Discrete Structure

# Chapter 2: Basic Structures: Sets, Functions

## Sets



## 2.1 Sets

DEFINITION 1

- A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that $a$ is an element of the set $A$. The notation $a \in A$ denotes that $a$ is not an element of the set $A$.

- a  A: a is an element of the set A.
- a  A: a is not an element of the set A.
- Note: lower case letters are used to denote elements.

# 2.1 Sets

- Ways to describe a set:
  - Use { … }
    - E.g. {a, b, c, d} – A set with four elements.
      - V = {a, e, i, o, u} – The set V of all vowels in English alphabet.
      - O = {1, 3, 5, 7, 9} – The set O of odd positive integers less than 10.
      - {1, 2, 3, …, 99} – The set of positive integers less than 100.
  - Use set builder notation: characterize all the elements in the set by stating the property or properties.
    - E.g. O = { x | x is an odd positive integer less than 10}
      - O = { $x \in Z^+$| x is odd and x < 10}



# 2.1 Sets

  - Commonly accepted letters to represent sets
    - **N** = {0, 1, 2, 3, …}, the set of natural numbers
    - **Z** = {…, -2, -1, 0, 1, 2, …}, the set of integers
    - **$Z^+$** = {1, 2, 3, …}, the set of positive integers
    - **Q** = {p/q | $p \in Z$, $q \in Z$, and $q \neq 0$}, the set of rational numbers
    - **R**, the set of real numbers
    - **R+**, the set of **positive real numbers**
    - **C**, the set of **complex numbers**.
- Sets can have other sets as members
  - Example: The set {N, Z, Q, R} is a set containing four elements, each of which is a set
    - The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers. .

# 2.1 Sets

DEFINITION 3
Two sets are equal if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B).$$

We write A = B if A and B are equal sets.

∃

- Example:
  - Are sets {1, 3, 5} and {3, 5,1} equal?
  - Are sets {1, 3, 3, 3, 5, 5, 5, 5} and {1, 3, 5} equal?

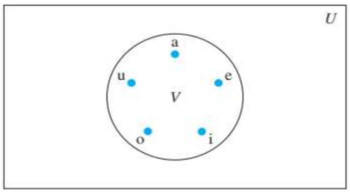
The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements. ◀

# Venn Diagrams

- Represent sets graphically
- The universal set U, which contains all the objects under consideration, is represented by a rectangle. The set varies depending on which objects are of interest.
- Inside the rectangle, circles or other geometrical figures are used to represent sets.
- Sometimes points are used to represent the particular elements of the set.

**EXAMPLE 7** Draw a Venn diagram that represents $V$, the set of vowels in the English alphabet.

*Solution:* We draw a rectangle to indicate the universal set $U$, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent $V$. Inside this circle we indicate the elements of $V$ with points (see Figure 1). ◀



# Empty Set and Singleton set

- Empty Set (null set): a set that has no elements, denoted by $\phi$ or {}.
- Example: The set of all positive integers that are greater than their squares is an empty set.
- Singleton set: a set with one element
- Compare: $\phi$ and $\{\phi\}$
  - $\Phi$: an empty set. Think of this as an empty folder
  - $\{\phi\}$: a set with one element. The element is an empty set. Think of this as an folder with an empty folder in it.

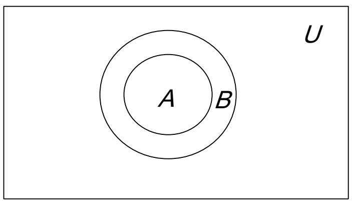


# Subsets

DEFINITION 4
The set A is said to be a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

- $A \subseteq B$ if and only if the quantification

  $\forall x(x \in A \rightarrow x \in B)$ is true




- $A \not\subseteq B$ if and only if the quantification
  - we need only find one element $x \in A$ with $x \notin B$

# 2.1 Sets

- every non-empty set S is guaranteed to have at least two subset, the empty set and the set S itself, that is $\phi \subseteq S$ and $S \subseteq S$.

THEOREM 1
For every set S,
(i) $\phi \subseteq S$ and (ii) $S \subseteq S$

- If A is a subset of B but $A \neq B$, then $A \subset B$ or A is a proper subset of B.
- For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A, i.e.

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

*Showing Two Sets are Equal* To show that two sets $A$ and $B$ are equal, show that $A \subseteq B$ and $B \subseteq A$.



# The Size of a Set

DEFINITION 5
Let *S* be a set. If there are exactly *n* distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is a *finite* set and that *n* is the *cardinality* of *S*. The cardinality of *S* is denoted by |*S*|.

- Example:
  - Let A be the set of odd positive integers less than 10. Then |A| = 5.
  - Let S be the set of letters in the English alphabet. Then |A| = 26.
  - Null set has no elements, | $\phi$ | = 0.

**Infinite Set**

DEFINITION 6
A set is said to be *infinite* if it is not finite.

- Example: The set of positive integers is infinite.

# Power Sets

**DEFINITION 7**
Given a set, the *power set* of $S$ is the set of all subsets of the set $S$. The power set of $S$ is denoted by $P(S)$.

- Example:
  - What is the power set of the set {0,1,2}?
    Solution: P({0,1,2}) = {$\phi$, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}}
  - What is the power set of the empty set? What is the power set of the set {$\phi$}?
    Solution: The empty set has exactly one subset, namely, itself.
    P($\phi$ ) = {$\phi$}
    The set {$\phi$} has exactly two subsets, namely, $\phi$ and the set {$\phi$}.
    P({$\phi$}) = {$\phi$, {$\phi$}}
- If a set has $n$ elements, its power set has $2^n$ elements.



# Cartesian Products

- Sets are unordered, a different structure is needed to represent an ordered collections – ordered n-tuples.

**DEFINITION 8**
The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as its second element, …, and $a_n$ as its $n$th element.

- Two ordered $n$-tuples are equal if and only if each corresponding pair of their elements is equal.
  - $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, \ldots, n$

## Cartesian Products

DEFINITION 9
Let $A$ and $B$ be sets. The *Cartesian product* of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}.$$

- Example:

  What is the Cartesian product of A = {1,2} and B = {a,b,c}?

  Solution:

  $A \times B$ = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}
- Cartesian product of $A \times B$ and $B \times A$ are not equal, unless $A = \phi$ or $B = \phi$ (so that $A \times B = \phi$ ) or $A = B$.

  $B \times A$ = {(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)}



## Cartesian Products

DEFINITION 10
The *Cartesian product* of sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i$ belongs to $A_i$ for $i = 1,2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1,2, \ldots, n\}.$$

- Example:

  What is the Cartesian product of A × B × C where A= {0,1}, B = {1,2}, and C = {0,1,2}?

  Solution:

  $A \times B \times$ C= {(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

# Cartesian Products

- Notes:
  - When A, B, and C are sets, (A × B) × C is not the same as A × B × C
  - We use the notation $A^2$ to denote $A \times A$, the Cartesian product of the set $A$ with itself. Similarly,
    - $A^3 = A \times A \times A$
    - $A^4 = A \times A \times A \times A$, and so on. More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

**EXAMPLE 20** Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. ◀



# Relation

- A subset $R$ of the Cartesian product $A \times B$ is called a **relation** from the set $A$ to the set $B$.
  - The elements of $R$ are ordered pairs, where the first element belongs to $A$ and the second to $B$.
  - For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$.
  - A relation from a set $A$ to itself is called a relation on $A$.

**EXAMPLE :**
What are the ordered pairs in the less than or equal to relation, which contains $(a, b)$ if $a \leq b$, on the set $\{0, 1, 2, 3\}$?
Solution:
$R$ are (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and (3, 3)

# Set Notation with Quantifiers

- Using Set Notation with Quantifiers
  - *$\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$*
  - *$\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \land P(x))$*
  - Example:

    What do the statements $\forall x \in R(x^2 \geq 0)$ mean?

    Solution:

    For every real number x, $x^2 \geq 0$. "The square of every real number is nonnegative.
- The truth set of *P* is the set of elements x in *D* for which *P(x)* is true. It is denoted by $\{x \in D \mid P(x)\}$.
  - Example: What is the truth set of the predicate *P(x)* where the domain is the set of integers and *P(x)* is "$|x| = 1$"?

    Solution: {-1,1}



# Set Notation with Quantifiers

**Example**

The truth set of $Q$, $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers $x$ for which $x^2 = 2$.

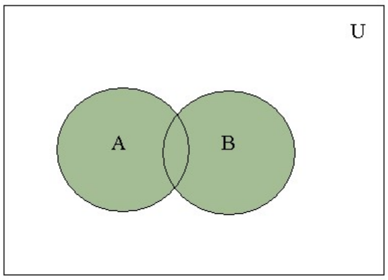
The truth set of $R$, $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of $R$ is $\mathbf{N}$, the set of nonnegative integers.

# 2.2 Set Operations

**The Union Of The Sets**

DEFINITION 1
Let $A$ and $B$ be sets. The *union* of the sets $A$ and $B$, denoted by $A \cup B$, is the set that contains those elements that are either in $A$ or in $B$, or in both.

- $A \cup B = \{ x \mid x \in A \lor x \in B \}$




- Shaded area represents $A \cup B$.



# 2.2 Set Operations

- Example:
  - The union of the sets {1,3,5} and {1,2,3} is the set {1,2,3,5}; that is {1,3,5} U {1,2,3} = {1,2,3,5}
  - The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both).
  - SQL command when retrieving data from the student database:

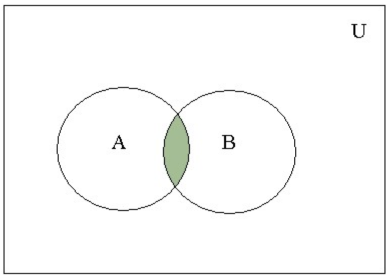
```
select * from student
where major ='cs'
UNION
select * from student
where major = 'math'
```

# 2.2 Set Operations

**The Intersection Of The Sets**

DEFINITION 2
Let $A$ and $B$ be sets. The *intersection* of the sets $A$ and $B$, denoted by $A \cap B$, is the set containing those elements in both $A$ and $B$.

- $A \cap B = \{ x \mid x \in A \wedge x \in B \}$




- Shaded area represents $A \cap B$.



# 2.2 Set Operations

- Example:
  - The intersection of the sets {1,3,5} and {1,2,3} is the set {1,3}; that is {1,3,5} ∩ {1,2,3} = {1,3}
  - The intersection of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are joint majors in mathematics and in computer science.
  - SQL command when retrieving data from the student database:

```
select *
from csMajor, mathMajor
where csMajor.studentID = mathMajor.studentID
```

# 2.2 Set Operations

**The Disjoint Of The Sets**

DEFINITION 3
Two sets are called *disjoint* if their intersection is the empty set.

- Example: Let $A = \{1,3,5,7,9\}$ and $B = \{2,4,6,8,10\}$. Because $A \cap B = \phi$, $A$ and $B$ are disjoint.

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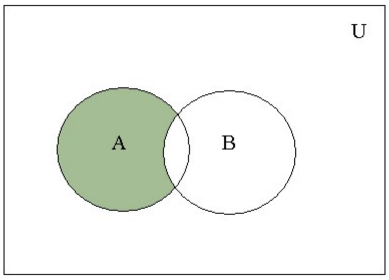
# 2.2 Set Operations

**The Difference of A and B,**

DEFINITION 4
Let $A$ and $B$ be sets. The *difference* of $A$ and $B$, denoted by $A - B$, is the set containing those elements that are in $A$ but not in $B$. The difference of $A$ and $B$ is also called the *complement* of $B$ *with respect to* $A$.

- $A - B = \{ x \mid x \in A \land x \notin B \}$




- $A - B$ is shaded.

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# 2.2 Set Operations

- Example:
  - {1,3,5} - {1,2,3} = {5}
  - {1,2,3} – {1,3,5} = {2}
  - The difference of the set of computer science majors at your school and the set of mathematics majors at your school is the set of all computer science majors at your school who are not mathematics majors.
  - SQL command when retrieving data from the student database:

```
select *
from csMajor
where csMajor.studentID NOT IN (select studentID from mathMajor)
```



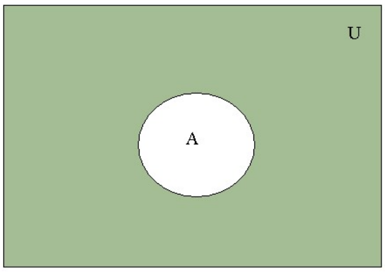
# 2.2 Set Operations

## The Complement Of The Set

DEFINITION 5

Let $U$ be the universal set. The *complement* of the set $A$, denoted by $\bar{A}$, is the complement of $A$ with respect to $U$. In other words, the containing those complement of the set $A$ is $U - A$.

- $\bar{A} = \{ x \mid x \notin A \}$




- $\bar{A}$ is shaded.

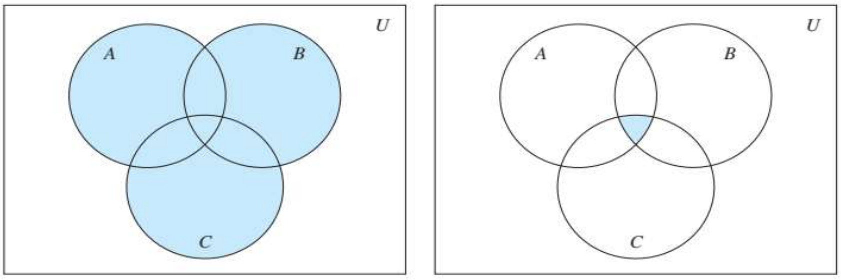
# 2.2 Set Operations

- Example:
  - Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers.) Then $\bar{A}$ = {1,2,3,4,5,6,7,8,9,10}



**TABLE 1 Set Identities.**

| *Identity* | *Name* |
|---|---|
| $A \cap U = A$<br>$A \cup \emptyset = A$ | Identity laws |
| $A \cup U = U$<br>$A \cap \emptyset = \emptyset$ | Domination laws |
| $A \cup A = A$<br>$A \cap A = A$ | Idempotent laws |
| $\overline{(\overline{A})} = A$ | Complementation law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ | Commutative laws |
| $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ | Associative laws |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributive laws |
| $\overline{A \cap B} = \overline{A} \cup \overline{B}$<br>$\overline{A \cup B} = \overline{A} \cap \overline{B}$ | De Morgan's laws |
| $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ | Absorption laws |
| $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \emptyset$ | Complement laws |




(a) $A \cup B \cup C$ is shaded.   (b) $A \cap B \cap C$ is shaded.

**FIGURE 5** **The Union and Intersection of $A$, $B$, and $C$.**

EXAMPLE

Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

*Solution:* The set $A \cup B \cup C$ contains those elements in at least one of $A$, $B$, and $C$. Hence,

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of $A$, $B$, and $C$. Thus,

$$A \cap B \cap C = \{0\}.$$



# 2.2 Set Operations

## Computer Representation of Sets

- Represent a subset $A$ of U with the bit string of length $n$, where the $i$th bit in the string is 1 if $a_i$ belongs to $A$ and is 0 if $a_i$ does not belong to $A$.
- Example:
  - Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, and the ordering of elements of $U$ has the elements in increasing order; that is $a_i = i$.

    What bit string represents the subset of all odd integers in $U$?

    Solution: 10 1010 1010

    What bit string represents the subset of all even integers in $U$?

    Solution: 01 010 10101

    What bit string represents the subset of all integers not exceeding 5 in $U$?

    Solution: 11 1110 0000

    What bit string represents the complement of the set {1,3,5,7,9}?

    Solution: 01 0101 0101

## 2.2 Set Operations

- The bit string for the union is the bitwise *OR* of the bit string for the two sets. The bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.
- Example:
  - The bit strings for the sets {1,2,3,4,5} and {1,3,5,7,9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

    Solution:

    Union:

    11 1110 0000 $\vee$ 10 1010 1010 = 11 1110 1010, {1,2,3,4,5,7,9}

    Intersection:

    11 1110 0000 $\wedge$ 10 1010 1010 = 10 1010 0000, {1,3,5}