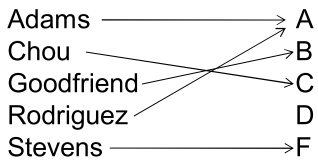
Chapter 2: Basic Structures: Sets, Functions

Functions

2.3 Functions

- Function: task, subroutine, procedure, method, mapping, …
- E.g. Find the grades of student A.

```
int findGrades(string name){
    //go to grades array,
    //find the name, and find the corresponding grades
    …
    return grades;
}
```



2.3 Functions

DEFINITION 1
Let A and B to be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \rightarrow B$.

- We can use a formula or a computer program to define a function.
  Example: $f(x) = x + 1$
  Or:

```
int increaseByOne(int x){
        x = x + 1;
        return x;
}
```

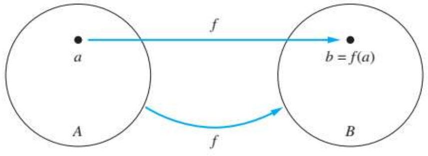


FIGURE 2 The Function f Maps A to B.

2.3 Functions

- A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B.
  Example:
  $R = \{(a,0),(a,1),(a,3),(b,1),(b,2),(c,0),(c,3)\}$ is a relation from the set $\{a,b,c\}$ to the set $\{0,1,2,3\}$.
- A relation from A to B that contains one and only one ordered pair (a,b) for every element $a \in A$, defines a function f from A to B.
  Example: $R=\{(a,2),(b,1),(c,3)\}$

2.3 Functions

DEFINITION 2
If *f* is a function from *A* to *B*, we say that *A* is the domain of *f* and *B* is the codomain of *f*. If *f(a)* = *b*, we say that *b* is the *image* of *a* and *a* is a *preimage* of *b*. The *range* of *f* is the set of all images of elements of *A*. Also, if *f* is a function from *A* to *B*, we say that *f maps A* to *B*.

- When we define a function, we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain. Two functions are equal when they have the same domain and codomain, and map elements of their common domain to the same elements in their common codomain. If we change either the domain or the codomain of a function, we obtain a different function. If we change the mapping of elements, we also obtain a different function.

2.3 Functions

- What are the domain, codomain, and range of the function that assigns grades to students described in the slide 13?
  Solution:
  domain: {Adams, Chou, Goodfriend, Rodriguez, Stevens}
  codomain: {*A, B, C, D, F*}
  range: {*A, B, C, F*}
- Let *f* be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, *f*(11010) = 10. Then, the domain of *f* is the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00,01,10,11}
- What is the domain and codomain of the function
  int floor(float real){…}?
  Solution: domain: the set of real numbers
  codomain: the set of integer numbers

2.3 Functions

DEFINITION 3
If f_1 and f_2 be functions from A to R. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to R defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

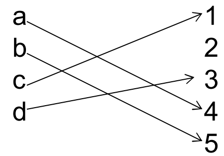
- Example: Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?
  Solution:
  $$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$
  $$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4$$

2.3 Functions

One-to-One and Onto Functions

DEFINITION 5
A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

- $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$ (If it's a different element, it should map to a different value.)
- Example: Determine whether the function f from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$ is one-to-one.



Solution: Yes.

2.3 Functions

- Example: Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.
  Solution: $f(1) = f(-1) = 1$, not one-to-one

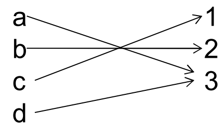
DEFINITION 6
A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \le f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \ge f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

- A function that is either strictly increasing or strictly decreasing must be one-to-one.

2.3 Functions

DEFINITION 7
A function f from A to B is called *onto*, or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called a *surjection* if it is onto.

- Example: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?



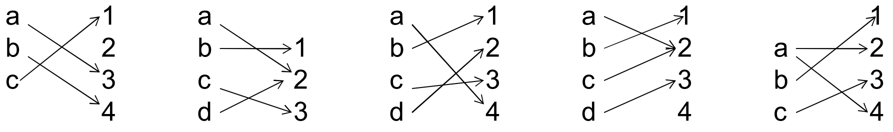
  Solution: Yes.
- Example: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
  Solution: No. There is no integer x with $x^2 = -1$, for instance.

2.3 Functions

DEFINITION 8

The function *f* is a *one-to-one correspondence* or a *bijection*, if it is both one-to-one and onto.

a. One-to-one, Not onto | b. Onto, not one-to-one | c. One-to-one, and onto | d. neither | d. Not a function



Inverse Functions

DEFINITION

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b. The inverse function of f is denoted by f −1. Hence, f −1(b) = a when f (a) = b.

- A one-to-one correspondence is called invertible because we can define an inverse of this function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

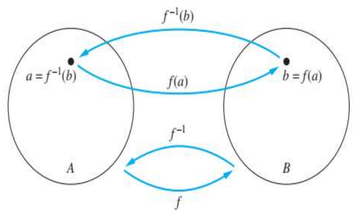


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

Inverse Functions

EXAMPLE 18 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. ◀

EXAMPLE 19 Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as follows from Examples 10 and 14. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$. ◀

EXAMPLE 20 Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.) ◀

Sometimes we can restrict the domain or the codomain of a function, or both, to obtain an invertible function, as Example 21 illustrates.

EXAMPLE 21 Show that if we restrict the function $f(x) = x^2$ in Example 20 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible.

Solution: The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both x and y are nonnegative, we must have $x = y$. So, this function is one-to-one. Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$. ◀

2.4 Composition Functions

DEFINITION 1

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g , denoted by f ∘ g , is defined by (f ∘ g) (a)= f(g(a))

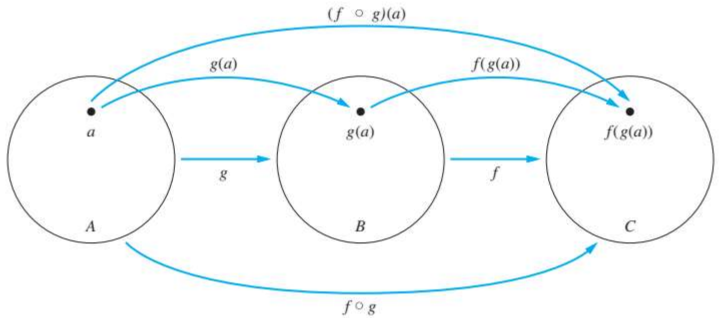


FIGURE 7 The Composition of the Functions *f* and *g*.

2.4 Composition Functions

EXAMPLE 22 Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g. ◀

EXAMPLE 23 Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

◀

Remark: Note that even though $f \circ g$ and $g \circ f$ are defined for the functions f and g in Example 23, $f \circ g$ and $g \circ f$ are not equal. In other words, the commutative law does not hold for the composition of functions.

Composition Functions

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$. Hence,

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,$$

and

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.