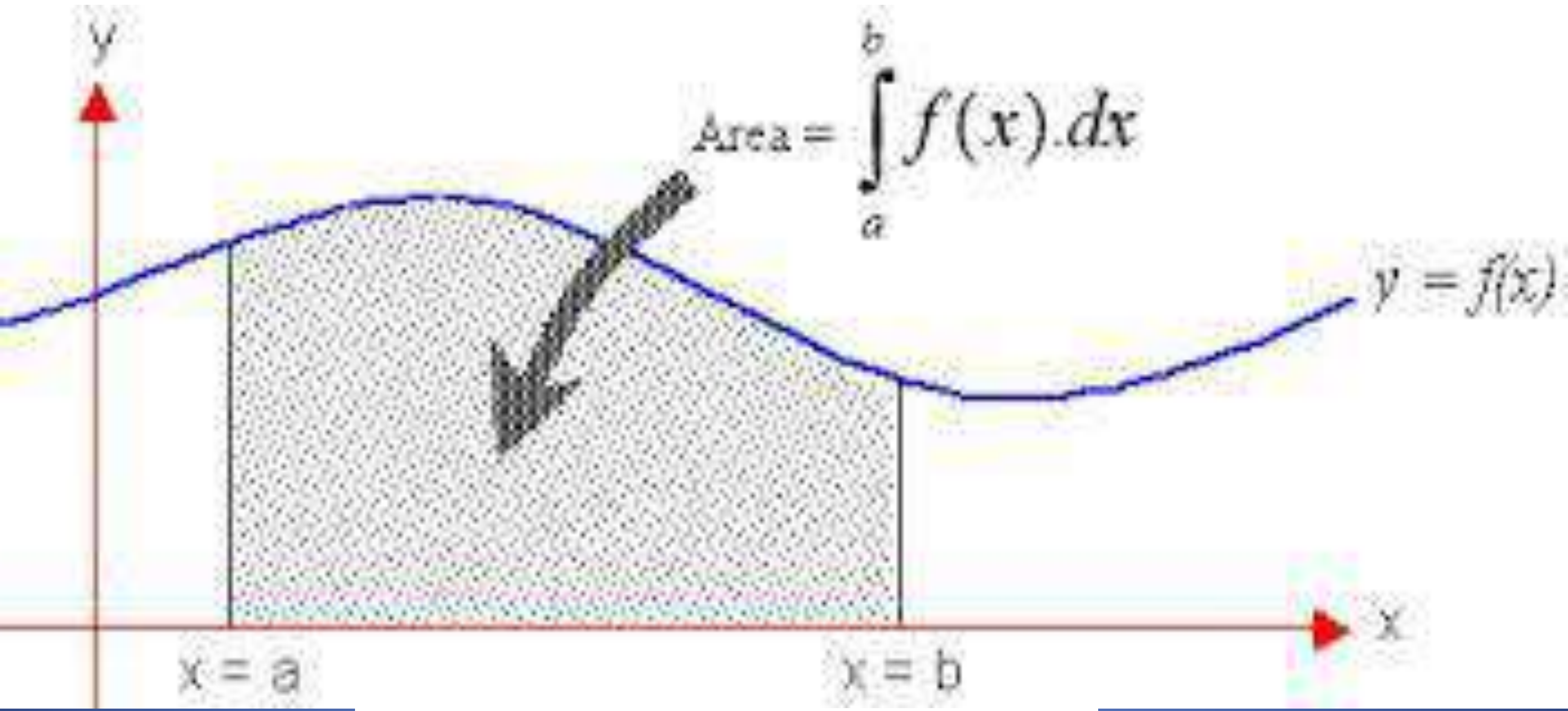

Integral Calculus

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Chaper 1: integrals

Antiderivatives

Definition: A function $F(x)$ is an **antidervative** of the function $f(x)$ on an interval I if $F'(x) = f(x)$ for every x in I .

Illustration:

1) $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because $F'(x) = \frac{d}{dx}(x^2) = 2x = f(x)$
2) $F(x) = x^2 + 4$ is an antiderivative of $f(x) = 2x$, because $F'(x) = \frac{d}{dx}(x^2 + 4) = 2x = f(x)$
3) $F(x) = x^2 - 7$ is an antiderivative of $f(x) = 2x$, because $F'(x) = \frac{d}{dx}(x^2 - 7) = 2x = f(x)$

In general, if C is any constant, then $x^2 + C$ is an antiderivative of $2x$, because

$$\frac{d}{dx}(x^2 + C) = 2x + 0 = 2x$$

The next theorem restates this result in the language of antiderivatves .

Theorem:

let F be antiderivative of f an antiderivative of on an interval I , if G is any antiderivative of f on I , then

$G(x) = F(x) + C$ for some constant C and every x in I.

Indefinite integrals

Definition: The indefinite integral of $f(x)$ with respect to x is denoted by $\int f(x)\,dx$ and is defined as

$$\boxed{\int f(x)\,dx = F(x) + C}$$

Where $F'(x) = f(x)$ and C is an arbitrary constant

Example: Construct the indefinite integrals of the following pairs

1) $f(x) = 6x$ ꕸ $g(x) = 3x^2$, 2) $h(x) = \sin x$ ꕸ $r(x) = \cos x$, 3) $f(x) = e^{x^2}$ ꕸ $h(x) = 2xe^{x^2}$

Solution:

1) Since $\frac{d}{dx}(3x^2) = 6x$ then $\int 6x\,dx = 3x^2 + c$
2) Since $\frac{d}{dx}(\sin x) = \cos x$ then $\int \cos x\,dx = \sin x + c$
3) Exercise

Example: Verify the formulas by differentiation

1) $\int x^4\,dx = \frac{1}{5}x^5 + C$, 2) $\int \cos x\,dx = \sin x + C$, 3) $\int x\cos x\,dx = x\sin x + \cos x + C$

Solution:

1) As $\frac{d}{dx}\left(\frac{1}{5}x^5 + C\right) = \frac{1}{5}(5x^4) + 0 = x^4$ then $\int x^4\,dx = \frac{1}{5}x^5 + C$

2) As $\frac{d}{dx}(\sin x + C) = \cos x$ then $\int \cos x\,dx = \sin x + C$

3) As $\frac{d}{dx}(x\sin x + \cos x + C) = \sin x + x\cos x - \sin x$ then

$\int x\cos x\,dx = x\sin x + \cos x + C$

Note **:** in general

$$1)\ \int \frac{d}{dx}\big(f(x)\big)\,dx = f(x) + C \qquad 2)\ \frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

For example $\int \frac{d}{dx}(\ln x)\,dx = \ln x + c$ & $\frac{d}{dx}\left(\int e^{\tan x}\,dx\right) = e^{\tan x}$

The previous result allows us to use any derivative formula to obtain a corresponding formula. Therefore we get the following standard forms:

1) $\int 1\,dx = \int dx = x + C \implies \int a\,dx = ax + c$, Where a is any real number

2) $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$, $r \neq -1$ 3) $\int \cos x\,dx = \sin x + C$

6) $\int \sin x\,dx = -\cos x + C$ 5) $\int sec^2 x\,dx = \tan x + C$

7) $\int \csc^2 x\ dx = -\cot x + C$ 8) $\int \sec x\ \tan x\,dx = \sec x + C$

9) $\int \csc x\ \cot x\,dx = -\csc x + C$

Example: Evaluate the integrals

1) $\int x^3\,dx$, 2) $\int \frac{1}{x^4}dx$, 3) $\int \sqrt{x}\,dx$, 4) $\int \sqrt[3]{x^2}\,dx$, 5) $\int \frac{\tan x}{\sec x}dx$, 6) $\int \sin x\ sec^2 x\,dx$, 7) $\int \frac{1}{\tan x\sin x}\,dx$

Solution:

1) $\int x^3\,dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$

2) $\int \frac{1}{x^4}dx = \int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1} + C = \frac{x^{-3}}{-3} + C = -\frac{x^{-3}}{3} + C = -\frac{1}{3x^3} + c$

3) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\ dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2\sqrt{x^3}}{3} + c$

4) $\int \sqrt[3]{x^2}\,dx = \int x^{\frac{2}{3}}\,dx == \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = 3\ \frac{x^{\frac{5}{3}}}{5} + C = \frac{3\sqrt[3]{x^5}}{5} + c$

5) $\int \frac{\tan x}{\sec x}dx = \int \frac{\sin x}{\cos x}\cos x\,dx = \int \sin x\,dx = -\cos x + C$

6) $\int \sin x\ sec^2 x\,dx = \int \sin x\ \frac{1}{\cos^2 x}dx = \int \frac{\sin x}{\cos x}\ \frac{1}{\cos x}dx = \int \tan x\sec x\,dx = \sec x + C$

7) $\int \frac{1}{\tan x\sin x}dx = \int \frac{1}{\tan x}\ \frac{1}{\sin x}dx = \int \cot x\csc x\,dx = -\csc x + C$

Theorem:

1) $\int cf(x)\,dx = c\int f(x)\,dx$ for any nonzero constant c

2) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example: Evaluate the integrals

1) $\int (5x^3 + 2\cos x)\,dx$, 2) $\int \frac{(x^2+x+1)}{\sqrt{x}}dx$, 3) $\int \tan^2 x\,dx$, 4) $\int \cot^2 x\,dx$, 5) $\int (\tan x + \cot x)^2\,dx$

Solution:

1) $\int (5x^3 + 2\cos x)\,dx = 5\int x^3\,dx + 2\int \cos x\,dx = \frac{5x^4}{4} + 2\sin x + C$

2) $\int \frac{(x^2+x+1)}{\sqrt{x}}dx = \int \frac{(x^2+x+1)}{x^{\frac{1}{2}}}dx = \int (x^2+x+1)x^{-\frac{1}{2}}\,dx = \int \left(x^{2-\frac{1}{2}} + x^{1-\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$

$= \int \left(x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx = \frac{2x^{\frac{5}{2}}}{5} + \frac{2x^{\frac{3}{2}}}{3} + 2x^{\frac{1}{2}} + C$

3) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \int \sec^2 x\,dx - \int dx = \tan x - x + C$

4) $\int \cot^2 x\,dx$ Exercise (Ans: $-\cot x - x + C$)

5) $\int (\tan x + \cot x)^2\,dx = \int (\tan^2 x + 2\tan x\ \cot x + \cot^2 x)dx$

$= \int (\tan^2 x + 2 + \cot^2 x)dx$

$= \tan x - x + 2x - \cot x - x + C = \tan x - \cot x + 2x - 2x + C = \tan x - \cot x + C$

<u>Important extension of the standard formula</u>

If x is replaced by $ax + b$ in the above standard, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants ,then the standard forms remain true provided the result on R.H.S is divided by a

For example $\int (ax+b)^n\,dx = \frac{1}{a}\frac{1}{n+1}(ax+b)^{n+1} + C$ *where* $n \neq -1$

In general: if $\int f(x)\,dx = \varphi(x) + C$, then $\int f(ax+b)\,dx = \frac{1}{a}\varphi(ax+b) + C$

1) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

2) $\int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b) + C$

3) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$

4) $\int \csc^2(ax+b)\,dx = \frac{-1}{a}\cot(ax+b) + C$

5) $\int \sec(ax+b)\ \tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

6) $\int \csc(ax+b)\ \cot(ax+b)\,dx = \frac{-1}{a}\csc(ax+b) + C$

Example: Evaluate the integrals

1) $\int (2x+1)^3\,dx$, 2) $\int \sin(5x)\,dx$, 3) $\int sec^2(4x+1)\,dx$, 4) $\int \sqrt[3]{x^2-2x+1}\,dx$

Solution:

1) $\int (2x+1)^3\,dx = \frac{1}{2}\,\frac{(2x+1)^4}{4} + C = \frac{(2x+1)^4}{8} + C$

2) $\int \sin(5x)\,dx = \frac{1}{5}\,(-\cos 5x) + C = -\frac{1}{5}\cos 5x + C$

3) $\int sec^2(4x+1)\,dx = \frac{1}{4}\tan(4x+1) + C$

4) $\int \sqrt[3]{x^2-2x+1}\,dx \int \sqrt[3]{x^2-2x+1}\,dx = \int \sqrt[3]{(x-1)^2}\,dx = \int (x-1)^{\frac{2}{3}}\,dx = \frac{3(x-1)^{\frac{5}{3}}}{5} + C = \frac{3\sqrt[3]{(x-1)^5}}{5} + C$

<u>Change variables in indefinite integral (substitution)</u>

Integration by substitution is used when the integration of the given function can't be obtained directly for example $\int (2x^3+1)^7 x^2\,dx$ here this integral is not in the standard form but we can transformed to another form by replacing $(2x^3+1)$ by a new variable as u

<u>Steps to integration by substitution</u>

The following are the steps that are helpful in performing this method as follows :

Step – 1 : Choose a new variable as u for the given function to be reduced

Step – 2 : Differentiate u

Step – 2 : Determine the value of dx

Step – 3 : Replace both u and dx in the integration

Step – 4 : Integrate the new function after the substitution

Step – 5 : Use the back substitution to get of answer in the initial variable

Example: Evaluate the integrals

1) $\int (2x^3+1)^7 x^2\,dx$ 2) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$, 3) $\int \frac{\cos 3x}{\sin^2 3x}\,dx$, 4) $\int \csc^6 x\,\cot x\,dx$

Solution:

1) Let $u = 2x^3+1 \;\Rightarrow\; du = 6x^2dx \;\Rightarrow\; dx = \frac{du}{6x^2}$

$\Rightarrow \int (2x^3+1)^7 x^2\,dx = \int u^7\,x^2\,\frac{du}{6x^2} = \frac{1}{6}\int u^7 du = \frac{1}{6}\;\frac{u^8}{8} + C = \frac{(2x^3+1)^8}{48} + C$

2) Let $u = \sqrt{x} \;\Rightarrow\; du = \frac{dx}{2\sqrt{x}} \;\Rightarrow\; dx = 2\sqrt{x}\,du$

$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\cos u}{\sqrt{x}}\,2\sqrt{x}\,du = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

3) Let $u=\sin 3x \quad \Rightarrow \quad du = 3\cos 3x\,dx \quad \Rightarrow \quad dx = \frac{du}{3\cos 3x}$

$$\Rightarrow \int \frac{\cos 3x}{\sin^2 3x}dx = \int \frac{\cos 3x}{u^2}\ \frac{du}{3\cos 3x} = \frac{1}{3}\int \frac{du}{u^2} = \frac{1}{3}\int u^{-2}\,du = \frac{1}{3}\ \frac{u^{-1}}{-1}+C = \frac{(\sin 3x)^{-1}}{-3}+C$$

4) Let $u=\csc x \quad \Rightarrow \quad du = -\csc x\ \cot x\,dx \quad \Rightarrow \quad dx = \frac{du}{-\csc x\ \cot x}$

$$\Rightarrow \int \csc^6 x\ \cot x\,dx = \int u^6\ \cot x \frac{du}{-\csc x\ \cot x} = -\int \frac{u^6}{u}du = -\int u^5 du = -\frac{u^6}{6}+C$$

$$= -\frac{(\csc x)^6}{6}+C = -\frac{\csc^6 x}{6}+C$$

Exercises

a) Construct the indefinite integrals of the following pairs

1) $f(x)=12x^2\ \ and\ \ g(x)=4x^3$ 2) $h(x)=\tan x^2\ \ and\ \ r(x)=2x\sec^2 x^2$

b) Verify the formula by differentiation.

1) $\int(7x+2)^3dx = \frac{(7x+2)^4}{28}+C$, 2) $\int x\sin 2x\,dx = \frac{\sin 2x}{4}-\frac{x\cos 2x}{2}+C$, 3) $\int \left(\frac{\sec x}{1+\tan x}\right)^2 dx = -\frac{1}{1+\tan x}+C$

c) Evaluate the following integrals.

1) $\int(4x+3)\,dx$, 2) $\int(9x^2-4x+3)\,dx$, 3) $\int\left(\frac{1}{x^3}-\frac{3}{x^2}\right)dx$, 4) $\int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$, 5) $\int(3x-1)^2\,dx$

6) $\int\frac{(8x-5)}{\sqrt[3]{x}}dx$, 7) $\int\frac{x^3-1}{x-1}dx$, 8) $\int\frac{7}{\csc x}dx$, 9) $\int(\sqrt{x}+\cos x)\,dx$, 10) $\int\frac{\sec x}{\cos x}dx$

11) $\int(\sec x\cos x\sec^2 x)\,dx$, 12) $\int\frac{\sec x\sin x}{\cos x}dx$, 13) $\int(5\sin^2 x+5\cos^2 x)dx$, 14) $\int\sin^2 x\,dx$

15) $\int\frac{d}{dx}\left(\sqrt{x^2+4}\right)dx$, 16) $\frac{d}{dx}\left(\int\left(\sin\sqrt[3]{x}\right)\right)dx$, 17) $\int\sqrt[7]{(9x^2-12x+4}\ dx$ 18) $\int(-7x+8)^3\,dx$,

19) $\int\csc^2(8x-1)\,dx$, 20) $\int\frac{8}{3}\cos(9x)\,dx$, 21) $\int\sqrt[3]{(2x+1)^5}\,dx$, 22) $\int\sec(-2x)\tan(-2)\,dx$

c) Evaluate the integral using substitution and express the answer in terms of x.

1) $\int x(2x^2+3)^{10}\,dx$, $u=2x^2+3$, 2) $\int\frac{x}{(x^2+5)^3}dx$, $u=x^2+5$, 3) $\int\frac{(1+\sqrt{x})^3}{\sqrt{x}}dx$, $u=1+\sqrt{x}$

4) $\int\sqrt{x}\cos\sqrt{x^3}\,dx$, $u=\sqrt{x^3}$, 5) $\int x^2\sqrt{x^3-1}\,dx$, 6) $\int\frac{x}{\sqrt{1-2x^2}}dx$, 7) $\int 3\sin 4x\,dx$

8) $\int\frac{\cos\sqrt[3]{x}}{\sqrt[3]{x^2}}dx$, 9) $\int\frac{\sin x}{\cos^4 x}dx$, 10) $\int\sin(3x)\sec^2(\cos(3x))\,dx$

d) Solve the initial value problems

1) $f'(x)=6x^2+x-5$, $f(1)=2$, 2) $\frac{dy}{dx}=-\pi\sin(\pi x)$, $y(0)=0$

3) $f''(x)=5\cos x+2\sin x$, $f(0)=3\ and\ f'(0)=4$

Answers

b) 1) $\frac{(4x+3)^2}{8}+C$, 2) $3x^3-2x^2+3x+C$, 3) $-\frac{1}{2x^2}+\frac{3}{x}+C$, 4) $2\sqrt{x^3}+2\sqrt{x}+C$, 5) $\frac{(3x-2)^3}{9}+C$

6) $\frac{24}{5}x^{5/3}-\frac{15}{2}x^{2/3}+C$, 7) $\frac{x^3}{3}+\frac{x^2}{2}+x+C$, 8) $-7\cos x+C$, 9) $\frac{2}{3}x^{3/2}+\sin x+C$

10) $\tan x+C$, 11) $\tan x+C$, 12) $\sec x+C$, 13) $5x+c$, 14) $\frac{1}{2}\left(1-\frac{1}{2}\sin 2x\right)+c$

15) $\sqrt{x^2+4}+C$, , 16) $\sin\sqrt[3]{x}$, 17) $\frac{7\sqrt[7]{(3x-2)^9}}{7}+c$, 18) $-\frac{(-7x+8)^4}{28}+C$, 19) $-\frac{1}{8}\cot(8x-1)+C$

20) $\frac{8}{27}\sin(9x)+C$, 21) $\frac{3\,(2x+1)^{8/3}}{16}+C$, 22) $-\frac{1}{2}\sec(-2x)+c$

c) 1) $\frac{(2x^2+3)^{11}}{44}+C$, 2) $\frac{-1}{4(x^2+5)^2}+C$, 3) $\frac{(1+\sqrt{x})^4}{2}+C$, 4) $\frac{2}{3}\sin\sqrt{x^3}+C$, 5) $\frac{2}{9}\,(x^3-1)^{3/2}+C$

6) $-\frac{1}{2}\sqrt{1-2x^2}+C$, 7) $-\frac{3}{4}\cos 4x+C$, 8) $3\sin\sqrt[3]{x}+C$, 9) $\frac{1}{3\cos^3 x}+C$, 10) $-\frac{1}{3}\tan(3x)+c$

d) 1) $f(x)=2x^3+\frac{x^2}{2}-5x+\frac{9}{2}$, 2) $y=\cos(\pi x)-1$, 3) $f(x)=-5\cos x-2\sin x+6x+8$

Summation notation

Definition: Sigma notation: Let $a_1, a_2, \ldots, a_n$ be n numbers, The sum $a_1+a_2+\cdots+a_n$ will be denoted in sigma notation by the symbol $\sum_{i=1}^{n} a_i = a_1+a_2+\cdots+a_n$. Which is read as sum of a sub i as i goes from 1 to n

Example: Evaluate 1) $\sum_{i=1}^{4}(i-3)$, 2) $\sum_{i=1}^{3} i^2$

Solution:

1) $\sum_{i=1}^{4}(i-3)=(1-3)+(2-3)+(3-3)+(4-3)=-2+(-1)+0+1=-2$

2) $\sum_{i=1}^{3} i^2 = 1^2+2^2+3^2=1+4+9=14$

Theorem:

1) $\sum_{i=1}^{n} c = nc$, 2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, 3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, 4) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: Evaluate 1) $\sum_{i=1}^{4}(i-3)$, 2) $\sum_{i=1}^{3} i^2$ (using the last theorem)

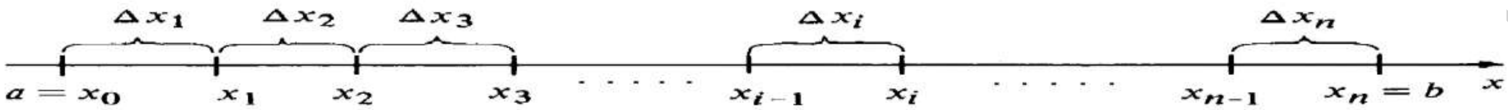
Solution:

1) $\sum_{i=1}^{4}(i-3)=\sum_{i=1}^{4} i-\sum_{i=1}^{4} 3=\frac{(4)(4+1)}{2}-(3)(4)=10-12=-2$

2) $\sum_{i=1}^{3} i^2 = \frac{(3)(3+1)(2(3)+1)}{6}=\frac{84}{6}=14$

The definite integral

Definition: Let f the function defined on a closed interval $[a,b]$ .Divide $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$ by the set of points $P=\{x_0,x_1,\ldots,x_n\}$ such that $x_0=a<x_1<x_2<\cdots<x_n=b$



Choose $w_i \in [x_{i-1},x_i]$ and put $\Delta x_i = x_i - x_{i-1}$

The definit integral of $f(x)$ with respect to x from a to b is denoted by $\int_a^b f(x)\,dx$ and is defined as

$\int_a^b f(x)\,dx = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x_i$ provided that the Limit is exist .

We remark the following notes about the above notations

1- P is called a partition of $[a,b]$

2- The sum $\sum_{i=1}^{n} f(w_i)\Delta x_i$ is called Riemann Sum of f for P and $\int_a^b f(x)\,dx$ is called Riemann Integral of f with respect to x from a to b

3- Since the $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x_i$ is independently of the choice of the points w_i ,we my take our choice in any convenient way ,so that we choose the sub intervals $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ all are equal and choose $w_i = x_i$,then $\Delta x_n = \frac{b-a}{n}$ and $w_i = a + i\left(\frac{b-a}{n}\right)$

4- If $f(x) \geq 0$ then

i) $\sum_{i=1}^{n} f(w_i)\Delta x_i$ is the area of inscribed rectangles , see figure (1)

ii) $\int_a^b f(x)\,dx$ is the area of the region under the graph of f from a to b , see figure (2)

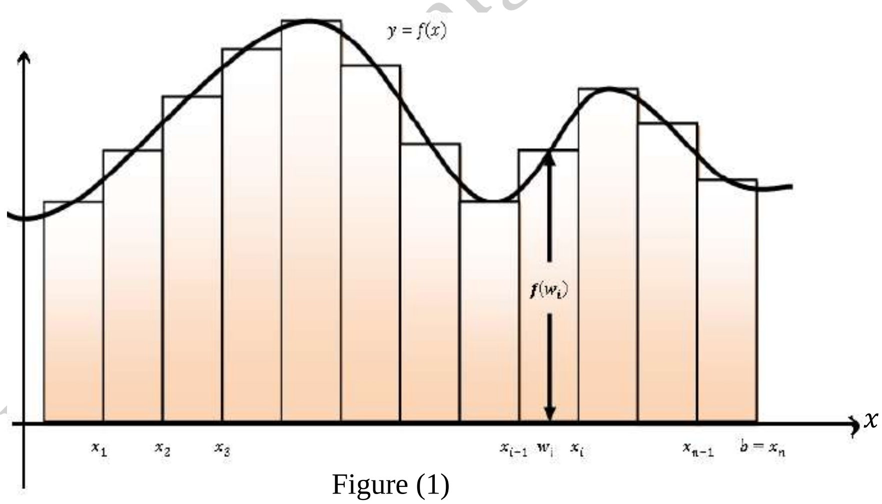


Figure (1)

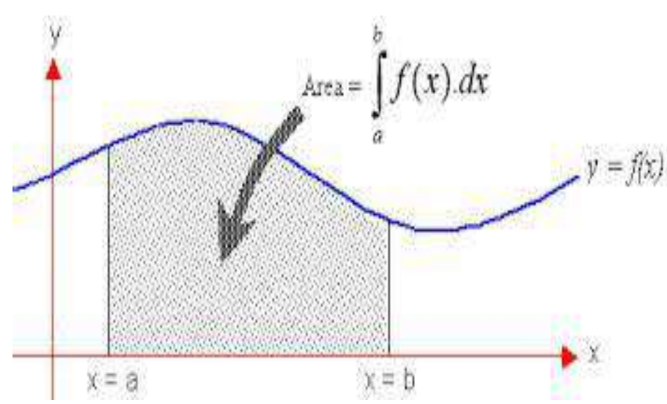


Figure (2)

Example : Evaluate the following integrals by definition integral

1) $\int_0^1 5x^2\,dx$, 2) $\int_1^4 (x^2+2x-5)\,dx$

Solution :

1) Since $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$ and $w_i = a + i\Delta x = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$

Then $\int_0^1 5x^2\,dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(w_i)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 5\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{5}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{5}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{5}{6}\frac{n(n+1)(2n+1)}{n^3}$$

$$= \frac{5}{6}\times 2 = \frac{5}{3}$$

2) Since $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$ and $w_i = a + i\Delta x = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$

Then $\int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left(\frac{9i^2}{n^2}+\frac{12i}{n}-2\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left(\frac{9}{n^2}i^2+\frac{12}{n}i-2\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2 + \frac{12}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{12}{n}\frac{n(n+1)}{2} - 2n\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{6}\frac{n(n+1)(2n+1)}{n^3} + \frac{36}{2}\frac{n(n+1)}{n^2} - \frac{6n}{n}\right]$$

$$= \frac{27}{6}\times 2 + 18\times 1 - 6\times 1 = 9 + 18 - 6 = 21$$

Properties of definite integral

Let f and g be continuous function , and let c be a constant , then

1) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
3) $\int_a^a f(x)\,dx = 0$. *For example* $\int_2^2 \sqrt[3]{x^6-1}\,\sin x^3\,dx = 0$
4) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
5) If a, b *and* c are numbers , $a < c < b$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Example: If $\int_{-2}^5 f(x)\,dx = 6$ *and* $\int_5^1 4\,f(x)\,dx = 12$ *find* $\int_{-2}^1 7\,f(x)\,dx$

Solution :

Since $-2 \le 1 \le 5$, then $\int_{-2}^5 f(x)\,dx = \int_{-2}^1 f(x)\,dx + \int_1^5 f(x)\,dx$ ……… $(*)$

Since $\int_5^1 4\,f(x)\,dx = 12$ then $\int_5^1 f(x)\,dx = 3 \Longrightarrow \int_1^5 f(x)\,dx = -3$

Now by (*) we get

$$6 = \int_{-2}^1 f(x)\,dx + (-3) \Longrightarrow \int_{-2}^1 f(x)\,dx = 6+3$$

$$\Longrightarrow \int_{-2}^1 7f(x)\,dx = 7 \times 9 = 63$$

Continue for the properties of definite integral

6) If $f(x) \ge 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)\,dx \ge 0$
7) If $f(x) \ge g(x)$, $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Example : Verify the inequalities without evaluating the integrals

1) $\int_1^2 (2x-4)\,dx \le 0$, 2) $\int_1^2 x^2\,dx \le \int_1^2 x^3\,dx$

Solution :

1) As $1 \le x \le 2$

$\Rightarrow 2 \le 2x \le 4$

$\Rightarrow 2-4 \le 2x-4 \le 4-4$

$\Rightarrow -2 \le 2x-4 \le 0$

$\Rightarrow 2x-4 \le 0 \quad \Longrightarrow \int_1^2 (2x-4)\,dx \le 0$

2) $x^2 - x^3 = x^2(1-x)$

And as $1 \le x \le 2 \quad , x^2 > 0$

$\Rightarrow -1 \ge -x \ge -2 \quad , x^2 > 0$

$\Rightarrow 1-1 \ge 1-x \ge 1-2 \quad , x^2 > 0$

$\Rightarrow 0 \ge 1-x \ge -1 \quad , x^2 > 0$

$\Rightarrow 1-x \le 0 \quad , x^2 > 0$

Then $x^2(1-x) \le 0$

$\Rightarrow x^2 - x^3 \le 0 \Rightarrow x^2 \le x^3$

$$\Rightarrow \int_1^2 x^2\,dx \le \int_1^2 x^3\,dx$$

Continue for the properties of definite integral

8) If m and M are two numbers and $m \le f(x) \le M , \forall x \in [a,b]$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

We called $\boldsymbol{m(b-a)}$ a lower bound of f and $\boldsymbol{M(b-a)}$ a upper bound of f

Example : Find the lower bound and upper bound of the following integrals

1) $\int_{-3}^{1} (5x^2 - 7)\,dx$, 2) $\int_1^3 \sqrt{2x^2+7}\,dx$

Solution :

1) $\int_{-3}^{1} (5x^2 - 7)\,dx$

Since $-3 \le x \le 1$, then $0 \le x^2 \le 9$

$\Longrightarrow 0 \le 5x^2 \le 45$

$\Longrightarrow -7 \le 5x^2 - 7 \le 38$

$$\Longrightarrow -7\big(1-(-3)\big) \le \int_{-3}^{1}(5x^2-7)\,dx \le 38\,\big(1-(-3)\big)$$

$$\Longrightarrow -28 \le \int_{-3}^{1}(5x^2-7)\,dx \le 152$$

Then the lower bound is -28 and the upper bound 152

2) $\int_1^3 \sqrt{2x^2+7}\,dx$ Exercise

The lower bound is 6 and the upper bound 10

Mean value theorem for definite integral

Definition : If f is continuous on a closed interval $[a,b]$, then there is a number c in the closed interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Example: Find the number c that satisfies the mean value theorem for the following integrals

1) $\int_0^1 5x^2\,dx$, 2) $\int_1^4 (x^2+2x-5)\,dx$

Solution:

1) From the previous example we found

$\int_0^1 5x^2\,dx = \frac{5}{3}$, then by mean value there exist a number c in $[0,1]$ such that

$\frac{5}{3} = 5c^2(1-0)$ (s.t $f(c) = 5c^2$)

$\Rightarrow \frac{5}{3} = 5c^2 \Rightarrow c^2 = \frac{1}{3} \Rightarrow c = \pm\frac{1}{\sqrt{3}}$

But $-\frac{1}{\sqrt{3}} \notin [0,1]$ then $c = \frac{1}{\sqrt{3}}$

2) From the previous example we found

$\int_1^4 (x^2+2x-5)\,dx = 21$ then by mean value there exist a number c in $[1,4]$ such that

$21 = (c^2+2c-5)(4-1)$ (s.t $f(c) = c^2+2c-5$)

$\Rightarrow 21 = 3(c^2+2c-5)$

$\Rightarrow c^2+2c-5 = 7$

$\Rightarrow c^2+2c-12 = 0$

$\Rightarrow c = \frac{-2\pm\sqrt{52}}{2}$

$\Rightarrow c \approx 2.61 \;\; or \;\; c \approx -4.6$

But $-4.6 \notin [1,4]$ then $c \approx 2.61$

The fundamental theorem of calculus

Suppose f is continuous on a closed interval $[a,b]$, then $\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$ Such that F is any antiderivative of f on $[a,b]$

Example: evaluate the following integrals

1) $\int_{-1}^{4}(6x^2-5)\,dx$, 2) $\int_0^{\pi}\sin x\,dx$, 3) $\int_0^9 \frac{dx}{(x+3)^2}$, 4) $\int_1^2 x\,(x^2-1)^5\,dx$

Solution:

1) $\int_{-1}^{4}(6x^2-5)\,dx = [2x^3-5x\,]_{-1}^{4} = [2(4)^3-5(4)] - [2(-1)^3-5(-1)]$

$$=[128-20]—[-2+5] = 105$$

2) $\int_0^{\pi}\sin x\,dx = [-\cos x]_0^{\pi} = [-\cos\pi]—\cos 0 = -(-1)+1 = 2$

3) $\int_0^9 \frac{dx}{(x+3)^2} = \int_0^9 (x+3)^{-2}\,dx = -(x+3)^{-1}|_0^9 = [-(9+3)^{-1}]—(0+3)^{-1}$

$$=\frac{-1}{12}+\frac{1}{3}=\frac{-1+4}{12}=\frac{3}{12}=\frac{1}{4}$$

4) $\int_1^2 x\,(x^2-1)^5\,dx$ Exercise (Ans : $\frac{243}{4}$)

Example : Find the number c that satisfies the mean value theorem for $\int_{-2}^{3}(3x^2+2x)\,dx$

Solution :

First we will find $\int_{-2}^{3}(3x^2+2x)\,dx$

$\int_{-2}^{3}(3x^2+2x)\,dx = [\,x^3+x^2]_{-2}^{3} = [27+9]-[-8+4] = 40$

Now by mean value there exist a number c in $[-2\,,3\,]$ such that

$$(3c^2+2c)(5) = 40$$

$$\Longrightarrow 3c^2+2c = 8$$

$$\Longrightarrow 3c^2+2c-8 = 0$$

$$\Longrightarrow (3c-4)(c+2) = 0$$

$$Ether\ 3c-4=0 \Longrightarrow c=\frac{4}{3}\in[-2\,,3\,]\ \ or\ c+2=0 \Longrightarrow c=-2\ \in[\,-2\,,3\,]$$

Then the values of c are -2 or $\frac{4}{3}$

Numerical integration

As we have seen, the ideal way to evaluate a definite integral $\int_a^b f(x)\,dx$ is to find a formula $F(x)$ for one of the antiderivatives of $f(x)$ and calculate the number $F(b) - F(a)$ .But some antiderivatives require considerable work to find, and still others, like the antiderivatives of $\sin(x^2)$, $\frac{1}{lnx}$ and $\sqrt{1+x^4}$ have no elementary formulas.

Another situation arises when a function is defined by a table whose entries were obtained experimentally through instrument readings. In this case a formula for the function may not even exist. Whatever the reason, when we cannot evaluate a definite integral with an antiderivative, we turn to numerical methods such as the $Trapezoidal\ Rule$ and $Simpson's\ Rule$.

The Trapezoidal Rule

$$I = \int_{x_0=a}^{x_n\ =b} f(x)\,dx = \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots + y_{n-1} + y_n)$$

$$= \frac{h}{2}[y_0 + y_n + 2(y_1 + y_2 + y_3 + y_4 + \cdots + y_{n-1})]$$

Where $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ... , $x_n = b$ and $h = \frac{b-a}{n}$

Example : Approximate $\int_0^1 \frac{1}{x^2+1}dx$ by using the Trapezoidal Rule with $n = 4$.(take 6 decimal places)

Solution : Since $h = \frac{b-a}{n}$ then $h = \frac{1-0}{4} = \frac{1}{4}$

i	0	1	2	3	4
x_i	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y_i = f(x_i)$	$\frac{1}{0+1} = 1$	$\frac{1}{\left(\frac{1}{4}\right)^2 + 1} = \frac{16}{17}$	$\frac{1}{\left(\frac{1}{2}\right)^2 + 1} = \frac{4}{5}$	$\frac{1}{\left(\frac{3}{4}\right)^2 + 1} = \frac{16}{25}$	$\frac{1}{(1)^2 + 1} = \frac{1}{2}$

Since Trapezoidal Rule formula as $\int_a^b f(x)\,dx = \frac{h}{2}(\ y_0 + 2y_1 + 2y_2 + \ .\ .\ . + 2y_{n-1} + y_n)$

Then $\int_0^1 \frac{1}{x^2+1}dx = \frac{\frac{1}{4}}{2}\left(\ 1 + 2\left(\frac{16}{17}\right) + 2\left(\frac{4}{5}\right) + 2\left(\frac{16}{25}\right) + \frac{1}{2}\right) = \frac{5323}{6800} = 0.782794$

The Simpson's Rule

$$I = \int_{x_0}^{x_n = x_0 + nh} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1\ + y_2 + y_2 + 4y_3 + y_4 + y_4 + \cdots + y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

Where $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ... , $x_n = b$ and $h = \frac{b-a}{n}$, n is even integer

Example : Use the Simpson's Rule method with $n = 4$ to estimate $\int_0^1 \frac{1}{x^2+1} dx$.(take 6 decimal places)

Solution : Again $h = \frac{1}{4}$ and $f(x) = \frac{1}{x^2+1}$ $\mathit{so\ we\ will\ use\ the\ table\ in\ the\ previouse\ example}$

i	0	1	2	3	4
x_i	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y_i = f(x_i)$	$\frac{1}{0+1} = 1$	$\frac{1}{\left(\frac{1}{4}\right)^2 + 1} = \frac{16}{17}$	$\frac{1}{\left(\frac{1}{2}\right)^2 + 1} = \frac{4}{5}$	$\frac{1}{\left(\frac{3}{4}\right)^2 + 1} = \frac{16}{25}$	$\frac{1}{(1)^2 + 1} = \frac{1}{2}$

Since Simpson's Rule formula as $\int_a^b f(x)\, dx = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$

Then $\int_0^1 \frac{1}{x^2+1} dx = \frac{\frac{1}{4}}{3}\left(1 + 4\left(\frac{16}{17}\right) + 2\left(\frac{4}{5}\right) + 4\left(\frac{16}{25}\right) + \frac{1}{2}\right) = \frac{8011}{10200} = 0.785392$

Exercise : Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\, dx$ by using 1) Trapezoidal Rule 2) Simpson's Rule from the following table.(take 5 decimal places)

x	-1.57080	-1.17810	-0.78540	-0.39270	0	0.39270	0.78540	1.17810	1.57080
y	0.0	0.99138	1.26906	1.05961	0.75	0.48821	0.28946	0.13429	0

Ans : 1) 1.95643 2) 2.00421

Exercises

a) Evaluate the sums.

1) $\sum_{i=1}^{5}(3i-10)$, 2) $\sum_{k=1}^{4}(k^2+1)$, 3) $\sum_{k=1}^{8} 2^k$

b) Evaluate the following integrals by the definition definite integral.

1) $\int_0^4 5dx$, 2) $\int_0^2 x^2dx$, 3) $\int_1^3(x^2+x+1)dx$, 4) $\int_0^2(x^2+3)dx$

c) If $\int_6^2 g(x)\,dx = \int_{-1}^3 6x^2\,dx$ and $\int_{-5}^6 4\,g(x)\,dx = 28$ find

1) $\int_2^6 g(x)\,dx$, 2) $\int_{-5}^6 g(x)\,dx$, 3) $\int_{-5}^2 g(x)\,dx$

d) Verify the inequalities without evaluating the integrals.

1) $\int_1^2(3x^2+4)dx \geq \int_1^2(2x^2+5)dx$, 2) $\int_1^3(x^2-1)\,dx \geq 0$

3) $\int_0^1 \frac{\cos\frac{\pi}{2}x}{x^3-3}dx \leq 0$, 4) $\int_1^4 x^2dx \leq \int_1^4 x^3dx$

e) Find the lower bound and upper bound of the following integrals

1) $\int_0^6 x^3\,dx$, 2) $\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1+\sin x}}dx$, 3) $\int_{-2}^{-1}\frac{dx}{2x^2-1}$, 4) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin 2x\,dx$

f) Find the number 'c' that satisfies the mean value theorem for integrals.

1) $\int_0^3 3x^2dx = 27$, 2) $\int_{-1}^7 \sqrt[3]{x+1}dx = 12$, 3) $\int_1^4(x^2+1)\,dx$, 4) $\int_{-1}^2(x^2-2x)\,dx$

g) Evaluating the following integrals.

1) $\int_1^4(x^2-4x-3)\,dx$, 2) $\int_{-2}^3 (8z^3+3z-1)\,dz$, 3) $\int_7^{12} dx$, 4) $\int_1^2\left(\frac{5}{8x^6}\right)dx$, 5) $\int_4^9 \frac{t-3}{\sqrt{t}}dt$

6) $\int_1^2 \frac{w+1}{\sqrt{w^2+2w}}dw$, 7) $\int_0^{\frac{\pi}{2}} 5\cos^2 x\sin x\,dx$, 8) $\int_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin\left(\frac{1}{x}\right)}{x^2}dx$, 9) $\int_0^{\frac{3\pi}{2}} |\cos x|dx$, 10) $\int_0^{\frac{\pi}{3}}(x-\sec x\,\tan x)dx$

11) $\int_1^2(2x+1)\sqrt{x^2+x+1}\,dx$, 12) $\int_{-1}^3|x-2|\,dx$, 13) $\int_0^3 f(x)\,dx$ where $f(x)=\begin{cases} x^2+1 & , x\geq 2 \\ 3x-1 & , x\leq 2\end{cases}$

h) Evaluate the following integrals by using the Trapezoidal Rule. (take 4 d.p)

1) $\int_0^2 \sin\sqrt{x}\,dx$, $n=2$, 2) $\int_1^2 \frac{2^x}{x}dx$, $n=3$, 3) $\int_1^3 \cos x^2\,dx$, $n=4$

i) Evaluate the following integrals by using the Simpson's Rule. (take 4 d.p)

1) $\int_0^1 \frac{dx}{x^3+1}$, $n=2$, 2) $\int_0^1 \frac{dx}{x^4+1}$. $n=4$

Answers

a) 1) 20 , 2) $\frac{8}{3}$, 3) $\frac{44}{3}$, 4) $\frac{26}{3}$

b) 1) -5 , 2) 34 , 3) 510 , **c)** 1) -56 , 2) 7 , 3) 63

e) 1) $L: 125$ $U: 216$, 2) $L: \frac{\pi}{2\sqrt{2}}$ $U: \frac{\pi}{2}$, 3) $L: \frac{1}{7}$ $U: 1$, 4) $L: \frac{\sqrt{13}\pi}{24}$, $U: \frac{\pi}{12}$

f) 1) $c = \sqrt{3}$, 2) $c \approx 2.38$, 3) $c = \sqrt{7}$, 4) $c = 0\,,\ 2$

g) 1) -18 , 2) $\frac{265}{2}$, 3) 5 , 4) $\frac{31}{256}$, 5) $\frac{20}{3}$, 6) $2\sqrt{2} - \sqrt{3} \approx 1.096$, 7) $\frac{5}{3}$, 8) 1

9) 3 , 10) $\frac{\pi^2}{18} - 1 \cong -0.45$, 11) $-2\sqrt{3} + \frac{14}{3}\sqrt{7} \cong 8.88$, 12) 5 , 13) $\frac{34}{3}$

h) 1) 1.3354 , 2) 1.9316 , 3) -0.2339 **i)** 1) 0.8426 2) 0.8671

Chapter 2: Transcendental functions

Logarithmic and exponential function

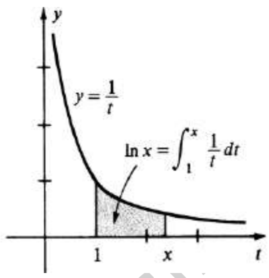


Figure 1

Definition: The **natural logarithm function,** denoted by ln,is defined by

$$\ln x = \int_1^x \frac{1}{t}dt \qquad \text{for every } x > 0$$

Theorem: If $u = g(x) \neq 0$ and g is differentiable .then

$$\int \frac{1}{u}du = \ln|u| + C$$

Note: $\int \frac{1}{u}du = \int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C \qquad (\ du = g'(x)dx \ \text{because} \ u = g(x) \ )$

Example: Evaluate the following integrals

1) $\int \frac{8x}{x^2-1}dx$, 2) $\int \frac{x}{9x^2+5}dx$, 3) $\int_2^4 \frac{1}{9-2x}dx$, 4) $\int_1^2 \frac{6x^2-2}{x^3-x+1}dx$

Solution:

1) $\int \frac{8x}{x^2-1}dx = 4\int \frac{2x}{x^2-1}dx = 4\ln|x^2-1| + C$

<u>Another solution by substitution</u> : Let $u = x^2 - 1 \implies du = 2x\,dx$

$$\implies dx = \frac{du}{2x}$$

$$\implies \int \frac{8x}{x^2-1}\,dx = \int \frac{8x}{u}\frac{du}{2x}$$

$$= \int \frac{4du}{u}$$

$$= 4\ln u + c = 4\ln|x^2-1| + c$$

2) $\int \frac{x}{9x^2+5}dx = \frac{1}{18}\int \frac{18x}{9x^2+5}dx = \frac{1}{18}\ln|9x^2+5| + C$

3) As $\int \frac{1}{9-2x}dx = \frac{1}{-2}\int \frac{-2}{9-2x}dx = \frac{1}{-2}\ln|9-2x| + C$, then

$$\int_2^4 \frac{1}{9-2x}dx = \left[\frac{-1}{2}\ln|9-2x|\right]_2^4 = \frac{-1}{2}[\ln|9-2(4)| - \ln|9-2(2)|] = \frac{-1}{2}[\ln|1| - \ln|5|]$$

$$= \frac{-1}{2}[0 - \ln|5|] = \frac{1}{2}\ln 5$$

4) As $\int \frac{6x^2-2}{x^3-x+1}dx = \int \frac{2(3x^2-1)}{x^3-x+1}dx = 2\int \frac{3x^2-1}{x^3-x+1}dx = 2\ln|x^3-x+1| + C$, then

$$\int_1^2 \frac{6x^2-2}{x^3-x+1}dx = [2\ln|x^3-x+1|]_1^2 = 2[\ln|(2)^3-2+1| - \ln|(1)^3-1+1|] = 2[\ln 7 - \ln 1]$$

$$= 2[\ln 7 - 0] = 2\ln 7$$

Theorem:

1) $\int e^{bx+d}\,dx = \frac{1}{b}e^{bx+d} + C$ 2) $\int e^{g(x)}\ g'(x)dx = \ e^{g(x)} + C$

3) $\int a^{bx+d}\,dx = \frac{1}{b\ln a}\ a^{bx+d} + C$ 4) $\int a^{g(x)}\ g'(x)dx = \frac{1}{\ln a}\ a^{g(x)} + C$

Example: Evaluate the following integrals

1) $\int e^{3x+7}dx$, 2) $\int_0^1 2^{3x}\,dx$, 3) $\int 3^{x^2}\,x\,dx$, 4) $\int_0^{\frac{\pi}{4}} \left(\frac{1}{3}\right)^{\tan x} \sec^2 x\,dx$

Solution:

1) $\int e^{3x+7}dx = \frac{1}{3}e^{3x+7} + c$

2) $\int_0^1 2^{3x}\,dx = \frac{1}{3\ln 2}2^{3x}\,|_0^1 = \frac{1}{3\ln 2}2^0 - \frac{1}{3\ln 2}2^3 = \frac{1}{3\ln 2}[1-8] = -\frac{7}{3\ln 2}$

3) $\int 3^{x^2}\,x\,dx = \frac{1}{2}\int 3^{x^2}\,2x\,dx = \frac{1}{2}\,\frac{1}{\ln 3}\,3^{x^2} + C = \frac{1}{2\ln 3}\,3^{x^2} + C$

<u>Another solution by substitution</u> Let $u = x^2$

$$\Rightarrow du = 2x\,dx$$
$$\Rightarrow dx = \frac{du}{2x}$$
$$\Rightarrow \int 3^{x^2}x\,dx = \int 3^u x\,\frac{du}{2x}$$
$$= \frac{1}{2}\int 3^u\,du$$
$$= \frac{1}{2\ln 3}3^u + c = \frac{1}{2\ln 3}3^{x^2} + c$$

4) $\int_0^{\frac{\pi}{4}} \left(\frac{1}{3}\right)^{\tan x} \sec^2 x\,dx = \left[\frac{1}{\ln\frac{1}{3}}\left(\frac{1}{3}\right)^{\tan x}\right]_0^{\pi/4} = \left[\frac{1}{-\ln 3}\left(\frac{1}{3}\right)^{\tan x}\right]_0^{\pi/4} = \left[\frac{1}{-\ln 3}\left(\frac{1}{3}\right)^{\tan\frac{\pi}{4}} - \frac{1}{-\ln 3}\left(\frac{1}{3}\right)^{\tan 0}\right]$

$$= \left[\frac{1}{-\ln 3}\left(\frac{1}{3}\right)^1 - \frac{1}{-\ln 3}\left(\frac{1}{3}\right)^0\right] = \left[\frac{-1}{\ln 3}\left(\frac{1}{3}\right) + \frac{1}{\ln 3}(1)\right]$$
$$= \frac{1}{\ln 3}\left[-\frac{1}{3} + 1\right] = \frac{1}{\ln 3}\left(\frac{2}{3}\right) = \frac{2}{3\ln 3}$$

Example: Evaluate

1) $\int \frac{\log_{10} x}{x}dx$, 2) $\int_0^2 \frac{\log_{10}(x+2)}{x+2}dx$

Solution:

1) $\int \frac{\log_{10} x}{x}dx$

Let $u = \log_{10} x \Rightarrow du = \frac{1}{x\ln 10}\,dx \Rightarrow dx = x\ln 10\;\,du$

$$\Rightarrow \int \frac{\log_{10} x}{x}dx = \int \frac{u}{x}x\ln 10\;\,du = \ln 10\int u\,du = \ln 10\,\frac{u^2}{2} + C$$
$$= \ln 10\,\frac{(\log_{10} x)^2}{2} + C = \frac{1}{2\ln 10}\,(\ln x)^2 + C$$

2) Let $u = \log_2(x+2) \Rightarrow du = \frac{1}{(x+2)\ln 2}dx \Rightarrow dx = (x+2)\ln 2\;du$

$$\Rightarrow \int_0^2 \frac{\log_2(x+2)}{x+2}dx = \int_{u(0)}^{u(2)} \frac{u}{x+2}(x+2)\ln 2\;du = \ln 2\int_{u(0)}^{u(2)} u\,du$$
$$= \ln 2\left[\frac{u^2}{2}\right]_{u(0)}^{u(2)}$$
$$= \ln 2\,[\,(\log_2(x+2))^2]_0^2$$
$$= \ln 2\left[\left(\frac{(\log_2 4)^2}{2} - \frac{(\log_2 2)^2}{2}\right)\right]$$
$$= \ln 2\left[\frac{4}{2} - 1\right]$$
$$= \frac{3}{2}\ln 2$$

Integral formulas for Inverse Trigonometric Functions

1) $\int \frac{g'(x)}{\sqrt{a^2-b^2(g(x))^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{b}{a}g(x)\right)+C$ such that $a,b>0$

2) $\int \frac{g'(x)}{a^2+b^2(g(x))^2}dx = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}g(x)\right)+C$ such that $a,b>0$

3) $\int \frac{g'(x)}{|g(x)|\sqrt{b^2(g(x))^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{b}{a}g(x)\right)+C$ such that $a,b>0$

Example: Find the following integrals

1) $\int \frac{dx}{\sqrt{1-9x^2}}$, 2) $\int \frac{dx}{|x|\sqrt{3x^2-5}}$, 3) $\int \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}d\theta$, 4) $\int \frac{x+1}{\sqrt{3-2x-x^2}}dx$, 5) $\int \frac{4}{5+2x+x^2}dx$

Solution:

1) $\int \frac{dx}{\sqrt{1-9x^2}}$ $a=1\,,b=3\,,g(x)=x\ \ and\ g'(x)=1$

$$\Rightarrow \int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\sin^{-1}(3x)+C$$

2) $\int \frac{dx}{|x|\sqrt{3x^2-5}}$ $a=\sqrt{5}\,,b=\sqrt{3}\,,g(x)=x\ \ and\ g'(x)=1$

$$\Rightarrow \int \frac{dx}{|x|\sqrt{3x^2-5}} = \frac{1}{\sqrt{5}}\sec^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}x\right)+C$$

3) $\int \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}d\theta$ $a=1\,,b=1\,,g(x)=\cos\theta\ \ and\ g'(x)=-\sin\theta$

$$\Rightarrow \int \frac{\sin\theta}{\sqrt{1-\cos^2\theta}}d\theta = -\int \frac{-\sin\theta}{\sqrt{1-\cos^2\theta}}d\theta = -\sin^{-1}(\cos\theta)+c$$

4) $\int \frac{x+1}{\sqrt{3-2x-x^2}}dx$

Let $u = 3-2x-x^2 \Rightarrow du = (-2-2x)dx \Rightarrow dx = \frac{du}{-2-2x} = \frac{du}{-2(1+x)}$

$$\Rightarrow \int \frac{x+1}{\sqrt{3-2x-x^2}}dx = \int \frac{x+1}{\sqrt{u}}\frac{du}{-2(1+x)} = \frac{1}{-2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}du = -\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+C = -u^{\frac{1}{2}}+C = -\sqrt{u}+C$$

$$= -\sqrt{3-2x-x^2}+C$$

5) $\int \frac{4}{5+2x+x^2}dx$

By completing the square of $5+2x+x^2$

Since $(x^2+2x+1)-1+5=(x+1)^2+4$

then $\int \frac{4}{5+2x+x^2}dx = \int \frac{4}{(x+1)^2+4}dx$ $where\ \ a=2\ ,\ b=1\ ,\ g(x)=x+1\ \ and\ g'(x)=1$

$$\Rightarrow \int \frac{4}{5+2x+x^2}dx = \int \frac{4}{(x+1)^2+4}dx$$

$$= 4\int \frac{1}{(x+1)^2+4}dx = 4\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

$$= 2\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

Integral formulas for hyperbolic functions

1) $\int \sinh x \, dx = \cosh x + c$

2) $\int \cosh x \, dx = \sinh x + c$

3) $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

4) $\int \operatorname{csch}^2 x \, dx = -\coth x + c$

5) $\int \operatorname{sech} x \, \tanh x \, dx = -\operatorname{sech} x + c$

6) $\int \operatorname{csch} x \, \coth x = -\operatorname{csch} x + c$

Note : If x is replaced by $ax+b$ in the above standard, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants ,then the standard forms remain true provided the result on R.H.S is divided by a for example $\int \sinh(ax+b)\, dx = \frac{1}{a}\cosh(ax+b) + C$

In general

if $\int f(x)\, dx = \varphi(x) + C$, then $\int f(ax+b)\, dx = \frac{1}{a}\varphi(ax+b) + C$

Example: Evaluate

1) $\int \operatorname{csch}^2(3-2x)\, \boldsymbol{dx}$ 2) $\int \cosh^2 x \, dx$ 3) $\int x^2 \operatorname{sech}^2 x^3 \, dx$ 4) $\int_0^{\ln 2} 4e^x \sinh x$

Solution:

1) $\int \operatorname{csch}^2(3-2x)\, dx = -\dfrac{1}{-2}\coth(3-2x) + c = \dfrac{1}{2}\coth(3-2x) + c$

1) $\int \cosh^2 x \, dx = \int \frac{\cosh 2x + 1}{2} dx = \frac{1}{2}\int (\cosh 2x + 1)\, dx = \frac{1}{2}\left(\frac{1}{2}\sinh 2x + x\right) + C$

2) $\int x^2 \operatorname{sech}^2 x^3 \, dx$

Let $u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$

$\Rightarrow \int x^2 \operatorname{sech}^2 x^3 \, dx = \int x^2 \operatorname{sech}^2 u \frac{du}{3x^2} = \frac{1}{3}\int \operatorname{sech}^2 u \, du = \frac{1}{3}\tanh u + C = \frac{1}{3}\tanh x^3 + C$

4) $\displaystyle\int_0^{\ln 2} 4\, e^x \sinh x \, dx = \int_0^{\ln 2} 4\, e^x \frac{e^x - e^{-x}}{2} dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx$

$= [\, e^{2x} - 2x \,]_0^{\ln 2}$

$= \left(e^{2\ln 2} - 2\ln 2\right) - (1 - 0)$

$= 4 - 2\ln 2 - 1 \cong 1.6137$

Integral formulas for inverse hyperbolic functions

1) $\int \frac{g'(x)}{\sqrt{a^2+b^2(g(x))^2}} dx = \frac{1}{b}\sinh^{-1}\left(\frac{b}{a}g(x)\right) + c \quad , \quad a > 0$

2) $\int \frac{g'(x)}{\sqrt{b^2(g(x))^2 - a^2}} dx = \frac{1}{b}\cosh^{-1}\left(\frac{b}{a}g(x)\right) + c \quad , \quad g(x) > a > 0$

3) $\int \frac{g'(x)}{a^2-b^2(g(x))^2} dx = \begin{cases} \frac{1}{ab}\tanh^{-1}\left(\frac{b}{a}g(x)\right) & if \ -a < u < a \\ \frac{1}{ab}\coth^{-1}\left(\frac{b}{a}g(x)\right) & if \ u > a \ or \ u < -a \end{cases}$

4) $\int \frac{g'(x)}{g(x)\sqrt{a^2-b^2(g(x))^2}} dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{b}{a}g(x)\right) \quad , \quad 0 < u < a$

5) $\int \frac{g'(x)}{g(x)\sqrt{a^2+b^2(g(x))^2}} dx = -\frac{1}{a}\operatorname{csch}^{-1}\left(\left|\frac{b}{a}g(x)\right|\right) \quad , \quad a \neq 0 \ and \ a > 0$

Example: Evaluate

1) $\int \frac{1}{\sqrt{9x^2+25}} dx$ 2) $\int \frac{e^x}{16-e^{2x}} dx$ 3) 3) $\int_{1/5}^{3/13} \frac{1}{x\sqrt{1-16x^2}} dx$

Solution :

1) $\int \frac{1}{\sqrt{9x^2+25}} dx \quad a = 5 \quad , \quad b = 3 \quad , \quad g(x) = x \ \ and \ g'(x) = 1$

$\Longrightarrow \int \frac{1}{\sqrt{9x^2+25}} dx = \frac{1}{3}\int \frac{3}{\sqrt{(3x)^2+5^2}} dx = \frac{1}{3}\sinh^{-1}\frac{3x}{5} + C$

2) $\int \frac{e^x}{16-e^{2x}} dx = \int \frac{e^x}{4^2-(e^x)^2} dx \quad a = 4 \quad , \quad b = 1 \quad , \quad g(x) = e^x \ \ and \ g'(x) = e^x$

$\Longrightarrow \int \frac{e^x}{16-e^{2x}} dx = \int \frac{e^x}{4^2-(e^x)^2} dx = \int \frac{e^x}{4^2-(e^x)^2} dx = \frac{1}{4}\tanh^{-1}\frac{e^x}{4} + C$

3) $\int_{1/5}^{3/13} \frac{1}{x\sqrt{1-16x^2}} dx$ *exercise* (*Ans:* $-\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$)

Exercises

a) Find the following integrals.

1) $\int \frac{2x}{x^2+4}dx$, 2) $\int \frac{x+3}{x}dx$, 3) $\int \frac{dx}{3-2x}$, 4) $\int \frac{e^{3x}}{e^{3x}+6}dx$, 5) $\int \frac{1+\cos 2x}{2x+\sin 2x}dx$, 6) $\int \frac{\csc^2(\ln x)}{x}dx$,

7) $\int_0^1 x\,10^{x^2}dx$, 8) $\int \tan x\ dx$, 9) $\int \cot x\ dx$, 10) $\int \sec x\ dx$, 11) $\int \csc x\ dx$, 12) $\int \frac{\cos(\ln 4x^2)}{x}dx$

13) $\int \frac{\sec^3 x+e^{\sin x}}{\sec x}dx$, 14) $\int \sinh\frac{x}{5}dx$, 15) $\int \coth\frac{x}{\sqrt{3}}dx$, 16) $\int \frac{\operatorname{sech}\sqrt{x}\ \coth\sqrt{x}}{\sqrt{x}}dx$

17) $\int_{-\ln 4}^{-\ln 2} 2e^x\cosh x\,dx$, 18) $\int_0^{\frac{\pi}{2}} 2\sinh(\sin x)\cos x\,dx$, **19)** $\int \sinh x\cosh x\,dx$

20) $\int \tanh^2 3x\,\operatorname{sech}^2 3x\,dx$, 21) $\int \frac{\operatorname{sech}^2 x}{1+2\tanh x}dx$, 22) $\int \frac{x}{\operatorname{sech} x^2}dx$, 23) $\int \frac{\sinh(\ln x)}{x}dx$

b) Find the following integrals.

1) $\int \frac{dx}{\sqrt{9-x^2}}$, 2) $\int \frac{dx}{16+x^2}$, 3) $\int \frac{dx}{4x^2+9}$, 4) $\int \frac{dx}{|x|\sqrt{x^2-4}}$, 5) $\int \frac{dx}{2x^2+4x+3}$, 6) $\int \frac{6x-3}{x^2-x-1}dx$

7) $\int \frac{\cos x}{\sqrt{1+\sin^2 x}}dx$, 8) $\int_0^{2\sqrt{3}} \frac{1}{\sqrt{4+x^2}}dx$, 9) $\int_{\frac{5}{4}}^{2} \frac{1}{1-x^2}dx$, 10) $\int_1^e \frac{1}{x\sqrt{1+(\ln x)^2}}dx$

Answers

a) 1) $\ln(x^2+4)+C$, 2) $x+3\ln|x|+C$, 3) $-\frac{1}{2}\ln|3-2x|+C$, 4) $\frac{1}{3}\ln(e^{3x}+6)+C$,

5) $\frac{1}{2}\ln|2x+\sin 2x|+C$, 6) $-\cot(\ln x)+C$, 7) $\frac{9}{2\ln 10}$, 8) $-\ln|\cos x|+C$ *or* $\ln|\sec x|+C$

9) $\ln|\sin x|+C$, 10) $\ln|\sec x+\tan x|+C$, 11) $-\ln|\csc x+\cot x|+C$, 12) $\frac{1}{2}\sin(\ln 4x^2)+C$

13) $\tan x+e^{\sin x}+C$, 14) $5\cosh\frac{x}{5}+C$, 15) $\sqrt{3}\ln|\sinh\frac{x}{\sqrt{3}}|+C$, 16) $-2\operatorname{sech}\sqrt{x}+C$

17) $\frac{3}{32}+\ln 2$, 18) $2\left(\frac{e+e^{-1}}{2}-1\right)$, 19) $\frac{1}{2}\sinh^2 x+C$ *or* $\frac{1}{2}\cosh^2 x+C$, 20) $\frac{1}{9}\tanh^3 3x+C$

21) $\frac{1}{2}\ln(1+2\tanh x)+C$, 22) $\frac{1}{2}\sinh x^2+C$, 23) $\cosh(\ln x)+C$

b) 1) $\sin^{-1}\left(\frac{x}{3}\right)+C$, 2) $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$, 3) $\frac{1}{6}\tan^{-1}\left(\frac{2}{3}x\right)+C$, 4) $\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right)+C$

5) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}(x+1)\right)+C$, 6) $3\ln|x^2-x-1|+C$, 7) $\sinh^{-1}(\sin x)+C$, 8) $\sinh^{-1}\sqrt{3}$

9) $\coth^{-1}2-\coth^{-1}\frac{5}{4}$, 10) $\sinh^{-1}1-\sinh^{-1}0$

Chapter 3: Techniques of integration

Integration by parts

Definition :

Integration by parts is a special method of integration which is often useful when two functions are multiplied together and it was impossible to use integration by substitution
You will see plenty of examples soon, but first let us see the rule

Integration by Parts Formula

If $u = f(x)$ *and* $v = g(x)$ *and if* f' *and* g' are continuous, then

$$\int u\,dv = uv - \int v\,du$$

Note : choosing the first function u(x), we have to see which of the following function comes first in the following order and then assume it as u.

- Logarithmic (L)
- Inverse trigonometric (I)
- Algebraic (A)
- Trigonometric (T)
- Exponential (E)

Example: Evaluate the following integrals

1) $\int xe^{2x}\,dx$, 2) $\int x^2e^{2x}\,dx$, 3) $\int x\sec^2 x\,dx$, 4) $\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx$

5) $\int \ln x\,dx$, 6) $\int e^x\cos x\,dx$, 7) $\int \sec^3 x\,dx$, 8) $\int x^5\,e^{x^3}dx$

Solution:

1) $\int xe^{2x}\,dx$

Let us choose $u = x$ $\qquad dv = e^{2x}dx$.

Then $du = dx$ $\qquad v = \frac{1}{2}e^{2x}$

Integrating by Parts gives us:

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

2) $\int x^2e^{2x}\,dx$

Let $u = x^2$ $\qquad dv = e^{2x}dx$.

Then $du = 2xdx$ $\qquad v = \frac{1}{2}e^{2x}$

Integrating by Parts gives us:

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \int\left(\frac{1}{2}e^{2x}\right)2x\,dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx$$
$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C \quad [\text{ by } (1)\ ]$$

3) $\int x\sec^2 x\,dx$

Let $\quad u = x \quad dv = \sec^2 x\,dx.$

Then $\quad du = dx \quad v = \tan x$

Integrating by Parts gives us:

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

4) From 2) we obtain $\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$

Then $\int_0^{\frac{\pi}{3}} x\sec^2 x\ dx = [x\tan x + \ln|\cos x|]_0^{\frac{\pi}{3}}$

$$= \left(\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\left|\cos\frac{\pi}{3}\right|\right) - (0 + \ln 1)$$
$$= \left(\frac{\pi}{3}\sqrt{3} + \ln\frac{1}{2}\right) - (0 + 0)$$
$$= \frac{\pi}{3}\sqrt{3} - \ln 2 \ \approx 1.12$$

5) $\int \ln x\,dx$

Let $\quad u = \ln x \quad dv = dx.$

Then $\quad du = \frac{1}{x}dx \quad v = x$

Integrating by Parts gives us:

$$\int \ln x\,dx = x\ln x - \int (x)\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$$

6) $\int e^x\cos x\,dx$

Let $\quad u = e^x \quad dv = \cos x\,dx.$

Then $\quad du = e^x dx \quad v = \sin x$

Integrating by Parts gives us:

$\int e^x\cos x\,dx = e^x\sin x - \int e^x\sin x\,dx$ ....(1)

We next apply integration by parts to the integral on the right side of equation (1)

Let $\quad u = e^x \quad dv = \sin x\,dx.$

Then $\quad du = e^x dx \quad v = -\cos x$

And integrating by parts , we have

$\int e^x\sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx$ ....(2)

Now we use equation (2) to substitute on the right side of equation (1) we obtain

$$\int e^x\cos x\,dx = e^x\sin x - \left[-e^x\cos x + \int e^x\cos x\,dx\right]$$

$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

Adding $\int e^x \cos x\, dx$ to both sides of the last equation gives us

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x\, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

7) $\int \sec^3 x\, dx$

Let $u = \sec x$ $\quad dv = \sec^2 x\, dx.$

Then $du = \sec x \tan x\, dx$ $\quad v = \tan x$

Integrating by Parts gives us:

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$$

or $\int \sec^3 x\, dx = \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\, dx$

or $\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$

$2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$

$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

8) $\int x^5\, e^{x^3} dx$ Exercise

Ans ($\frac{1}{3} e^{x^3}(x^3 - 1) + c$)

Exercises

Evaluate the following integrals (use integration by parts)

1) $\int x\, e^{-x} dx$, 2) $\int x^2\, e^{3x} dx$, 3) $\int x \sec x \tan x\, dx$, 4) $\int x^2 \cos x\, dx$, 5) $\int \tan^{-1} x\, dx$

6) $\int \sqrt{x} \ln x\, dx$, 7) $\int e^{-x} \sin x\, dx$, 8) $\int \sin x \ln \cos x\, dx$, 9) $\int \csc^3 x\, dx$, 10) $\int_0^1 \frac{x^3}{\sqrt{x^2+1}} dx$

11) $\int x\,(2x+3)^{99} dx$, 12) $\int (\ln x)^2\, dx$, 13) $\int \cos\sqrt{x}\, dx$, 14) $\int x\, 2^x dx$

Answers

1) $-(x+1)e^{-x} + C$, 2) $\frac{1}{27} e^{3x}(9x^2 - 6x + 2) + C$, 3) $x \sec x - \ln|\sec x + \tan x| + C$

4) $x^2 \sin x + 2x \cos x - 2 \sin x + C$, 5) $x \tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + C$ 6) $\frac{2}{9} x^{3/2}(3 \ln x - 2) + C$

7) $-\frac{1}{2} e^{-x}(\sin x + \cos x) + C$, 8) $\cos x\,(1 - \ln \cos x) + C$, 9) $-\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$

10) $\frac{1}{3}\left(2 - \sqrt{2}\right) \approx 0.20$, 11) $\frac{1}{40400}(2x + 3)^{100}(200x - 3) + C$, 12) $x(\ln x)^2 - 2x \ln x + 2x + C$

13) $2\sqrt{x} \sin\sqrt{x} + 2 \cos\sqrt{x} + C$, 14) $\frac{1}{\ln 2}\, x2^x - \frac{1}{(\ln 2)^2} 2^x + C$

Trigonometric integrals

Guidelines for Evaluating $\int \sin^m x \cos^n x\ dx$

1 if m is an odd integer : Write the integral as

$$\int \sin^m x \cos^n x\ dx = \int \sin^{m-1} x \cos^n x\ \sin x\, dx$$

and express $\sin^{m-1} x$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. Make the substitution $u = \cos x$, $du = -\sin x\ dx$
and evaluate the resulting integral .

Example: Evaluate $\int \sin^5 x\, dx$

Solution:

$$\int \sin^5 x\, dx = \int \sin^4 x\ \sin x\, dx$$

$$= \int (\sin^2 x)^2\ \sin x\, dx$$

$$= \int (1 - \cos^2 x)^2\ \sin x\, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x)\ \sin x\, dx$$

If we substitute $u = \cos x$, $du = -\sin x\ dx$

We obtain

$$\int \sin^5 x\, dx = -\int (1 - 2\cos^2 x + \cos^4 x)\,(-\sin x)\, dx$$

$$= -\int (1 - 2u^2 + u^4)\, du$$

$$\Longrightarrow \int \sin^5 x\, dx = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

2 if n is an odd integer : Write the integral as

$$\int \sin^m x \cos^n x \; dx = \int \sin^m x \cos^{n-1} x \; \cos x\, dx$$

and express $\cos^{n-1} x$ in terms of $\sin x$ by using the trigonometric identity $\cos^2 x = 1 - \sin^2 x$.
Make the substitution $u = \sin x$, $du = \cos x \; dx$
and evaluate the resulting integral .

Example: Evaluate $\int \cos^3 x \; \sin^4 x\, dx$

Solution:

$$\int \cos^3 x \; \sin^4 x\, dx = \int \cos^2 x \; \sin^4 x \cos x\, dx$$

$$= \int (1 - \sin^2 x) \; \sin^4 x \cos x\, dx$$

If we let $u = \sin x$, *then* $du = \cos x \; dx$, and the integral may be written

$$\int \cos^3 x \sin^4 x\, dx = \int (1 - u^2)\, u^4\, du = \int (u^4 - u^6) \; du$$

$$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + \mathrm{C} = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + \mathrm{C}$$

3 if m *and* n are even : Use half-angle formula for $\sin^2 x$ *and* $\cos^2 x$ to reduce the exponents by one – half

Example: Evaluate $\int \cos^2 x\, dx$

Solution: using a half – angle formula, we have

$$\int \cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Example: Evaluate $\int \sin^4 x\, dx$

Solution:

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx$$

We apply a half – angle formula again and write

$$\cos^2 2x = \frac{1}{2}(1 + \cos 4x) = \frac{1}{2} + \frac{1}{2}\cos 4x$$

$$\int \sin^4 x\, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$\Longrightarrow \int \sin^4 x\, dx = \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Guidelines for Evaluating $\int \tan^m x \sec^n x \; dx$

1 **if m is an odd integer:** Write the integral as

$$\int \tan^m x \sec^n x \; dx = \int \tan^{m-1} x \sec^{n-1} x \; \sec x \tan x \, dx$$

and express $\tan^{m-1} x$ in terms of $\sec x$ by using the trigonometric identity $\tan^2 x = \sec^2 x - 1$. Make the substitution $u = \sec x$, $du = \sec x \tan x \; dx$
and evaluate the resulting integral .

Example: Evaluate $\int \tan^3 x \sec^5 x \, dx$

Solution:

$$\int \tan^3 x \sec^5 x \, dx = \int \tan^2 x \sec^4 x (\sec x \tan x) \, dx = \int (\sec^2 x - 1) \sec^4 x (\sec x \tan x) \, dx$$

Substituting $u = \sec x$ *and* $du = \sec x \tan x \; dx$, we obtain

$$\int \mathrm{Tan}^3 x \sec^5 x \, dx = \int (u^2 - 1)\, u^4 \, du = \int (u^6 - u^4) \; du = \frac{1}{7} u^7 - \frac{1}{5} u^5 + \mathrm{C}$$

$$= \frac{1}{7} \sec^7 x - \frac{1}{5} \tan^5 x + \mathrm{C}$$

2 **if n is an even integer:** Write the integral as

$$\int \tan^m x \sec^n x \; dx = \int \tan^m x \sec^{n-2} x \; \sec^2 x \, dx$$

and express $\sec^{n-2} x$ in terms of $\tan x$ by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x$.
Make the substitution $u = \tan x$, $du = \sec^2 x \; dx$ and evaluate the resulting integral .

Example: Evaluate $\int \tan^2 x \sec^4 x \, dx$

Solution:

$$\int \tan^2 x \; \sec^4 x \, dx = \int \tan^2 x \; \sec^2 x \; \sec^2 x \, dx = \int \tan^2 x \, (\tan^2 x + 1) \; \sec^2 x \, dx$$

If we let $u = \tan x$, *then* $du = \sec^2 x \; dx$, and

$$\int \tan^2 x \; \sec^4 x \, dx = \int u^2 (u^2 + 1) \; du = \int (u^4 + u^2) \; du$$

$$= \frac{1}{5} u^5 + \frac{1}{3} u^3 + \mathrm{C} = \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + \mathrm{C}$$

3 **if m is even and n is odd :** There is no standard method the trigonometric identity .
Possibly use integration by parts.

Exercises

Evaluate the integral.

1) $\int \cos^3 x\,dx$, 2) $\int \sin^3 x \cos^2 x\,dx$, 3) $\int \tan^3 x \sec^4 x\,dx$, 4) $\int \tan^3 x \sec^3 x\,dx$

5) $\int \tan^6 x\,dx$, 6) $\int \sin 5x \sin 3x\,dx$, 7) $\int_0^{\pi/2} \sin 3x \cos 2x\,dx$, 8) $\int \cos 5x \cos 3x\,dx$

Answers

1) $\sin x - \frac{1}{3}\sin^3 x + C$, 2) $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$, 3) $\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C$, 4) $\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$, 5) $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$, 6) $\frac{1}{2}\left(\frac{1}{2}\sin 2x - \frac{1}{8}\sin 8x\right) + C$, 7) $\frac{5}{2}$, 8) $\frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$

Trigonometric substitutions

Expression in integrand	Trigonometric substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

Example: Evaluate $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

Solution: The integrand contains $\sqrt{16-x^2}$, which is of the form $\sqrt{a^2-x^2}$ with $a = 4$
We let $x = 4\sin\theta$ for $-\pi/2 < \theta < \pi/2$. It follows that

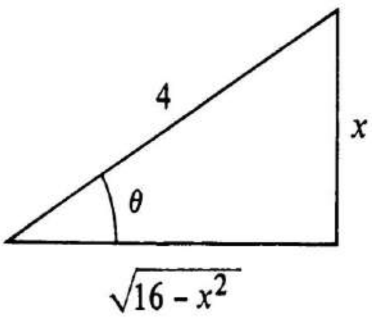


Figure 2

$$\sqrt{16-x^2} = \sqrt{16-(4\sin\theta)^2} = \sqrt{16-16\sin^2\theta}$$

$$= 4\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\cos\theta$$

Since $x = 4\sin\theta$, we have $dx = 4\cos\theta\,d\theta$. Substituting in the given integral yields

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = \int \frac{1}{(16\sin^2\theta)4\cos\theta}4\cos\theta\,d\theta = \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{16}\cot\theta + C$$

Using $\sin\theta = \frac{x}{4} \Longrightarrow \theta = \sin^{-1}\left(\frac{x}{4}\right)$, in case of $0 < \theta < \pi/2$,we sketch the triangle in **Figure 1** ,from which we obtain $\cot\theta = \frac{\sqrt{16-x^2}}{x}$ (also in case of $-\pi/2 < \theta < 0$, $\cot\theta = \frac{\sqrt{16-x^2}}{x}$) , then

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + C = -\frac{\sqrt{16-x^2}}{16x} + C$$

Example: Evaluate $\int \frac{1}{\sqrt{4+x^2}}\,dx$

Solution: The denominator of the integrand has the forms $\sqrt{a^2+x^2}$ with $a = 2$

We make the substitution $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$ for $-\pi/2 < \theta < \pi/2$. It follows that

$\Longrightarrow \sqrt{4+x^2} = \sqrt{4+(2\tan\theta)^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2\sqrt{\sec^2\theta} = 2\sec\theta$

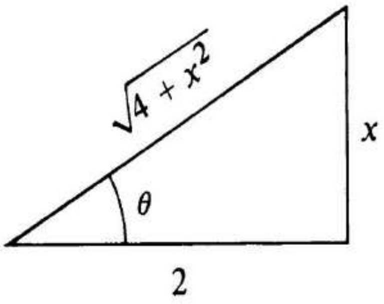


Figure 3

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{1}{2\sec\theta} 2\sec^2\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

Using $\tan\theta = \frac{x}{2} \Longrightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$, in case of $0 < \theta < \pi/2$, we sketch the triangle in **Figure 2** , from which we obtain $\sec\theta = \frac{\sqrt{4+x^2}}{2}$

(also in case of $-\pi/2 < \theta < 0$, $\sec\theta = \frac{\sqrt{4+x^2}}{2}$), then

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C = \ln\left|\sqrt{4+x^2} + x\right| - \ln 2 + C$$

Since $\sqrt{4+x^2} + x > 0\ for\ every\ x$, the absolute value sign is unnecessary. If we also let $D = -\ln 2 + C$, then

$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \ln\left(\sqrt{4+x^2} + x\right) + D$$

Exercises

Evaluate the integral.

1) $\int \frac{1}{x\sqrt{4-x^2}}dx$, 2) $\int \frac{1}{x\sqrt{9+x^2}}dx$, 3) $\int \frac{1}{x^2\sqrt{x^2-25}}dx$, 4) $\int \frac{x}{\sqrt{4-x^2}}dx$, 5) $\int \frac{1}{(x^2-1)^{3/2}}dx$

6) $\int \frac{1}{(36+x^2)^2}dx$, 7) $\int \frac{1}{\sqrt{9-x^2}}dx$, 8) $\int \frac{1}{(16-x^2)^2}dx$, 9) $\int \frac{x^3}{\sqrt{9x^2+49}}dx$, 10) $\int \frac{1}{x^4\sqrt{x^2-3}}dx$

Answers

1) $\frac{1}{2}\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + C$, 2) $\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}}{x} - \frac{3}{x}\right| + C$, 3) $\frac{\sqrt{x^2-25}}{25x} + C$, 4) $-\sqrt{4-x^2} + C$

5) $-\frac{x}{\sqrt{x^2-1}} + C$, 6) $\frac{1}{432}\left[\tan^{-1}\left(\frac{x}{6}\right) + \frac{6x}{x^2+36}\right] + C$, 7) $\sin^{-1}\left(\frac{x}{3}\right) + C$, 8) $\frac{1}{2(16-x^2)} + C$

9) $\frac{1}{243}(9x^2+49)^{3/2} - \frac{49}{81}\sqrt{9x^2+49} + C$, 10) $\frac{(3+2x^2)\sqrt{x^2-3}}{27x^3} + C$

Integrals of rational function

Guidelines for Partial Fraction Decompositions of $f(x)/g(x)$

1 If the degree of $f(x)$ is not lower than the degree of $g(x)$, use long division to obtain the proper form.

2 Express $g(x)$ as a product of linear factors $ax+b$ or irreducible quadratic factors ax^2+bx+c, and collect repeated factors so that $g(x)$ is a product of *different* factors of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$ for nonnegative integer n.

3 Apply the following rules.

Rule a For each factor $(ax+b)^n$ with $n \geq 1$, the partial fraction decomposition contains a sum of n partial fraction of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$

where each numerator A_k is a real number

Rule b For each factor $(ax^2+bx+c)^n$ with $n \geq 1$, and with ax^2+bx+c irreducible, the partial fraction decomposition contains a sum of n partial fraction of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where each A_k and B_k is a real number

Example: Evaluate $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}dx$

Solution**:** We may factor the denominator of the integrand as follows:

$$x^3+2x^2-3x = x(x^2+2x-3) = x(x+3)(x-1)$$

Then $\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$

Multiplying by the lowest common denominator gives us:

$$4x^2+13x-9 = A(x+3)(x-1)+Bx(x-1)+Cx(x+3) \ldots\ldots\ldots (*)$$

If we let $x=0\ in\ (*)$, then $-9=-3A$, or $A=3$

Letting let $x=1\ in\ (*)$ gives us $8=4C$, or $C=2$

Finally, if $x=-3\ in\ (*)$, then $-12=12B$, or $B=-1$

Therefore, $\frac{4x^2+13x-9}{x^3+2x^2-3x}=\frac{3}{x}+\frac{-1}{x+3}+\frac{2}{x-1}$

Integrating yields

$$\int \frac{4x^2+13x-9}{x^3+2x^2-3x}dx = \int \frac{3}{x}dx+\int \frac{-1}{x+3}dx+\int \frac{2}{x-1}dx$$

$$= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + K$$

$$= \ln|x^3| - \ln|x+3| + \ln|x-1|^2 + K$$

$$= \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + K$$

Note: Another technique for finding $A, B, and\ C$ is to expand the right – hand side of $(*)$ and collect like powers of x as follows:

$$4x^2 + 13x - 9 = (A + B + C)x^2 + (2A - B + 3C)x - 3A$$

coefficients of x^2 : $\quad A + B + C = 4$

coefficients of x : $\quad 2A - B + 3C = 13$

constant terms : $\quad -3A \qquad = -9$

Then $A = 3\,, B = -1\,, and\ C = 2$

Example: Evaluate $\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}dx$

Solution: The partial fraction decomposition has the form

$$\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Multiplying by $(x+1)(x-2)^3$ gives us:

$$3x^3 - 18x^2 + 29x - 4 = A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + D(x+1) \ldots\ldots\ldots (*)$$

If we let $x = 2\ in\ (*)$, then $6 = 3D$, or $D = 2$

Letting let $x = -1\ in\ (*)$ gives us $-54 = -27A$, or $A = 2$

coefficients of x^3 : $\quad 3 = A + B$

Since $A = 2$, it follows that $B = 1$

Finally, we compare the constant terms $in\ (*)$ by letting $x = 0$, then

constant terms : $\quad -4 = -8A + 4B - 2C + D$

$$\Longrightarrow -4 = -16 + 4 - 2C + 2 \qquad \Longrightarrow C = -3$$

Therefore, $\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} = \frac{2}{x+1} + \frac{1}{x-2} + \frac{-3}{(x-2)^2} + \frac{2}{(x-2)^3}$

Integrating yields

$$\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}dx = \int \frac{2}{x+1}dx + \int \frac{1}{x-2}dx + \int \frac{-3}{(x-2)^2}dx + \int \frac{2}{(x-2)^3}dx$$

$$= \ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{2}{(x-2)^2} + K$$

$$= \ln[\,(x+1)^2\,|x-2|\,] + \frac{3}{x-2} - \frac{2}{(x-2)^2} + K$$

Example: Evaluate $\int \frac{x^2-x-21}{2x^3-x^2+8x-4}dx$

Solution: The denominator may be factored by grouping as follows:

$$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1) = (x^2+4)(2x-1)$$

Then $\frac{x^2-x-21}{2x^3-x^2+8x-4} = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}$

Multiplying by $(x^2+4)(2x-1)$ gives us:

$$x^2 - x - 21 = (Ax + B)(2x - 1) + C(x^2 + 4) \dots\dots\dots (*)$$

We can find one constant easily. substituting $x = \frac{1}{2}$ $in\ (*) gives\ us$

$$-\frac{85}{4} = \frac{17}{4}C \quad , \quad or\ \ C = -5.$$

The remaining constants may be found by comparing coefficients $in\ (*)$:

coefficients of x^2: $1 = 2A + C$

coefficients of x : $-1 = -A + 2B$

constant terms : $-21 = -B + 4C$

Since $C = -5$, it follows from $1 = 2A + C$ that $A = 3$. Similarly, using the coefficients of x with $A = 3$ gives us $-1 = -3 + 2B$, $or\ B = 1$.

Thus, the partial fraction decomposition of the integrand is

$$\frac{x^2-x-21}{2x^3-x^2+8x-4} = \frac{3x+1}{x^2+4} + \frac{-5}{2x-1}$$

$$= \frac{3x}{x^2+4} + \frac{1}{x^2+4} - \frac{5}{2x-1}$$

Integrating yields

$$\int \frac{x^2-x-21}{2x^3-x^2+8x-4}dx = \int \frac{3x}{x^2+4}dx + \int \frac{1}{x^2+4}dx - \int \frac{5}{2x-1}dx$$

$$= \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{5}{2}\ln|2x-1| + K$$

Example: Evaluate $\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx$

Solution:

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Multiplying by $(x^2+1)^2$ gives us:

$$5x^3 - 3x^2 + 7x - 3 = (Ax + B)(x^2 + 1) + Cx + D$$

$$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Bx^2 + (A + C)x + (B + D)$$

We next compare coefficients as follows:

coefficients of x^3 : $5 = A$

coefficients of x^2 : $-3 = B$

coefficients of x : $7 = A + C$

constant terms : $-3 = B + D$

We now have $A = 5$, $B = -3$, $C = 7 - A = 2$, $and\ D = -3 - B = 0$; therefore,

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{5x-3}{x^2+1} + \frac{2x+0}{(x^2+1)^2} = \frac{5x}{x^2+1} - \frac{3}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

Integrating yields

$$\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx = \int \frac{5x}{x^2+1}dx - \int \frac{3}{x^2+1}dx + \int \frac{2x}{(x^2+1)^2}dx$$

$$= \frac{5}{2}\ln(x^2+1) - 3\tan^{-1}x - \frac{1}{x^2+1} + K$$

Exercises

Evaluate the integral.

1) $\int \frac{5x-12}{x(x-4)}dx$, 2) $\int \frac{37-11x}{(x+1)(x-2)(x-3)}dx$, 3) $\int \frac{6x-11}{(x-1)^2}dx$, 4) $\int \frac{x+16}{x^2+2x-8}dx$, 5) $\int \frac{5x^2-10x-8}{x^3-4x}dx$

6) $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}dx$, 7) $\int \frac{9x^4+17x^3+3x^2-8x+3}{x^5+3x^4}dx$, 8) $\int \frac{x^3+6x^2+3x+16}{x^3+4x}dx$, 9) $\int \frac{x^2+3x+1}{x^4+5x^2+4}dx$, 10) $\int \frac{x^3+3x-2}{x^2-x}dx$

Answers

1) $\ln[|x|^3(x-4)^2]+C$, 2) $\ln\left[\frac{|x-3|(x+1)^4}{|x-2|^5}\right]+C$, 3) $6\ln|x-3|+\frac{5}{x-1}+C$, 4) $\ln\left[\frac{|x-2|^3}{(x+4)^2}\right]+C$

5) $\ln\left[\frac{x^2(x+2)^4}{|x-2|}\right]+C$, 6) $\ln\left[\frac{|x+1|^5}{|x-5|^3}\right]-\frac{1}{x+1}+C$, 7) $\ln\left[\frac{|x|^5}{(x+3)^4}\right]-\frac{2}{x}+\frac{3}{2x^2}-\frac{1}{3x^3}+C$,

8) $x+\ln[x^4(x^2+4)]-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$, 9) $\ln\left[\frac{\sqrt{x^2+1}}{\sqrt{x^2+4}}\right]+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$, 10) $\frac{1}{2}x^2+x+\ln[x^2(x-1)^2]+C$

Quadratic expressions and miscellaneous substitutions

INTEGRALS INVOLVING QUADRATIC EXPRESSIONS

Partial fraction decompositions may lead t integrands containing an irreducible quadratic expression ax^2+bx+c . If $b \neq 0$, it is sometimes necessary to complete the square as follows:

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c$$

$$= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}$$

The substitution $u = x+b/(2a)$ may then lead to an integrable form.

Example: Evaluate $\int \frac{2x-1}{x^2-6x+13}dx$

Solution: Note that the quadratic expression $x^2-6x+13$ is irreducible, since $b^2-4ac=-16<0$. We complete the square as follows:

$$x^2-6x+13 = (x^2-6x \qquad)+13$$

$$= (x^2-6x+9)+13-9 = (x-3)^2+4$$

Thus

$$\int \frac{2x-1}{x^2-6x+13}dx = \int \frac{2x-1}{(x-3)^2+4}dx$$

We now make the substitution

$$u = x - 3 \quad , \quad x = u + 3 \quad , \quad dx = du$$

Thus

$$\int \frac{2x-1}{x^2-6x+13}\,dx = \int \frac{2(u+3)-1}{u^2+4}\,du$$

$$= \int \frac{2u+5}{u^2+4}\,du$$

$$= \int \frac{2u}{u^2+4}\,du + 5\int \frac{1}{u^2+4}\,du$$

$$\Longrightarrow \int \frac{2x-1}{x^2-6x+13}\,dx = \ln(u^2+4) + \frac{5}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \ln(x^2-6x+13) + \frac{5}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$$

Note that: We may also use the technique of completing the square if a quadratic expression appears under a radical sign. In the next example, we make a trigonometric substitution after completing the square.

Example: Evaluate $\int \frac{1}{\sqrt{x^2+8x+25}}\,dx$

Solution: We complete the square for the quadratic expression as follows:

$$x^2 + 8x + 25 = (x^2 + 8x \quad\ ) + 25$$

$$= (x^2 + 8x + 16) + 25 - 16 = (x+4)^2 + 9$$

Thus

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx$$

If we make the trigonometric substitution

$$x + 4 = 3\tan\theta \quad , \quad dx = 3\sec^2\theta\, d\theta \quad \text{for } -\pi/2 < \theta < \pi/2$$

then $\sqrt{(x+4)^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\tan^2\theta + 1} = 3\sec\theta$

and $\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{3\sec\theta} 3\sec^2\theta\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

As $\tan\theta = \frac{x+4}{3}$ then $\sec\theta = \pm\sqrt{\tan^2\theta + 1} = \sqrt{\tan^2\theta + 1}$ $(because\ -\pi/2 < \theta < \pi/2\ then\ \sec\theta > 0)$

$$= \sqrt{\left(\frac{x+4}{3}\right)^2 + 1} = \sqrt{\frac{x^2+8x+25}{9}} = \frac{\sqrt{x^2+8x+25}}{3}$$

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C$$

$$= \ln\left|\sqrt{x^2+8x+25} + x + 4\right| - \ln 3 + C$$

$$= \ln\left|\sqrt{x^2+8x+25} + x + 4\right| + K$$

$$with\ K = -\ln 3 + C$$

Miscellaneous substitutions

We now consider substitutions that are useful for evaluating certain types of integrals. The next examples illustrates that if an integral contains an expression of the form $\sqrt[n]{f(x)}$, then one of the substitutions $u = \sqrt[n]{f(x)}$ or $u = f(x)$ may simplify the evaluating.

Example: Evaluate $\int \frac{x^3}{\sqrt[3]{x^2+4}}dx$

Solution 1:

Let $u = \sqrt[3]{x^2+4}$, $u^3 = x^2+4$, $x^2 = u^3 - 4$ $then$ $2xdx = 3u^2du$, or $xdx = \frac{3}{2}u^2du$

We now substitute as follows:

$$\int \frac{x^3}{\sqrt[3]{x^2+4}}dx = \int \frac{x^2}{\sqrt[3]{x^2+4}}xdx = \int \frac{u^3-4}{u}\frac{3}{2}u^2du = \frac{3}{2}\int (u^4 - 4u)\,du$$

$$= \frac{3}{2}\left(\frac{1}{5}u^5 - 2u^2\right) + C = \frac{3}{10}u^2(u^3 - 10) + C$$

$$= \frac{3}{10}(x^2+4)^{\frac{2}{3}}(x^2 - 6) + C$$

***Solution 2*:**

Let $u = x^2 + 4$, $or\ x^2 = u - 4$ $then\ 2xdx = du$, or $xdx = \frac{1}{2}du$

$$\int \frac{x^3}{\sqrt[3]{x^2+4}}dx = \int \frac{x^2}{\sqrt[3]{x^2+4}}xdx$$

$$= \int \frac{u-4}{u^{1/3}}\frac{1}{2}du = \frac{1}{2}\int \left(u^{2/3} - 4u^{-1/3}\right)du$$

$$= \frac{1}{2}\left(\frac{3}{5}u^{5/3} - 6u^{2/3}\right) + C = \frac{3}{10}u^{2/3}(u - 10) + C$$

$$= \frac{3}{10}(x^2+4)^{\frac{2}{3}}(x^2 - 6) + C$$

Example: Evaluate $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx$

Solution:

Let $u = x^{1/6}$, or $x = u^6$ $.Then$ $dx = 6u^5du$, $x^{1/2} = (u^6)^{1/2} = u^3$, $x^{1/3} = (u^6)^{1/3} = u^2$

and, therefore,

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = \int \frac{1}{u^3+u^2}6u^5du = 6\int \frac{u^3}{u+1}du$$

By long division,

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}$$

Consequently,

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)du$$

$$= 6\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right) + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left(\sqrt[6]{x}+1\right)$$

Theorem: If an integrand is a rational expression in $\sin x$ *and* $\cos x$, the following substitutions will produce a rational expression in u:

$$\sin x = \frac{2u}{1+u^2}\ , \qquad \cos x = \frac{1-u^2}{1+u^2}\ , \qquad dx = \frac{2}{1+u^2}du\ ,$$

Where $u = \tan\frac{x}{2} \quad for\ -\pi < x < \pi$.

Example: Evaluate $\int \frac{1}{4\sin x - 3\cos x}dx$

Solution:

$$\int \frac{1}{4\sin x - 3\cos x}dx = \int \frac{1}{4\left(\frac{2u}{1+u^2}\right) - 3\left(\frac{1-u^2}{1+u^2}\right)}\ \frac{2}{1+u^2}du$$

$$= \int \frac{2}{8u - 3(1-u^2)}du$$

$$= 2\int \frac{1}{3u^2+8u-3}du$$

$$= 2\int \frac{1}{10}\left(\frac{3}{3u-1} - \frac{1}{u+3}\right)du$$

$$= \frac{1}{5}\int \left(\frac{3}{3u-1} - \frac{1}{u+3}\right)du$$

$$= \frac{1}{5}(\ln|3u-1| - \ln|u+3|) + C$$

$$= \frac{1}{5}\ln\left|\frac{3u-1}{u+3}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{3\tan(x/2)-1}{\tan(x/2)+3}\right| + C$$

Exercises

a) Evaluate the integral.

1) $\int \frac{1}{(x+1)^2+4}dx$, 2) $\int \frac{1}{x^2-4x+8}dx$, 3) $\int \frac{2x+3}{\sqrt{9-8x-x^2}}dx$, 4) $\int \frac{1}{(x^2+4x+5)^2}dx$, 5) $\int \frac{1}{2x^2-3x+9}dx$,

6) $\int \frac{e^x}{e^{2x}+3e^x+2}dx$, 7) $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx$, 8) $\int \frac{e^{2x}}{e^x+4}dx$, 9) $\int \frac{\sin 2x}{\sin^2 x-2\sin x-8}dx$, 10) $\int \frac{e^x}{e^{2x}-1}dx$, 11) $\int \frac{1}{e^x+e^{-x}}dx$

b) Use the last theorem to evaluate the integral.

1) $\int \frac{1}{2+\sin x}dx$, 2) $\int \frac{1}{1+\sin x+\cos x}dx$, 3) $\int \frac{\sec x}{4-3\tan x}dx$

Answers

a) 1) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$, 2) $\frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right)+C$, 3) $-2\sqrt{9-8x-x^2}-5\sin^{-1}\left(\frac{x+4}{5}\right)+C$

4) $\frac{1}{2}\left[\tan^{-1}(x+2)+\frac{x+2}{x^2+4x+5}\right]+C$, 5) $\frac{2}{3\sqrt{7}}\tan^{-1}\left(\frac{4x-3}{3\sqrt{7}}\right)+C$, 6) $\ln\left(\frac{e^x+1}{e^x+2}\right)+C$

7) $\frac{6}{7}x^{7/6}-\frac{6}{5}x^{5/6}+2x^{1/2}-6x^{1/6}+6\tan^{-1}\left(x^{1/6}\right)+C$, 8) $e^x-4\ln(e^x+4)+C$

9) $\ln\sqrt[3]{(4-\sin x)^4(\sin x+2)^2}+C$, 10) $\ln\sqrt{\frac{|e^x-1|}{e^x+1}}+C$, 11) $\tan^{-1}(e^x)+C$

b) 1) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(x/2)+1}{\sqrt{3}}\right)+C$, 2) $\ln\left|\tan\frac{x}{2}+1\right|+C$, 3) $\ln\sqrt[5]{\frac{|\tan(x/2)+2|}{|2\tan(x/2)-1|}}+C$

Improper integrals and **Taylor's formula**

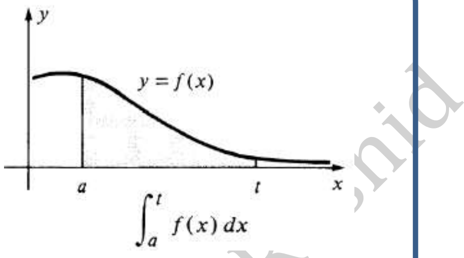
1 - Integrals with infinite limit of integration

Definition:

$(\boldsymbol{i})$ If f is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

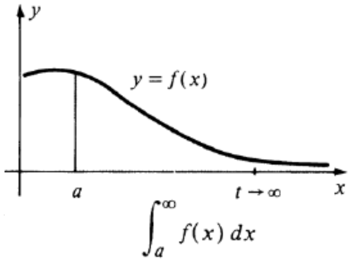
Provided the limit exists.



$(\boldsymbol{ii})$ If f is continuous on$(-\infty,a]$, then

$$\int_{-\infty}^a f(x)dx = \lim_{t\to-\infty}\int_t^a f(x)dx$$

Provided the limit exists.



Note:

The expressions in the last Definition are **improper integrals**. An improper integral is said to **converge** if the limit exists, and the limit is the **value** of the improper integral. If the limit does not exist, the improper integral **diverges.**

Example: Determine whether the integral converges or diverges, and if it converges, find its value.

$$1)\ \int_2^\infty \frac{1}{(x-1)^2}dx \quad , \quad 2)\ \int_2^\infty \frac{1}{x-1}dx$$

Solution:

1) $\int_2^\infty \frac{1}{(x-1)^2}dx = \lim\limits_{t\to\infty}\int_2^t \frac{1}{(x-1)^2}dx = \lim\limits_{t\to\infty}\left[\frac{-1}{x-1}\right]_2^t = \lim\limits_{t\to\infty}\left(\frac{-1}{t-1}+\frac{1}{2-1}\right) = 0+1 = 1$

Thus, the integral converges and has the value 1.

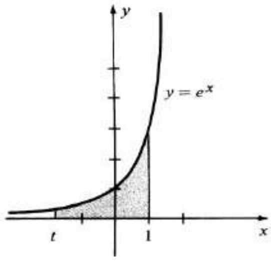
2) $\int_2^\infty \frac{1}{x-1}dx = \lim\limits_{t\to\infty}\int_2^t \frac{1}{x-1}dx = \lim\limits_{t\to\infty}[\ln(x-1)]_2^t = \lim\limits_{t\to\infty}[\ln(t-1)+\ln(2-1)] = \lim\limits_{t\to\infty}\ln(t-1) = \infty$

Since the limit does not exist, the improper integral diverges.

Example: Assign an area to the region that lies under the graph of $y = e^x$, over the x −axis, and to the left of $x = 1$.

Solution: The region bounded by the graphs of $y = e^x\ , y = 0\ , x = 1\ , and\ x = t\ , for\ t < 1$, is sketched in **Figure**.

The area of the *unbounded* region to the left of $x = 1$ is



$$\int_{-\infty}^1 e^x dx = \lim_{t\to-\infty}\int_t^1 e^x dx = \lim_{t\to-\infty}[e^x]_t^1 = \lim_{t\to-\infty}(e-e^t) = e-0 = e$$

Definition:

Let f be continuous for every x. If a is any real number, then

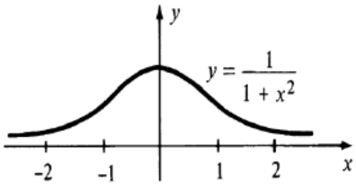
$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

Provided both of the improper integrals on the right converges

.

Note: If either of the integrals on the right in the last definition diverges, then $\int_{-\infty}^{\infty} f(x)dx$ is said to **diverges.**

Example: Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$



Solution: using the last definition, with $a = 0$, yields

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx.$$

$$\int_{0}^{\infty} \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} dx = \lim_{t\to\infty} [\tan^{-1} x]_0^t = \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Also

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2} dx = \lim_{t\to-\infty} [\tan^{-1} x]_t^0 = \lim_{t\to-\infty} (\tan^{-1} 0 - \tan^{-1} t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Consequently, the given improper integral converges and has the value $(\pi/2) + (\pi/2) = \pi$

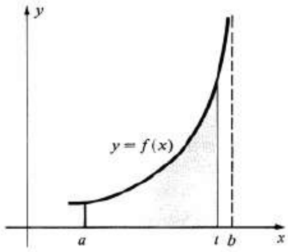
2 - Integrals with discontinuous integrands

Definition:

(i) If f is continuous on $[a\,, b\,)$, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to b^-} \int_{a}^{t} f(x)dx$$

Provided the limit exists.

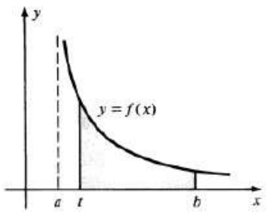


(i) $\int_a^t f(x)\,dx$

(ii) If f is continuous on$(\,a\,, b\,]$, then

$$\int_{a}^{b} f(x)dx = \lim_{t\to a^+} \int_{t}^{a} f(x)dx$$

Provided the limit exists.



(ii) $\int_t^b f(x)\,dx$

Example : Evaluate $\int_0^3 \frac{1}{\sqrt{3-x}} dx$

Solution :

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx = \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}} dx$$

$$= \lim_{t\to 3^-} \left[-2\sqrt{3-x}\right]_0^t$$

$$= \lim_{t\to 3^-} [-2\sqrt{3-t} + 2\sqrt{3}]$$

$$= 0 + 2\sqrt{3} = 2\sqrt{3}.$$

Example: Determine whether $\int_0^1 \frac{1}{x}\ dx$ converges or diverges

Solution :

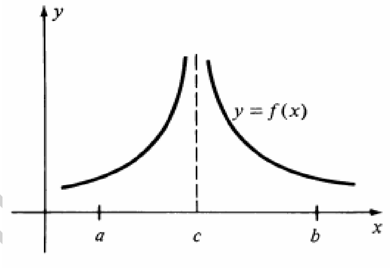
$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\Big[\ln x\Big]_t^1$$
$$= \lim_{t\to 0^+}[0-\ln t] = \infty.$$

Since the limit does not exist, the improper integral diverges.

***Definition*:**

If f has a discontinuity at a number c in the open interval $(\,a\,,b\,)$ but is continuous elsewhere on $[\,a\,,b\,]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$



Provided both of the improper integrals on the right converges.

If both converge , then the value of the improper integral $\int_a^b f(x)dx$ is the sum of the two values.

Example : Determine whether the following integrals are converges or diverges

1) $\int_0^4 \frac{1}{(x-3)^2}dx$ 2) $\int_{-2}^7 \frac{1}{(x+1)^{2/3}}dx$

Solution :

1) $\int_0^4 \frac{1}{(x-3)^2}dx$

The integrand is undefined at $x=3$, thus we apply the previous definition with $c=3$ as follows:

$$\int_0^4 \frac{1}{(x-3)^2}dx = \int_0^3 \frac{1}{(x-3)^2}dx + \int_0^4 \frac{1}{(x-3)^2}dx$$

$$\int_0^3 \frac{1}{(x-3)^2}dx = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}dx = \lim_{t\to 3^-}\int_0^t (x-3)^{-2}\,dx$$
$$= \lim_{t\to 3^-} -(x-3)^{-1}\Big|_0^t$$
$$= \lim_{t\to 3^-} -\frac{1}{x-3}\Big|_0^t$$
$$= \lim_{t\to 3^-}\left(-\frac{1}{t-3}-\frac{1}{3}\right)$$
$$= \infty$$

Since $\int_0^3 \frac{1}{(x-3)^2}dx$ is diverge ,

Then $\int_0^4 \frac{1}{(x-3)^2}dx$ is diverge

2) $\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx$

The integrand is undefined at $x = -1$, thus we apply the previous definition with $c = -1$ as follows:

$$\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx = \int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx + \int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx$$

$$\int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx = \lim_{t\to -1^-} \int_{-2}^{t} \frac{1}{(x+1)^{2/3}}\,dx$$

$$= \lim_{t\to -1^-} \int_{-2}^{t} (x+1)^{-2/3}\,dx$$

$$= \lim_{t\to -1^-} 3(x+1)^{\frac{1}{3}} \Big|_{-2}^{t}$$

$$= 3\lim_{t\to -1^-} (t+1)^{\frac{1}{3}} - (-1)^{\frac{1}{3}} = 3(0+1) = 3$$

Also $\int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx = \lim_{t\to -1^+} \int_{t}^{7} \frac{1}{(x+1)^{2/3}}\,dx$

$$= \lim_{t\to -1^+} \int_{t}^{7} (x+1)^{-2/3}\,dx$$

$$= \lim_{t\to -1^+} 3(x+1)^{\frac{1}{3}} \Big|_{t}^{7}$$

$$= 3\lim_{t\to -1^-} (8)^{\frac{1}{3}} - (t+1)^{\frac{1}{3}} = 3(2-0) = 6$$

Since the integrals $\int_{-2}^{-1} \frac{1}{(x+1)^{2/3}}\,dx$, $\int_{-1}^{7} \frac{1}{(x+1)^{2/3}}\,dx$ are converges , then $\int_{-2}^{7} \frac{1}{(x+1)^{2/3}}\,dx$ is converge and has the value 3 + 6 = 9

Exercises

Determine whether the integral converges or diverges, and if it converges, find its value.

1) $\int_{1}^{\infty} \frac{1}{x^{4/3}}\,dx$, 2) $\int_{1}^{\infty} \frac{1}{x^{3/4}}\,dx$, 3) $\int_{-\infty}^{2} \frac{1}{5-2x}\,dx$, 4) $\int_{0}^{\infty} e^{-2x}\,dx$, 5) $\int_{-\infty}^{-1} \frac{1}{x^3}\,dx$, 6) $\int_{-\infty}^{0} \frac{1}{(x-8)^{2/3}}\,dx$

7) $\int_{0}^{\infty} \frac{\cos x}{1+\sin^2 x}\,dx$, 8) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$, 9) $\int_{1}^{\infty} \frac{\ln x}{x}\,dx$, 10) $\int_{-\infty}^{\pi/2} \sin 2x\,dx$, 11) $\int_{0}^{\infty} \cos x\,dx$

12) $\int_{-\infty}^{\infty} \operatorname{sech} x\,dx$, 13) $\int_{-\infty}^{0} \frac{1}{x^2-3x+2}\,dx$, 14) $\int_{0}^{8} \frac{1}{\sqrt[3]{x}}\,dx$, 15) $\int_{-3}^{1} \frac{1}{x^2}\,dx$, 16) $\int_{0}^{\pi/2} \sec^2 x\,dx$

17) $\int_{0}^{4} \frac{1}{(x-4)^{\frac{3}{2}}}\,dx$, 18) $\int_{0}^{4} \frac{1}{(x-4)^{\frac{2}{3}}}\,dx$, 19) $\int_{-2}^{2} \frac{1}{(x-4)^3}\,dx$, 20) $\int_{-2}^{0} \frac{1}{\sqrt{4-x^2}}\,dx$, 21) $\int_{-1}^{2} \frac{1}{x}\,dx$, 22) $\int_{0}^{1} x\ln x\,dx$

23) $\int_{0}^{\pi/2} \tan x\,dx$, 24) $\int_{2}^{4} \frac{x-2}{x^2-5x+4}\,dx$, 25) $\int_{-1}^{2} \frac{1}{x^2}\cos\frac{1}{x}\,dx$, 26) $\int_{0}^{\pi} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$, 27) $\int_{0}^{\infty} \frac{1}{(x-4)^2}\,dx$

Answers

1) C ; 3 , 2) D , 3) D , 4) C ; $\frac{1}{2}$, 5) C ; $-\frac{1}{2}$, 6) D , 7) D , 8) C ; 0 , 9) D , 10) D

11) D , 12) C ; π , 13) C ; ln 2 , 14) C ; 6 , 15) D , 16) D , 17) D , 18) C ; $3\sqrt[3]{4}$, 19) D , 20) C ; $\frac{\pi}{2}$
21) D , 22) C ; -1/4 , 23) D , 24) D , 25) D 26) C ; 0 27) D

Taylor's formula

Introduction :

In this section we develop a method of constructing polynomial approximations to functions that are not polynomials. The higher derivatives of all orders will play a pivotal role.

Definition :

If the function f has derivative through order n at a , then the $\boldsymbol{nth-order\ Taylor\ polynomial}$ of f at a is defined as

$$P_n(x\,,a) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Where $n! = n(n-1)(n-2)(n-3)\ldots \times 3 \times 2 \times 1$

Example : Compute the fourth – order Taylor polynomial at $a = 1$ for the function $f(x) = \frac{1}{x}$

Solution:

Since $n = 4$, then we need the first four derivatives of $f(x) = \frac{1}{x}$ at $x = 1$ as follows

$$f^{(1)}(x) = -\frac{1}{x^2} \quad \Rightarrow \quad f^{(1)}(1) = -1$$

$$f^{(2)}(x) = \frac{2}{x^3} \quad \Rightarrow \quad f^{(2)}(1) = 2$$

$$f^{(3)}(x) = -\frac{6}{x^4} \quad \Rightarrow \quad f^{(3)}(1) = -6$$

$$f^{(4)}(x) = \frac{24}{x^5} \quad \Rightarrow \quad f^{(4)}(1) = 24$$

Since $P_n(x\,,a) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Then $P_4(x\,,1) = f(1) + f^{(1)}(1)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$

$$= 1 + (-1)(x-1) + \frac{2}{2}(x-1)^2 + \frac{-6}{6}(x-1)^3 + \frac{24}{24}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

Example : Find the Taylor polynomial $P_5(x\,,0)$ for the function $f(x) = \sin x$

Solution :

Since $n = 5$, then we need the first four derivatives of $f(x) = \sin x$ at $x = 1$ as follows

$$f^{(1)}(x) = \cos x \quad \Rightarrow \quad f^{(1)}(0) = 1$$

$$f^{(2)}(x) = -\sin x \quad \Rightarrow \quad f^{(2)}(0) = 0$$

$$f^{(3)}(x) = -\cos x \quad \Rightarrow \quad f^{(3)}(0) = -1$$

$$f^{(4)}(x) = \sin x \quad \Rightarrow \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad \Rightarrow \quad f^{(5)}(0) = 1$$

Since $P_n(x\,,a) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Then $P_5(x\,,0) = f(0) + f^{(1)}(0)(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \frac{f^{(5)}(0)}{5!}(x-0)^5$

$$= 0 + x + \frac{0}{2}x^2 - \frac{1}{6}x^3 + \frac{0}{24}x^4 + \frac{1}{120}x^5$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120}$$

Exercises

Compute the given Taylor polynomials $P_n(x\,,a)$.

1) $P_2(x\,,0)$ $for\ f(x)=\frac{1}{x+1}$, 2) $P_3(x\,,1)$ $for\ f(x)=\ln x$, 3)$P_3\left(x\,,\frac{\pi}{2}\right)$ $for\ f(x)=\sin x$

4)$P_3(x\,,4)$ $for\ f(x)=\sqrt{x}$, , 5)$P_4\left(x\,,\frac{\pi}{4}\right)$ $for\ f(x)=\tan x$, 6)$P_2(x\,,1)$ $for\ f(x)=\tan^{-1}x$

7)$P_4(x\,,-1)$ $for\ f(x)=xe^x$, 8)$P_4(x\,,0)$ $for\ f(x)=\ln(x+1)$, 9)$P_5(x\,,0)$ $for\ f(x)=\frac{1}{(x-1)^2}$

10)$P_4(x\,,0)$ $for\ f(x)=2x^4-5x^3+x^2-3x+7$

Answers

1) $P_2(x\,,0)=1-x+x^2$, 2) $P_3(x\,,1)=x-1-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3$

3) $P_3\left(x\,,\frac{\pi}{2}\right)=1-\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2$, 4) $P_3(x\,,4)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$

5) $P_4\left(x\,,\frac{\pi}{4}\right)=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3+\frac{10}{3}\left(x-\frac{\pi}{4}\right)^4$, 6)$P_2(x\,,1)=\frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2$

, 7)$P_4(x\,,-1)=-\frac{1}{e}+\frac{1}{2e}(x+1)^2+\frac{1}{3e}(x+1)^3+\frac{1}{8e}(x+1)^4$ 8)$P_4(x\,,0)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$

, 9)$P_5(x\,,0)=1+2x+3x^2+4x^3+5x^4+6x^5$, 10)$P_4(x\,,0)=7-3x+x^2-5x^3+2x^4$

General rules : قواعد عامة

1) $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$ 2) $\frac{a}{b} + 1 = \frac{a+b}{b}$ 3) $\frac{1}{x^n} = x^{-n}$ 4) $\frac{x^n}{x^m} = x^{n-m}$ 5) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

Rules for finding derivative

Let $f(x)$, $g(x)$ and $h(x)$ are functions differentiable , n any real number and c is a constant value , then

1) If $(x) = c \implies f'(x) = 0$.
2) If $f(x) = x^n \implies f'(x) = n\,x^{n-1}$.
3) If $f(x) = c\,g(x) \implies f'(x) = c\,g'(x)$
4) If $f(x) = g(x) \pm h(x) \implies f'(x) = g'(x) \pm h'(x)$.
5) If $f(x) = g(x) \cdot h(x) \implies f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$.
6) If $f(x) = \frac{g(x)}{h(x)}$ provided $h(x) \neq 0 \implies f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$
7) If $f(x) = [g(x)]^n \implies f'(x) = n[g(x)]^{n-1} \cdot g'(x)$
8) If $f(x) = \sqrt[n]{g(x)} \implies f'(x) = \frac{g'(x)}{n\sqrt[n]{(g(x))^{n-1}}}$

Derivative of trigonometric functions

If $g(x)$ is a differentiable function . then by chain rule , we get the following

1) $\frac{d}{dx}\left(\sin\left(g(x)\right)\right) = g'(x)\ \cos\left(g(x)\right)$ 2) $\frac{d}{dx}\left(\cos\left(g(x)\right)\right) = -g'(x)\sin\left(g(x)\right)$
3) $\frac{d}{dx}\left(\tan\left(g(x)\right)\right) = g'(x)\ \sec^2\left(g(x)\right)$ 4) $\frac{d}{dx}\left(\cot\left(g(x)\right)\right) = -g'(x)\ \csc^2\left(g(x)\right)$
5) $\frac{d}{dx}\left(\sec\left(g(x)\right)\right) = g'(x)\ \sec\left(g(x)\right)\tan\left(g(x)\right)$ 6) $\frac{d}{dx}\left(\csc\left(g(x)\right)\right) = -g'(x)\ \csc\left(g(x)\right)\cot\left(g(x)\right)$

Derivative of logarithmic and exponential functions

1) $\frac{d}{dx}[\log_a g(x)] = \frac{g'(x)}{g(x)\ln x}$ 2) $\frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}$
3) $\frac{d}{dx}\left(a^{g(x)}\right) = (g'(x)\ln a)a^{g(x)}$ 4) $\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)\ e^{g(x)}$

Derivative of Inverse Trigonometric function

If $g(x)$ is a differentiable function . then by chain rule , we get the following

1) $\frac{d}{dx}[\sin^{-1}(g(x))] = \frac{g'(x)}{\sqrt{1-(g(x))^2}}$, 2) $\frac{d}{dx}[\cos^{-1}(g(x))] = -\frac{g'(x)}{\sqrt{1-(g(x))^2}}$
3) $\frac{d}{dx}[\tan^{-1}(g(x))] = \frac{g'(x)}{1+(g(x))^2}$, 4) $\frac{d}{dx}[\cot^{-1}(g(x))] = -\frac{g'(x)}{1+(g(x))^2}$
5) $\frac{d}{dx}[\sec^{-1}(g(x))] = \frac{g'(x)}{g(x)\sqrt{(g(x))^2-1}}$, 6) $\frac{d}{dx}[\csc^{-1}(g(x))] = -\frac{g'(x)}{g(x)\sqrt{(g(x))^2-1}}$

Derivative of the hyperbolic functions

If $g(x)$ is a differentiable function . then by chain rule , we get the following

1) $\frac{d}{dx}\left(\sinh\left(g(x)\right)\right) = g'(x)\ \cos h\left(g(x)\right)$

2) $\frac{d}{dx}\left(\cosh\left(g(x)\right)\right) = g'(x) \sinh\left(g(x)\right)$

3) $\frac{d}{dx}\left(\tanh\left(g(x)\right)\right) = g'(x)\ \operatorname{sech}^2\left(g(x)\right)$

4) $\frac{d}{dx}\left(\coth\left(g(x)\right)\right) = -g'(x)\ \operatorname{csch}^2\left(g(x)\right)$

5) $\frac{d}{dx}\left(\operatorname{sech}\left(g(x)\right)\right) = -g'(x) \operatorname{sech}\left(g(x)\right) \tanh\left(g(x)\right)$

6) $\frac{d}{dx}\left(\operatorname{csch}\left(g(x)\right)\right) = -g'(x)\ \csc h\left(g(x)\right) \cot h\left(g(x)\right)$

Derivative of the inverse hyperbolic functions

If $g(x)$ is a differentiable function . then by chain rule , we get the following

1) $\frac{d}{dx}(\sinh^{-1}(g(x)) = \frac{g'(x)}{\sqrt{1+(g(x))^2}}$,

2) $\frac{d}{dx}(\cosh^{-1}(g(x)) = \frac{g'(x)}{\sqrt{(g(x))^2-1}}$, $g(x) > 1$

3) $\frac{d}{dx}(\tanh^{-1}(g(x)) = \frac{g'(x)}{1-(g(x))^2}$, $|g(x)| > 1$,

4) $\frac{d}{dx}(\coth^{-1}(g(x)) = \frac{g'(x)}{1-(g(x))^2}$, $|g(x)| > 1$

5) $\frac{d}{dx}(\operatorname{sech}^{-1}(g(x)) = \frac{-g'(x)}{g(x)\sqrt{1-(g(x))^2}}$, $0 < g(x) < 1$,

6) $\frac{d}{dx}(\operatorname{csch}^{-1}(g(x)) = \frac{-g'(x)}{|g(x)|\sqrt{1+(g(x))^2}}$, $g(x) \neq 1$

Fundamental Trigonometric Identities

$$\sin x = \frac{1}{\csc x} \quad , \quad \cos x = \frac{1}{\sec x} \quad , \quad \tan x = \frac{1}{\cot x}$$

$$\csc x = \frac{1}{\sin x} \quad , \quad \sec x = \frac{1}{\cos x} \quad , \quad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \quad , \quad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1 \ , \quad \tan^2 x = \sec^2 x - 1 \ , \quad \cot^2 x = \csc^2 x - 1$$

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \Longrightarrow \sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \\ &= 2\cos^2 x - 1 \Longrightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x) \end{aligned}$$

Identities for hyperbolic functions

1) $\sinh x = \frac{e^x - e^{-x}}{2}$

2) $\cosh x = \frac{e^x + e^{-x}}{2}$

3) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

4) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

5) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

6) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

7) $\cosh^2 x - \sinh^2 x = 1$

8) $\sinh 2x = 2 \sinh x \cosh x$

9) $\cosh 2x = \cosh^2 x + \sinh^2 x$

10) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

11) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

12) $\tanh^2 x = 1 - \operatorname{sech}^2 x$

13) $\coth^2 x = 1 + \operatorname{csch}^2 x$