Predicates and Quantification

Predicates

- statement involving one or more variables,
- e.g.: x-3 > 5.
- Let us call this propositional function P(x), where P is the predicate and x is the variable.

What is the truth value of P(2) ? false

What is the truth value of P(8) ? false

What is the truth value of P(9) ? true

Predicates

- Let us consider the propositional function Q(x, y, z) defined as:
- x + y = z.
- Here, Q is the predicate and x, y, and z are the variables.

What is the truth value of Q(2, 3, 5) ? true

What is the truth value of Q(0, 1, 2) ? false

What is the truth value of Q(9, -9, 0) ? true

Predicates

- Let Q(x, y) denote the statement “x = y + 3.” What are the truth values of the propositions
- Q(1, 2) and Q(3, 0) defined as:
- Here, Q is the predicate and x, y are the variables.

What is the truth value of Q(3, 0) ? true

What is the truth value of Q(1, 2) ? false

What is the truth value of Q(5, 2) ? true

Quantifiers

- **Quantification** is important way to create a proposition from a propositional function.
- Quantification expresses the extent to which a predicate is true over a range of elements.
- In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications.
- We will focus on two types of quantification here:
  - universal quantification, which tells us that a predicate is true for every element under consideration, and
  - existential quantification, which tells us that there is one or more element under consideration for which the predicate is true.
- The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Universal Quantification

•Let P(x) be a propositional function.

•**Universally quantified sentence**:

•For all x in the universe of discourse P(x) is true.

•Using the universal quantifier $\forall$:

•$\forall$x P(x) "for all x P(x)" or "for every x P(x)"

•(Note: $\forall$x P(x) is either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Definition

The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

Universal Quantification

•Example:

•S(x): x is a UMBC student.

•G(x): x is a genius.

•What does $\forall$x (S(x) $\rightarrow$ G(x)) mean ?

•"If x is a UMBC student, then x is a genius."

•or

•"All UMBC students are geniuses."

Example

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus

$$\forall x Q(x)$$

is false. ◀

Example

What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false. ◀

Example

What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 \ngeq \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$. ◀

Existential Quantification

- **Existentially quantified sentence**:
- There exists an x in the universe of discourse for which P(x) is true.

- Using the existential quantifier ∃:
- ∃x P(x) "There is an x such that P(x)."
- "There is at least one x such that P(x)."

- (Note: ∃x P(x) is either true or false, so it is a proposition, but no propositional function.)

Existential Quantification

Definition

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

Existential Quantification

- Example:
- P(x): x is a UMBC professor.
- G(x): x is a genius.

- What does ∃x (P(x) ∧ G(x)) mean ?

- "There is an x such that x is a UMBC professor and x is a genius."
- or
- "At least one UMBC professor is a genius."

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Example

What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is $\{1, 2, 3, 4\}$, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true. ◀

Example

Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false. ◀

Disproof by Counterexample

- A counterexample to ∀x P(x) is an object c so that P(c) is false.

- Statements such as ∀x (P(x) → Q(x)) can be disproved by simply providing a counterexample.

Statement: "All birds can fly."
Disproved by counterexample: Penguin.

Negation

- ¬(∀x P(x)) is logically equivalent to ∃x (¬P(x)).
- ¬(∃x P(x)) is logically equivalent to ∀x (¬P(x)).

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain.

Nested Quantifiers

- Let the universe of discourse be the real numbers.
- What does ∀x∃y (x + y = 320) mean ?
- "For every x there exists a y so that x + y = 320."

Is it true? yes

Is it true for the natural numbers? no

Nested Quantifiers

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x\forall y P(x, y)$ $\forall y\forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x\exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x\forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x\exists y P(x, y)$ $\exists y\exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Proof Methods For Implications

- for proving implication p →q, we have :
  1. Direct Proof: Assume P is true and proof q
  2. Indirect proof: Assume ¬q and proof ¬p. it is called proof by **Contraposition**

Basic Number Theory Definitions
from Chapters 1.6, 2

- Z = Set of all Integers
- Z+ = Set of all Positive Integers
- N = Set of Natural Numbers (Z+ and Zero)
- R = Set of Real Numbers
- Addition and multiplication on integers produce integers. $(a,b \in Z) \rightarrow [(a+b) \in Z] \wedge [(ab) \in Z]$

Direct Proof Example

- **Definition:** An integer n is called *odd* if and only if $n=2k+1$ for some integer k; n is *even* if and only if $n=2k$ for some k.
- **Theorem:** (For all numbers n) If n is an odd integer, then n^2 is an odd integer.
- **Proof:** If n is odd, then $n = 2k+1$ for some integer k. Thus, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore n^2 is of the form $2j + 1$ (with j the integer $2k^2 + 2k$), thus n^2 is odd. □

Indirect Proof Example

- **Theorem:** (For all integers n)
  If $3n+2$ is odd, then n is odd.
- **Proof:** Suppose that the conclusion is false, *i.e.*, that n is even. Then $n=2k$ for some integer k. Then $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$. Thus $3n+2$ is even, because it equals $2j$ for integer $j = 3k+1$. So $3n+2$ is not odd. We have shown that ¬(n is odd)→¬($3n+2$ is odd), thus its contra-positive ($3n+2$ is odd) → (n is odd) is also true. □

Example

Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution: Because there is no obvious way of showing that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ directly from the equation $n = ab$, where a and b are positive integers, we attempt a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false. That is, we assume that the statement $(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false. Using the meaning of disjunction together with De Morgan's law, we see that this implies that both $a \leq \sqrt{n}$ and $b \leq \sqrt{n}$ are false. This implies that $a > \sqrt{n}$ and $b > \sqrt{n}$. We can multiply these inequalities together (using the fact that if $0 < s < t$ and $0 < u < v$, then $su < tv$) to obtain $ab > \sqrt{n} \cdot \sqrt{n} = n$. This shows that $ab \neq n$, which contradicts the statement $n = ab$.

Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. ◀