Chapter 3 : Solutions Of Linear Systems

Introduction

In this chapter we study methods of solving large linear systems of equations. Linear systems are used in many problems in engineering and science, as well as with applications of mathematics to the social sciences and the quantitative study of business and economic problems.

Consider a system of n linear algebraic equations in n unknowns

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{matrix} \qquad \ldots\ldots\ldots (1)$$

Where a_{ij} where $i, j = 1 , 2 , \ldots \ , n$ are the known coefficients , b_i where $i = 1 , 2 , \ldots \ , n$ are the known values and x_i where $i = 1 , 2 , \ldots \ , n$ are the unknowns (variables) to be determined.

The previous system (1) can be written in the following form

$$\mathrm{AX} = \mathrm{B} \quad \ldots (2)$$

Where $A = \begin{bmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ a_{21} a_{22} & \ddots & a_{2n} \\ \vdots \quad \vdots & & \vdots \\ a_{n1} a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

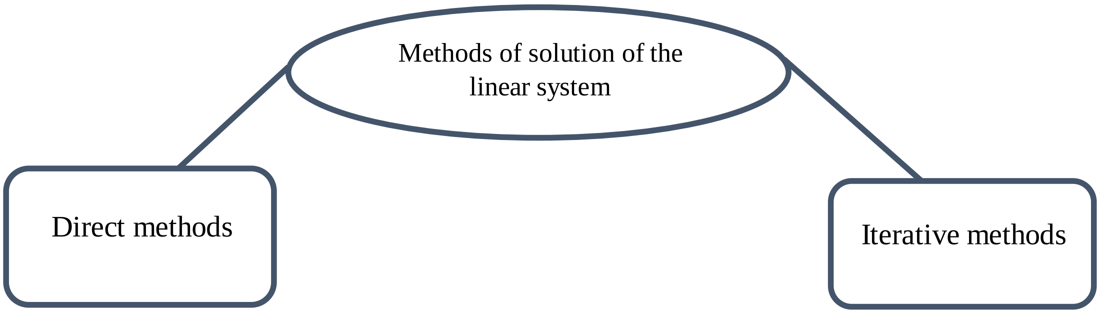
Notes:

1- $\mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ called thee vector solution

2- The linear system (1) written in the augmented matrix as $\begin{bmatrix} a_{11} a_{12} & \cdots & a_{1n} = b_1 \\ a_{21} a_{22} & \ddots & a_{2n} = b_2 \\ \vdots \quad \vdots & & \vdots \quad \vdots \\ a_{n1} a_{n2} & \cdots & a_{nn} = b_n \end{bmatrix}$

Methods of solution of the linear system

The methods of solution of the linear system (1) may broadly be classified into tow types as following figure



i – Direct methods

These methods produce the exact solution after a finite number of steps and most important these method

1) Inversion method 2) Gauss Elimination method 3) LU Decomposition method

1 – Inversion method

Let $AX = B$ be a linear system where A is a nonsingular matrix ($|A| \neq 0$) .

$$\boxed{X = A^{-1}B \Longrightarrow X = \frac{adjA}{\det A}\,B}$$

Example : Solve the following system by using inversion method

$$\begin{gathered} 3x + y + 2z = 3 \\ 2x - 3y - z = -3 \\ x + 2y + z = 4 \end{gathered}$$

Solution :

First , we will found $adjA$

$$\text{Since } A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \quad , B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} \quad and \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then $A_{11} = \begin{vmatrix} -3 & -1 \\ 2 & 1 \end{vmatrix} = -3 - (-2) = -1 \, , A_{12} = -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -(2 — (-1)) = -3$

$$A_{13} = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 4 — 3 = 7 \, , A_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -(1 - 4) = 3 \, , A_{22} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1$$

$$A_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -(6 - 1) = -5 \ , A_{31} = \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} = -1 - (-6) = 5$$

$$A_{32} = -\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -\big(-3 - (-4)\big) = -1 \ , A_{33} = \begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} = -9 - (2) = -11$$

$$\Longrightarrow adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & -1 \\ 7 & -5 & -11 \end{bmatrix}$$

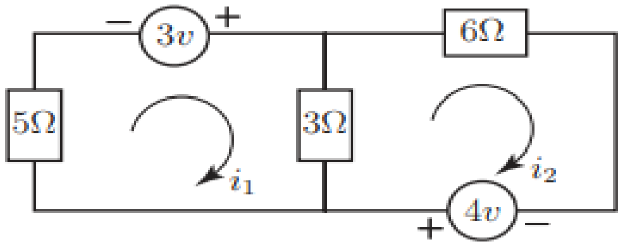
Second , we will found $\det A$

$$\det A = |A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3A_{11} + A_{12} + 2A_{13} = 3(-1) - 3 + 2(7) = 8$$

Now $X = A^{-1}B = \frac{adjA}{\det A} B = \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & -1 \\ 7 & -5 & -11 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

$$= \frac{1}{8} \begin{bmatrix} -3 - 9 + 20 \\ -9 - 3 + 28 \\ 21 + 15 - 44 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example : In the circuit shown find the currents (i_1 , i_2) in the loops by using inversion method



Solution :

We note that the current across the 3 Ω resistor (travelling top to bottom in the diagram) is given by (i_1 , i_2). With this proviso we can apply Kirchhoff's law:

In the left-hand loop $3(i_1 - i_2) + 5i_1 = 3 \quad \Rightarrow 8i_1 - 3i_2 = 3$

In the Right-hand loop $3(i_2 - i_1) + 6i_2 = 4 \quad \Rightarrow -3i_1 + 9i_2 = 4$

So to get i_1 and i_2 , we have to solve the system

$$\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The invers of $\begin{bmatrix} 8 & -3 \\ -3 & 9 \end{bmatrix}$ is $\frac{1}{63}\begin{bmatrix} 9 & 3 \\ 3 & 8 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 9 & 3 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 27+12 \\ 9+32 \end{bmatrix} = \frac{1}{63}\begin{bmatrix} 39 \\ 41 \end{bmatrix} = \begin{bmatrix} \frac{39}{63} \\ \frac{41}{63} \end{bmatrix}$$

$$\text{So} \quad i_1 = \frac{39}{63} \text{ and } i_2 = \frac{41}{63}$$

2) Cramer's method

Let $AX = B$ be a linear system where A is a nonsingular matrix ($|A| \neq 0$). Then the solution to the linear system $AX = B$ is given by

$$x_1 = \frac{\det A_1}{\det A} \quad , \quad x_2 = \frac{\det A_2}{\det A} \quad , \quad x_3 = \frac{\det A_3}{\det A} \quad , \dots \quad , \quad x_n = \frac{\det A_n}{\det A}$$

Where A_j is the matrix obtained by replacing the entries in the j^{th} column of A by the entries in the matrix

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example : Solve the following systems by Cramer's method

$$1) \quad \begin{matrix} 2x + 4y = 2 \\ 2x + y = -1 \end{matrix} \qquad\qquad 2) \quad \begin{matrix} x + 5y - 3z = -36 \\ x + 4y + 2z = -11 \\ 2x - y \quad\quad = 7 \end{matrix}$$

Solution :

1) $\begin{matrix} 2x+4y=2 \\ 2x+y=-1 \end{matrix}$

Since $A=\begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, $B=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Then $A_1=\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$, $A_2=\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$. Therefore

$\det A=2-8=-6$, $\det A_1=2+4=6$ and $\det A_2=-2-4=-6$

$\Longrightarrow x=\frac{\det A_1}{\det A}=-1$ *and* $y=\frac{\det A_2}{\det A}=1 \Longrightarrow X=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2) $\begin{matrix} x+5y-3z=-36 \\ x+4y+2z=-11 \\ 2x-y \quad\quad =7 \end{matrix}$

Since $A=\begin{bmatrix} 1 & 5 & -3 \\ 1 & 4 & 2 \\ 2 & -1 & 0 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} -36 \\ -11 \\ 7 \end{bmatrix}$

Then $A_1=\begin{bmatrix} -36 & 5 & -3 \\ -11 & 4 & 2 \\ 7 & -1 & 0 \end{bmatrix}$, $A_1=\begin{bmatrix} 1 & -36 & -3 \\ 1 & -11 & 2 \\ 2 & 7 & 0 \end{bmatrix} A_1=\begin{bmatrix} 1 & 5 & -36 \\ 1 & 4 & -11 \\ 2 & -1 & 7 \end{bmatrix}$. Therefore

$$\det A = (0+2)-5(0-4)-3(-1-8)=49$$

$$\det A_1=-36\,(0+2)-5(0-14)-3(11-28)=49$$

$$\det A_2= (0-14)+36(0-4)-3(7+22)=-245$$

$$\det A_3= (28-11)-5(7+22)-36(-1-8)=196$$

$$\Longrightarrow x=\frac{\det A_1}{\det A}=1 \ , \ y=\frac{\det A_2}{\det A}=-5 \ \textit{and}\ z=\frac{\det A_3}{\det A}=4 \Longrightarrow X=\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

3) Gauss Elimination method

Engineers often need to solve large systems of linear equations; for example in determining the forces in a large framework or finding currents in a complicated electrical circuit. The method of Gauss elimination provides a systematic approach to their solution as follows.

Step 1 : Write the given linear system in augmented matrix form

Step 2 : Convert the given linear system to upper triangle matrix

Step 3 : Use the back substitution to find thee vector solution

Example : Solve the following linear system by using Gauss Elimination method

$$\begin{matrix} x+y-z=-2 \\ 2x-y+z=5 \\ -x+2y+2z=1 \end{matrix}$$

Solution :

Step 1 : First we write the system in augmented matrix form as follows

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Step 2 : Convert the system to upper triangle matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 \\ R_2^{(1)} = R_2 - 2R_1 \\ R_3^{(1)} = R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 0 & 4 & 8 \end{array}\right] \begin{array}{l} R_1 \\ R_2^{(1)} \\ R_3^{(2)} = R_3^{(1)} + R_2^{(1)} \end{array}$$

Step 3 : Use the back substitution to find thee vector solution

First, we extract the equations as follows

$$x + y - z = -2 \quad \dots (1)$$
$$-3y + 3z = 9 \quad \dots (2)$$
$$4z = 8 \quad \dots (3)$$

From Eq (3) $4z = 8 \quad \Longrightarrow z = 2$

From Eq(2) $-3y + 3z = 9 \quad \Longrightarrow -3y + 6 = 9 \Longrightarrow -3y = 3 \Longrightarrow y = -1$

Form Eq(1) $x + y - z = -2 \Longrightarrow x - 1 - 2 = -2 \Longrightarrow x - 3 = -2 \Longrightarrow x = 1$

Then the vector solution is $X = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

ii – Iterative methods

These methods give a sequence of approximate solutions , which converge when the number of steps tend to infinity , and in these method , we order the given system are diagonally dominant system

Diagonally dominant matrix (D.D.M)

Definition : A matrix A of order (size) $n \times n$ is called diagonally dominant matrix if

$$| a_{ii} | \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} | a_{ij} | \ s.t \ i,j = 1 , 2 , \dots \ , n$$

Example : Decide whether the following matrices are D.D.M

$$1)\ A = \begin{bmatrix} -7 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 6 & 10 \end{bmatrix} \qquad 2)\ B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -3 & 1 \\ 8 & -1 & 5 \end{bmatrix}$$

Solution :

$$1)\ A = \begin{bmatrix} -7 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 6 & 10 \end{bmatrix}$$

Since $|-7| > |3| + |1|$, $|5| > |2| + |0|$ and $|10| > |3| + |6|$, then A is D.D.M

$$2)\ B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -3 & 1 \\ 8 & -1 & 5 \end{bmatrix}$$ Exercise (Ans : not D.D.M)

Test of the convergence

1) If $X_t = (x_1 \ , \ x_2 \ , \ x_3 \ , \dots \ \ , \ x_n)^T$ is exat solution of the system $AX = B$, then

$X^{(k)} = \left(x_1{}^{(k)} \ , \ x_2{}^{(k)} \ , \ x_3{}^{(k)} \ , \dots \ \ , \ x_n{}^{(k)}\right)^T$ is approximate root with accuracy **ε** if

$$\left\| X_t - X^{(k)} \right\|_\infty < \varepsilon \ \text{,where}$$

$$\left\| X_t - X^{(k)} \right\|_\infty = \max\left\{ \left|x_1 - x_1{}^{(k)}\right| , \left|x_2 - x_2{}^{(k)}\right| , \left|x_3 - x_3{}^{(k)}\right| , \dots , \left|x_n - x_n{}^{(k)}\right| \right\}$$

2) If x_t is not given then $x^{(k+1)} = \left(x_1{}^{(k+1)} \ , \ x_2{}^{(k+1)} \ , \ x_3{}^{(k+1)} \ , \dots \ \ , \ x_n{}^{(k+1)}\right)^T$

is approximate root with accuracy ε if

$$\left\| X^{(k+1)} - X^{(k)} \right\|_\infty < \varepsilon \ \text{,where}$$

$$\left\| X^{(k+1)} - X^{(k)} \right\|_\infty = \max\left\{ \left|x_1{}^{(k+1)} - x_1{}^{(k)}\right| , \left|x_2{}^{(k+1)} - x_2{}^{(k)}\right| , \left|x_3{}^{(k+1)} - x_3{}^{(k)}\right| , \dots , \left|x_n{}^{(k+1)} - x_n{}^{(k)}\right| \right\}$$

Example :

If $X^{(5)} = (-0.0012\ ,\ 0.0580\ ,\ 1.2020\ )^T$ and $X^{(4)} = (\ 0.0002\ ,0.0087\ ,\ 1.2073\ )^T$ find

$$\left\| X^{(5)} - X^{(4)} \right\|_\infty$$

Solution :

Since $\left\| X^{(5)} - X^{(4)} \right\|_\infty = \max\{\ \left|{x_1}^{(5)} - {x_1}^{(4)}\right| , \left|{x_2}^{(5)} - {x_2}^{(4)}\right| , \left|{x_3}^{(5)} - {x_3}^{(4)}\right|\ \}$

Then $\left\| X^{(5)} - X^{(4)} \right\|_\infty = \max\{\ |-0.0012\ - 0.0002| , |0.0580 - 0.0087| , |1.2020 - 1.2073|\ \}$

$$\Longrightarrow \left\| X^{(5)} - X^{(4)} \right\|_\infty = \max\{\ 0.0014\quad ,\quad 0.0493\quad ,\quad 0.0053\ \} = 0.0493$$

In this section we will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

1) Jacobi's method

Consider the following system in the diagonally dominant system form

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{gathered}$$

Solve the previous equations for $x_1\ ,\ x_2\ , \ldots ,\ x_n$ respectively , we get

$$\begin{gathered} x_1 = \frac{1}{a_{11}}\ [b_1 - a_{12}x_2 - \cdots - a_{1n}x_n] \\ x_2 = \frac{1}{a_{22}}\ [b_2 - a_{21}x_1 - \cdots - a_{2n}x_n] \\ \vdots \\ x_n = \frac{1}{a_{nn}}[b_n - a_{n1}x_1 - \cdots - a_{n(n-1)}x_{n-1}] \end{gathered}$$

Now , we construct the iteration formula

$$\begin{gathered} x_1^{(k+1)} = \frac{1}{a_{11}}\ \left[b_1 - a_{12}x_2^{(k)} - \cdots - a_{1n}x_1^{(k)}\right] \\ x_2^{(k+1)} = \frac{1}{a_{22}}\ [b_2 - a_{21}x_1^{(k)} - \cdots - a_{2n}x_n^{(k)}] \\ \vdots \\ x_n^{(k+1)} = \frac{1}{a_{nn}}[b_n - a_{n1}x_1^{(k)} - \cdots - a_{n(n-1)}x_{n-1}^{(k)}] \end{gathered}$$

Where $k = 0\ , 1\ , 2\ , \ldots\ \ , N_{max}$, N_{max} is the number of iterations allowed and

$\mathrm{X}^{(0)} = (\ x_1^{(0)}, x_2^{(0)}\ , \ldots\ \ ,\ x_n^{(0)}\ )^T$ is given guess vector otherwise $\mathrm{X}^{(0)} = (\ 0\ , 0\ , \ldots\ \ ,\ 0)^T$

Example : Use Jacobi's method to approximate solution of the following system to third iterations

$$2x_1 - 3x_2 + 20x_3 = 25$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$20x_1 + x_2 - 2x_3 = 17$$

Solution :

First , we will convert the system to D.D.S as follows

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 + 20x_3 = 25$$

$$\begin{aligned} &x_1 = \frac{1}{20}(17 - x_2 + 2x_3) \\ \Rightarrow\ &x_2 = \frac{1}{20}(-18 - 3x_1 + x_3) \\ &x_3 = \frac{1}{20}(25 - 2x_1 + 3x_2) \end{aligned}$$

Now , we construct the iterations formulas

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{20}\left(17 - x_2^{(k)} + 2x_3^{(k)}\right) \\ x_2^{(k+1)} &= \tfrac{1}{20}\left(-18 - 3x_1^{(k)} + x_3^{(k)}\right) \quad \dots (*) \\ x_3^{(k+1)} &= \tfrac{1}{20}\left(25 - 2x_1^{(k)} + 3x_2^{(k)}\right) \end{aligned}$$

Consider $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})^T = (0, 0, 0)^T$

1^{st}iteration at $k = 0$ in $(*)$

$$x_1^{(1)} = \frac{1}{20}\left(17 - x_2^{(0)} + 2x_3^{(0)}\right) = \frac{1}{20}\left(17 - (0) + 2(0)\right) = 0.85$$
$$x_2^{(1)} = \frac{1}{20}\left(-18 - 3x_1^{(0)} + x_3^{(0)}\right) = \frac{1}{20}\left(-18 - 3(0) + (0)\right) = -0.9$$
$$x_3^{(1)} = \frac{1}{20}\left(25 - 2x_1^{(0)} + 3x_2^{(0)}\right) = \frac{1}{20}\left(25 - 2(0) + 3(0)\right) = 1.25$$

$\Rightarrow X^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)})^T = (0.85, -0.9, 1.25)^T$

2^{nd}iteration at $k = 1$ in $(*)$

$$x_1^{(2)} = \frac{1}{20}\left(17 - x_2^{(1)} + 2x_3^{(1)}\right) = \frac{1}{20}\left(17 - (-0.9) + 2(1.25)\right) = 1.02$$
$$x_2^{(2)} = \frac{1}{20}\left(-18 - 3x_1^{(1)} + x_3^{(1)}\right) = \frac{1}{20}\left(-18 - 3(0.85) + (1.25)\right) = -0.965$$
$$x_3^{(2)} = \frac{1}{20}\left(25 - 2x_1^{(1)} + 3x_2^{(1)}\right) = \frac{1}{20}\left(25 - 2(0.85) + 3(-0.9)\right) = 1.03$$

$\Rightarrow X^{(2)} = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)})^T = (1.02, -0.965, 1.03)^T$

3^{rd}iteration at $k = 2$ in $(*)$

$$x_1^{(3)} = \frac{1}{20}\left(17 - x_2^{(2)} + 2x_3^{(2)}\right) = \frac{1}{20}\left(17 - (-0.965) + 2(1.03)\right) = 1.00125$$
$$x_2^{(3)} = \frac{1}{20}\left(-18 - 3x_1^{(2)} + x_3^{(2)}\right) = \frac{1}{20}\left(-18 - 3(1.02) + (1.03)\right) = -1.0015$$
$$x_3^{(3)} = \frac{1}{20}\left(25 - 2x_1^{(2)} + 3x_2^{(2)}\right) = \frac{1}{20}\left(25 - 2(1.02) + 3(-0.965)\right) = 1.00325$$

$$\Longrightarrow \mathrm{X}^{(3)} = (\, x_1^{(3)}, x_2^{(3)}\,,\, x_3^{(3)}\,)^T \; = (\, 1.00125 \;, -1.0015 \;\;,\;\; 1.00325 \;)^T$$

Then the vector solution is $\mathrm{X}^{(3)} = (\, 1.00125 \;, -1.0015 \;,\; 1.00325 \;)^T$ to three iterations

Example : Use Jacobi's method to solve the following system

$$\begin{aligned} 5x_1 + 10x_2 - 4x_3 \qquad &= 25 \\ -x_3 + 5x_4 &= -11 \\ -4x_2 + 8x_3 - x_4 &= -11 \\ 10x_1 + 5x_2 \qquad\qquad &= 6 \end{aligned}$$

Where $\mathrm{X}^{(0)} = \left(\, x_1^{(0)}, x_2^{(0)}\,,\, x_3^{(0)},\;\; x_4^{(0)} \right)^T = (\, -0.1500 \;,\; 2.6500 \;, -0.9375 \;, -2.3000\,)^T$

Repeat until $\left\|\mathrm{X}^{(k+1)} - \mathrm{X}^{(k)}\right\|_\infty \le 3.0 \times 10^{-1}$

Solution :

First , we will convert the system to D.D.S as follows

$$\begin{aligned} 10x_1 + 5x_2 \qquad\qquad &= 6 \\ 5x_1 + 10x_2 - 4x_3 \qquad &= 25 \\ -4x_2 + 8x_3 - x_4 &= -11 \\ -x_3 + 5x_4 &= -11 \end{aligned}$$

$$\Longrightarrow \begin{aligned} x_1 &= \tfrac{1}{10}(6 - 5x_2) \\ x_2 &= \tfrac{1}{10}(25 - 5x_1 + 4x_3) \\ x_3 &= \tfrac{1}{8}(-11 + 4x_2 + x_4) \\ x_4 &= \tfrac{1}{5}(-11 + x_3) \end{aligned}$$

Now , we construct the iterations formulas

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{10}\left(6 - 5x_2^{(k)}\right) \\ x_2^{(k+1)} &= \tfrac{1}{10}\left(25 - 5x_1^{(k)} + 4x_3^{(k)}\right) \\ x_3^{(k+1)} &= \tfrac{1}{8}(-11 + 4x_2^{(k)} + x_4^{(k)}) \\ x_4^{(k+1)} &= \tfrac{1}{5}\left(-11 + x_3^{(k)}\right) \end{aligned} \qquad \ldots (*\,,*)$$

1^{st}iteration at $k = 0$ in (*,*)

$$x_1^{(1)} = \frac{1}{10}\left(6 - 5x_2^{(0)}\right) = \frac{1}{10}(6 - 5(2.6500)) = -0.7250$$
$$x_2^{(1)} = \frac{1}{10}\left(25 - 5x_1^{(0)} + 4x_3^{(0)}\right) = \frac{1}{10}\left(25 - 5(-0.15000) + 4(0.9375)\right) = 2.2$$
$$x_3^{(1)} = \frac{1}{8}\left(-11 + 4x_2^{(0)} + x_4^{(0)}\right) = \frac{1}{8}\left(-11 + 4(2.6500) + (-2.3000)\right) = -0.3375$$
$$x_4^{(1)} = \frac{1}{5}\left(-11 + x_3^{(0)}\right) = \frac{1}{5}\left(-11 + (-0.9375)\right) = -2.3875$$

$$\Rightarrow \mathrm{X}^{(1)} = (\, x_1^{(1)}, x_2^{(1)}\,,\, x_3^{(1)}\,,\, x_4^{(1)}\,)^T = (-0.7250\ ,2.2000\ \ ,\ -0.3375\ ,-2.3875\ )^T$$

Since

$$\left\|\mathrm{X}^{(1)} - \mathrm{X}^{(0)}\right\|_\infty = \max\{\, |-0.250 + 0.1500|\,, |2.2000 - 2.6500|\,, |-0.3375 + 0.9375|\,,\ |-2.3875 + 2.3000|\}$$
$$= \max\{\, 0.5750\,, 0.4500\ \,, 0.6000\,, 0.0875\,\} = 0.6000 > 3.0 \times 10^{-1}$$

Then , we will continue

2^{nd}iteration at $k = 1$ in (*,*)

$$x_1^{(2)} = \frac{1}{10}\left(6 - 5x_2^{(1)}\right) = \frac{1}{10}(6 - 5(2.2000)) = -0.5000$$
$$x_2^{(2)} = \frac{1}{10}\left(25 - 5x_1^{(1)} + 4x_3^{(1)}\right) = \frac{1}{10}\left(25 - 5(-0.7250) + 4(-0.3375)\right) = 2.7275$$
$$x_3^{(2)} = \frac{1}{8}\left(-11 + 4x_2^{(1)} + x_4^{(1)}\right) = \frac{1}{8}\left(-11 + 4(2.2000) + (-2.3875)\right) = -0.5734$$
$$x_4^{(2)} = \frac{1}{5}\left(-11 + x_3^{(1)}\right) = \frac{1}{5}\left(-11 + (-0.3375)\right) = -2.2675$$

$$\Rightarrow x^{(2)} = (\, x_1^{(2)}, x_2^{(2)}\,,\, x_3^{(2)}\,,\, x_4^{(2)}\,)^T = (-0.5000\ ,2.7275\ \ ,-0.5734\ ,-2.2675\ )^T$$

Since

$$\left\|\mathrm{X}^{(2)} - \mathrm{X}^{(1)}\right\|_\infty = \max\{\, |-0.5000 + 0.7250|\,, |2.7275 - 2.2000|\,, |-0.5734 + 0.3375|\,,\ |-2.2675 + 2.3875|\}$$
$$= \max\{\, 0.22500\,, 0.5275\ \,, 0.2359\,, 0.1200\,\} = 0.5275 > 3.0 \times 10^{-1}$$

Then , we will continue

3^{rd}iteration at $k = 2$ in (*,*)

$$x_1^{(3)} = \frac{1}{10}\left(6 - 5x_2^{(2)}\right) = \frac{1}{10}(6 - 5(2.7275)) = -0.7638$$
$$x_2^{(3)} = \frac{1}{10}\left(25 - 5x_1^{(2)} + 4x_3^{(2)}\right) = \frac{1}{10}\left(25 - 5(-0.5000) + 4(-0.5734)\right) = 2.5206$$
$$x_3^{(3)} = \frac{1}{8}\left(-11 + 4x_2^{(2)} + x_4^{(2)}\right) = \frac{1}{8}\left(-11 + 4(2.7275) + (-2.2675)\right) = -0.2947$$
$$x_4^{(3)} = \frac{1}{5}\left(-11 + x_3^{(2)}\right) = \frac{1}{5}\left(-11 + (-0.5734)\right) = -2.3147$$

$$\Rightarrow \mathrm{X}^{(3)} = (x_1^{(3)}, x_2^{(3)} , x_3^{(3)} , x_4^{(3)})^T = (-0.7638 , 2.5206 , -0.2947 , -2.3147)^T$$

Since

$\left\|\mathrm{X}^{(3)} - \mathrm{X}^{(2)}\right\|_\infty = \max\{ |-0.7638 + 0.1500| , |2.5206 - 2.7275| , |-0.2947 + 0.5734| , |-2.3147 + 2.2675|\}$

$$= \max\{ 0.2638 , 0.2069 , 0.2787 , 0.0472 \} = 0.2069 > 3.0 \times 10^{-1}$$

Then the vector solution is $\mathrm{X}^{(3)} = (x_1^{(3)}, x_2^{(3)} , x_3^{(3)} , x_4^{(3)})^T = (-0.7638 , 2.5206 , -0.2947 , -2.3147)^T$

2) Gauss – Seidel method

Consider the following system in the diagonally dominant system form

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{gathered}$$

Solve the previous equations for x_1 , x_2 , ..., x_n respectively , we get

$$\begin{gathered} x_1 = \frac{1}{a_{11}} [b_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4 - \cdots - a_{1n}x_n] \\ x_2 = \frac{1}{a_{22}} [b_2 - a_{21}x_1 - a_{23}x_3 - a_{24}x_4 - \cdots - a_{2n}x_n] \\ \vdots \\ x_n = \frac{1}{a_{nn}} [b_n - a_{n1}x_1 - a_{n3}x_3 - a_{n4}x_4 - \cdots - a_{n(n-1)}x_{n-1}] \end{gathered}$$

Now , we construct the iteration formula

$$\begin{gathered} x_1^{(k+1)} = \frac{1}{a_{11}} \left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)} - \cdots - a_{1n}x_1^{(k)}\right] \\ x_2^{(k+1)} = \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - a_{24}x_4^{(k)} - \cdots - a_{2n}x_n^{(k)}] \\ x_3^{(k+1)} = \frac{1}{a_{33}} [b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)} - \cdots - a_{2n}x_n^{(k)}] \\ \vdots \\ x_n^{(k+1)} = \frac{1}{a_{nn}} [b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - a_{n3}x_n^{(k+1)} - \cdots - a_{n(n-1)}x_{n-1}^{(k+1)}] \end{gathered}$$

Where $k = 0 , 1 , 2 , \ldots , N_{max}$, N_{max} is the number of iterations allowed and

$\mathrm{X}^{(0)} = (x_1^{(0)}, x_2^{(0)} , \ldots , x_n^{(0)})^T$ is given guess vector otherwise $\mathrm{X}^{(0)} = (0 , 0 , \ldots , 0)^T$

Example : Use Gauss – Seidel method to find the exact solution of the following system (Take 4 d.p)

$$2x_1 + 10x_2 + 3x_3 = 19$$
$$9x_1 + x_2 + x_3 = 10$$
$$3x_1 + 4x_2 + 11x_3 = 0$$

Solution :

First , we will convert the system to D.D.S as follows

$$9x_1 + x_2 + x_3 = 10$$
$$2x_1 + 10x_2 + 3x_3 = 19$$
$$3x_1 + 4x_2 + 11x_3 = 0$$

$$\Longrightarrow \begin{aligned} x_1 &= \tfrac{1}{9}(10 - x_2 - x_3) \\ x_2 &= \tfrac{1}{10}(19 - 2x_1 - 3x_3) \\ x_3 &= \tfrac{1}{11}(-3x_1 - 4x_2) \end{aligned}$$

Now , we construct the iterations formulas

$$\begin{aligned} x_1^{(k+1)} &= \tfrac{1}{9}\left(10 - x_2^{(k)} - x_3^{(k)}\right) \\ x_2^{(k+1)} &= \tfrac{1}{10}\left(19 - 2x_1^{(k+1)} - 3x_3^{(k)}\right) \\ x_3^{(k+1)} &= \tfrac{1}{11}\left(-3x_1^{(k+1)} - 4x_2^{(k+1)}\right) \end{aligned} \qquad \ldots (*,*,*)$$

$$\text{Consider } X^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})^T = (0, 0, 0)^T$$

1^{st}iteration at $k = 0$ in $(*,*,*)$

$$x_1^{(1)} = \frac{1}{9}\left(10 - x_2^{(0)} - x_3^{(0)}\right) = \frac{1}{9}(10 - 0 - 0) = 1.1111$$
$$x_2^{(1)} = \frac{1}{10}\left(19 - 2x_1^{(1)} - 3x_3^{(0)}\right) = \frac{1}{10}\left(19 - 2(1.1111) - 3(0)\right) = 1.6778$$
$$x_3^{(1)} = \frac{1}{11}\left(-3x_1^{(1)} - 4x_2^{(1)}\right) = \frac{1}{11}\left(-3(1.1111) - 4(1.6778)\right) = -0.9131$$

$$\Longrightarrow X^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right)^T = (1.1111, 1.6778, -0.9131)^T$$

2^{nd}iteration at $k = 1$ in $(*,*,*)$

$$x_1^{(2)} = \frac{1}{9}\left(10 - x_2^{(1)} - x_3^{(1)}\right) = \frac{1}{9}(10 - 1.6778 + 0.9131) = 1.0261$$
$$x_2^{(2)} = \frac{1}{10}\left(19 - 2x_1^{(2)} - 3x_3^{(1)}\right) = \frac{1}{10}\left(19 - 2(1.0261) - 3(-0.9131)\right) = 1.9687$$
$$x_3^{(2)} = \frac{1}{11}\left(-3x_1^{(2)} - 4x_2^{(2)}\right) = \frac{1}{11}\left(-3(1.0261) - 4(1.9687)\right) = -0.9957$$

$$\Longrightarrow X^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right)^T = (1.0261, 1.9687, -0.9957)^T$$

3^{rd}iteration at $k = 2$ in $(*,*,*)$

$$x_1^{(3)} = \frac{1}{9}\left(10 - x_2^{(2)} - x_3^{(2)}\right) = \frac{1}{9}(10 - 1.9687 + 0.9957) = 1.003$$

$$x_2^{(3)} = \frac{1}{10}\left(19 - 2x_1^{(3)} - 3x_3^{(2)}\right) = \frac{1}{10}\big(19 - 2(1.003) - 3(-0.9957)\big) = 1.9981$$

$$x_3^{(3)} = \frac{1}{11}\left(-3x_1^{(3)} - 4x_2^{(3)}\right) = \frac{1}{11}\big(-3(1.003) - 4(1.9981)\big) = -1.0001$$

$$\Longrightarrow \mathrm{X}^{(3)} = \left(x_1^{(3)}, x_2^{(3)}, x_3^{(3)} \right)^T = (1.003\,, 1.9981\,, -1.0001\,)^T$$

4^{th}iteration at $k = 3$ in $(*,*,*)$

$$x_1^{(4)} = \frac{1}{9}\left(10 - x_2^{(3)} - x_3^{(3)}\right) = \frac{1}{9}(10 - 1.9981 + 1.0001) = 1.002$$

$$x_2^{(4)} = \frac{1}{10}\left(19 - 2x_1^{(4)} - 3x_3^{(3)}\right) = \frac{1}{10}\big(19 - 2(1.002) - 3(-1.0001)\big) = 1.9996$$

$$x_3^{(4)} = \frac{1}{11}\left(-3x_1^{(4)} - 4x_2^{(4)}\right) = \frac{1}{11}\big(-3(1.002) - 4(1.9996)\big) = -1.0004$$

$$\Longrightarrow \mathrm{X}^{(4)} = \left(x_1^{(4)}, x_2^{(4)}, x_3^{(4)} \right)^T = (1.002\,, 1.9996\,, -1.0004\,)^T$$

5^{th}iteration at $k = 4$ in $(*,*,*)$

$$x_1^{(5)} = \frac{1}{9}\left(10 - x_2^{(4)} - x_3^{(4)}\right) = \frac{1}{9}(10 - 1.9996 + 1.0004) = 1.0001$$

$$x_2^{(5)} = \frac{1}{10}\left(19 - 2x_1^{(5)} - 3x_3^{(4)}\right) = \frac{1}{10}\big(19 - 2(1.0001) - 3(-1.0004)\big) = 2.0001$$

$$x_3^{(5)} = \frac{1}{11}\left(-3x_1^{(5)} - 4x_2^{(5)}\right) = \frac{1}{11}\big(-3(1.0001) - 4(2.0001)\big) = -1.0000$$

$$\Longrightarrow \mathrm{X}^{(5)} = \left(x_1^{(5)}, x_2^{(5)}, x_3^{(5)} \right)^T = (1.0001\,, 2.0001\,, -1.0000\,)^T$$

6^{th}iteration at $k = 5$ in $(*,*,*)$

$$x_1^{(6)} = \frac{1}{9}\left(10 - x_2^{(5)} - x_3^{(5)}\right) = \frac{1}{9}(10 - 2.0001 + 1.0000) = 1.0000$$

$$x_2^{(6)} = \frac{1}{10}\left(19 - 2x_1^{(6)} - 3x_3^{(5)}\right) = \frac{1}{10}\big(19 - 2(1.0000) - 3(-1.0000)\big) = 2.0000$$

$$x_3^{(6)} = \frac{1}{11}\left(-3x_1^{(6)} - 4x_2^{(6)}\right) = \frac{1}{11}\big(-3(1.0000) - 4(2.0000)\big) = -1.0000$$

$$\Longrightarrow \mathrm{X}^{(6)} = \left(x_1^{(6)}, x_2^{(6)}, x_3^{(6)} \right)^T = (1.0000\,, 2.0000\,, -1.0000\,)^T$$

7^{th}iteration at $k = 6$ in $(*,*,*)$

$$x_1^{(7)} = \frac{1}{9}\left(10 - x_2^{(6)} - x_3^{(6)}\right) = \frac{1}{9}(10 - 2.0000 + 1.0000) = 1.0000$$
$$x_2^{(7)} = \frac{1}{10}\left(19 - 2x_1^{(7)} - 3x_3^{(6)}\right) = \frac{1}{10}\left(19 - 2(1.0000) - 3(-1.0000)\right) = 2.0000$$
$$x_3^{(7)} = \frac{1}{11}\left(-3x_1^{(7)} - 4x_2^{(7)}\right) = \frac{1}{11}\left(-3(1.0000) - 4(2.0000)\right) = -1.0000$$

$$\Rightarrow \mathrm{X}^{(7)} = \left(x_1^{(7)}, x_2^{(7)}, x_3^{(7)} \right)^T = (1.0000\,, 2.0000\,, -1.0000\,)^T$$

Then the exact solution is $(1.0000\,, 2.0000\,, -1.0000\,)^T$

Example : Use Jacobi's method and Gauss Seidel to solve the following system (Take tow d.p)

Let $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)} \right)^T = (0.500\,, 1.500\,)^T$

$$\begin{gathered} 4x_1 + 10x_2 = 24 \\ 10x_1 + x_2 = 12 \end{gathered}$$

Solution:

First , we will convert the system to D.D.S as follows

$$\begin{gathered} 10x_1 + x_2 = 12 \\ 4x_1 + 10x_2 = 24 \end{gathered}$$

$$\Rightarrow \begin{aligned} x_1 &= \frac{1}{10}(12 - x_2) \\ x_2 &= \frac{1}{10}(24 - 4x_1) \end{aligned}$$

Now , we construct the iterations formulas اكتب المعادلة هنا.

Jacobi's method

$$x_1^{(k+1)} = \frac{1}{10}(12 - x_2^{(k)})$$
$$x_2^{(k+1)} = \frac{1}{10}\left(24 - 4x_1^{(k)}\right)$$

1^{st}iteration at $k = 0$

$$x_1^{(1)} = \frac{1}{10}\left(12 - x_2^{(0)}\right) = \frac{1}{10}\left(12 - (1.50)\right) = 1.05$$
$$x_2^{(1)} = \frac{1}{10}\left(24 - 4x_1^{(0)}\right) = \frac{1}{10}\left(24 - 4(0.5)\right) = 2.2$$
$$\Longrightarrow x^{(1)} = (1.05\,, 2.2\,)^T$$

2^{nd}iteration at $k = 1$

$$x_1^{(2)} = \frac{1}{10}\left(12 - x_2^{(1)}\right) = \frac{1}{10}\left(12 - (2.2)\right) = 0.98$$
$$x_2^{(2)} = \frac{1}{10}\left(24 - 4x_1^{(1)}\right) = \frac{1}{10}\left(24 - 4(1.05)\right) = 1.98$$
$$\Longrightarrow x^{(2)} = (0.98\,, 1.98\,)^T$$

3^{rd}iteration at $k = 2$

$$x_1^{(3)} = \frac{1}{10}\left(12 - x_2^{(2)}\right) = \frac{1}{10}\left(12 - (1.98)\right) = 1.00$$
$$x_2^{(3)} = \frac{1}{10}\left(24 - 4x_1^{(2)}\right) = \frac{1}{10}\left(24 - 4(0.98)\right) = 2.01$$
$$\Longrightarrow x^{(3)} = (1.00\,, 2.01\,)^T$$

3^{rd}iteration at $k = 3$

$$x_1^{(4)} = \frac{1}{10}\left(12 - x_2^{(3)}\right) = \frac{1}{10}\left(12 - (2.01)\right) = 1.00$$
$$x_2^{(4)} = \frac{1}{10}\left(24 - 4x_1^{(3)}\right) = \frac{1}{10}\left(24 - 4(1.00)\right) = 2.00$$
$$\Longrightarrow x^{(4)} = (1.00\,, 2.00\,)^T$$

Then the vector solution is $(1.00\,, 2.00\,)^T$

Gauss – Seidel method

$$x_1^{(k+1)} = \frac{1}{10}(12 - x_2^{(k)})$$
$$x_2^{(k+1)} = \frac{1}{10}\left(24 - 4x_1^{(k+1)}\right)$$

$$x_1^{(1)} = \frac{1}{10}\left(12 - x_2^{(0)}\right)\frac{1}{10}\left(12 - (1.50)\right) = 1.05$$
$$x_2^{(1)} = \frac{1}{10}\left(24 - 4x_1^{(1)}\right) = \frac{1}{10}(24 - 4(1.05) = 1.98$$
$$x^{(1)} = (1.05\,, 1.98\,)^T$$

$$x_1^{(2)} = \frac{1}{10}\left(12 - x_2^{(1)}\right)\frac{1}{10}\left(12 - (1.98)\right) = 1.00$$
$$x_2^{(2)} = \frac{1}{10}\left(24 - 4x_1^{(2)}\right) = \frac{1}{10}(24 - 4(1.002) = 2.00$$
$$\Longrightarrow x^{(2)} = (1.00\,, 2.00\,)^T$$

Then the vector solution is $(1.00\,, 2.00\,)^T$

Note : From the previous example , we note that in Gauss – Seidel method the convergence to the exact solution after tow iterations ,but in Jacobi's method the convergence after four iterations this means Gauss – Seidel method in most cases faster from Jacobi's method

Exercise (3)

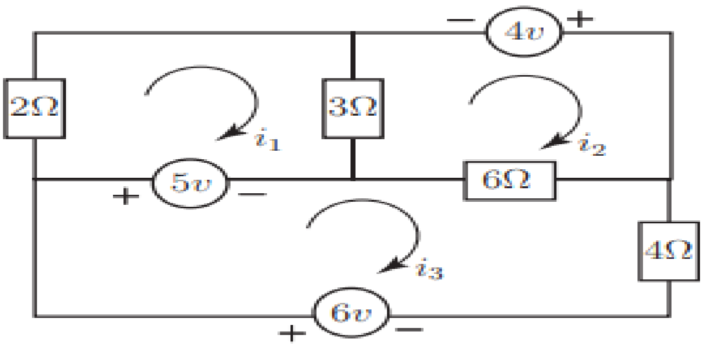
1) Solve the following systems

a) $\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}$ by Inversion method and Gauss Elimination method $\left(Ans\ X = \begin{bmatrix} 7 \\ -9 \\ 5 \end{bmatrix} \right)$

b) $\begin{aligned} 30x - 20y - 10z &= 0 \\ -20x + 55y - 10z &= 0 \\ -10x - 10y + 50z &= 200 \end{aligned}$ by Inversion method and Gauss Elimination method $\left(Ans\ X = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right)$

c) $\begin{aligned} -x + z + 3u &= 4 \\ 2x + y - z + u &= 1 \\ 3x - y - z + 2u &= -3 \\ -x + 2y + 3z + u &= 6 \end{aligned}$ by Gauss Elimination method $\left(Ans\ X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

2) In the circuit shown find the currents (i_1 , i_2 , i_3)in the following loops by using Gauss Elimination method



Ans ($i_1 = \frac{34}{15}$, $i_2 = \frac{19}{9}$, $i_3 = \frac{41}{30}$)

3) Use Cramer's method to solve the following systems

a) $\begin{aligned} 2x + 10y &= 42 \\ -x + 3y &= 19 \end{aligned}$ $\left(Ans:\ \mathrm{X} = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \right)$

b) $\begin{aligned} 2x + y - z &= 8 \\ 3x + 6y - 8z &= 32 \\ 2x + 4y - 6z &= 22 \end{aligned}$ $\left(Ans:\ \mathrm{X} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right)$

c) $\begin{aligned} x + 4y + 3z &= 10 \\ 2x + y - z &= -1 \\ 6x - 2y + 2z &= 22 \end{aligned}$ $\left(Ans:\ \mathrm{X} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)$

4) Use Jacobi's method to find approximate solution of the following system until $X^{(3)}$ (take 2 d.p)

$$\begin{matrix} 10x + y + z = 16 \\ 2x + 10y + 3z = 13 \\ 2x + 2y + 10z = 14 \end{matrix} \qquad \left(Ans: \; X = \begin{bmatrix} 0.93 \\ 0.92 \\ 0.90 \end{bmatrix} \right)$$

5) Use a) Jacobi's method , b) Gauss seidel method to find approximate solution of the following system

$$\begin{matrix} 9x + y + z = 16 \\ 2x + 10y + 3z = 44 \\ 3x + 4y + 11z = 5 \end{matrix} \quad ,\boldsymbol{where} \; X^{(0)} = (1\,,3\,,4\,)^T$$

$$Repeat\ until \left\| X^{(k+1)} - X^{(k)} \right\|_\infty \le 0.5 \times 10^{-4} \quad (\,take\ 5d.p)$$

$$\text{Ans : } a)\ X^{(15)} = \begin{bmatrix} 1.44626 \\ 4.59381 \\ -1.61042 \end{bmatrix} \; b)\ X^{(8)} = \begin{bmatrix} 1.44627 \\ 4.59387 \\ -1.61039 \end{bmatrix}$$

6) Use Gauss seidel method to find approximate solution of the following system

$$\begin{matrix} -x - y + 2z + 10u = -9 \\ 10x - 2y - z - u = 3 \\ -x - y + 10z - 2u = 27 \\ -2x + 10y - z - u = 15 \end{matrix}$$

$$Repeat\ until \left\| X_t - X^{(k)} \right\|_\infty \le 3.1 \times 10^{-3}\,, \qquad X_t = (1\,,2\,,3\;0\,)^T \quad (\,take\ 4d.p)$$

Ans: $X^{(4)}=(0.9968\,,1.9982\ \,,2.9987\ 0.0008\,)^T$