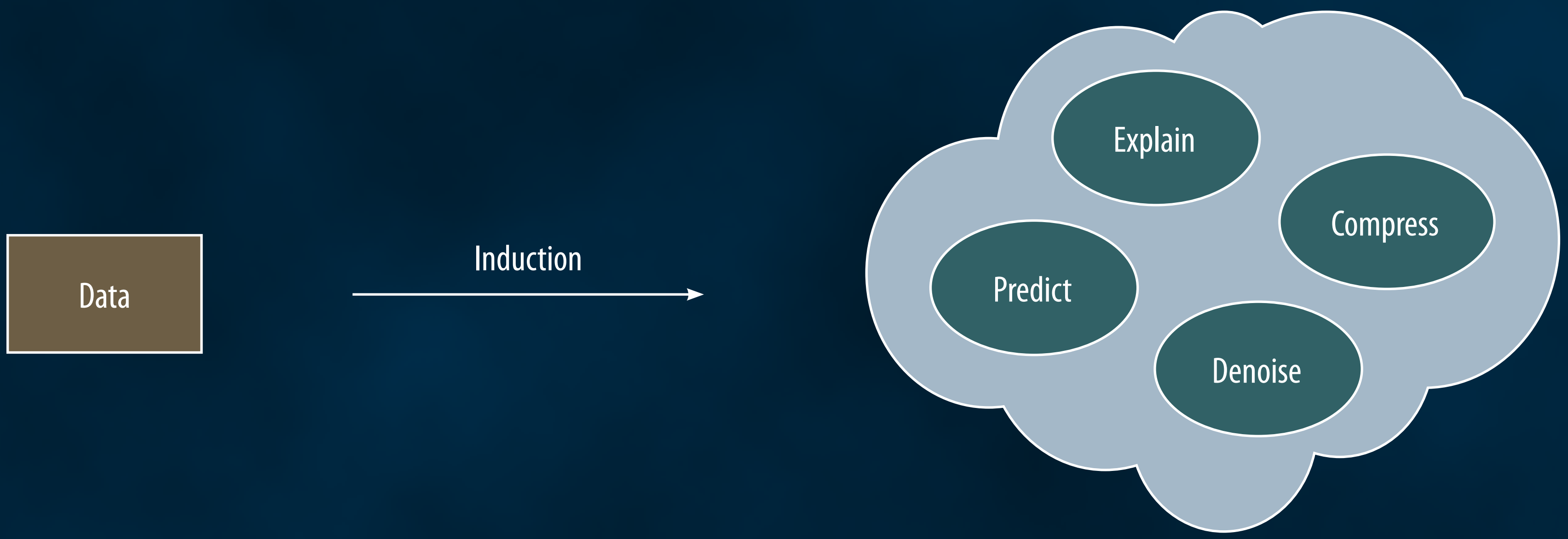
Minimum Message Length and Kolmogorov Complexity

C. S. Wallace and D. L. Dowe

Presentation by Alkis Gkotovos

Data


Data
Explain
Compress
Predict
Denoise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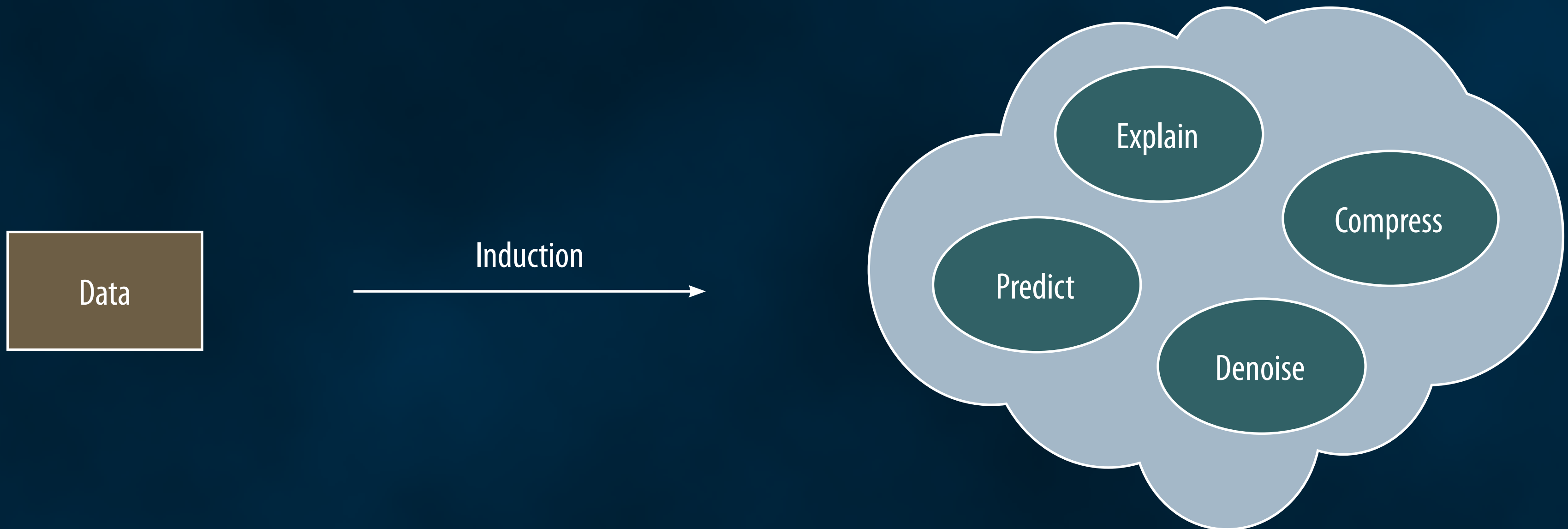

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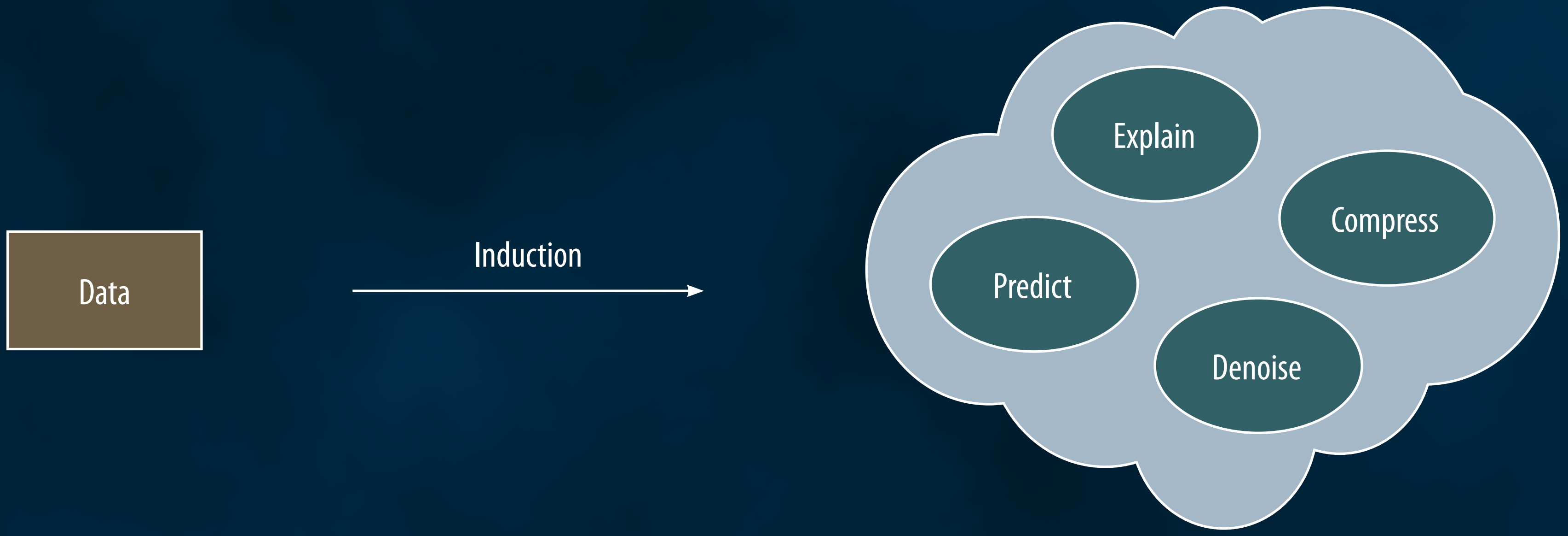

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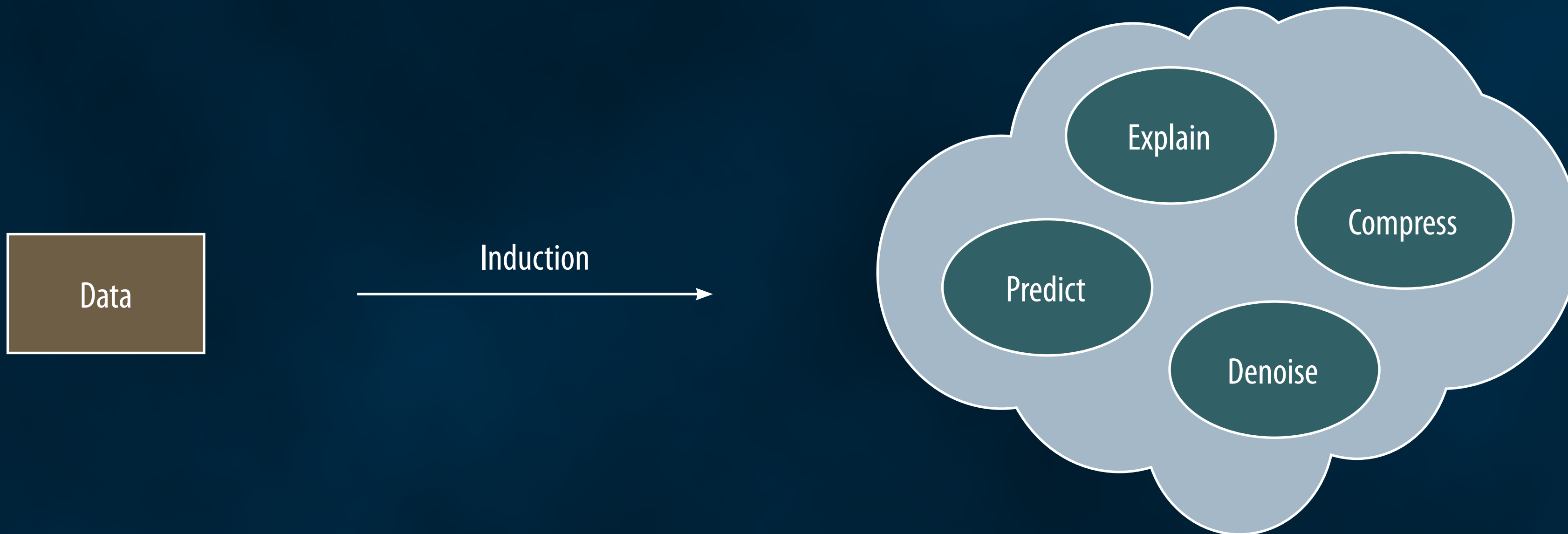
  - Information theory (e.g. MDL, MML)



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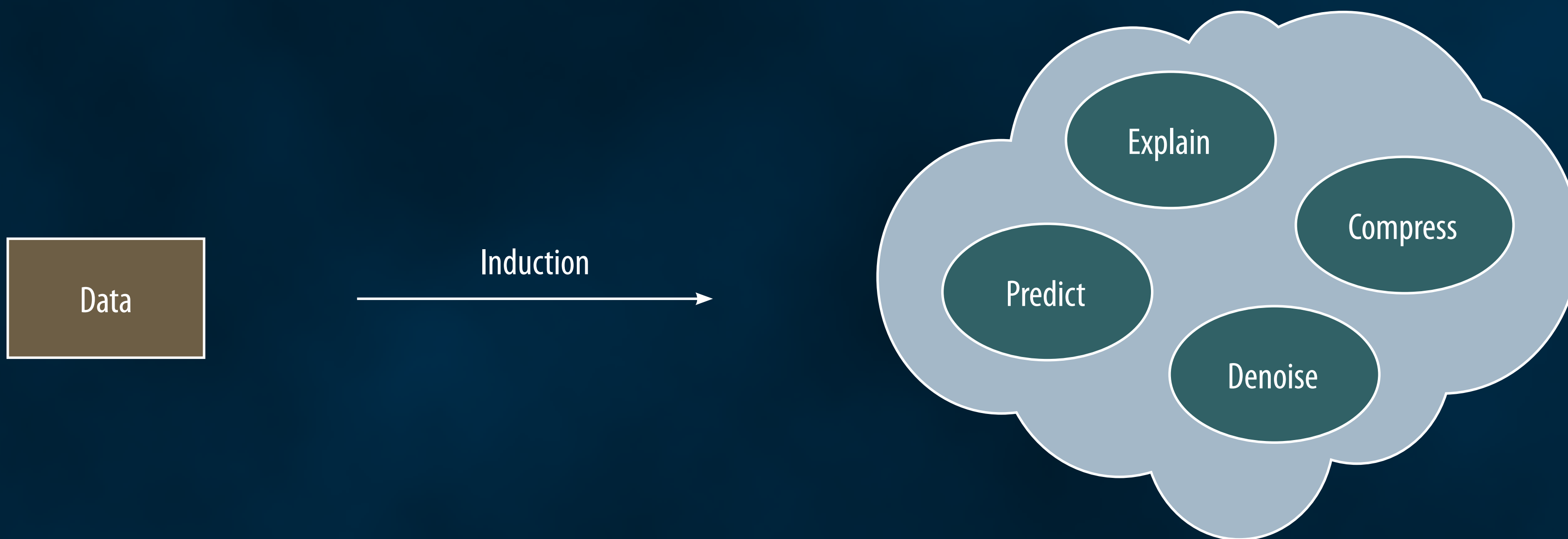

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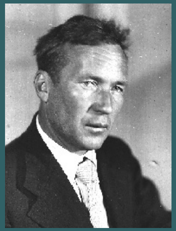
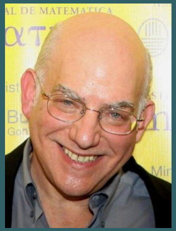
Kolmogorov complexity

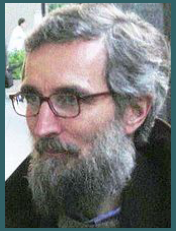
Quantify complexity of binary strings via Turing Machines (early '60s)



A. Kolmogorov



G. Chaitin



P. Martin-Löf

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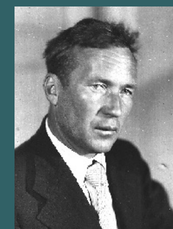

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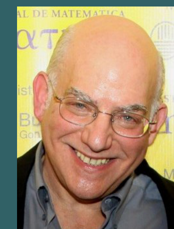

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Define algorithmic probability via Turing Machines and use it for induction (early '60s)

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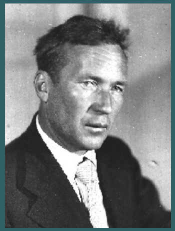
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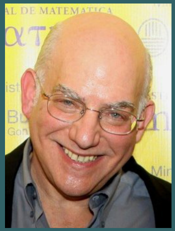
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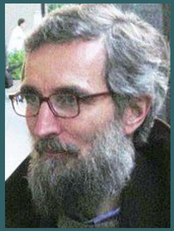
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MML/MDL

Infer a hypothesis about the data via two-part coding (late '60s and '70s)


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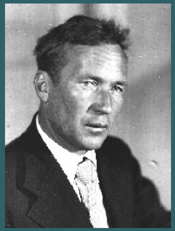
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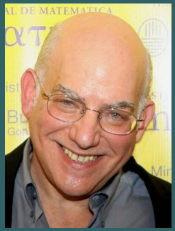
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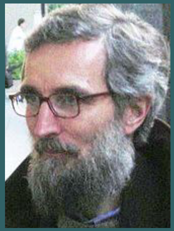
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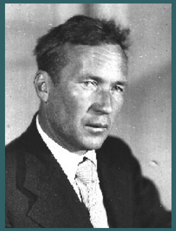
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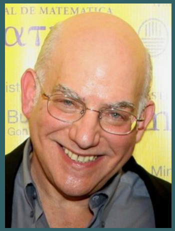
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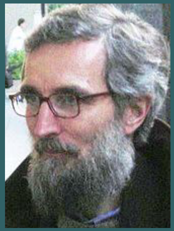
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Infer a hypothesis about the data via two-part coding (late '60s and '70s)


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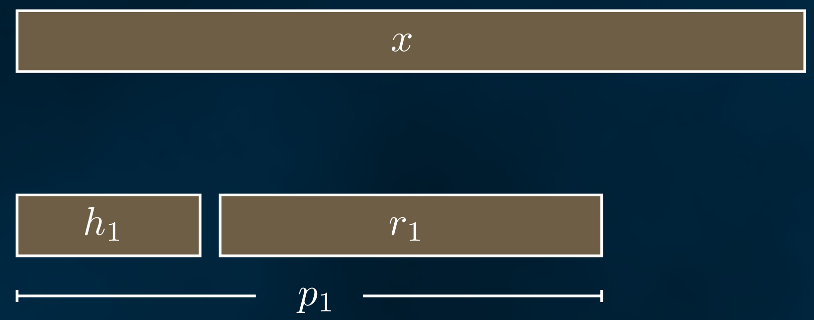
Data string

$\boxed{x}$

Data string

Encode x using a *two-part* scheme



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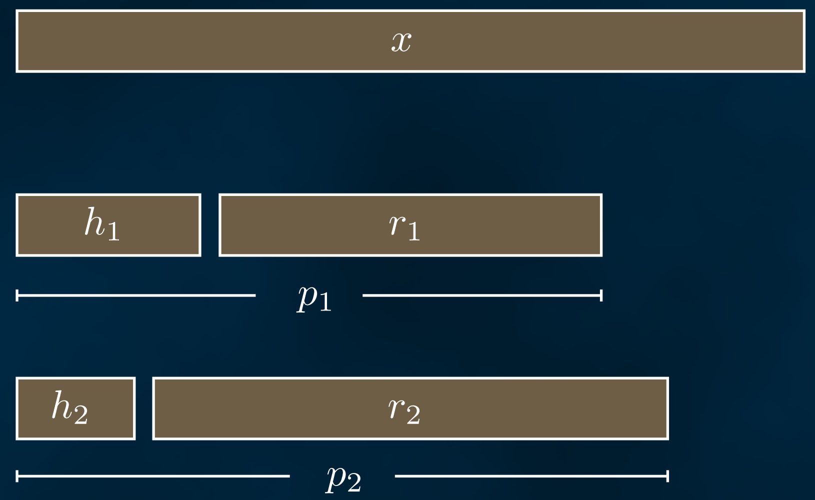

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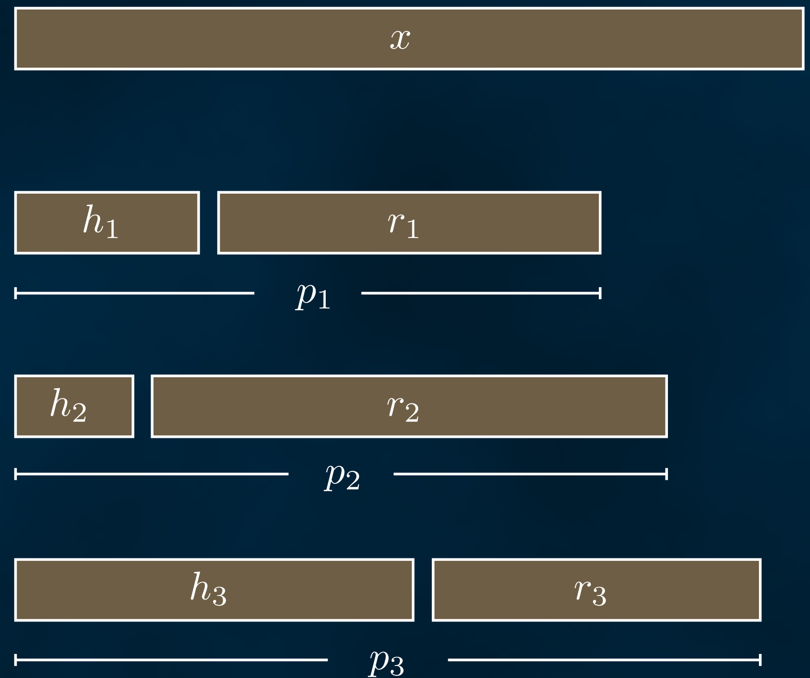

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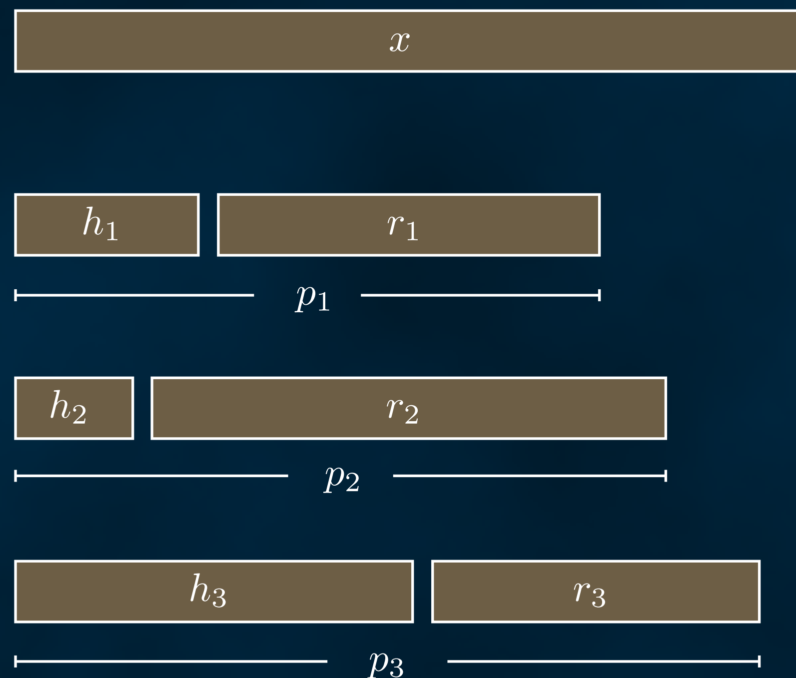
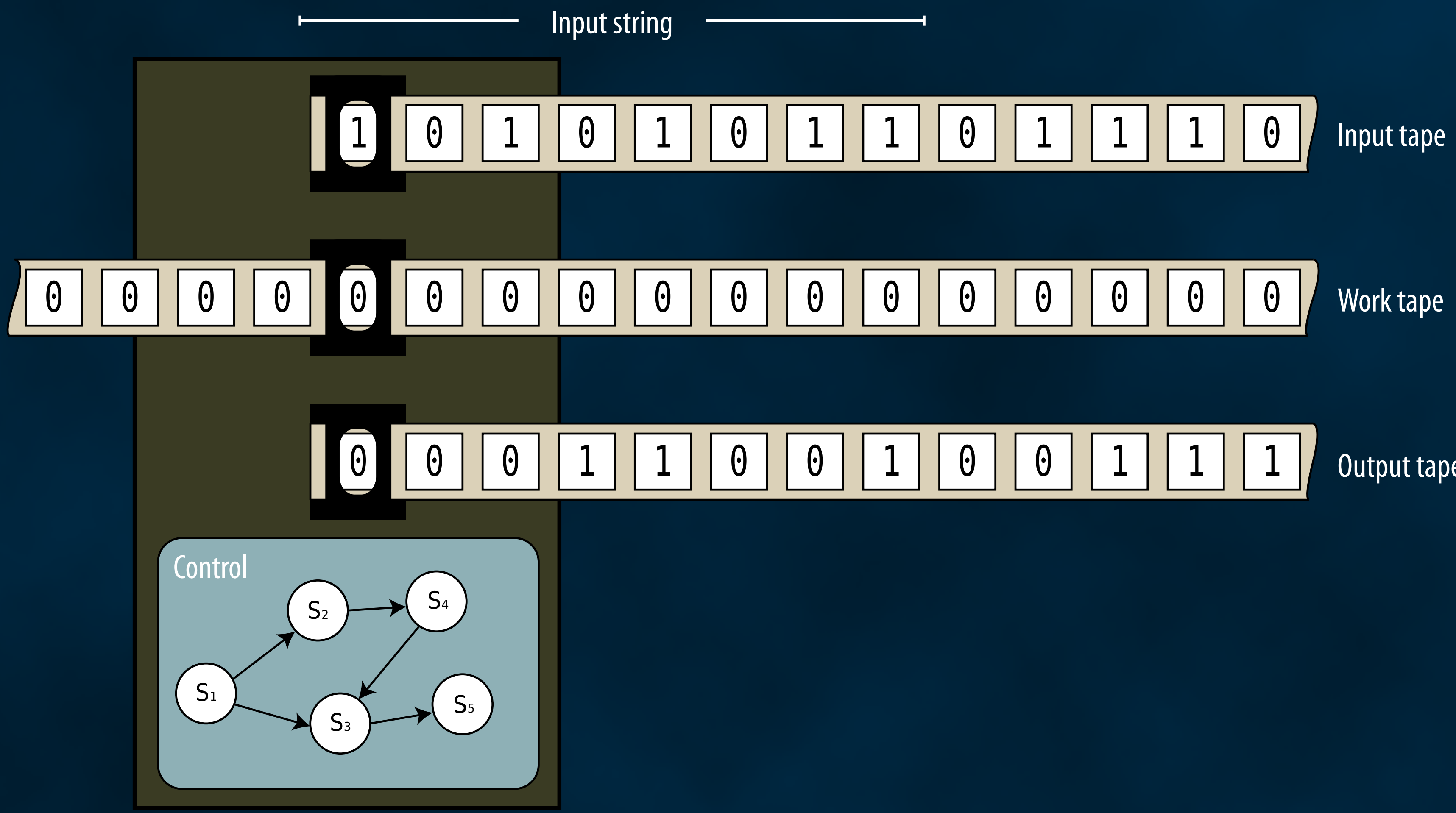
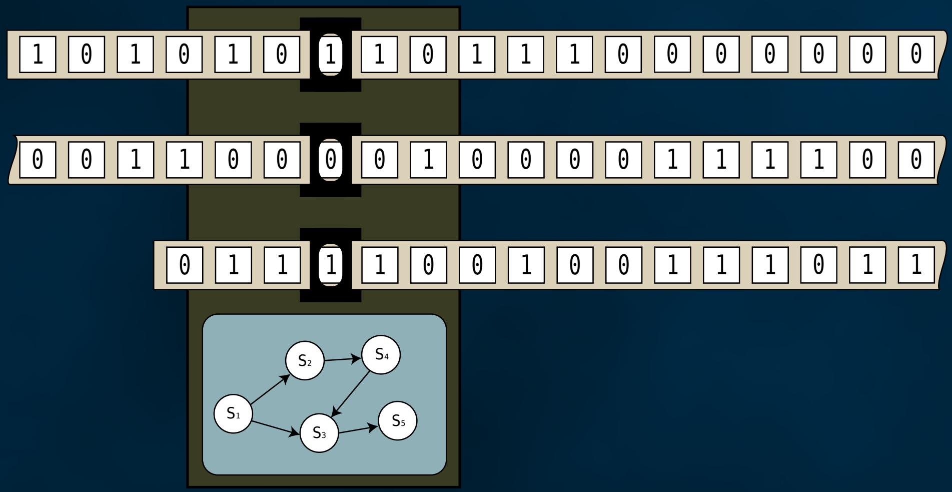


Pick the hypothesis that results in the minimum encoding length

$$l(p_i) = l(h_i) + l(r_i) = -\log_2(p_H(h_i)) - \log_2(p_X(x \mid h_i))$$


Input string
1 0 1 0 1 0 1 1 0 1 1 1 0
Input tape
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
Work tape
0 0 0 1 1 0 0 1 0 0 1 1 1
Output tape
Control
S1
S2
S3
S4
S5

$T(p) = x$ if


1 0 1 0 1 0 1 1 0 1 1 1 0 0 0 0 0 0 0
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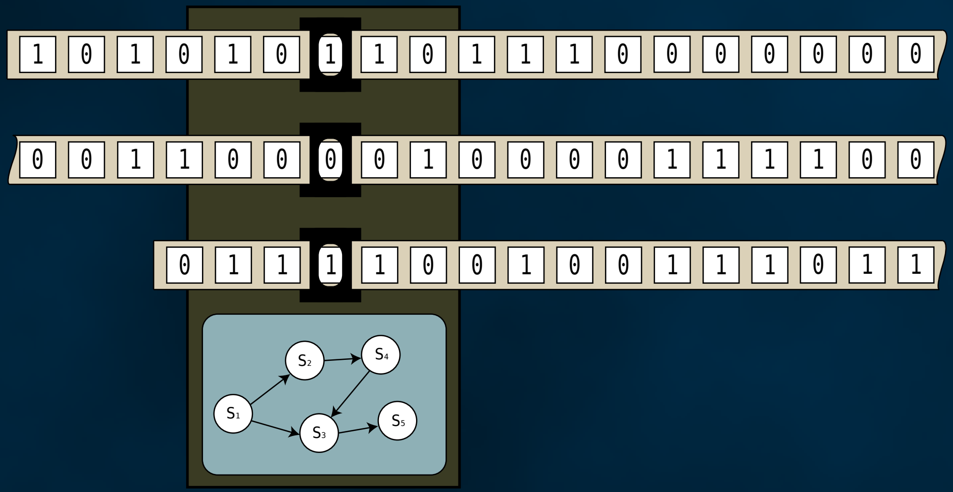

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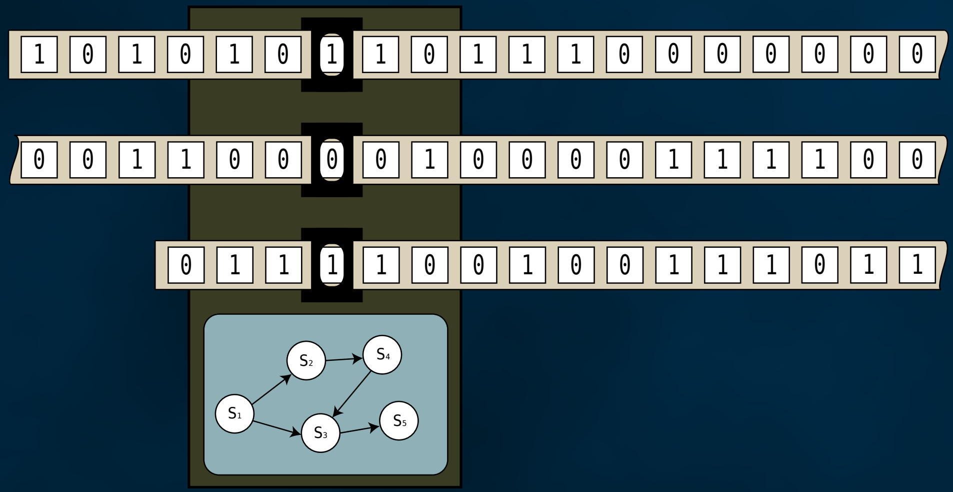

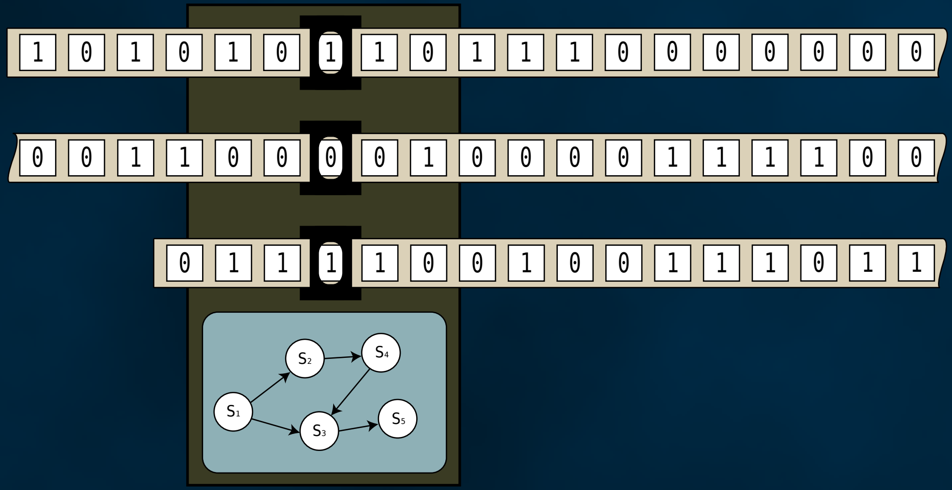

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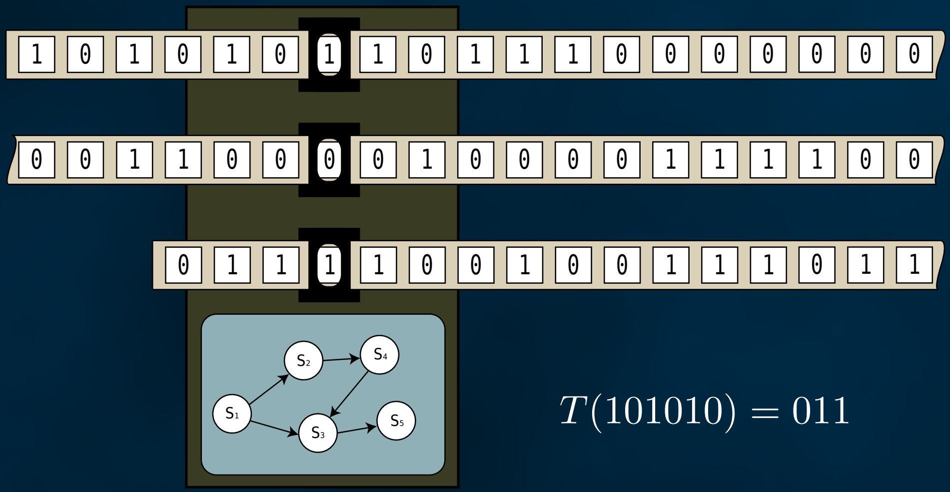

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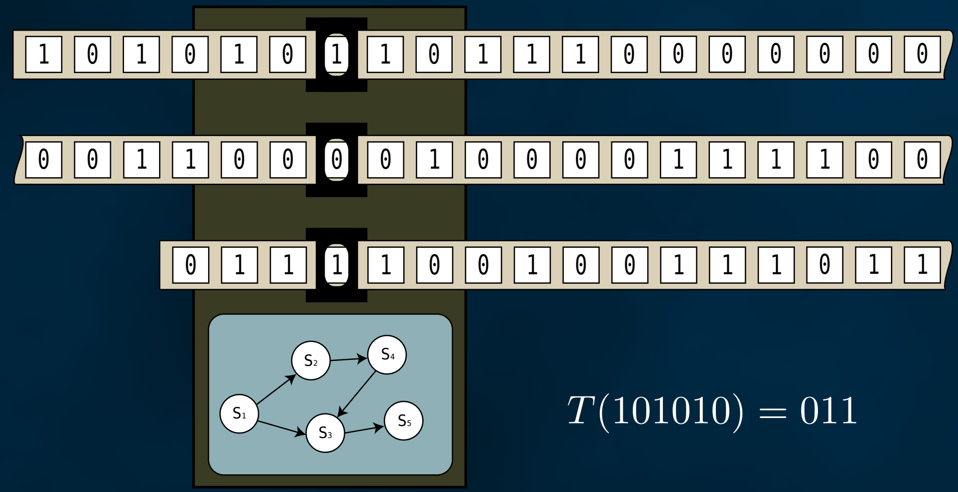

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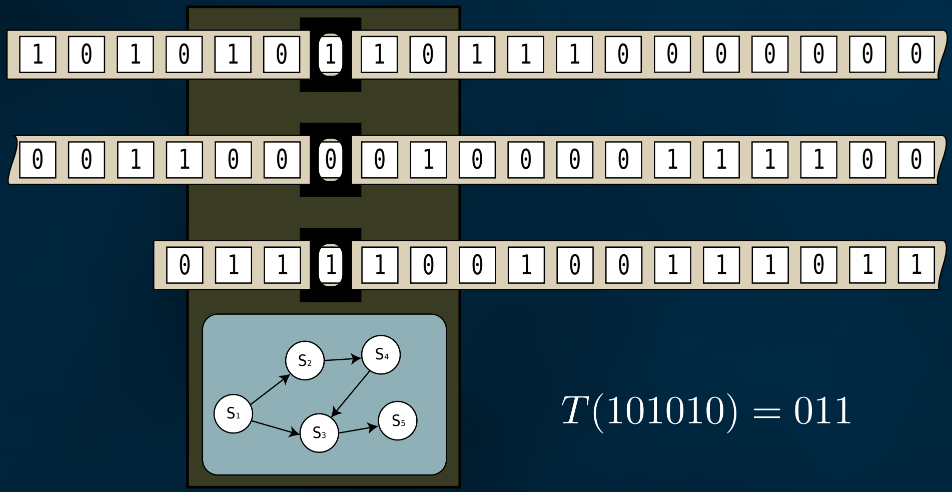
T is a decoder of a prefix code (why prefix?)



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Universal (prefix) TM = TM that can emulate any other (prefix) TM, e.g. $T(\langle i, p \rangle) = T_i(p)$

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$$K_T(x) = \min\{l(p) \mid T(p) = x\}$$

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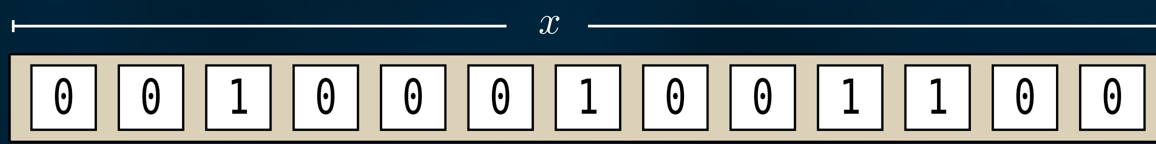
(Hereafter: "probability" $\longleftrightarrow$ semimeasure)

Data string x is a representation of observational data from a real world phenomenon



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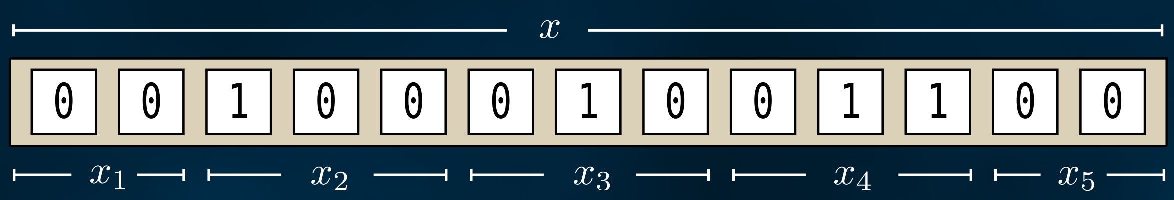

$$L = \{00, 100, 010, 011\}$$

- "Sentences" $x_i \in L$, where L is a prefix-free set (data "language")

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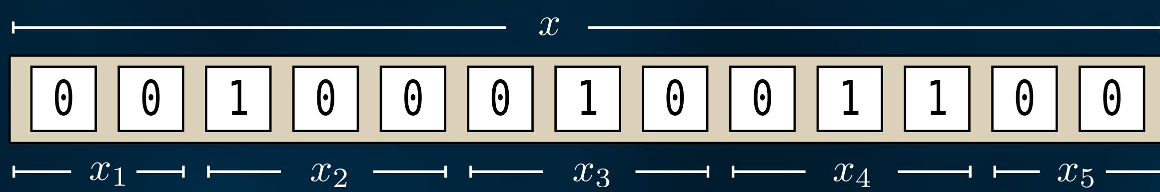

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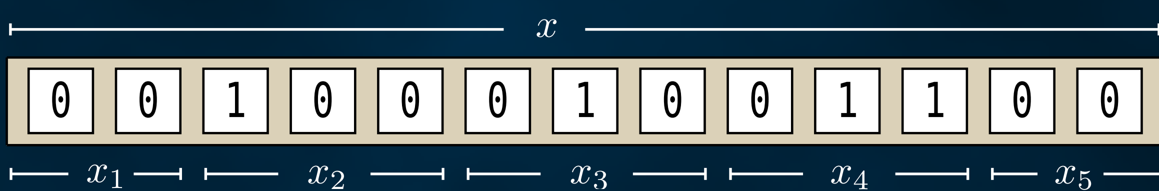

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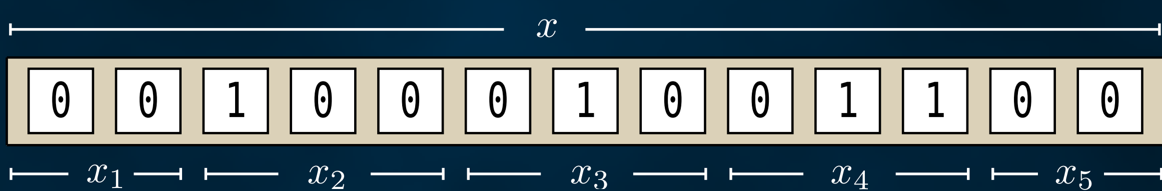

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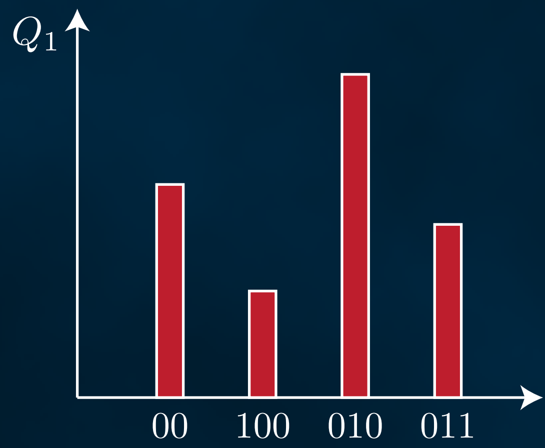

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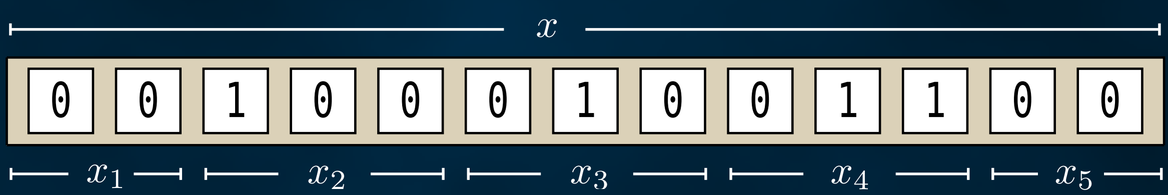

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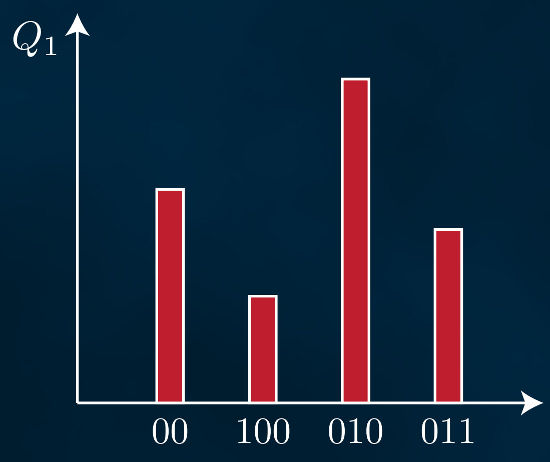

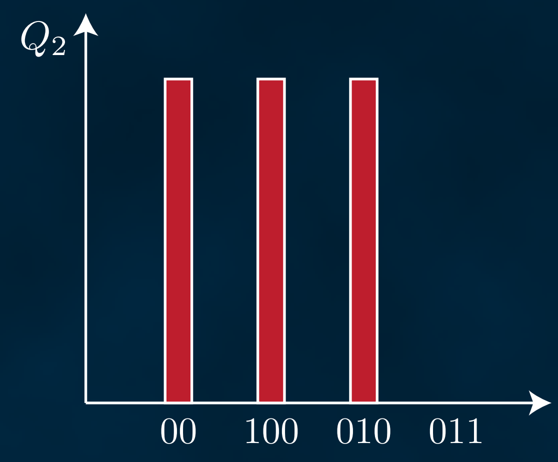

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Idea

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- Force intended two-part encoding by imposing conditions on p

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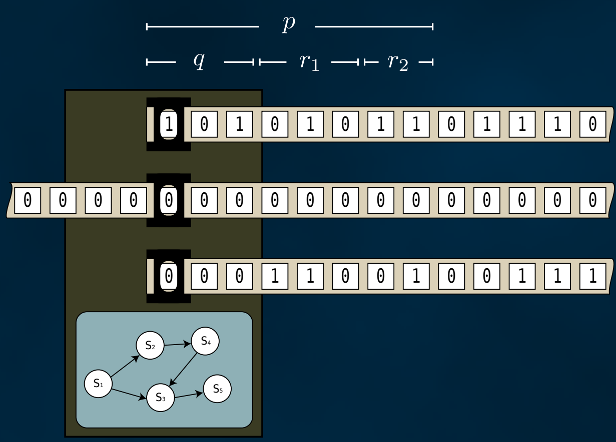
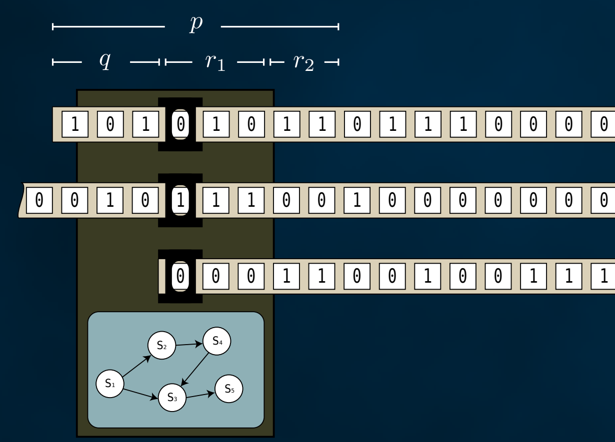
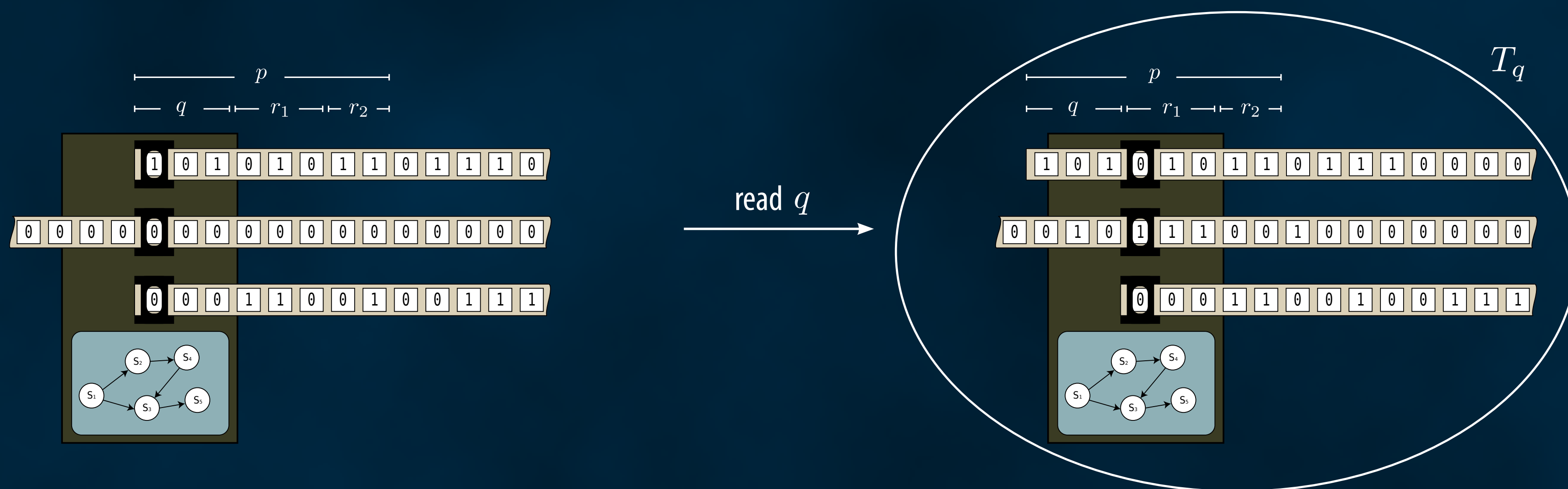


read q $\longrightarrow$




p
q
r1
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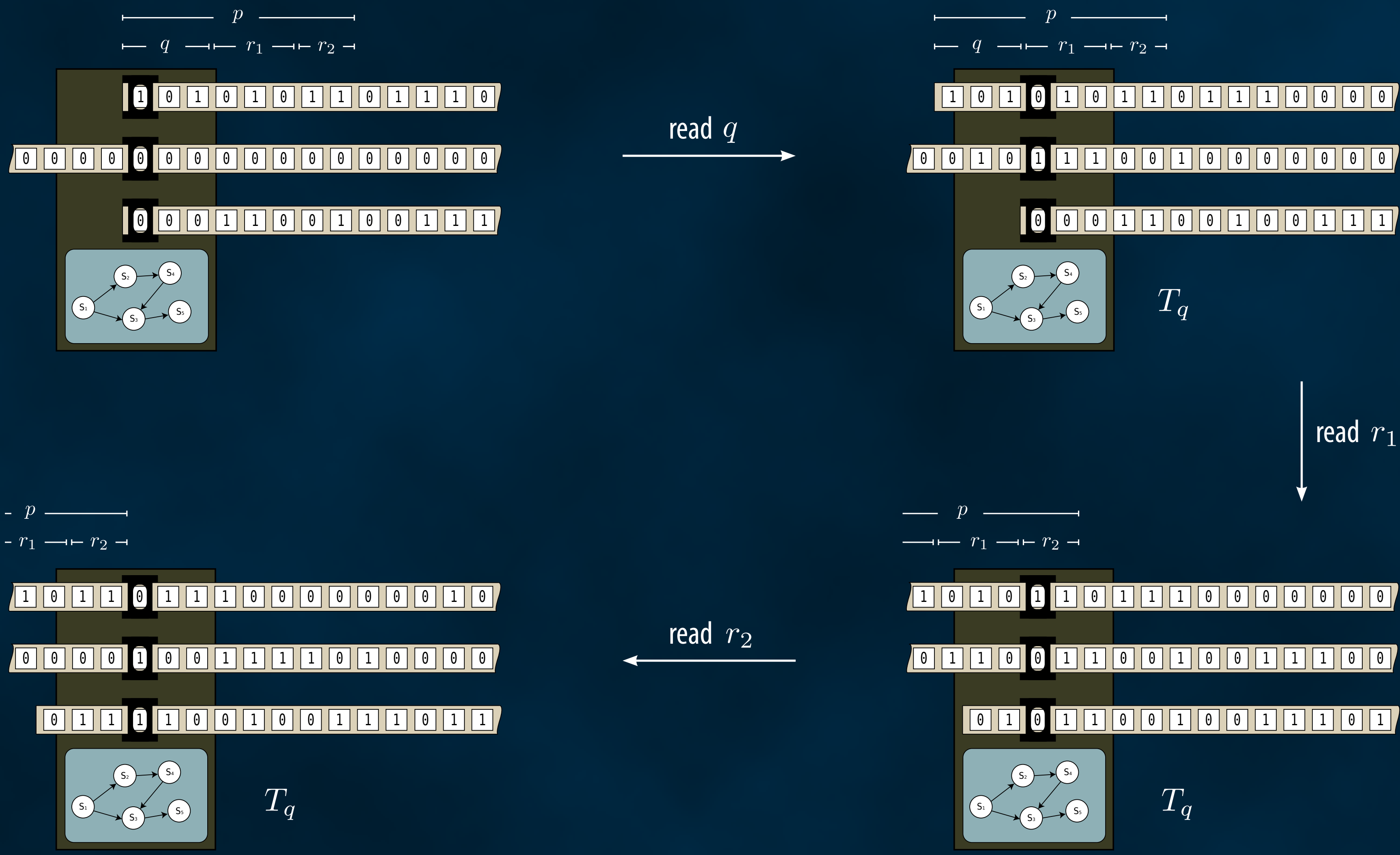
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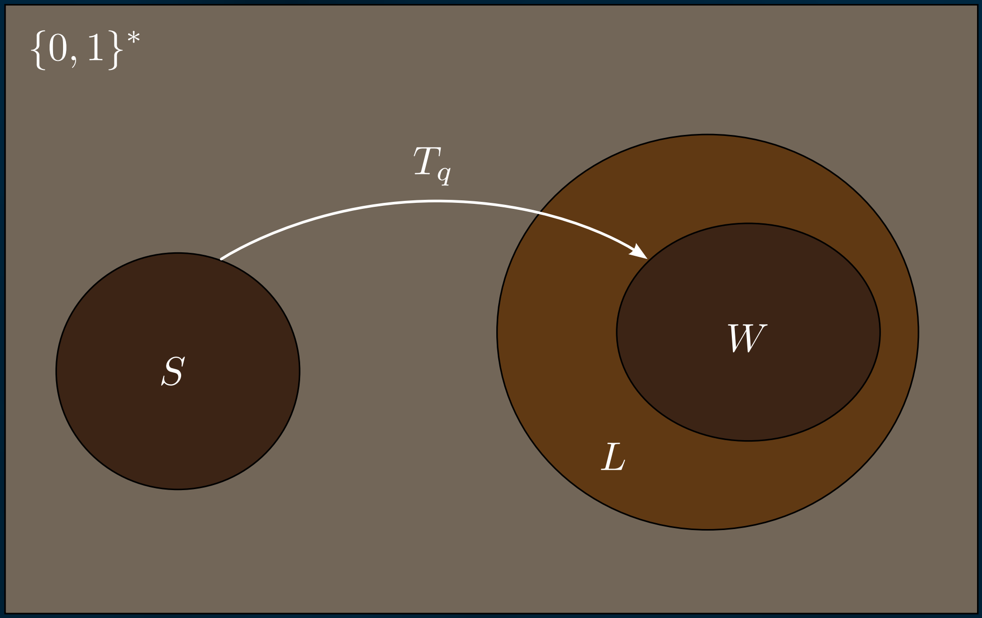

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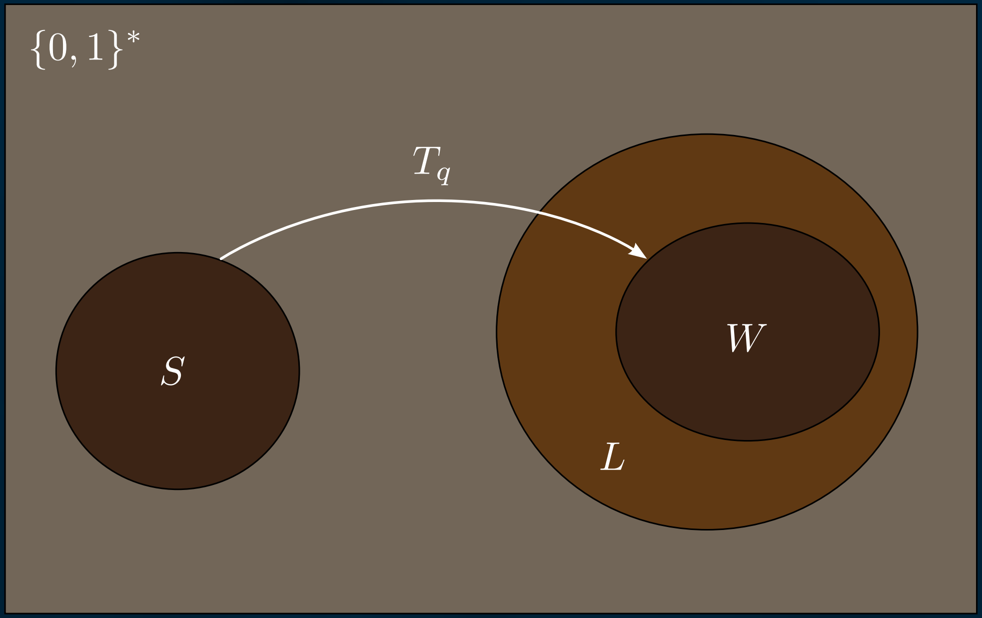
In fact, T_q decodes a prefix code (why?)



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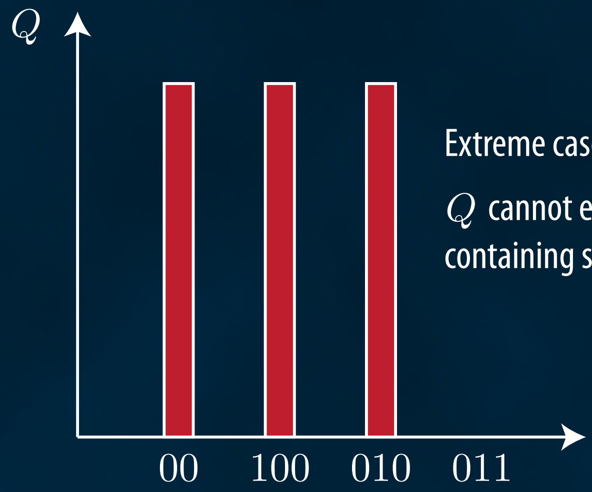

Extreme case:

Q cannot encode *any* data string containing sentence 011.

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where p_1 and p_2 are shortest input strings for their respective hypotheses

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> *[...] Bayes/Solomonoff is the Gold standard for prediction, but MML/MDL is (often) a good "approximation/simplification" for explanation and understanding.*
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> M. Hutter

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  ART

  Handcraft priors,
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- Remember $\log_2\left(\frac{\Pr(Q_1 \mid x)}{\Pr(Q_2 \mid x)}\right) = l(p_2) - l(p_1)$

  Example $l(p_2) - l(p_1) = 10 \Rightarrow \Pr(Q_1 \mid x) = 1024 \times \Pr(Q_2 \mid x)$

- In practice, one has to choose priors very carefully in order to avoid unwanted biases (authors claim that MML school is more considerate in this regard than Kolmogorov and MDL ones)

- Maybe time to change the paradigm?

ART		SCIENCE
Handcraft priors, parameter encodings, features, etc.	$\longrightarrow$	Automatically select suitable priors, encodings, features, etc.

T H A N K Y O U !