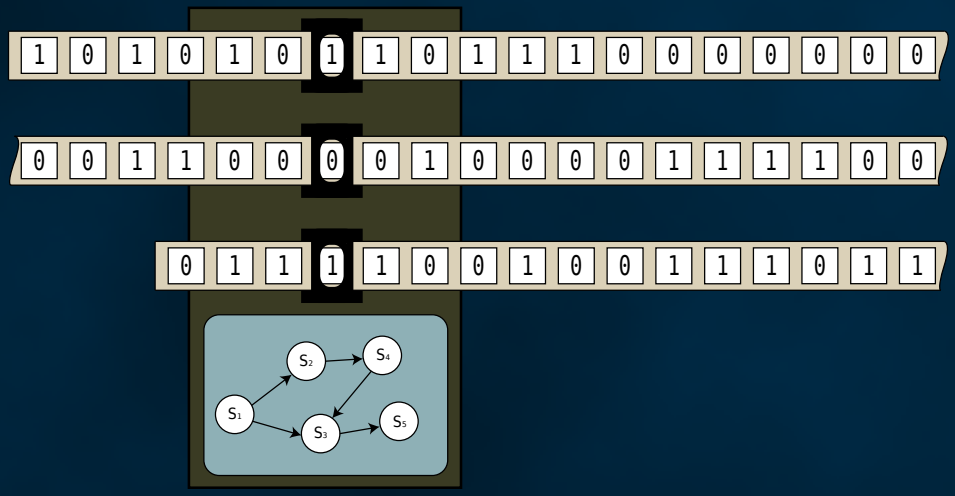


Pick the hypothesis that results in the minimum encoding length

$$l(p_i) = l(h_i) + l(r_i) = -\log_2(p_H(h_i)) - \log_2(p_X(x \mid h_i))$$


Input string
1 0 1 0 1 0 1 1 0 1 1 1 0
Input tape
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
Work tape
0 0 0 1 1 0 0 1 0 0 1 1 1
Output tape
Control
S1
S2
S3
S4
S5

$T(p) = x$ if


1 0 1 0 1 0 1 1 0 1 1 1 0 0 0 0 0 0 0
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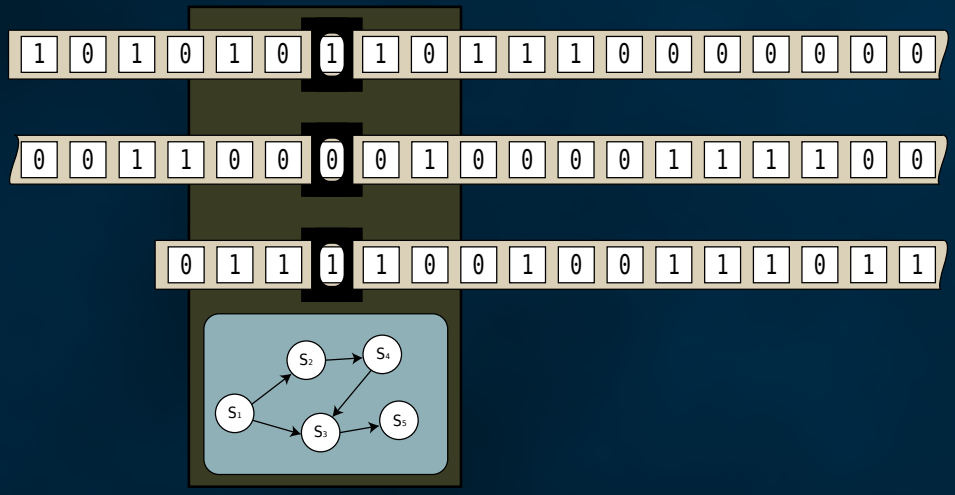
S2
S3
S4
S5

$T(p) = x$ if

- T reads all of the input p



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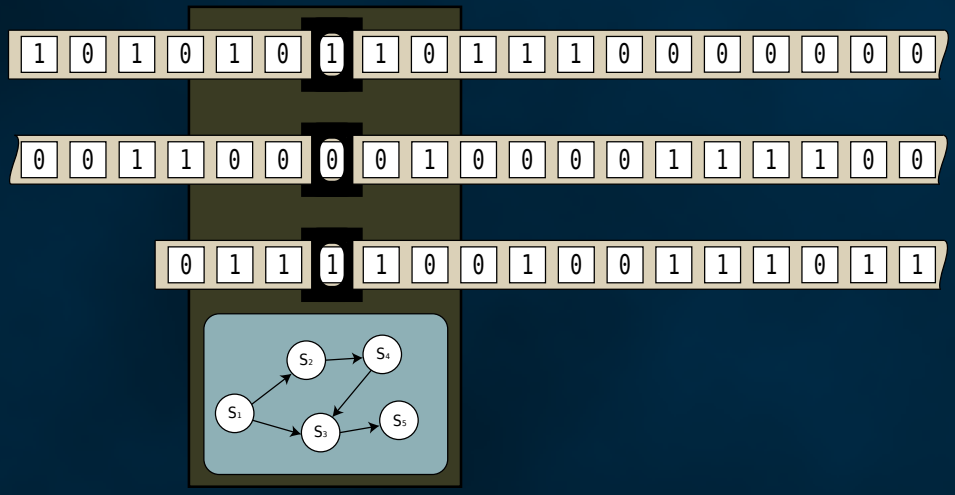

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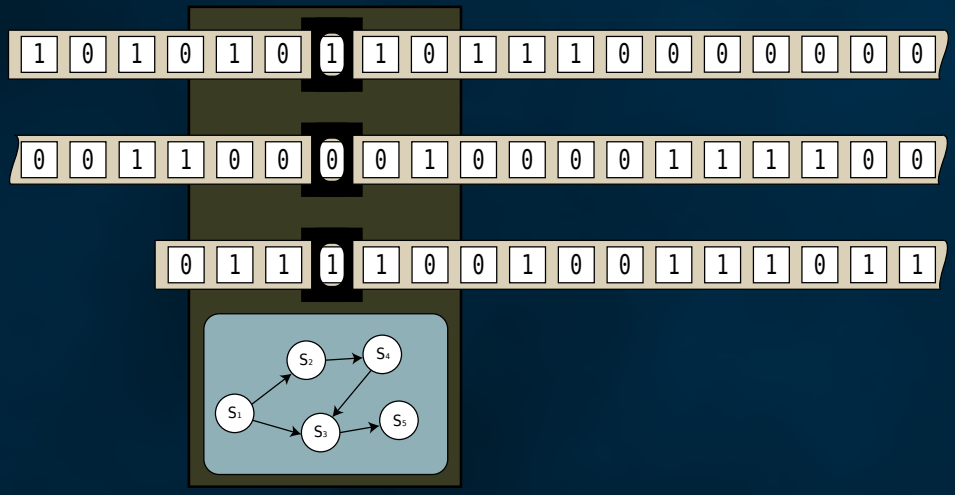

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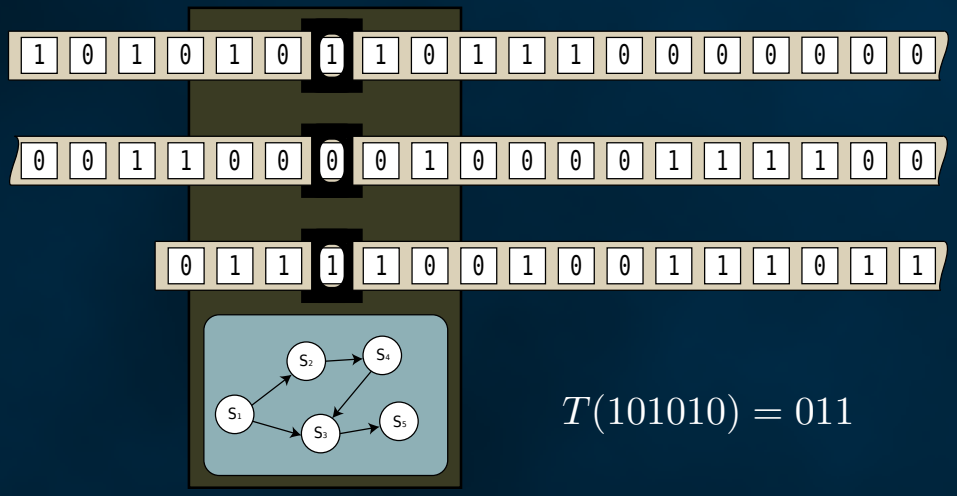

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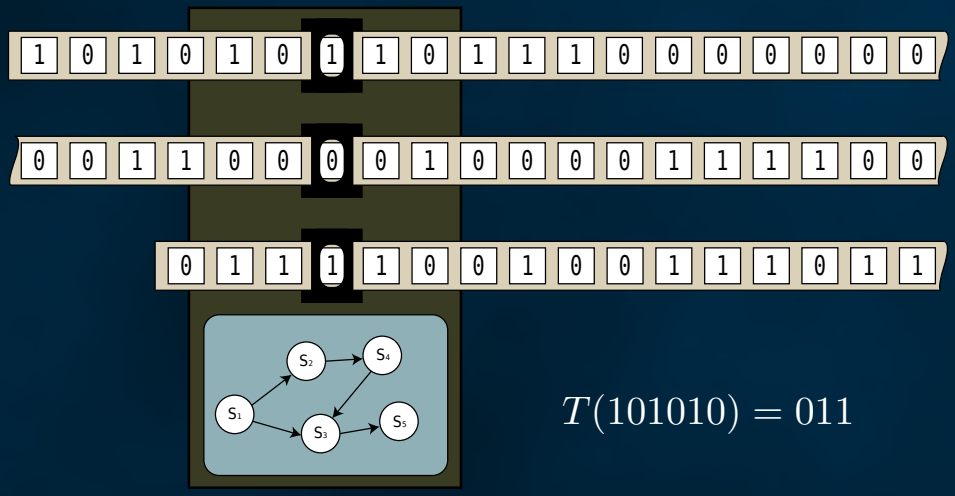

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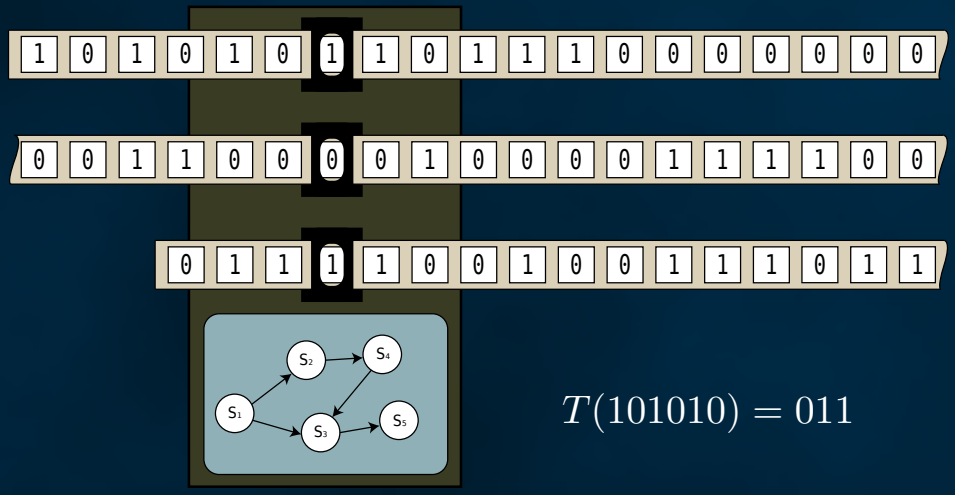

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Universal (prefix) TM = TM that can emulate any other (prefix) TM, e.g. $T(\langle i, p \rangle) = T_i(p)$

(Prefix) Kolmogorov complexity of x = Length of the shortest input string required to output x

$$K_T(x) = \min\{l(p) \mid T(p) = x\}$$

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$$P_T(x) = 2^{-K_T(x)} \qquad \text{with} \quad \sum_x P_T(x) \leq 1 \qquad \text{(why?)}$$

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Why don't we normalize?

(Prefix) Kolmogorov complexity of x = Length of the shortest input string required to output x

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Definition dependent on T! Does this make any sense?

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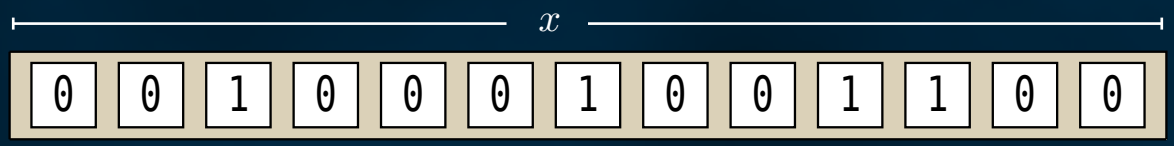
Data string x is a representation of observational data from a real world phenomenon



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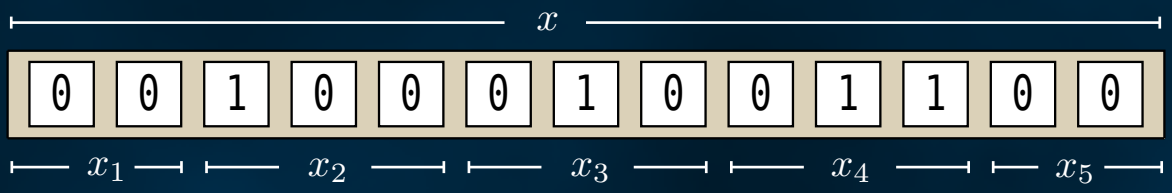

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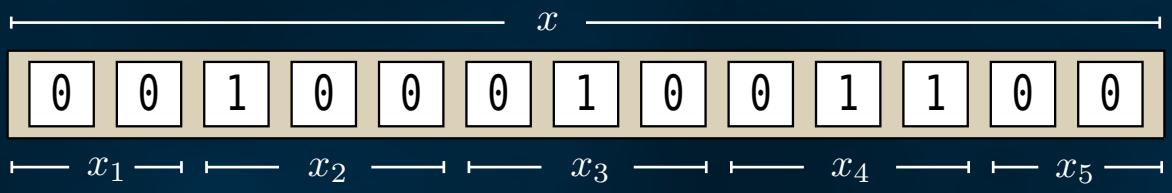

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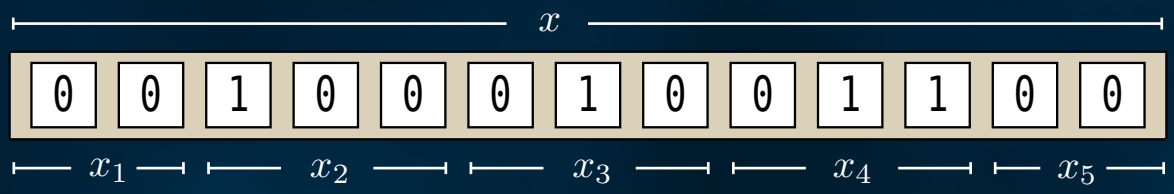

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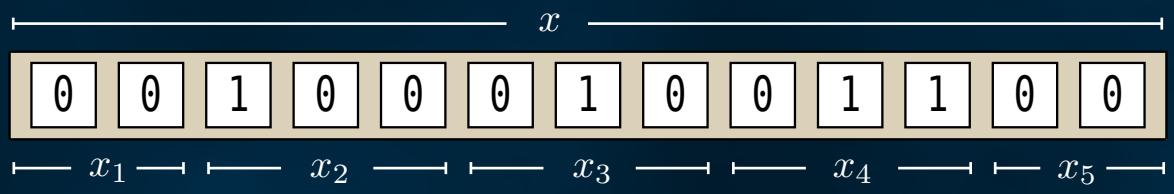

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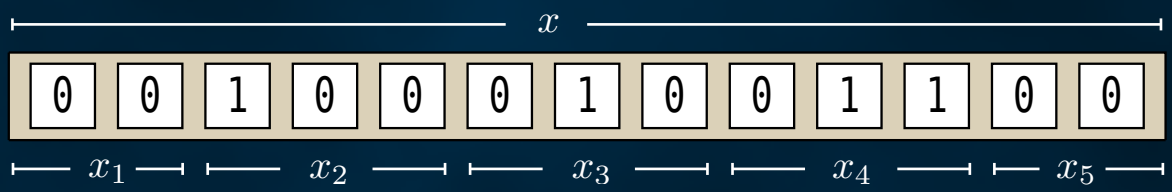

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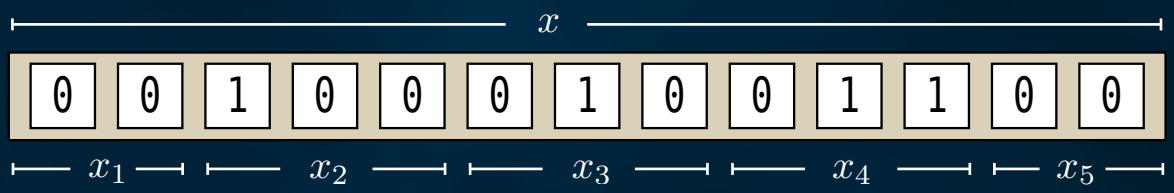

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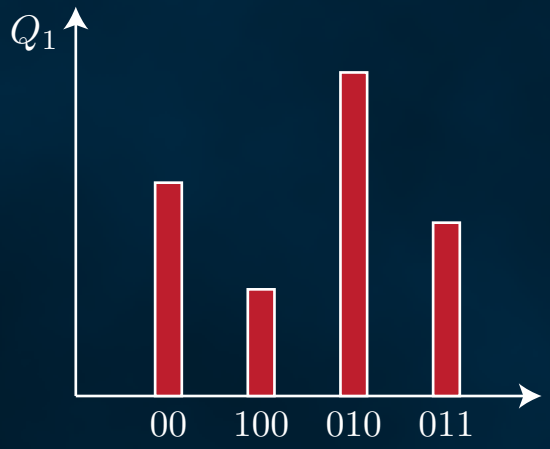

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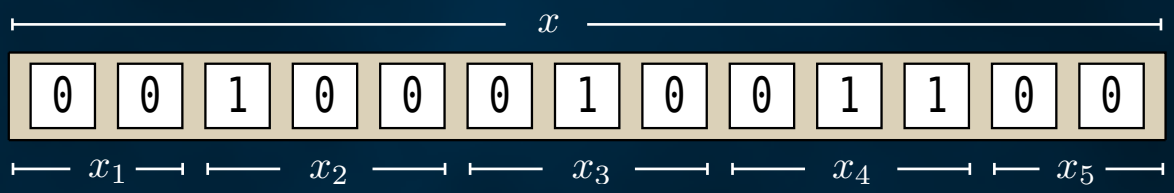

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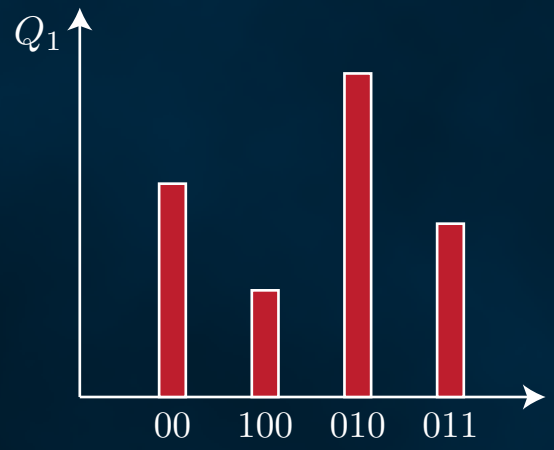

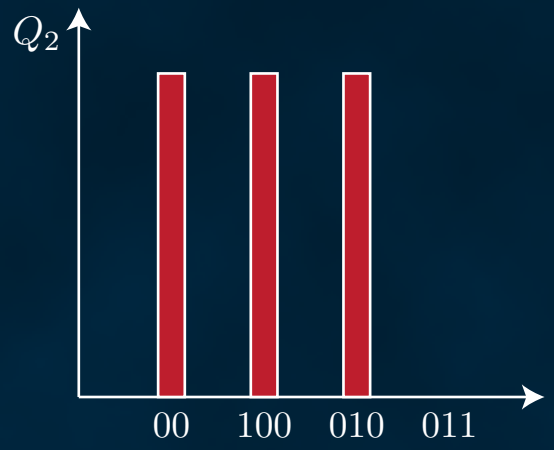

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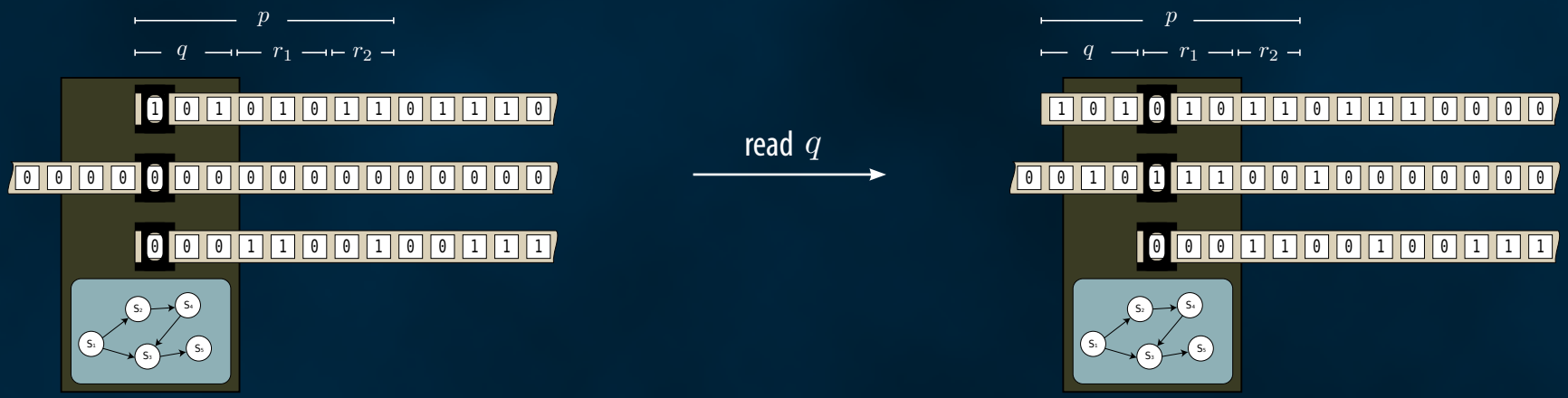
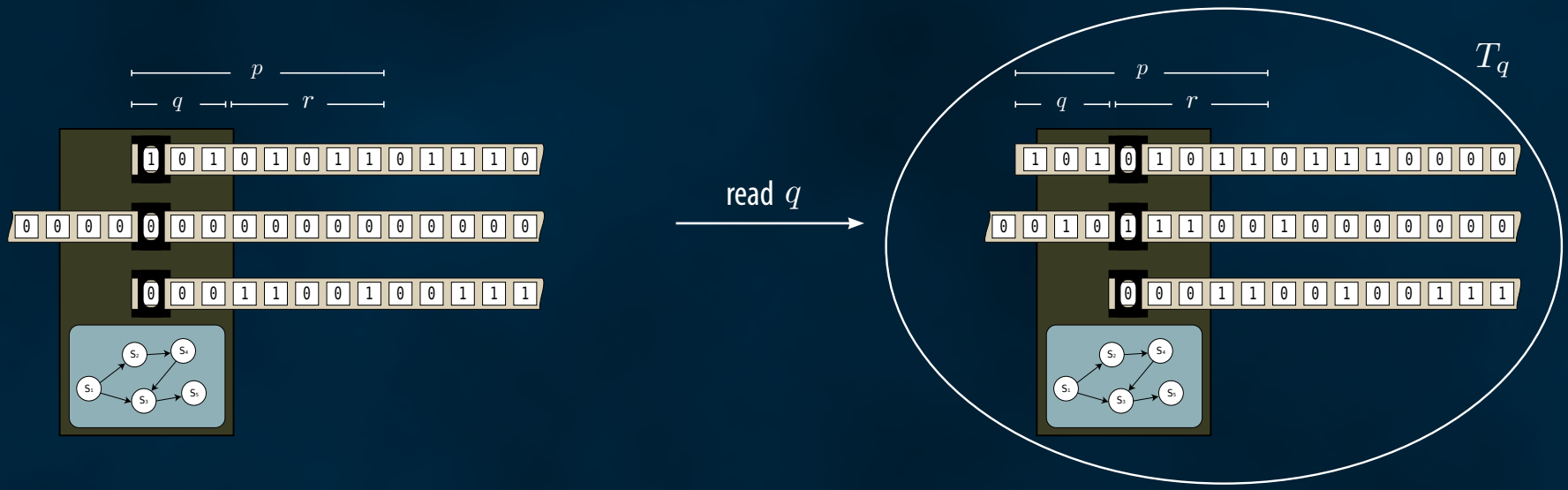

p
q
r1
r2
read q
p
q
r1
r2


p
q
r
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T_q

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q
r1
r2
read q
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Tq
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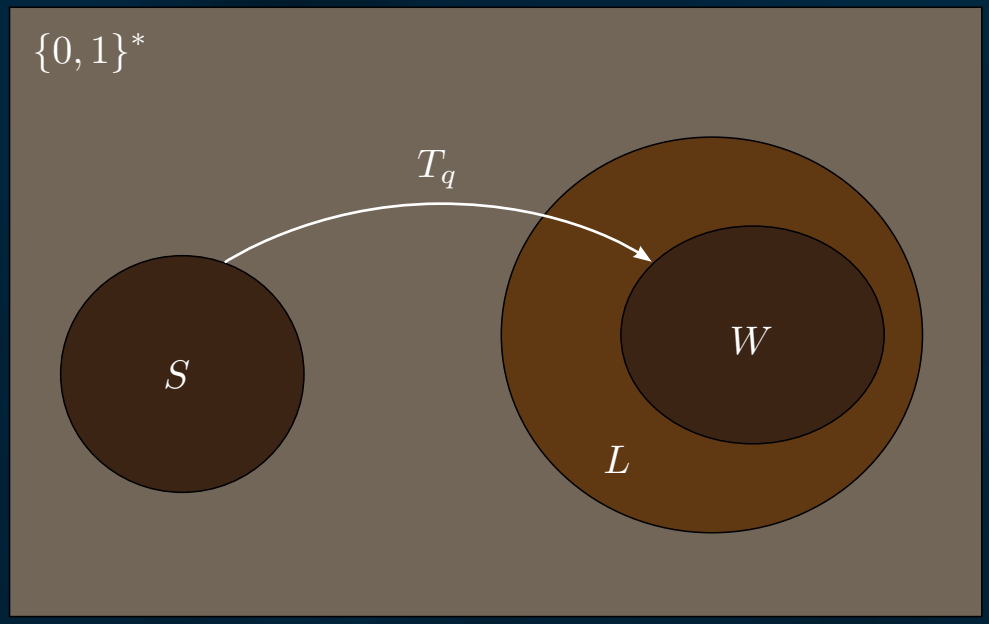

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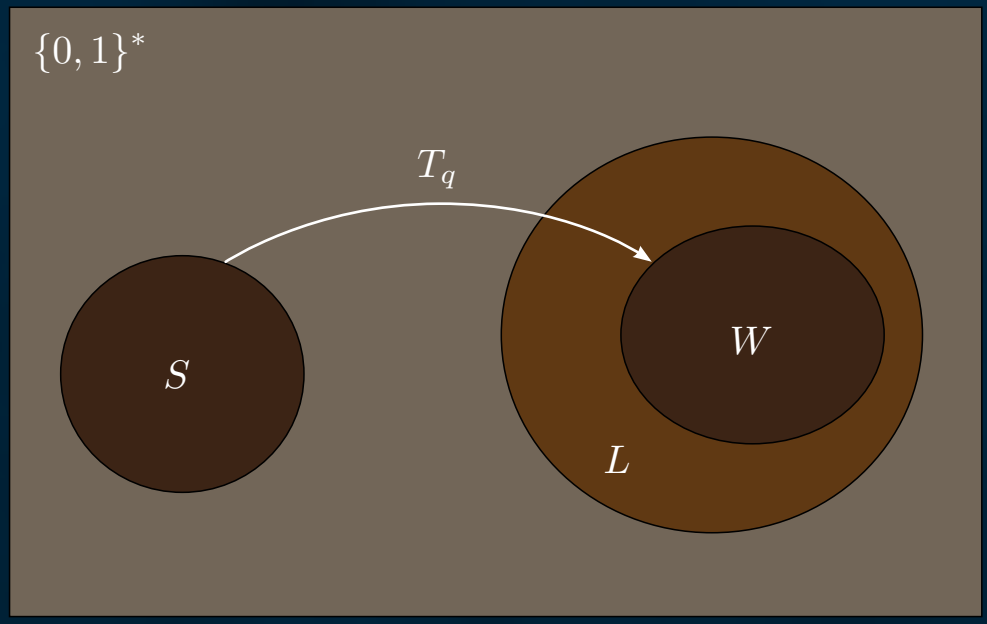
In fact, T_q decodes a prefix code (why?)



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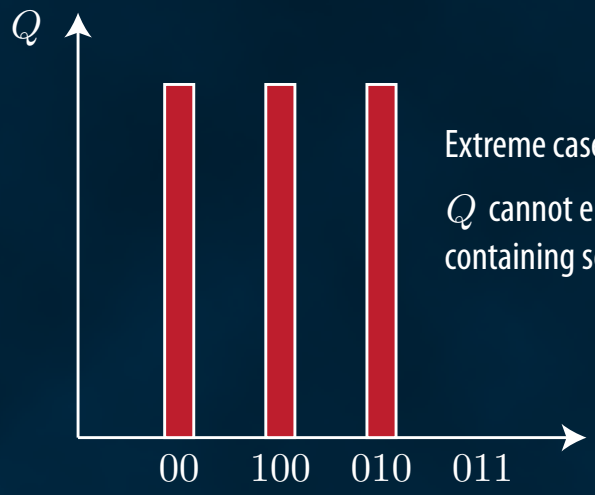
(L is a complete prefix code for "data facts")



Extreme case:

Q cannot encode *any* data string containing sentence 011.

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The log posterior odds ratio of two hypotheses is

$$\log_2\left(\frac{\Pr(Q_1 \mid x)}{\Pr(Q_2 \mid x)}\right) = l(p_2) - l(p_1)$$

where p_1 and p_2 are shortest input strings for their respective hypotheses

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> *[...] Bayes/Solomonoff is the Gold standard for prediction, but MML/MDL is (often) a good "approximation/simplification" for explanation and understanding.*
>
> M. Hutter

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ART	→	SCIENCE
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The End