Eigenvalues and Eigenvectors

Eigenvalues

Definition: If A is a square matrix then the roots of the characteristic equation ($|A-\lambda I|=0$) are called Eigenvalues of A where λ is a scalar

Example : Find the Eigenvalues of the following matrices

$$1)\ A=\begin{bmatrix}2 & 2\\ 5 & -1\end{bmatrix} \qquad 2)\ A=\begin{bmatrix}7 & 0 & -3\\ -9 & -2 & 3\\ 18 & 0 & -8\end{bmatrix}$$

Solution :

$$1)\ A=\begin{bmatrix}2 & 2\\ 5 & -1\end{bmatrix}$$

The eigenvalues are those λ for which $\det(\mathbf{A}-\lambda\mathbf{I})=0$. Now

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= \det\left(\begin{bmatrix}2 & 2\\ 5 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\\ &= \det\left(\begin{bmatrix}2 & 2\\ 5 & -1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)\\ &= \begin{vmatrix}2-\lambda & 2\\ 5 & -1-\lambda\end{vmatrix}\\ &= (2-\lambda)(-1-\lambda)-10\\ &= \lambda^2-\lambda-12.\end{aligned}$$

Now put $\det(A-\lambda I)=0$

$\Longrightarrow (\lambda-4)(\lambda+3)=0$

$\Longrightarrow \lambda=4$ or $\lambda=-3$

$$2)\ A=\begin{bmatrix}7 & 0 & -3\\ -9 & -2 & 3\\ 18 & 0 & -8\end{bmatrix}$$

First we compute $\det(\mathbf{A}-\lambda\mathbf{I})$ via a cofactor expansion along the second column:

$$\begin{aligned}\begin{vmatrix}7-\lambda & 0 & -3\\ -9 & -2-\lambda & 3\\ 18 & 0 & -8-\lambda\end{vmatrix} &= (-2-\lambda)(-1)^4\begin{vmatrix}7-\lambda & -3\\ 18 & -8-\lambda\end{vmatrix}\\ &= -(2+\lambda)[(7-\lambda)(-8-\lambda)+54]\\ &= -(\lambda+2)(\lambda^2+\lambda-2)\\ &= -(\lambda+2)^2(\lambda-1).\end{aligned}$$

Now put $\det(A-\lambda I)=0$

$\Longrightarrow -(\lambda+2)^2(\lambda-1)=0$

$\Longrightarrow \lambda=-2,-2$ or 1

Note : Called Eigenvalues sometimes by proper values and characteristic values called latent values

Eigenvectors

Definition: Let A is a square matrix .If there exist a non zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$, then the vector X is called an Eigenvector of A corresponding to the Eigenvalue λ

Example: Decide whether the following vectors are eigenvectors

$$1)\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ for } \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \quad 2)\ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ for } \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \quad 3)\ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ for } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution:

1) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

Since $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6+2 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$,then $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigenvector where $\lambda = 4$

2) $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ for $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Since $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0-2 \\ -2+2+1 \\ -2+0+3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ then $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is eigenvector where $\lambda = 1$

3) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ for $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$ Exercise

Example: Find the eigenvalues and associated eigenvectors of the following matrices

$$1)\ A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \quad 2)\ A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix} \quad 3)\ A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$$

Solution :

1) $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

The eigenvalues for A are $\lambda = 2$ and $\lambda = 3$ … check that

Now to find the eigenvectors we will find the vectors X that's $(A - \lambda I)X = 0$ when $\lambda = 2$ and $\lambda = 3$

At $\lambda = 2$ we get $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1^{(1)} = -R_1 \\ R_2^{(1)} = -2R_1 + R_2 \end{array}$$

$$x - y = 0$$

Since y is free put $y = t$ where t is any scalar , then $x = t$

Therefore the eigenspace when $\lambda = 2$ is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since t is any scalar take $t = 2$,then the eigenvector is $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

At $\lambda = 3$ we get $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{l} R_1^{(1)} = -\frac{1}{2}R_1 \\ R_2^{(1)} = -R_1 + R_2 \end{array}$$

$$\Rightarrow x - \tfrac{1}{2}y = 0$$

Since y is free put $y = t$ where t is any scalar, then $x = \frac{1}{2}t$

Therefore the eigenspace when $\lambda = 3$ is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Since t is any scalar take $t = -4$, then the eigenvector is $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$

The eigenvalues for A are $\lambda = 1$, $\lambda = -2$ and $\lambda = 3$ … check that

Now to find the eigenvectors we will find the vectors X that's $(A - \lambda I)X = 0$ when $\lambda = 1$, $\lambda = -2$ and $\lambda = 3$

At $\lambda = 1$ we get $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 2 & -3 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} -x + 2y = 0 \\ 3x + 2y - 3z = 0 \end{array}$$

Since z is free put $x = t$ where t is any scalar, then $y = \frac{1}{2}t \Rightarrow$

$$\Rightarrow 3t + 2(\frac{1}{2}\text{t}) - 3\text{z} = 0 \Rightarrow z = \frac{4}{3}\text{t}$$

Therefore the eigenspace when $\lambda = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{2}t \\ \frac{4}{3}t \end{bmatrix} = t\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{3} \end{bmatrix}$

Since t is any scalar take $t = -1$, then the eigenvector is $\begin{bmatrix} -1 \\ -\frac{1}{2} \\ -\frac{4}{3} \end{bmatrix}$

At $\lambda = -2$ we get $\begin{bmatrix} 3 & 0 & 0 \\ -1 & 5 & 0 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} 3x = 0 \\ \Longrightarrow -x + 5y = 0 \Longrightarrow x = y = 0 \\ 3x + 2y = 0 \end{array}$$

Since z is free put $z = t$ where t is any scalar

Therefore the eigenspace when $\lambda = -2$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since t is any scalar take $t = -2$,then the eigenvector is $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

At $\lambda = 3$ we get $\begin{bmatrix} -2 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & 2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 2 & -5 & 0 \end{array}\right]$$

$$\begin{array}{c} -2x = 0 \\ \Longrightarrow \quad -x = 0 \quad \Longrightarrow x = 0 \\ 3x + 2y - 5z = 0 \end{array}$$

Since y is free put $y = t$ where t is any scalar , then $3(0) + 2\mathrm{t} - 5\mathrm{z} = 0 \Longrightarrow \mathrm{z} = \frac{2}{5}t$

Therefore the eigenspace when $\lambda = 3$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} == \begin{bmatrix} 0 \\ t \\ \frac{2}{5}t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \end{bmatrix}$

Since t is any scalar take $t = 5$,then the eigenvector is $\begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

3) $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$

From the previous example we found $\lambda = -2 , -2$ or 1

At $\lambda = 1$ we get $\begin{bmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 6 & 0 & -3 & 0 \\ -9 & -3 & 3 & 0 \\ 18 & 0 & -9 & 0 \end{array}\right] \end{array}\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & -3 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} R_1^{(1)} = (1/6)R_1 \\ R_2^{(1)} = 9R_1^{(1)} + R_2 \\ R_3^{(1)} = -18R_1^{(1)} + R_3 \end{array}$$

$$\Longrightarrow \begin{array}{c} x - \frac{1}{2}z = 0 \\ -3y - \frac{3}{2}z = 0 \end{array}$$

Since z is free put $z = t$ where t is any scalar, then $x = \frac{1}{2}t$

$$\Longrightarrow -3\mathrm{y} = \frac{3}{2}\mathrm{t} \Longrightarrow \mathrm{y} = -\frac{1}{2}\mathrm{t}$$

Therefore the eigenspace when $\lambda = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Since t is any scalar take $t = -2$, then the eigenvector is $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

At $\lambda = -2$ we get $\begin{bmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$\Longrightarrow \left[\begin{array}{ccc|c} 9 & 0 & -3 & 0 \\ -9 & 0 & 3 & 0 \\ 18 & 0 & -6 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1^{(1)} = (1/9)R_1 \\ R_2^{(1)} = 9R_1^{(1)} + R_2 \\ R_3^{(1)} = -18R_1^{(1)} + R_3 \end{array}$$

$$\Longrightarrow x - \frac{1}{3}z = 0 \Longrightarrow x = \frac{1}{3}z$$

Since z and y are free put $z = t$ and $y = s$ where t and s are any scalar, then $x = \frac{1}{3}t$

Therefore the eigenspace when $\lambda = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = t\begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Since t and s are any scalar take $t = 3$ and $s = 1$, then the eigenvectors are $\left\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

Properties of Eigenvalues and Eigenvectors

1) If A is triangular, then the diagonal elements of A are the eigenvalues of A

For example the eigenvalues of $\begin{bmatrix} 2 & 0 & 0 \\ -9 & -1 & 0 \\ 5 & 1 & 3 \end{bmatrix}$ are $\lambda = 2, -1$ and 3

2) If λ is an eigenvalue of A with eigenvector X, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with eigenvector X

Example : Find the eigenvalues of A^{-1} $if\ A = \begin{bmatrix} 1 & 7 & 4 \\ 0 & -6 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

Solution :

Since the eigenvalues of A $are\ \lambda = 1, -6\ and -2$, then the eigenvalues of A^{-1} $are\ \lambda = 1, -\frac{1}{6}\ and -\frac{1}{2}$

3) If λ is an eigenvalue of A then λ is an eigenvalue of A

For example if $\lambda = \frac{4}{2}\ and - 6$ are tow eigenvalues of matrix B, then $\lambda = \frac{4}{2}\ and - 6$ of B^T

4) The sum of the eigenvalues of A is equal to the trace of $A(\ tr(A)\ )$ where $tr(A) = \sum_{i=1}^{n} a_{ii}$ (Diagonal elements).

4) The product of the eigenvalues of A is the equal to $\det(A)$, the determinant of A

Example: Find the sum & product of the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$

Solution :

First : Sum of the eigenvalues $= tr(A) = 2 + 3 + (-6) = -1$

Second : Since $\det A = 2\begin{vmatrix} 3 & 1 \\ 1 & -6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -6 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 2(-19) - (-8) + 2(-5) = -40$

Then the product of the eigenvalues $= -40$

Cayley- Hamilton Theorem :

Every square matrix satisfies its own characteristic equation , that's mean

$|\lambda I_n - A| = 0$, where A is a square matrix

Example : Show that the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation

Solution :

First : we will find the characteristic equation

The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$ ……… Check that

Now put $\lambda = A$ to proof that $A^2 - 2A + 5\, I_2 = O$ s.t O is zero matrix

$$L.H.S = A^2 - 2A + 5 = \left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\right)^2 - 2\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + (-4) & -2 + (-2) \\ 2 + 2 & -4 + 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O = R.H.S$$

<u>Uses of Cayley-Hamilton theorem:</u>

(1) To calculate the positive integral powers of A
(2) To calculate the inverse of a square matrix A

Example: Find A^4 and A^{-1} of the following matrices

$$1)\ A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \qquad 2)\ B = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution:

$$1)\ A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation is $\lambda^2 - 5 = 0$ ... ... Check that

Now by **<u>Cayley-Hamilton</u>** we get $A^2 - 5I_2 = 0$

To find A^4

Since $A^2 - 5I_2 = 0$

Then $A^2 = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ Multiplying by A^2 both sides

$$\Rightarrow A^4 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^2$$

$$\Rightarrow A^4 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1+4 & 2+(-2) \\ 2+(-2) & 4+1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

To find A^{-1}

Since $A^2 - 5I_2 = 0$ Multiplying by A^{-1} both sides

Then $A - 5A^{-1} = 0$

$\Rightarrow -5A^{-1} = -A$

$$\Rightarrow A^{-1} = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

2) $B = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Exercise (The characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$ *and*

$$B^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix} \qquad B^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$