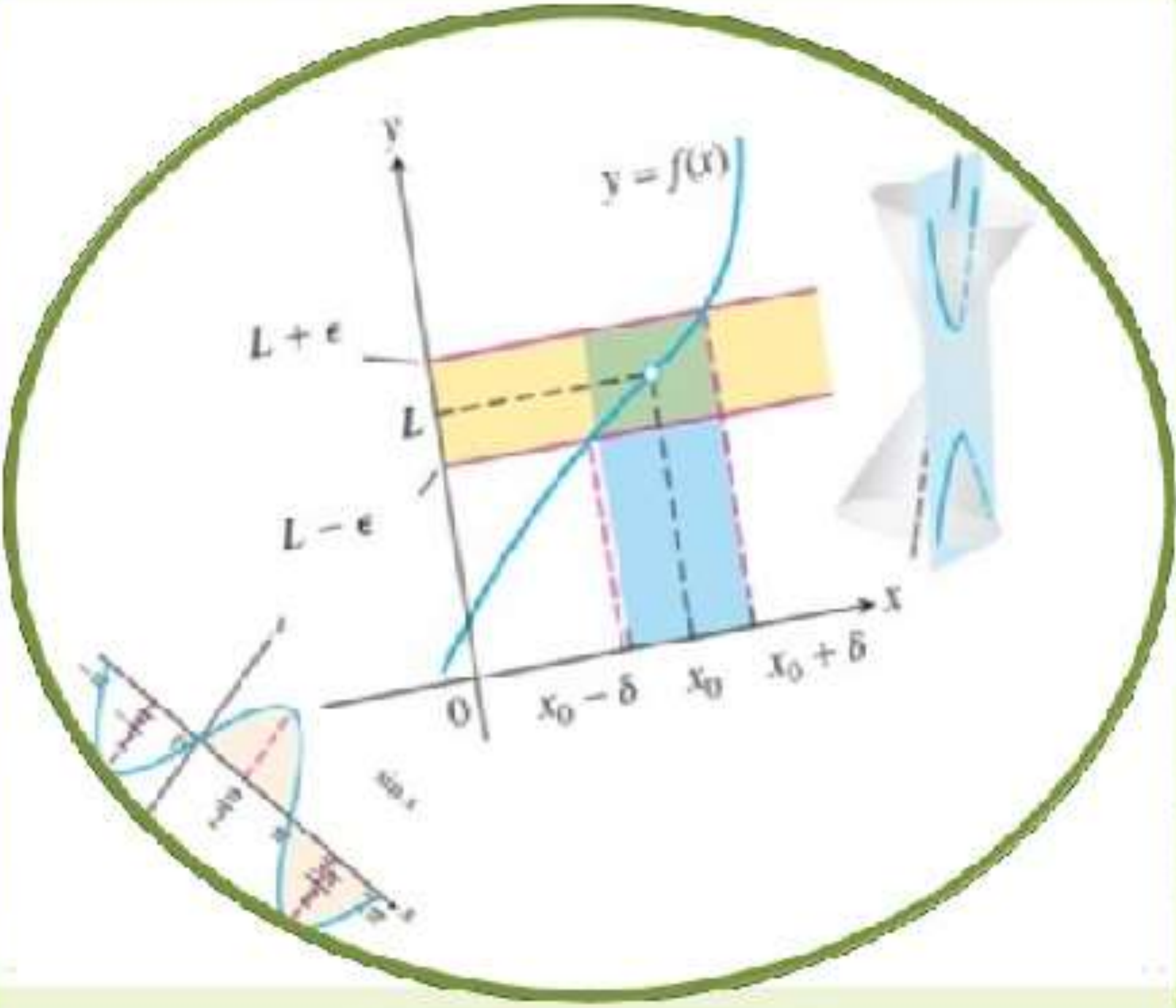
Differential Calculus



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CHAPTER 1: PREREQUISITES FOR CALCULUS

Inequalities

Def: **Inequality** is a sentence built form expressions using one or more the symbols $<$, $>$, $\leq$ or $\geq$ for example $x + y < 1$, $-1 \leq 3x \leq 2$

Properties of inequalities:

Let $a, b, c, d \in \mathbb{R}$

1) If $a < b$ and $c < d$ then $a + c < b + d$.
2) If $a < b$ then $a \pm c < b \pm c$.
3) If $a < b$ and $c \in \mathbb{R}^{+}$ then $ac < bc$ and $a/c < b/c$.
4) If $a < b$ and $c \in \mathbb{R}^{-}$ then $ac > bc$ and $a/c > b/c$.
5) If $ab > 0$. then $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.
6) If $ab < 0$. then $a > 0$ and $b < 0$ or $a > 0$ and $b < 0$.
7) If $a < b$ and $a, b \in \mathbb{R}^{+}$ or $a, b \in \mathbb{R}^{-}$ then $1/a > 1/b$.
8) If $a > 0$ then $\frac{1}{a} > 0$.

Types of intervals:

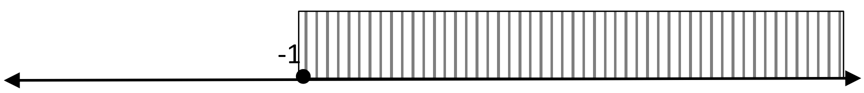
	Notation	Set description	Type
Finite	(a, b)	$\{x: \quad a < x < b\}$	Open
	$[a, b]$	$\{x: \quad a \leq x \leq b\}$	Closed
	$[a, b)$	$\{x: \quad a \leq x < b\}$	Half open or Half closed
	$(a, b]$	$\{x: \quad a < x \leq b\}$	Half open or Half closed
infinite	(a, ∞)	$\{x: \quad x > a\}$	Open
	$[a, \infty)$	$\{x: \quad x \geq a\}$	Closed
	$(-\infty, b)$	$\{x: \quad x < b\}$	Open
	$(-\infty, b]$	$\{x: \quad x \leq b\}$	Closed

Example: Write the following inequalities by notations and show their solution sets on the real line.

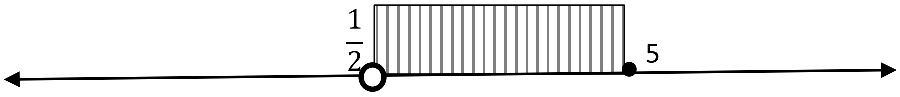
1) $x \geq -1$, 2) $\frac{1}{2} < x \leq 5$, 3) $6 \geq x$, 4) $-3 < x < 4$ and $x \geq 1$, 5) $x < 5$ $or -4 \leq x \leq 7$

Solution:

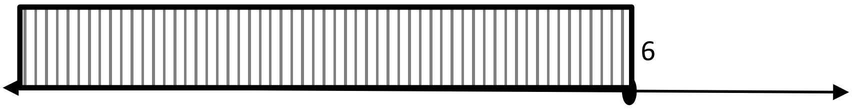
1) $x \geq -1 \Rightarrow x \in [-1, \infty)$



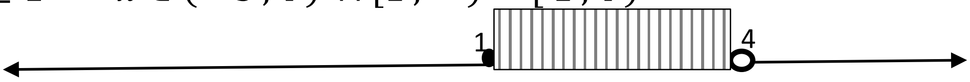
2) $\frac{1}{2} < x \leq 5 \Rightarrow x \in \left(\frac{1}{2}\ ,5\right]$



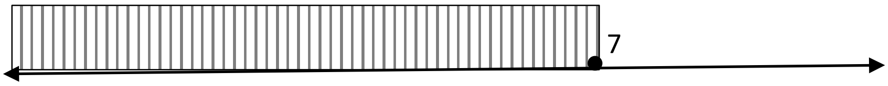
3) $6 \geq x \Rightarrow x \in (-\infty ,6\,]$



4) $-3 < x < 4\ \ and\ \ x \geq 1 \Rightarrow x \in (-3\,,4\,) \cap [1\,,\infty\,) = [\,1\,,4\,)$



5) $x < 5\ or - 4 \leq x \leq 7 \Rightarrow (-\infty\,,5\,) \cup [-4\,,7] = (-\infty\,,7\,]$



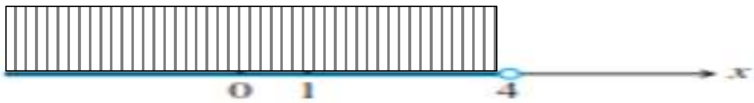
Example: Solve the following inequalities and show their solution sets on the real line

1) $2x - 1 < x + 3$, 2) $-3 \leq 6x - 1 < 3$, 3) $\frac{6}{x-1} \geq 5$, 4) $x^2 - 2x - 3 > 0$

Solution:

1) $2x - 1 < x + 3 \Rightarrow 2x - x < 3 + 1 \Rightarrow x < 4$

Then the set of solution is $(-\infty, 4)$

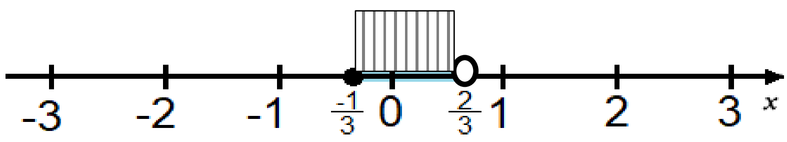


2) $-3 \leq 6x - 1 < 3 \Rightarrow -3 + 1 \leq 6x < 3 + 1$

$$\Rightarrow -2 \leq 6x < 4$$

$$\Rightarrow \frac{-2}{6} \leq x < \frac{4}{6}$$

$$\Rightarrow \frac{-1}{3} \leq x < \frac{2}{3}$$

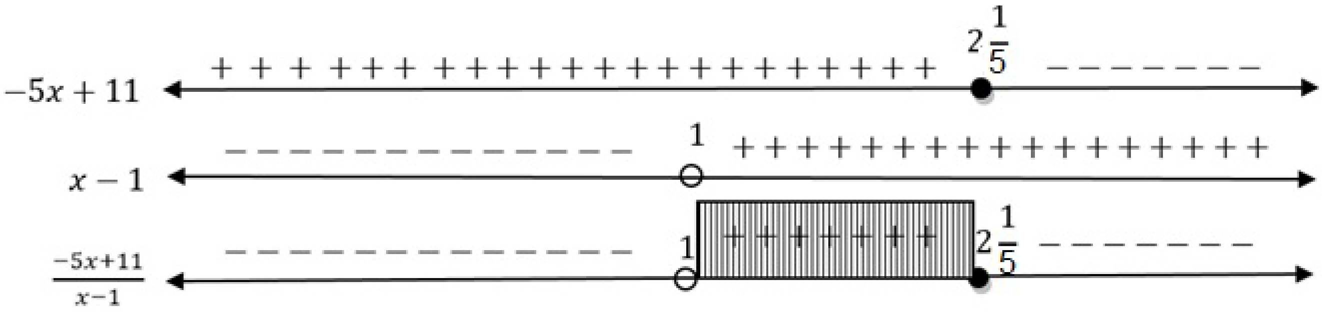


Then the set of solution is $\left[\frac{-1}{3}, \frac{2}{3}\right)$

3) $\frac{6}{x-1} \geq 5 \Rightarrow \frac{6}{x-1} - 5 \geq 0 \Rightarrow \frac{6-5x+5}{x-1} \geq 0 \Rightarrow \frac{-5x+11}{x-1} \geq 0$

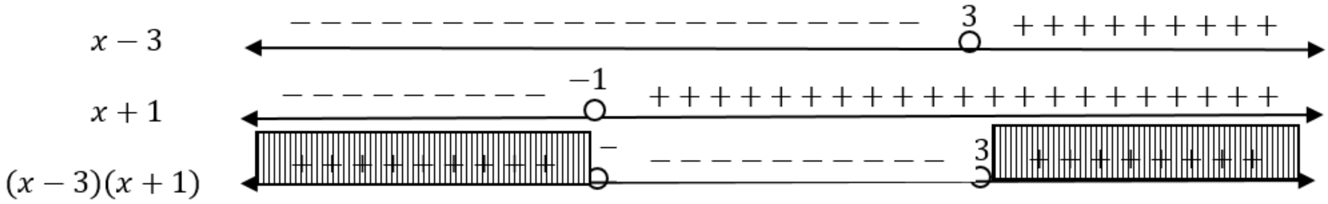
First let $-5x + 11 = 0 \Rightarrow x = \frac{11}{5} \Rightarrow x = 2\frac{1}{5}$

And let $x - 1 = 0 \quad \Longrightarrow \quad x = 1$



Then the set of solution is $\left(1, \frac{11}{5}\right]$

4) $x^2 - 2x - 3 > 0 \quad \Longrightarrow \quad (x-3)(x+1) > 0$
Let $x - 3 = 0 \quad \Longrightarrow \quad x = 3 \quad and\ let\ x + 1 = 0 \Longrightarrow x = -1$



Then the set of solution is $(-\infty, -1) \cup (3, \infty)$

Absolute value

The absolute value of a number , denoted by $|x|$, is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

For example : 1) $|3| = 3$, 2) $|-3| = 3$, 3) $|0| = 0$, 4) $|2-\sqrt{2}| = 2-\sqrt{2}$

5) $|\sqrt{2}-2| = 2-\sqrt{2}$ *because* $\sqrt{2}-2 < 0$.

Properties of absolute value:

Let $a, b \in \mathbb{R}$ then

1) $|-a| = |a|$ 2) $|ab| = |a| \cdot |b|$

3) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ 4) $|a \pm b| \leq |a| + |b|$

5) $\sqrt{a^2} = |a|$ 6) $if\ |x| = |a|\ then\ x = a\ or\ x = -a$

7) $if\ |x| = -a\ s.t\ a \in \mathbb{R}^+\ then\ the\ equation\ has\ not\ solution$

Example: Solve the following equations

1) $|3x+12|+7 = 7$, 2) $|x-3|-2 = -5$,

3) $|x+5| = |2x-1|$

Solution:

1) $|3x+12|+7 = 7 \Rightarrow |3x+12| = 0$

$\Rightarrow 3x+12 = 0$

$\Rightarrow x = -4$

Then the set of solutions is $\{-4\}$

2) $|x-3|-2 = -5 \Rightarrow |x-3| = -3$

Since -3 is a negative number then the equation has not solution.

3) $|x+5| = |2x-1|$ $\Rightarrow x+5 = 2x-1 \quad or \quad x+5 = -(2x-1)$

$\Rightarrow x+5 = 2x-1 \quad or \quad x+5 = -2x+1$

$\Rightarrow x-2x = -1-5 \quad or \quad x+2x = 1-5$

$\Rightarrow -x = -6 \quad or \quad 3x = -4$

$\Rightarrow x = 6 \quad or \quad x = \frac{-4}{3}$

Then the set of solution is $\left\{6, \frac{-4}{3}\right\}$

Remark: Let $a \in \mathbb{R}^+$ then

1) $|x| \le a \quad \Leftrightarrow \quad (if\ and\ only\ if) \quad -a \le x \le a$
2) $|x| \ge a \quad \Leftrightarrow \quad (if\ and\ only\ if) \quad x \le -a \quad or \quad x \ge a$

Example: Solve each inequality

1) $|x-5| \le 2$, 2) $x^2 - 6x + 2 > -1$

Solution:

1) $|x-5| \le 2 \quad \Longrightarrow -2 \le x-5 \le 2$

$\Longrightarrow 3 \le x \le 7$

Then the set of solution is $[3,7]$

2) Since $x^2 - 6x + 2 > -1$

By completing the square add $\left(\frac{6}{2}\right)^2$ to each side

Then $x^2 - 6x + 9 + 2 > -1 + 9 \quad \Longrightarrow \quad x^2 - 6x + 9 > 8 - 2$

$$\Longrightarrow \quad (x-3)^2 > 6 \qquad (\sqrt{\ })$$

$$\Longrightarrow \quad |x-3| > \sqrt{6}$$

$$\Longrightarrow \quad x - 3 > \sqrt{6} \quad or \quad x - 3 < -\sqrt{6}$$

$$\Longrightarrow \quad x > 3 + \sqrt{6} \quad or \quad x < 3 - \sqrt{6}$$

Then the set of solution is $\left(-\infty, 3-\sqrt{6}\right) \cup \left(3+\sqrt{6}, \infty\right)$

Exercises

In exercises (1-4) solve the following equations:

1) $|x+3| = 5$ (Ans: $\{-8,\ 2\}$)

2) $\left|\frac{4y+3}{y+2}\right| = 5$ (Ans: $\left\{-7,\ \frac{-13}{9}\right\}$)

3) $|2x-3| = |5x+4|$ (Ans: $\left\{\frac{-7}{3},\ \frac{-1}{7}\right\}$)

4) $|2x-1| = x+2$ (Ans: $\left\{3,\ \frac{-1}{3}\right\}$)

In exercises (5-10) solve each inequality. Write solution in interval notation:

5) $-x+3 \geq 2(5x+1)$ (Ans: $\left(-\infty,\ \frac{1}{11}\right]$)

6) $-2 < \frac{2x-1}{3} \leq 5$ (Ans: $\left(\frac{-5}{2}, 8\right]$)

7) $\frac{3}{x+2} > \frac{2}{x-4}$ (Ans: $(-2, 4) \cup (16,\ \infty)$)

8) $|3x+7| - 2 > 3$ (Ans: $(-\infty, -4) \cup \left(\frac{-2}{3}, \infty\right)$)

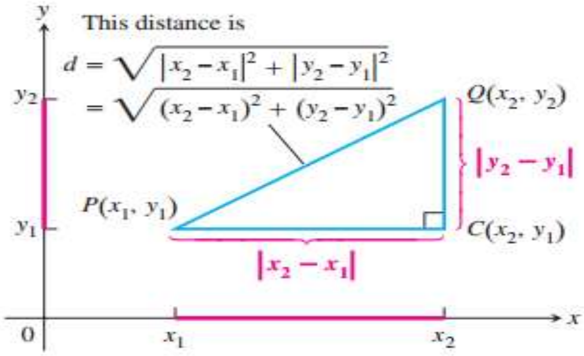
9) $\left|\frac{7-3x}{2}\right| \leq 1$ (Ans: $\left[\frac{5}{3}, 3\right]$)

10) $x^2 + 4x + 1 \geq -1$ (Ans: $\left(-\infty, -2-\sqrt{2}\right] \cup \left[-2+\sqrt{2}, \infty\right)$)

Distance formula

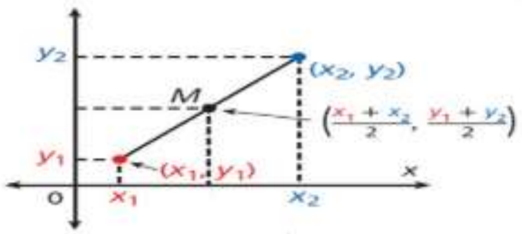
The distance formula $d(P_1, P_2)$ between two points $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ in coordinate plane is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Midpoint formula

The midpoint of the segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$



Example: Find the distance $d(A, B)$ and the coordinate of the midpoint of segment $\overline{AB}$ if

1) $A(-3,2)$, $B(-1, -2)$, 2) $A(-7, -5)$, $B(5,6)$

Solution:

1) $d(A, B) = \sqrt{(-1+3)^2 + (-2-2)^2}$

$$= \sqrt{2^2 + (-4)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

The midpoint of $\overline{AB} = \left(\frac{-3-1}{2}, \frac{2-2}{2}\right) = (-2,0)$

2) Exercise (Ans: $d(A, B) = \sqrt{265}$, The midpoint of $\overline{AB} = \left(-1, \frac{1}{2}\right)$)

Example: Find the distance $d(A, B)$ of segment $\overline{AB}$ if $A(\ 2\ ,3\ )$ and the midpoint between A *and* B *is* $(\ -1\ ,4)$

Solution: First , we will found the endpoint $B\ (x_2, y_2)$

Since the midpoint of $\overline{AB} = \left(\frac{2+x_2}{2}\ ,\ \frac{3+y_2}{2}\right) = (\ -1\ ,4)$

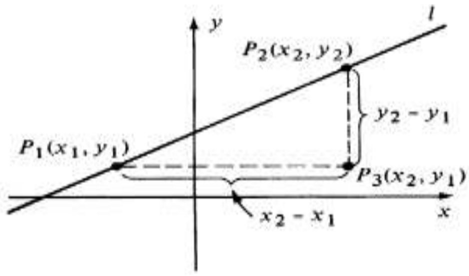
$\Rightarrow \frac{2+x_2}{2} = -1 \Rightarrow 2 + x_2 = -2 \Rightarrow x_2 = -4$, Also $\frac{3+y_2}{2} = 4 \Rightarrow y_2 = 5$

$\Rightarrow\ B\ (-4\ ,5)$

$\Rightarrow d(A, B) = \sqrt{(-4-2)^2 + (5-3)^2} = \sqrt{36+4} = \sqrt{40}$

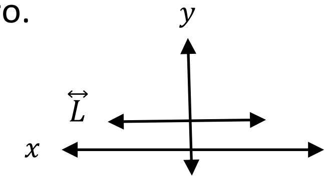
Lines

The slope of a nonvertical line $(\overleftrightarrow{L})$ whine passes through the points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$ $where\ x_1 \neq x_2$

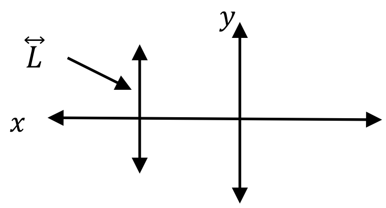


Notes:

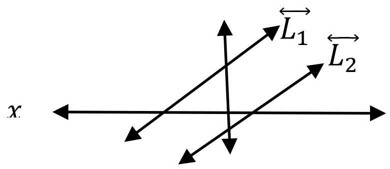
1) The slope of any horizontal line $\overleftrightarrow{L}: y = b$ s.t. b is y –intercept equal to zero.



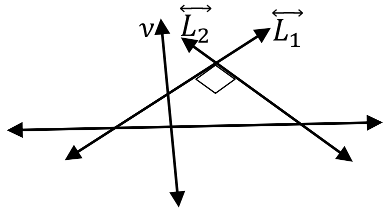
2) The slope of any vertical line $\overleftrightarrow{L}: x = a$ s.t. a is x –intercept undefined.



3) Let $\overleftrightarrow{L_1}$, $\overleftrightarrow{L_2}$ are two lines with slopes m_1, m_2 a respectively then
   a) $\overleftrightarrow{L_1}$ and $\overleftrightarrow{L_2}$ are parallel $\left(\overleftrightarrow{L_1}\ //\ \overleftrightarrow{L_2}\right)$ if and only if $m_1 = m_2$.



   b) $\overleftrightarrow{L_1}$ and $\overleftrightarrow{L_2}$ are perpendicular if and only if $m_1 \cdot m_2 = -1$ or $m_1 = \frac{-1}{m_2}$



Example: Find the slope of the following lines

1) $\overleftrightarrow{L}$ passes through $A(1, 3), B(5, -2)$ 2) $\overleftrightarrow{L} : x = 4$ 3) $\overleftrightarrow{L} : y = -6$

Solution:

1) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-3}{5-1} = -\frac{5}{4}$

2) $m\ is\ undefined$ because the line is vertical

3) $m = 0$ because the line is horizontal

Equations of a line

General form of the equation of a line:

The general equation or standard equation of straight line is given by

$$ax + by + c = 0 \quad , \quad where\ \ a, b, c \in \mathbb{R}\ \ and\ \ either\ a \neq 0\ or\ b \neq 0.$$

$*$ There exist other forms to write an equation line.

1) ***The slope – intercept equation***

   An equation of the line with slope m and y-intercept b is
   $$y = mx + b \ldots\ldots\ldots\ldots (*)$$

Notes:

a) From $(*)$ if
   i) $b = 0\ then\ y = mx$.
   ii) $m = 0$ (the line is a horizontal) then $y = b$.
b) The general equation $ax + by + c = 0$ can be converted to the slope – intercept form by solving for y: $y = \left(\frac{-a}{b}\right)x - \frac{c}{b}$ except for the spaical case $b = 0$ so this equation of straight line is slope intercept form with a slope of $\left(\frac{-a}{b}\right)$ and y −intercept of y is $\left(-\frac{c}{b}\right)$.

2) ***The point-slope equation***

   An equation of the line passes through the point (x_1, y_1) and has a slope m is
   $$y = y_1 + m(x - x_1)$$

Example: Find the equation of the line passes through the points $(1,7)$, $B(-3,2)$.

Solution: Since the slope m of the line is $m = \frac{y_2 - y_1}{x_2 - x_1}$

Then we get $m = \frac{7-2}{1-(-3)} = \frac{5}{4}$

Now, we can use the coordinates of ether $A\ or\ B$ in the points-slope equation . Using $A(1,7)$ we get

$$y - 7 = \frac{5}{4}(x - 1)$$
$$\Longrightarrow 4y - 28 = 5x - 5 \quad or \quad 5x - 4y + 23 = 0$$

Example: Find the equation of a line passes through the points $(5, -7)$ that is parallel to the line $6x + 3y = 4$.

Solution:

Let $\overleftrightarrow{L_1} : 6x + 3y = 4$ and let $\overleftrightarrow{L_2}$: the required line

Since $3y = -6x + 4 \Longrightarrow y = -2x + \frac{4}{3}$

$\Longrightarrow m_1 = -2$

Since $\overleftrightarrow{L_1} \ // \ \overleftrightarrow{L_2} \Longrightarrow m_2 = -2$

Now using the point-slope equation $y - y_1 = m(x - x_1)$ we get

$$y - (-7) = -2(x - 5)$$

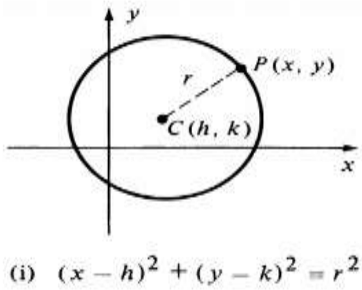
$$\Longrightarrow y + 7 = -2x + 10$$

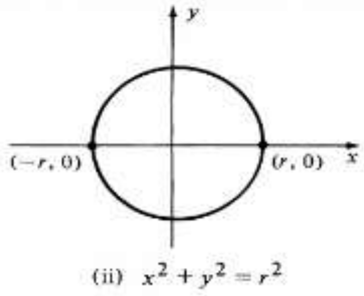
$$\Longrightarrow y = -2x + 3 \ \ or \ 2x + y - 3 = 0$$

Exercise : Find the equation of the above line when is horizontal to the line $6x + 3y = 4$

(Ans : $2y - x + 12 = 0$)

The circle





Standard form of the equation of the circle

The circle with center (h, k) and radius r has equation

$$(x-h)^2+(y-k)^2=r^2$$

Is the standard form of the equation of the circle. As a special case

$$x^2+y^2=r^2$$

Is the equation of the circle with radius r and center at the origin.

Example: Find the center and radius at the circle

$$x^2+y^2+4x-6y-3=0$$

Solution:

Since $x^2+y^2+4x-6y-3=0$ (add $\left(\frac{4}{2}\right)^2$ and $\left(\frac{6}{2}\right)^2$ to each side)

$\Longrightarrow (x^2+4x+4)+(y^2-6y+9)=3+4+9$

$\Longrightarrow (x+2)^2+(y-3)^2=16$

Then the center is $(-2,3)$ and the radius is $\sqrt{16}=4$

Note: It is possible to write the standard form of the equation circle in the form $x^2+y^2+Dx+Ey+F=0 \quad s,t, \ D,E,F \in \mathbb{R}$ and $D=-2h$, $E=-2k$, $F=h^2+k^2-r^2$

Illustration: From the previous example we find

$D=4 \ \Longrightarrow \ h=-2$, $\quad E=-6 \Longrightarrow \ k=3$

$F=-3 \ \Longrightarrow r=\sqrt{h^2+k^2-F}=\sqrt{4+9+3}=\sqrt{16}=4$

Example: Find the equation of the circle with center $(-8,6)$ and radius 5.

Solution: The standard form of the equation of the circle is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Longrightarrow \ (x+8)^2+(y-6)^2=5^2$$

$$\Longrightarrow \ x^2+16x+64+y^2-12y+36=25$$

$$\Longrightarrow \ x^2+y^2+16x-12y+75=0$$

Exercises

In exercises (1 – 2) find (a) the distace d(A , B) between the points A and B , and (b) the midpoint of segment $\overline{AB}$

1) A(6 , -2) ,B(2 ,1) (Ans: a)5 b) $(4 , -\frac{1}{2})$)

2)A(0 , -7), B(-1 , -2) (Ans : a) $\sqrt{26}$ $, b)(-\frac{1}{2} , -\frac{9}{2})$)

3) Prove that the triangle with vertices A(-3 , 4) , B(2 , -1) and C(9 , 6) is a right triangle and find its area

(Hint: For any **triangle** with sides a, b, c, $a^2 + b^2 = c^2$, *area* of right *triangle* $= \frac{1}{2} \times$ base$\times$ height)

(Ans : area= 35)

4) For what value of a is the distance between (a , 3) and (5, 2a) greater than $\sqrt{26}$?

(Ans : $a > 4$ or $a < \frac{2}{3}$))

In exercises (5 – 9) find an equation of a circle satisfying the given conditions.

5) Center (3, -2) , radius 4 (Ans : $x^2 + y^2 - 6x + 4y - 3 = 0$)

6) Center at the origin , passing through (-3 ,5) (Ans ; $x^2 + y^2 = 34$)

7) Center (-4 ,2) , tangent to the x-axis (Ans : $x^2 + y^2 + 8x - 4y + 16 = 0$)

8) Endpoints of a diameter A(4 ,-3) , B(-2 , 7) (Ans : $x^2 + y^2 - 2x - 4y - 29 = 0$)

9) Tangent to both axes , center in the first quadrant , radius 2

(Ans : $x^2 + y^2 - 4x - 4y + 4 = 0$)

In exercises (10 – 11) find the center and radius of the circle with given equation .

10) $x^2 + y^2 + 4x - 6y + 4 = 0$ (Ans : $(-2 , 3), 3$)

11) $2x^2 + 2y^2 - x + y - 3 = 0$ (Ans : $\left(\frac{1}{4} , -\frac{1}{4}\right), \frac{\sqrt{26}}{4}$)

12) If three vertices of parallelogram are $A(-1 , -3), B(4, 2)$ and $C(-7, 5)$ find the fourth vertex.

In exercises (13 – 18) find an equation for the line satisfying the given conditions.

13) Through $A(2 , -6), m = \frac{1}{2}$ (Ans : $x - 2y - 14 = 0$)

14) Through $A(-5 , -7), B(3 , -4)$ (Ans : $5x - 8y - 41 = 0$)

15) Through $A(8 , -2)$, $y - intercept - 3$ (Ans : $x - 8y - 24 = 0$)

16) Through $A(10 , -6)$, $parallel\ to\ a)\ y - axis\ \ b)\ x - axis$ (Ans : $x = 10$ $b)\ y = -6$)

17) Through $A(7,-3)$ $perpendicular$ to the line with equation $2x-5y=8$

(Ans : $5x+2y-29=0$)

18)Given $A(3,-1)$ and $B(-2,6)$, find an equation for the perpendicular bisector of the segment $\overline{AB}$ (Ans: $5x-7y+15=0$)

In exercises(19 – 20) find the slope and y-intercept of the line with the given equation .

19) $3x-4y+8=0$ (Ans : $m=\frac{3}{4}$, $y-intercept$: 2)

20) $y=4$ (Ans : $m=0$, $y-intercept$: 4)

21) Find a real number k such that the point $(-1,2)$ is on the line $kx+2y-7=0$

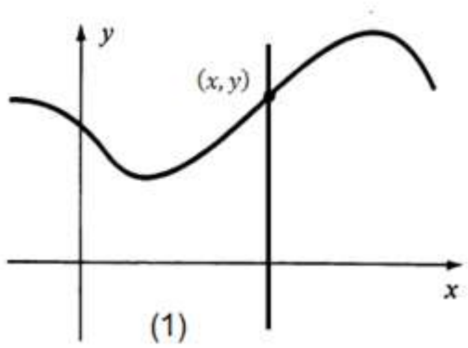
(Ans : $k=-3$)

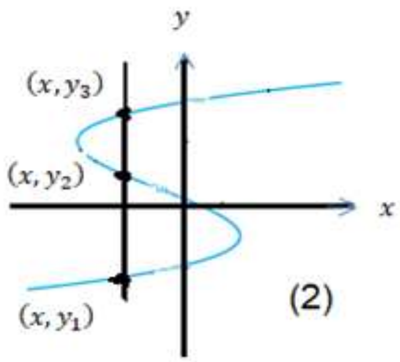
Functions

Def: A set f of order pairs is said to be a function from a set X to a set Y if $f \subseteq X * Y$ and for each x in X, there exists a unique $y \in Y$ such that $(x, y) \in f$.

In sense that f is a relation from X to Y.

Thus, the graph in figure 1 represent a function but the graph in figure 2 cannot represent a function because more than on pair (x, y) have the first element x.





Note: Let f be a function form X to Y, if $(x, y) \in f$ then

1) y is called the image of x under f and denoted by $f(x)$.
2) The set X is called the domain of the function and denoted by D_f.
3) The set Y is called the codomain of the function.
4) The all image of element of X called the range of the function and denoted by R_f.

Remarks:

1) If the domain of the function is not given, it will be assumed to be the largest possible set of the real numbers. Such that
   $D_f = \{x \in \mathbb{R} : f(x) \in \mathbb{R}\}$.
2) To find the Domain of given function we must keep the following two restrictions in mind
   i) The denominator $\neq 0$.
   ii) The quantity under to the n^{th} root $(\sqrt[n]{\ })$ must be greater than or equal to zero s.t. n is an even integer.

Example: Find the domain for the functions

$$1)\ f(x) = x^2 \qquad , \qquad 2)\ f(x) = \sqrt{-x^2 - 2x + 3}$$

Solution:

1) $f(x) = x^2$ Since any number can be squared then the Domain of f is $\mathbb{R}$

<u>Note:</u> Let $f(x)$ is a polynomial function s.t.

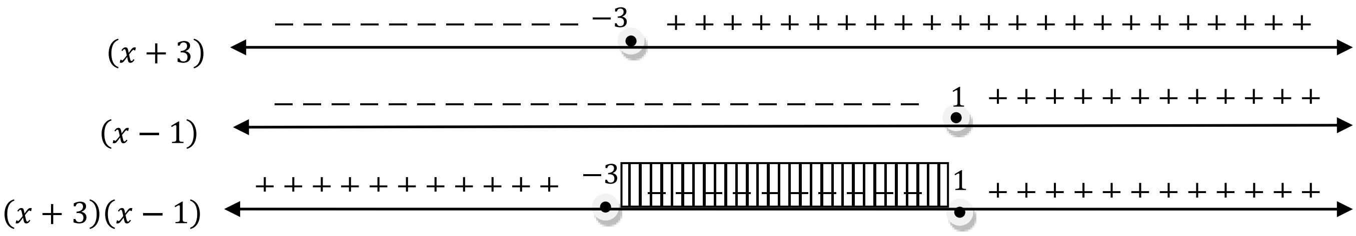
$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : \quad a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}, n \in \mathbb{Z}^+ \text{and } a_n \neq 0$

Then $D_f = \mathbb{R}$.

2) $f(x) = \sqrt{-x^2 - 2x + 3}$

The Domain is those values of x s.t.

$$-x^2 - 2x + 3 \geq 0 \quad \Longrightarrow \quad x^2 + 2x - 3 \leq 0 \quad \Longrightarrow \quad (x+3)(x-1) \leq 0$$



Then the domain is $[-3,1]$

<u>Note:</u> The Domain of

$$1)\ f(x) \pm g(x) \quad , \quad 2)\ \left. f(x) \middle/ g(x) \right. : g(x) \neq 0 \quad , \quad 3)\ f(x) \cdot g(x)$$

Is $D_f \cap D_g$

Example: Find D_f of $f(x) = \sqrt{x^2 - 4} + \frac{1}{\sqrt{2x+6}}$

Solution:

First , we will found the domain of $\left(\sqrt{x^2 - 4}\right)$ thus let

$x^2 - 4 \geq 0$

$\Longrightarrow x^2 \geq 4$

$\Longrightarrow |x| \geq 2 \Longrightarrow x \geq 2 \ or \ x \leq -2$

then the domain of $\left(\sqrt{x^2 - 4}\right) = (-\infty, 2] \cup [2, \infty)$

Second , we will found the domain of $\left(\frac{1}{\sqrt{2x+6}}\right)$ thus let

$2x + 6 > 0 \quad \ldots why \quad > 0 \ not \ \geq 0\,?$

$\Longrightarrow 2x > -6 \quad \Longrightarrow x > -3$

Then the domain of $\left(\frac{1}{\sqrt{2x+6}}\right) = \mathbb{R} \cap (-3, \infty) = (-3, \infty)$

Finally , the domain of $f(x) = ((-\infty, -2] \cup [2, \infty)) \cap (-3, \infty)$

$$\Longrightarrow D_f = (-3, -2) \cup [2, \infty)$$

Example: Find the range of the following functions

$$1)\ f(x) = x^2 \qquad , \qquad 2)\ f(x) = \sqrt{-x^2 - 2x + 3}$$

Solution:

1) $f(x) = x^2$

To find the range , first solve the equation for x , then let

$$y = f(x) \implies y = x^2 \implies x = \pm\sqrt{y}$$

Since $y \geq 0$ then the range of $f(x)$ is $[0, \infty)$.

2) $f(x) = \sqrt{-x^2 - 2x + 3}$

To find the range let $y = f(x)$

$$\Longrightarrow y = \sqrt{-x^2 - 2x + 3} \ : y \geq 0$$

$$\Longrightarrow y^2 = -x^2 - 2x + 3 \ : y \geq 0$$

$$\Longrightarrow x^2 + 2x - 3 + y^2 = 0$$

By quadratic formula we find

$$a = 1 \quad , \quad b = 2 \quad , \quad c = -3 + y^2$$

Since $\Delta = b^2 - 4ac$

Then $\Delta = 4 - 4(-3 + y^2) = 4 + 12 - 4y^2 = 16 - 4y^2$

Now let $\Delta \geq 0 \ \Longrightarrow 16 - 4y^2 \ \geq 0 \quad : y \geq 0$

$\Longrightarrow -4y^2 \ \geq -16 \quad : y \geq$

$\Longrightarrow y^2 \ \leq 4 \quad : y \geq 0$

$\Longrightarrow |y| \leq 2 \quad : y \geq 0 \ \Longrightarrow -2 \leq y \leq 2 \quad : y \geq 0$

Then the range is $[-2,2] \cap [0, \infty) = [0,2]$

<u>Even function and odd function</u>

Def: Even function and odd function

A function $y = f(x)$ is an

even function of $\boldsymbol{x}$ if $f(-x) = f(x)$

odd function of $\boldsymbol{x}$ if $f(-x) = -f(x)$

for every x in the function's domain.

Example: Determine whether f is even , odd , neither even nor odd

1) $f(x) = x^2 + 1$ 2) $f(x) = 2x^3 - x$ 3) $f(x) = x^5 + 5x^4$

Solution:

1) Since $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$ then f is an even function.
2) Since $f(-x) = 2(-x)^3 - (-x) = -2x^3 + x = -(2x^3 - x) = -f(x)$ then f is an odd function.
3) Since $f(-x) = (-x)^5 + 5(-x)^4(-x) = -x^5 + 5x^4 \neq f(x)$
$\neq -f(x)$
then f is neither even nor odd.

Note: The graph of an even function is **symmetric about the $\boldsymbol{y}$ −axis** . Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph. A reflection across the y −axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin** . Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph. Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged.

Graphs of functions

The graph of a function f is the graph of the equation $y = f(x)$; that is the set of all points (x, y) whose coordinates satisfy the equation $y = f(x)$.

Example: Sketch the following function.

$$1)\ f(x) = 3 - x^2 \quad ,\quad 2)\ f(x) = \sqrt{x-1} \quad ,3)\ f(x) = |x|\ ,$$
$$4)\ f(x) = [\![x]\!] \quad (or\ \lfloor x \rfloor)\ ,\quad 5)\ f(x) = \lceil x \rceil$$

Solution:

1) $f(x) = 3 - x^2$ (quadratic function) $D_f = \mathbb{R}$ ..... why?

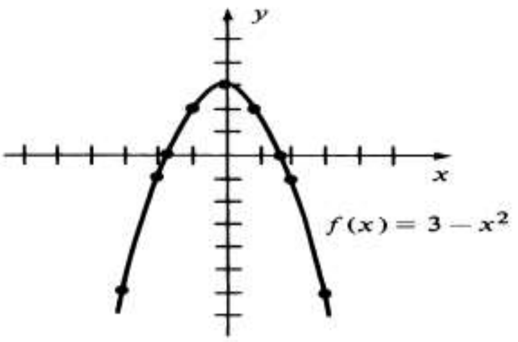
Now we will construct a table includes some points $(x, f(x))$ s.t. $x \in D_f$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-6	-1	2	3	2	-1	-6

To find x −intercept let

$f(x) = 0 \Longrightarrow x^2 = 3$

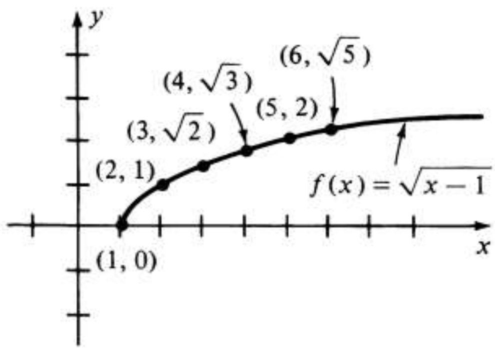
$\Longrightarrow x = \pm\sqrt{3} \cong \pm 1.7$



- From the graph we find that, the range of $f(x)$ is $(-\infty, 3]$

2) $f(x) = \sqrt{x-1}$ $D_f = [1, \infty)$ ......why?
(the square root function)

x	1	2	3	4	5	6
$f(x)$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$

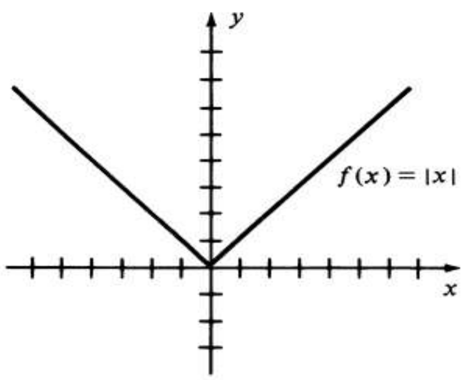


*from the graph we find that the range of $f(x)$ is $[0, \infty)$

3) $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ $D_f = \mathbb{R}$ ..... why?

(the absolute value function)

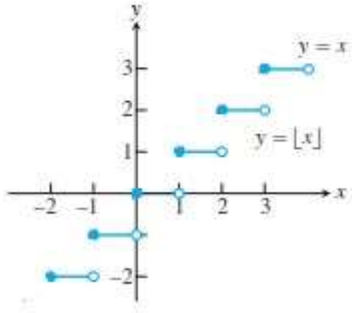
x	-3	-2	-1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3



4) $f(x) = [\![x]\!]$ or $\lfloor x \rfloor$ (the greatest integer function or integer floor function)

The function whose value at any number x is the greatest integer less than or equal to x.

If x is any real number, then there exist consecutive integers n and $n+1$ such that $n \le x < n+1$. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined as follows: If $n \le x < n+1$, then $f(x) = n$. Sketch the graph of f



$$\text{Since } [\![x]\!] = \lfloor x \rfloor = \begin{cases} \vdots \\ -2 & , \ -2 \le x < -1 \\ -1 & , \ -1 \le x < 0 \\ 0 & , \ 0 \le x < 1 \\ 1 & , \ 1 \le x < 2 \\ 2 & , \ 2 \le x < 3 \\ \vdots \end{cases}$$

■$D_f = \mathbb{R}$ ■the range of $f(x)$ is $\mathbb{Z}$ (integer numbers)

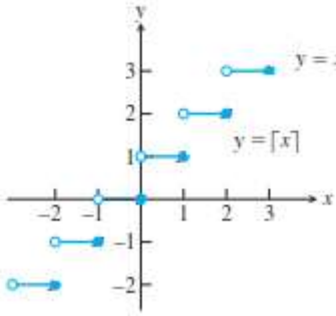
$\lfloor 2.4 \rfloor=2$	$\lfloor 1.9 \rfloor=1$	$\lfloor 0 \rfloor=0$	$\lfloor -1.2 \rfloor=-2$
$\lfloor 2 \rfloor=2$	$\lfloor 0.2 \rfloor=0$	$\lfloor -0.3 \rfloor=-1$	$\lfloor -2 \rfloor=-2$

5) $f(x) = \lceil x \rceil$ (the least integer function or integer ceiling function)

The function whose value at any number x is the least integer greater than or equal to x.

If x is any real number, then there exist consecutive integers $n-1$ and n such that $n-1 < x \le n$. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined as follows: If $n-1 < x \le n$, then $f(x) = n$. Sketch the graph of f

$$\text{Since } \lceil x \rceil = \begin{cases} \vdots \\ -2 & , \ -3 < x \le -2 \\ -1 & , \ -2 < x \le -1 \\ 0 & , \ -1 < x \le 0 \\ 1 & , \ 0 < x \le 1 \\ 2 & , \ 1 < x \le 2 \\ \vdots \end{cases}$$



■$D_f = \mathbb{R}$ ■the range of $f(x)$ is $\mathbb{Z}$ (integer numbers)

$\lceil 2.4 \rceil=3$	$\lceil 1.9 \rceil=2$	$\lceil 0 \rceil=0$	$\lceil -1.2 \rceil=-1$
$\lceil 2 \rceil=2$	$\lceil 0.2 \rceil=1$	$\lceil -0.3 \rceil=0$	$\lceil -2 \rceil=-2$

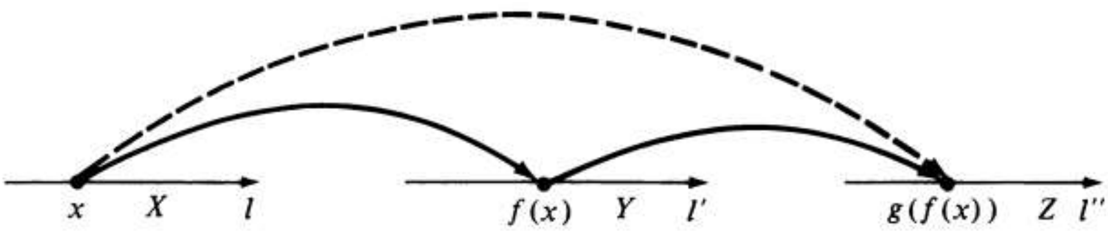
Exercises: Sketch the graph of the following function

$$1)\ f(x) = -\sqrt{x+2} \quad , \quad 2)\ f(x) = \begin{cases} 2x+3 & , \ x < 0 \\ x^2 & , \ 0 \le x < 2 \\ 1 & , \ x \ge 2 \end{cases}$$

Composition of functions

Def: If $f(x)$ is a function from X to Y and $g(x)$ is a function from Y to Z, then the composition function $(gof)(x)$ is a function form X to Z defined by

$$(gof)(x) = g\big(f(x)\big) \quad , \quad \forall x \in X$$



Note : In the composition function $(gof)(x)$ called $f(x)$ inside (input) function .

Example: If $f(x) = x - 2$ and $g(x) = 5x + \sqrt{x}$, find

$$1)\ (gof)(x)\ , \qquad 2)\ (gof)(6)$$

Solution:

1) $(gof)(x) = g\big(f(x)\big) = g(x-2) = 5(x-2) + \sqrt{x-2}$

$$= 5x + \sqrt{x-2} - 10$$

2) $(gof)(6) = 5(6) + \sqrt{6-2} - 10 = 30 + \sqrt{4} - 10 = 22$

Another solution

$$(gof)(6) = g\big(f(6)\big) = g(4) = 5(4) + \sqrt{4} = 20 + 2 = 22$$

Example: Let If $f(x) = 4x + 1$ and If $g(x) = 2x^2 + 5x$, find each of the following

$$1)\ (gof)(x)\ , \quad 2)\ (fog)(x)\ , \quad 3)\ (fof)(x)$$

Solution:

1) $(gof)(x) = g\big(f(x)\big) = g(4x+1) = 2(4x+1)^2 + 5(4x+1)$
$= 2(16x^2 + 8x + 1) + 20x + 5$
$= 32x^2 + 16x + 2 + 20x + 5$
$= 32x^2 + 36x + 7$

2) $(fog)(x) = f\big(g(x)\big) = f(2x^2 + 5x) = 4(2x^2 + 5x) + 1 = 8x^2 + 20x + 1$

Note that in the previous example $(gof)(x) \neq (fog)(x)$

3) $(fof)(x) = f\big(f(x)\big) = f(4x+1) = 4(4x+1) + 1 = 16x + 5$

Domain of the composition function

$$D_{(fog)(x)} = \{\, x : x \in D_{g(x)} \;\; and \;\; g(x) \in D_f \,\}$$

To find the domain of the composition function we work the following

1) Find the domain of inside (input) function

2) Find the domain of new function of the composition

3) Find the intersection of the sets in (1) and (2)

Example : Find $(fog)(x)$ and $(gof)(x)$ and domain of each , where

$$f(x) = \sqrt{x-2} \;\; \text{and} \;\; g(x) = \sqrt{x^2-1}$$

Solution :

1) $(fog)(x) = f\big(\, g(x)\big) = f\big(\, \sqrt{x^2-1}\,\big) = \sqrt{\sqrt{x^2-1}-2}$ (new function)

First : we will found the domain of inside function thus let

$$x^2 - 1 \geq 0 \Rightarrow x^2 \geq 1 \Rightarrow |\, x \,| \geq 1 \Rightarrow x \in (\, -\infty \,, -1\,] \cup [\, 1 \,, \infty\,) \ldots \;(1)$$

Second : we will found the domain of new function thus let

$$\sqrt{x^2-1} - 2 \geq 0 \Rightarrow \sqrt{x^2-1} \geq 2 \Rightarrow x^2 - 1 \geq 4$$

$$\Rightarrow x^2 \geq 5 \Rightarrow |\, x \,| \geq \sqrt{5} \Rightarrow x \in \big(-\infty \,, -\sqrt{5}\,\big] \cup \big[\,\sqrt{5} \,, \infty\,\big) \;\ldots (\,2\,)$$

Finally : we will found the intersection of (1) and (2) thus

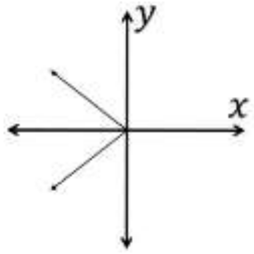
$$D_{(fog)(x)} = \big((\,-\infty \,, -1\,] \cup [\,1 \,, \infty\,)\big) \cap \Big((\,-\infty \,, -\sqrt{5}\,\big] \cup \big[\,\sqrt{5} \,, \infty\,)\Big)$$

$$= \big(\,-\infty \,, -\sqrt{5}\,\big] \cup \big[\,\sqrt{5} \,, \infty\,\big)$$

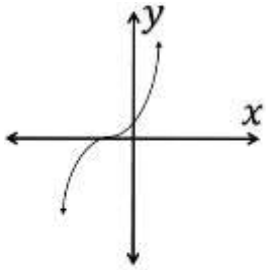
2) Exercise (Ans : $(gof)(x) = \sqrt{x-3}$, $D_{(gof)(x)} = [\,3 \,, \infty\,)$)

Exercises

In exercises (1-2) decide whether the following relations is a function:

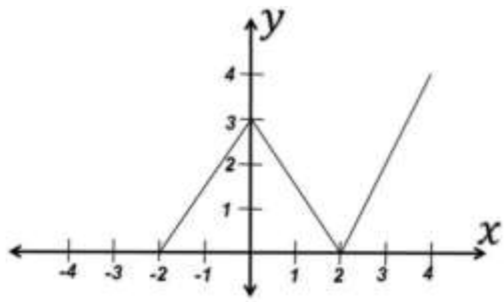
1) 

(Ans: no)

2) 

(Ans: yes)

3)for of the following graph (a) find $f(-2), f(0)$ and $f(4)$, (b) find the $x - value$ that correspond to the following function values: $3, 0$ and 2



(Ans: $\begin{cases} (a)\ f(-2) = 0,\ f(0) = 3, f(4) = 4 \\ (b)\ \{0\}, \{-2, 2\}, \{-1, 1,3\} \end{cases}$)

4) let $f(x) = 3 - 2x^2$, $g(x) = \sqrt{4x - 2}$ and $h(x) = \frac{x}{2-x}$. Find each of the following:

(a) $h(3)$, (b) $g(5)$, (c) $f(4) - 5$, (d) $[g(1)]^2$,

(e) $f(\frac{1}{a})$, (f) $\frac{1}{f(a)}$, (g) $\frac{f(5a)}{g(a)}$.

(Ans: (a) -3, (b) $\sqrt{18}$, (c) -34, (d) 2, (e) $3 - \frac{2}{a^2}$, (f) $\frac{1}{3-2a^2}$, (g) $\frac{3-50a^2}{4a-2}$)

In exercises (5-6) find the domain of the given function:

5) $f(x) = \sqrt{x^2 + x - 6}$ (Ans: $(-\infty, -3] \cup [2, \infty)$)

6) $f(x) = \frac{5}{3-\sqrt{x-1}}$ (Ans: $[1, \infty)\backslash\{10\}$)

In exercises (7-11) find the domain and range of the given function:

7) $f(x) = -\sqrt{5}$ (Ans: $\mathbb{R}$, $\{-\sqrt{5}\}$)

8) $f(x) = \frac{x}{x-1}$ (Ans: $\mathbb{R}\backslash\{1\}$, $\mathbb{R}\backslash\{1\}$)

9) $f(x) = \sqrt{9 - x^2} + 6$ (Ans: $[-3,3]$, $[6 , 9\,]$)

10) $f(x) = \frac{x^2}{x^2+9}$ (Ans : $\mathbb{R}$, $[\,0\ ,\ 1\,)$)

11) $f(x) = \frac{x^2+5}{x+6}$ (Ans: $\mathbb{R}\backslash\{-6\}$, $\left(-\infty, -12 - 2\sqrt{41}\right] \cup \left[-12 + 2\sqrt{41}, \infty\right)$)

In exercises (12 – 16) determine whether f is even , odd , or neither even nor odd

12) $f(x) = 3x^3 - 4x$ (Ans: odd)

13) $f(x) = 9 - 5x^2$ (Ans: even)

14) $f(x) = 2$ (Ans: even)

15) $f(x) = 2x^2 - 3x + 4$ (Ans: neither)

16) $f(x) = |x| + 5$ (Ans: even)

In exercises (17 – 26) sketch the graph of following functions.

17) $f(x) = -4x + 3$

18) $f(x) = \lceil x - 2 \rceil$

19) $f(x) = -3$

20) $f(x) = \lfloor x + 3 \rfloor$

21) $f(x) = 4 - x^2$

22) $f(x) = |x - 4| + 2$

23) $f(x) = \sqrt{4 - x^2}$

24) $f(x) = \frac{1}{x}$

25) $f(x) = \sqrt{4 - x}$

26) $f(x) = \begin{cases} -x , & x < 0 \\ 3 , & 0 \leq x \leq 1 \\ x^2 , & x > 1 \end{cases}$

In exercises (27 – 29) find $(fog)(x)$ and $(gof)(x)$ and domain of each of the following

27) $f(x) = x + 3$, $g(x) = \sqrt{9 - x^2}$

Ans : $f\big(g(x)\big) = \sqrt{9 - x^2} + 3$, $D_{(fog)(x)} = [\,-3 , 3\,]$

$g\big(f(x)\big) = \sqrt{-x^2 - 6x}$, $D_{(gof)(x)} = [\,-6 , 0\,]$)

28) $f(x) = \frac{2}{x-3}$, $g(x) = \frac{5}{x+2}$

Ans : $f\big(g(x)\big) = -\frac{2(x+2)}{3x+1}$, $D_{(fog)(x)} = \mathbb{R} - \left\{ -2 , -\frac{1}{3} \right\}$

$g\big(f(x)\big) = \dfrac{5(x-3)}{2x-4}$, $D_{(gof)(x)} = \mathbb{R} - \{3 , 2\}$

29) $f(x) = -\frac{3}{x}$, $g(x) = \frac{x}{x-2}$

Ans : $f\big(g(x)\big) = -\frac{3(x-2)}{x}$, $D_{(fog)(x)} = \mathbb{R} - \{\,2 , 0\,\}$

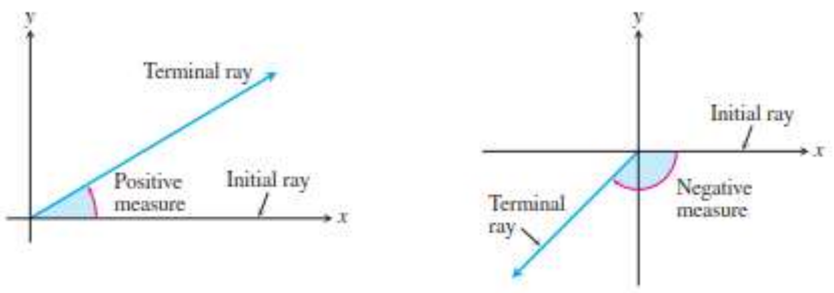
$g\big(f(x)\big) = \dfrac{3}{2x+3}$, $D_{(gof)(x)} = \mathbb{R} - \left\{ 0 , -\frac{3}{2} \right\}$

Trigonometry

Angles and their measure

An angle is formed by rotating a ray around its endpoint.

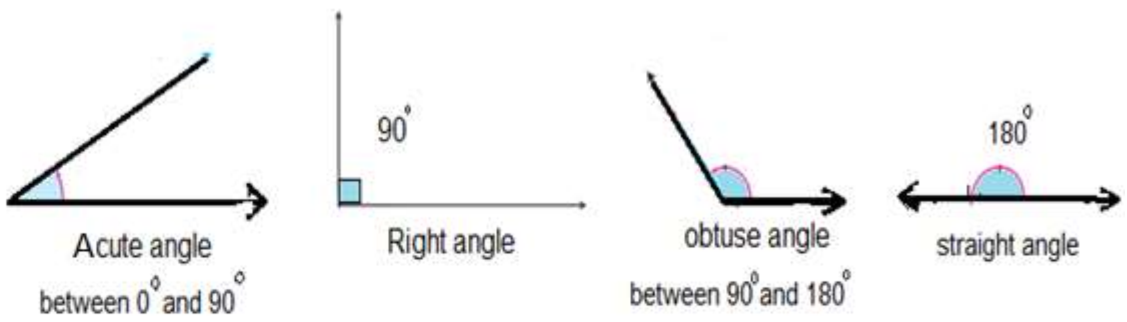
The initial position of the ray is the initial side of the angle, while the location of the ray at the end of its rotation is the terminal side of the angle. The end point of the ray is the vertex of the angle



Remarks:

1) If the rotation of an angle is counterclockwise , the angle is positive , if the rotation is clockwise , the angle is negative.
2) An angle is in standard position if its vertex is at origin $(0,0)$ of a coordinate system and its initial side is along the positive x −axis.
3) Angle can be measured two ways i) Degrees ii) Radians.
4) A circle is comprised of 360 degrees $(360°)$ or 2π radians (2π rad).

Some special angles



Notes:

1) To convert degrees to radians, multiply degrees by $\frac{\pi rad}{180°}$.
2) To convert radians to degrees , multiply radians by $\frac{180°}{\pi rad}$.

Example: Convert the following angles in radians to degrees

$$1)\ \frac{5\pi}{6}rad \quad , \quad 2)\ \ 2rad$$

Solution:

1) $\frac{5\pi}{6}rad \ = \frac{5\pi}{6}rad \ \times \frac{180°}{\pi rad} = \frac{5\times180°}{6} = 150°$
2) $2rad = 2rad \ \times \frac{180°}{\pi rad} = \frac{360°}{\pi} \cong 114.59°$

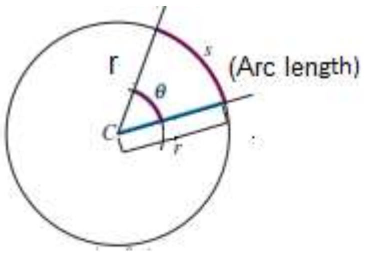
Example: Convert each angle in degrees to radians $1)\ 60° \ \ ,\ \ 2)\ -135°$

Solution:

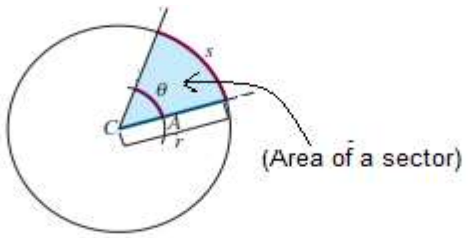
1) $60° = 60° \times \frac{\pi rad}{180°} = \frac{\pi}{3}rad$
2) $-135° = -135° \times \frac{\pi rad}{180°} = -\frac{3\pi}{4}rad$

Arc length and Area of a sector

If θ (in radians) is a central angle in a circle with radians r, then



1) The length of the arc cut off by θ is given by
$$S = r\theta$$
2) The area of the sector formed by angle θ is given by
$$A = \frac{1}{2}r^2\theta$$



Example: Find the length of the arc intercepted by central angle $240°$ on a circle with radius 4 inches.

Solution: To use the formula $= r\theta$, first convert $240°$ to radian measure

$$240° = 240° \times \frac{\pi rad}{180°} = \frac{4\pi}{3}rad$$

Then $\ \ S = 4 \times \frac{4\pi}{3} = \frac{16\pi}{3} \cong 16.76$ inches.

Example: Find the area of the sector of the circle with radius 20 cm and central angle $\frac{\pi}{4}rad$.

Solution: Since $A = \frac{1}{2}r^2\theta = \frac{1}{2}(20)^2 \ \times \frac{\pi}{4} = 50\pi \cong 157.1\ cm^2$

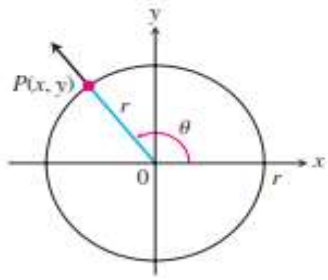
Trigonometric function

Def: Let (x, y) be a point other than the origin on the terminal side of an angle θ in standard position. Let r be the distance the origin to (x, y). Then the trigonometric function of θ are defined as follows (Assume no denominators are zero)

sine: $\sin\theta = \dfrac{y}{r}$ **cosecant**: $\csc\theta = \dfrac{r}{y}$

cosine: $\cos\theta = \dfrac{x}{r}$ **secant**: $\sec\theta = \dfrac{r}{x}$

tangent: $\tan\theta = \dfrac{y}{x}$ **cotangent**: $\cot\theta = \dfrac{x}{y}$



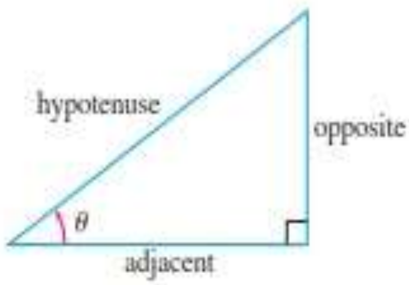
Remark:

If θ is an acute angle of a right triangle , then

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$



Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad , \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \ , \quad 1 + \tan^2\theta = \sec^2\theta \ , \ 1 + \cot^2\theta = \csc^2\theta \ .$$

Double Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned}$$

Example: Verify the following identities

1) $\tan^2\theta(1+\cot^2\theta) = \dfrac{1}{1-\sin^2\theta}$, 2) $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

Solution:

1) $\tan^2\theta(1+\cot^2\theta) = \tan^2\theta\csc^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}\dfrac{1}{\sin^2\theta} = \dfrac{1}{\cos^2\theta} = \dfrac{1}{1-\sin^2\theta}$

2) $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\sin\theta}{1+\cos\theta}\dfrac{1-\cos\theta}{1-\cos\theta}$

$$= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}$$

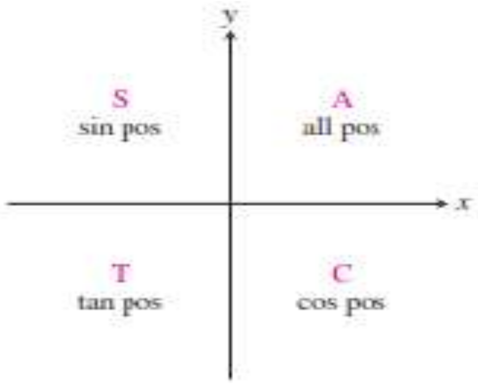
Some properties of trigonometric functions

Negative angle identities

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$ $\sec(-\theta) = \sec\theta$ $\cot(-\theta) = -\cot\theta$

Signs of values of trigonometric functions in quadrants



Trigonometric values of common angle

Values of sin θ, cos θ, and tan θ for selected values of θ

Degrees	**−180**	**−135**	**−90**	**−45**	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**270**	**360**
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
cos θ	-1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	0	1
tan θ	0	1	undefined	-1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0	undefined	0

Example: Given that θ is an acute angle and $\cos\theta = \frac{5}{8}$, find 1) $\sin\theta$, 2) $\tan\theta$

Solution:

1) Since $\sin^2\theta + \cos^2\theta = 1$

$$\Longrightarrow \sin^2\theta + \left(\frac{5}{8}\right)^2 = 1$$

$$\Longrightarrow \sin^2\theta = 1 - \frac{25}{64} = \frac{39}{64}$$

$$\Longrightarrow \sin\theta = \frac{\sqrt{39}}{8}$$ (because θ is acute angle)

2) Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Then $\tan\theta = \frac{\sqrt{39}/8}{5/8} = \frac{\sqrt{39}}{5}$

Example: If $\tan\theta = -\frac{5}{3}$ and θ in quadrant , find the values of the other trigonometric functions.

Solution: since $\sec^2\theta = 1 + \tan^2\theta$

Then $\sec^2\theta = 1 + \left(\frac{-5}{3}\right)^2 = 1 + \frac{25}{9} = \frac{34}{9}$

$\Longrightarrow \sec\theta = -\frac{\sqrt{34}}{3}$ (because θ is in quadrant II)

Since $\cos\theta = \frac{1}{\sec\theta}$ then $\cos\theta = \frac{-3}{\sqrt{34}}$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ then $\sin\theta = \tan\theta \cdot \cos\theta = -\frac{5}{3} \cdot \frac{-3}{\sqrt{34}} = \frac{5}{\sqrt{34}}$

Since $\csc\theta = \frac{1}{\sin\theta}$ then $\csc\theta = \frac{\sqrt{34}}{5}$

Since $\cot\theta = \frac{1}{\tan\theta}$ then $\cot\theta = \frac{-3}{5}$

Example: Find the exact value of each function

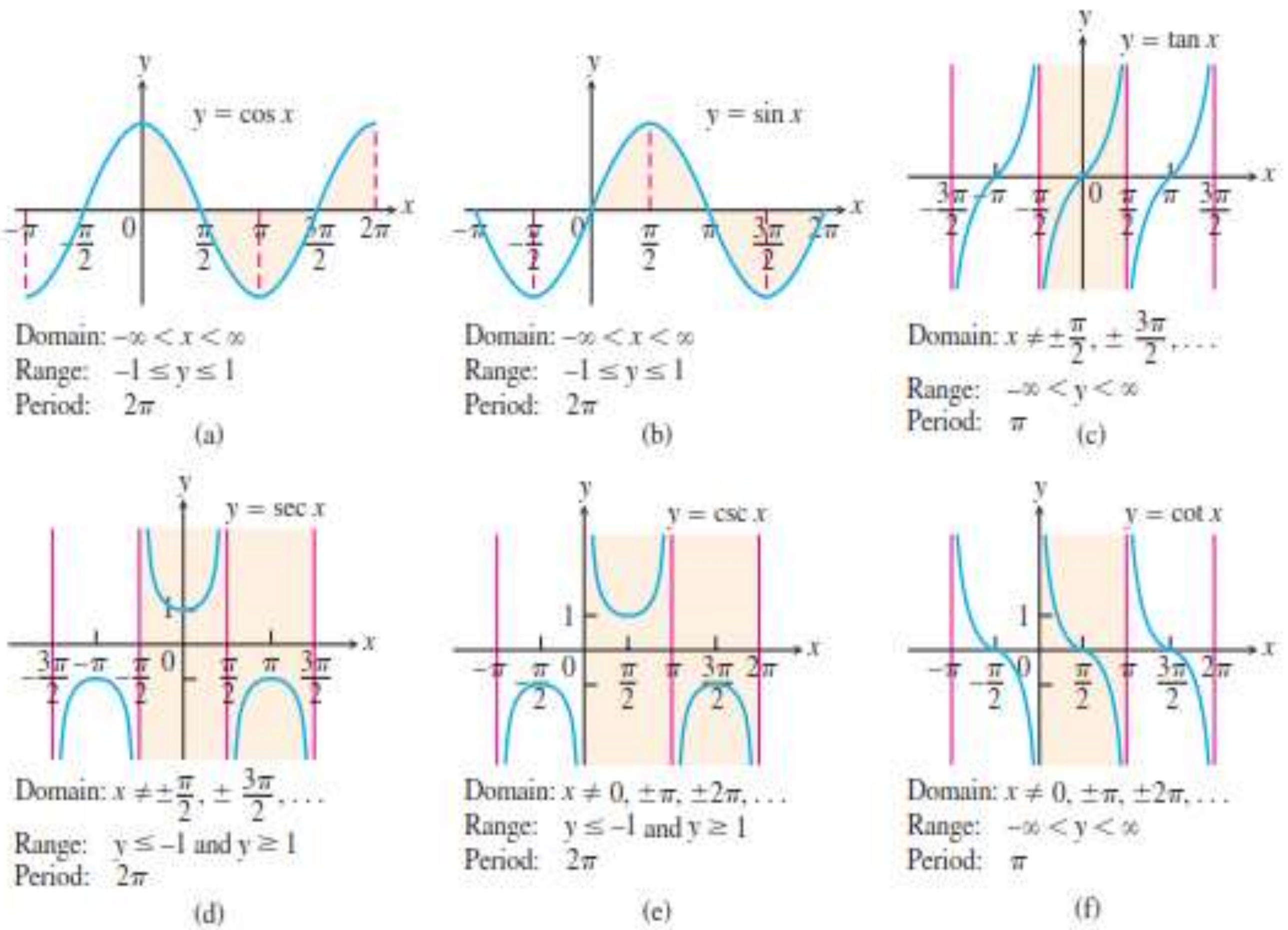
1) $\sin\left(-\frac{\pi}{4}\right)$ 2) $\tan\left(\frac{-5\pi}{4}\right)$ 3) $\sec\left(\frac{5\pi}{3}\right)$

4) $\cos\left(\frac{-7\pi}{6}\right)$ 5) $\cot(210°)$ 6) $\csc\left(\frac{9\pi}{4}\right)$

Solution:

1) $\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$
2) $\tan\left(\frac{-5\pi}{4}\right) = -\tan\left(\frac{5\pi}{4}\right) = -\tan\left(\pi + \frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1.$
3) $\sec\left(\frac{5\pi}{3}\right) = \sec\left(2\pi - \frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right) = 2.$
4) $\cos\left(\frac{-7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$
5) $\cot(210°) = \cot(180° + 30°) = \cot(30°) = \sqrt{3}\,.$
6) $\csc\left(\frac{9\pi}{4}\right) = \csc\left(\frac{9\pi}{4} - 2\pi\right) = \csc\left(\frac{\pi}{4}\right) = \sqrt{2}\,.$

<u>Graphs of the trigonometric functions</u>



Exercises

In exercises (1- 4) convert radian measure to degrees, and degree measure to radian:

1) $\frac{4\pi}{5}\ rad$ 2) $\frac{-11\pi}{18}\ rad$ 3) $270°$ 4) $1020°$

Ans: 1) $144°$ 2) $-110°$ 3) $\frac{3\pi}{2}\ rad$ 4) $\frac{17\pi}{3}\ rad$

5) Let $r = 12\ ft, \quad \theta = 90°$. Find S and A (Ans: $S = 6\pi\ ft, \quad A = 36\pi\ ft^2$)

In exercisers (6-9) show that following identities:

6) $\frac{\sin^2 x}{\cos x} = \sec x - \cos x$

7) $\frac{1}{\sec\theta+\tan\theta} = \sec\theta - \tan\theta$

8) $\sec 2y = \frac{\sec^2 y}{2-\sec^2 y}$

9) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

10) If $\csc\theta = \frac{-\sqrt{13}}{2}$ and $\theta \in III$. Find the values of the other trigonometric function.

(Ans: $\sin\theta = -\frac{2}{\sqrt{13}}$, $\cos\theta = -\frac{3}{\sqrt{13}}$, $\tan\theta = \frac{2}{3}$, $\cot\theta = \frac{3}{2}$, $\sec\theta = -\frac{\sqrt{13}}{3}$)

11) If $x = 3\tan\theta$ and $\in I$, find $\sec\theta + \tan\theta$ (Ans: $\frac{x+\sqrt{9+x^2}}{3}$)

Exponential and Logarithmic function

Definition of exponential function

The function $f(x)$ defined by

$$f(x) = a^x \quad , \quad a > 0 \; and \; a \neq 1$$

Is the exponential function with base a

Note:

The exponential function defined by $f(x) = e^x$ s.t. $e \cong 2.72$ is called the natural exponential function.

Some properties of the exponential function

1) If $a^{f(x)} = a^{g(x)}$ then $f(x) = g(x) \quad \forall a > 0 \; and \; a \neq 1.$
2) If $a^{f(x)} = b^{f(x)}$ then $f(x) = 0 \quad \forall a, b > 0 \; and \; a \neq 1 \neq b.$

Example: Solve the following equation $e^{-x^2} = e^{-3x-4}$.

Solution:

Since $e^{-x^2} = e^{-3x-4}$

Then $-x^2 = -3x - 4$

$\Longrightarrow x^2 - 3x - 4 = 0$

$\Longrightarrow (x+1)(x-4) = 0$

Either $x - 4 = 0 \Longrightarrow x = 4 \quad or \quad x + 1 = 0 \Longrightarrow x = -1$

Then the set of solution is $\{-1,4\}$.

Definition of $\log_a x$

For $a > 0$, $a \neq 1$ and > 0 , $\log_a x$ is the power to which a must be raised to get x.

Remark:

$$y = \log_a x \quad \Leftrightarrow \quad x = a^y \quad \forall y \in \mathbb{R} \;, \forall a, x \in \mathbb{R}^+ \; and \; a \neq 1$$

For example $2 = \log_5 25 \quad \Leftrightarrow \quad 25 = 5^2$

Definition of Logarithmic function

If $a > 0$, $a \neq 1$ and $x > 0$ then the function $f(x)$ defined by

$$f(x) = \log_a x$$

Is the Logarithmic function with base a.

Note: The Logarithmic function $f(x) = \log_a x$ is called

1) The natural Logarithmic function in case $a = e$ and denoted by $\ln x$.
2) The common Logarithmic function in case $a = 10$ and denoted by $\log x$.

Theorem: If $a > 0$, $a \neq 1$ then

$$1)\ \log_a 1 = 0 \quad , \quad 2)\ \log_a a = 1 \quad , \quad 3)\ \log_a a^x = x$$

Also , if $x > 0$ then $a^{\log_a x} = x$.

Properties of Logarithmic

If $a, x, y \in \mathbb{R}^+$ and $a \neq 1$ then

1) $\log_a(xy) = \log_a x + \log_a y$.
2) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3) $\log_a x^n = n \log_a x \quad , \quad \forall\, n \in \mathbb{R}$.
4) $\log_a x = \log_a y \quad \Leftrightarrow \quad x = y$.
5) $\log_a x = r \quad \Leftrightarrow \quad \log_a x = \log_a a^r \quad , \forall\, n \in \mathbb{R}$.

Change – of – base theorem

If $x \in \mathbb{R}$ $\ and\ a, b \in \mathbb{R}^+ , a \neq 1 \neq b$ then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example: Solve each equation

$$1)\ 3 = 5(1 - e^x) \quad , \quad 2)\ 3(2^x) = 42$$

Solution:

1) $3 = 5(1 - e^x) \quad \Rightarrow 1 - e^x = \frac{3}{5}$

$$\Rightarrow e^x = 1 - \frac{3}{5}$$

$$\Rightarrow e^x = \frac{2}{5}$$

$$\Rightarrow \ln e^x = \ln \frac{2}{5}$$

$$\Rightarrow x = \ln \frac{2}{5}$$

Then the solution is $x = \ln\frac{2}{5} \cong -0.916$

2) $3(2^x) = 42 \quad \Longrightarrow 2^x = 14$

$$\Longrightarrow \ln 2^x = \ln 14$$

$$\Longrightarrow x \ln 2 = \ln 14$$

$$\Longrightarrow x = \frac{\ln 14}{\ln 2}$$

Then the solution is $x = \frac{\ln 14}{\ln 2} \cong 3.807$

Example: Solve each equation

1) $\log_2(3x-5) = 3$ 2) $\ln(3x+8) = \ln(2x+2) + \ln(x-2)$

3) $\log_{(x+1)}(2x+5) = 2$ 4) $\log 5x + \log(x-1) = 2$

Solution:

1) $\log_2(3x-5) = 3 \quad \Longrightarrow \log_2(3x-5) = \log_2 2^3$

$$\Longrightarrow 3x - 5 = 2^3 = 8$$

$$\Longrightarrow 3x = 13 \quad \Longrightarrow x = \frac{13}{3}$$

Then the solution is $x = \frac{13}{3}$

2) $\ln(3x+8) = \ln(2x+2) + \ln(x-2)$

$\Longrightarrow \ln(3x+8) = \ln(2x+2)(x-2)$

$\Longrightarrow 3x + 8 = (2x+2)(x-2)$

$\Longrightarrow 3x + 8 = 2x^2 - 2x - 4$

$\Longrightarrow 2x^2 - 2x - 4 - 3x - 8 = 0$

$\Longrightarrow 2x^2 - 5x - 12 = 0$

$\Longrightarrow (2x+3)(x-4) = 0$

Either $2x + 3 = 0 \;\Longrightarrow\; x = -\frac{3}{2}$ (is not solution)

or $x - 4 = 0 \;\Longrightarrow x = 4$

Then the solution is $x = 4$

3) $\log_{(x+1)}(2x+5) = 2 \quad \Longrightarrow 2x + 5 = (x+1)^2$

$$\Longrightarrow 2x + 5 = x^2 + 2x + 1$$

$$\Longrightarrow x^2 + 2x + 1 - 2x - 5 = 0$$

$$\Longrightarrow x^2 - 4 = 0$$

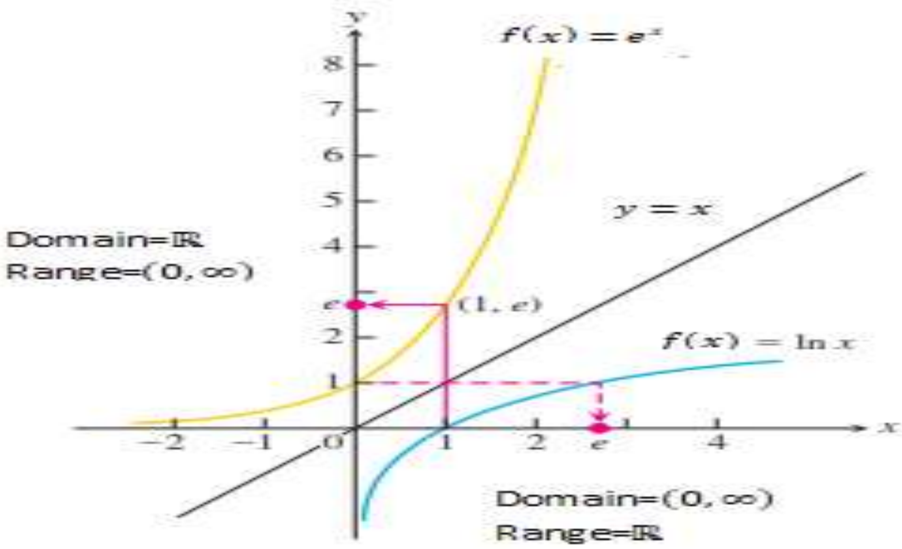
$$\Longrightarrow x^2 = 4 \quad \Longrightarrow x = \pm 2$$

But $x = -2$ is not solution

Then the solution is $x = 2$.

4) $\log 5x + \log(x-1) = 2$ Exercise (Ans: The solution is $x = 5$)

The graphs of e^x and $\ln x$:



Exercises

In exercises (1-7) solve each equation:

1) $5^r = 625$ (Ans: 4)

2) $\left(\frac{1}{2}\right)^r = 8$ (Ans: -3)

3) $3^{2x-5} = \frac{1}{9}$ (Ans: $\frac{3}{2}$)

4) $8^{2x} = 2^{x+3}$ (Ans:$\frac{3}{2}$)

5) $\left(\frac{1}{2}\right)^{-x} = \left(\frac{1}{4}\right)^{x+1}$ (Ans:$\frac{-2}{3}$)

6) $2^{|x|} = 128$ (Ans: ± 7)

7) $e^x - 5 + 6e^{-x} = 0$ (Ans: $\ln 2, \ln 3$)

In exercises (8- 12) find the value of each of the following:

8) $\log_5 25$ (Ans:2)

9) $10^{\log 2}$

10) $\ln \sqrt[3]{e}$ (Ans:$\frac{1}{3}$)

11) $(e^5)^{\ln 2}$

12) $e^{\ln 3 + \ln 5}$ (Ans:15)

In exercises (13- 19) find the value of each of the following:

13) $\log_x 256 = 8$ (Ans:2)

14) $5^{\log_5(x+1)}$=9 (Ans:8)

15) $\ln(y+1) = \ln(y-2) + \ln 2$ (Ans:5)

16) $\log_x(5x-6) = 2$ (Ans:2,3)

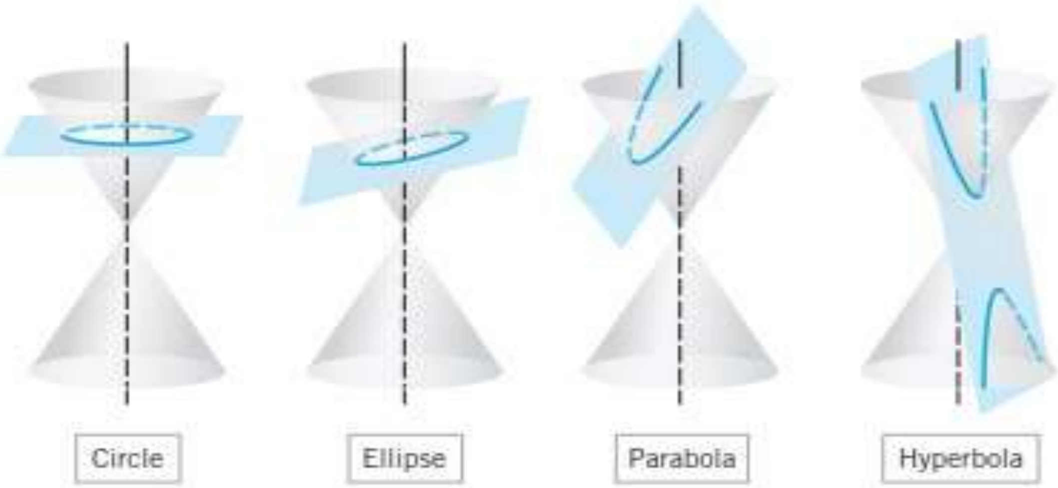
17) $7^{2x \log_7 4} = 256$ (Ans:2)

18) $\log z = \sqrt{\log z}$ (Ans:1,10)

19) $\ln(3+2y) - \ln(1+y) = \ln 10$ (Ans: $-\frac{7}{8}$)

Conic sections

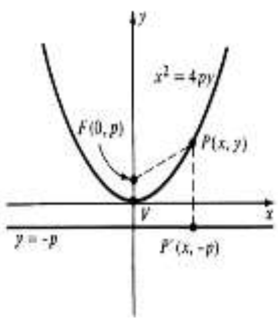
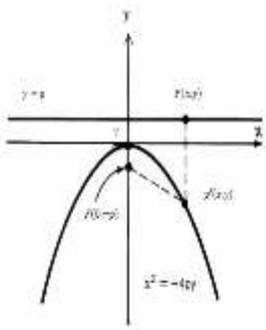
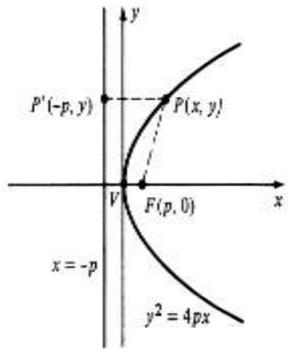
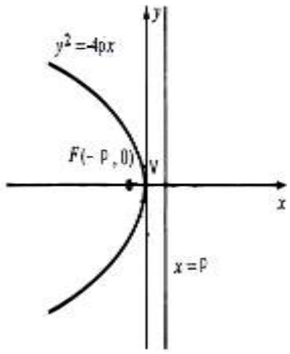
The intersection of a plane and the surface of a double cone is called a conic section (Circle – Parabola – Ellipse – Hyperbola)



1) ***Parabola***

A parabola is the set of points in a plane equidistant from a fixed point (focus) and a fixed line (directrix)

Standard equation of a parabola (vertex at origin)

Equation	$x^2 = 4py$	$x^2 = -4py$	$y^2 = 4px$	$y^2 = -4px$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Focus	$(0, p)$	$(0, -p)$	$(p, 0)$	$(-p, 0)$
Directrix	$y = -p$	$y = p$	$x = -p$	$x = p$
Graph				

Example: Find the focus and the directrix of the parabola and sketch its graph

$$1)\ x^2 = 8y \quad , \quad 2)\ y^2 = -28x \quad , \quad 3)\ \frac{1}{2}x^2 + y = 0$$

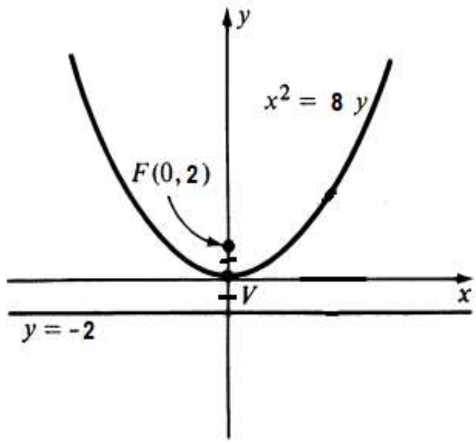
Solution:

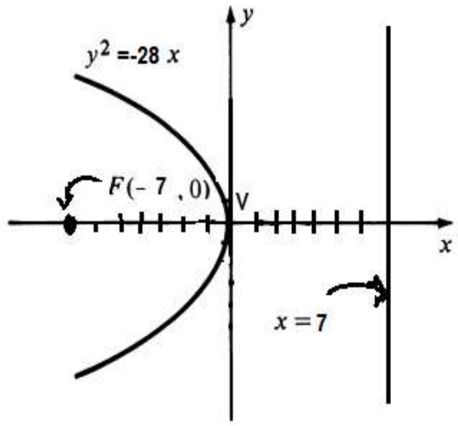
1) The equation of the form $x^2 = 4py$

$\Longrightarrow 4p = 8 \quad \Longrightarrow \quad p = 2$

Then , the focus $(0,2)$,

the dircetrix $y = -2$





2) The equation of the form $y^2 = -4px$

$\Longrightarrow 4p = 28 \quad \Longrightarrow \quad p = 7$

Then , the focus $(-7,0)$,

the dircetrix $x = 7$

3) Exercise..

Example: Write an equation for each of the following parabolas with vertex at the origin

1) Focus at $(-2,0)$, 2) Directrix $y = 1$

3) passes through the point $(-2,-2)$, vertical axis.

Solution:

1) the equation of the form $y^2 = -4px$, and as the focus is $(-2,0) \Longrightarrow p = 2$, then the equation is $y^2 = -8x$.

2) the equation of the form $x^2 = -4py$, and as the directrix

$y = 1 \Longrightarrow p = 1$ then the equation is $x^2 = -4y$.

3) Since the parabola passes through the point $(-2,-2)$ and the graph of equation on vertical axis. Then the equation of the form $x^2 = -4py$

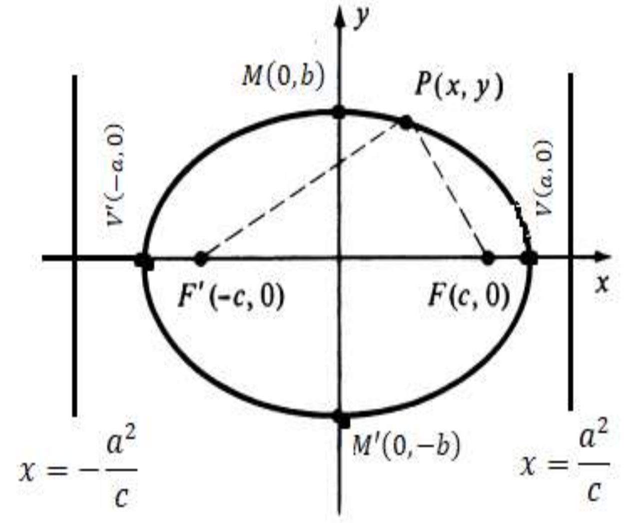
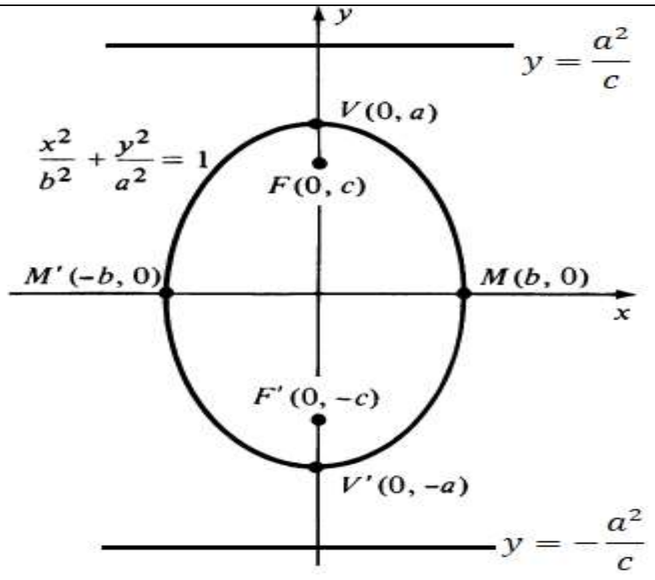
$$\Longrightarrow (-2)^2 = -4(-2)p \Longrightarrow 4 = 8p \Longrightarrow p = \frac{1}{2}$$

Then the equation is $x^2 = -2y$.

2) Ellipse:

An ellipse is the set of all points in a plane the sum of whose distances from two fixed points (foci) is constant.

Standard equation for Ellipses(Center at origin)

Equation	Vertices	Foci	Directrices	Graph
$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $a>b$	$(\pm a,0)$	$(\pm c,0)$	$x=\pm\dfrac{a^2}{c}$ or $x=\pm\dfrac{a}{e}$	
$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$ $a>b$	$(0,\pm a)$	$(0,\pm c)$	$y=\pm\dfrac{a^2}{c}$ or $y=\pm\dfrac{a}{e}$	

Notes:

1) center – to –focus distance $c=\sqrt{a^2-b^2}$.

2) The length of the major axis is $2a$.

3) The length of the minor axis is $2b$.

4) The eccentricity $e=\frac{c}{a}<1$.

Example: Given the ellipse equation. find the eccentricity, the coordinates of the vertices , the foci and the equations of the directrices

$$1)\ \frac{x^2}{9}+\frac{y^2}{2}=1 \quad , \quad 2)\ 4x^2+y^2=36$$

Solution:

1) $\frac{x^2}{9}+\frac{y^2}{2}=1$

$$a^2=9 \implies a=3 \quad , \quad b^2=2 \implies b=\sqrt{2}$$

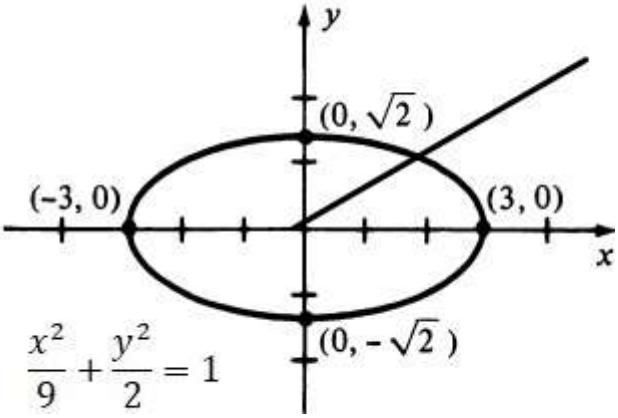
Since $c^2=a^2-b^2=9-2=7$ then $c=\sqrt{7}$

Since $e=\frac{c}{a}$ then $e=\frac{\sqrt{7}}{3}$

Vertices : $(\pm a,0)=(\pm 3,0)$

Foci : $(\pm c,0)=\left(\pm\sqrt{7},0\right)$

Directrices: $x=\pm\frac{a^2}{c} \implies x=\pm\frac{9}{\sqrt{7}}$

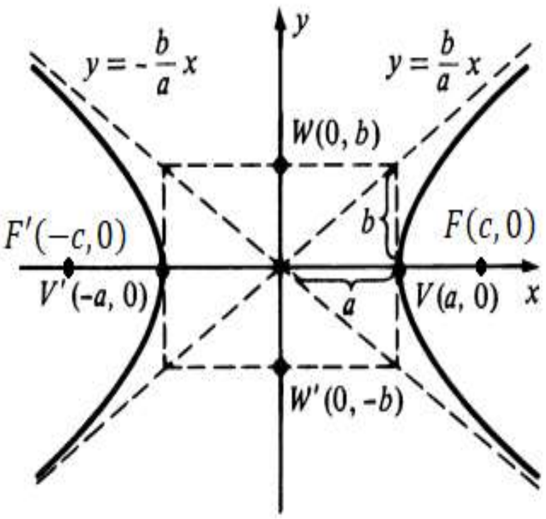
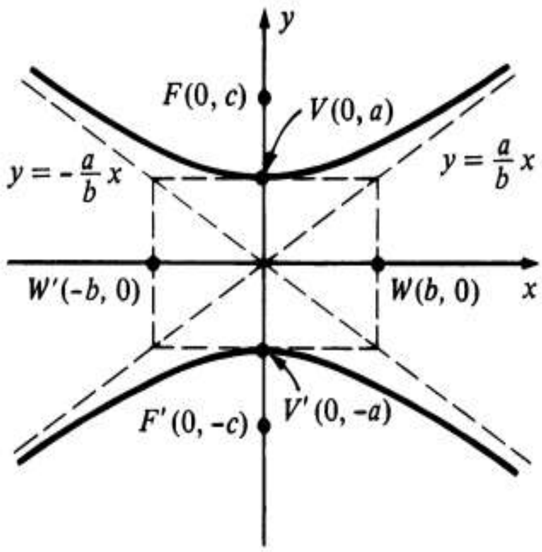


2) Exercise.

3) Hyperbola

A hyperbola is the set of all points in the plane the difference between whose distances from two points (Foci) is constant.

Standard equation for Hyperbolas (Center at origin)

Equation	Vertices	Foci	Directrices	Asymptotes	Graph
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$(\pm a,0)$	$(\pm c,0)$	$x=\pm\frac{a^2}{c}$ or $x=\pm\frac{a}{e}$	$y=\pm\frac{b}{a}x$	
$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	$(0,\pm a)$	$(0,\pm c)$	$y=\pm\frac{a^2}{c}$ or $y=\pm\frac{a}{e}$	$y=\pm\frac{a}{b}x$	

Notes:

1) a, b *and* c are related by the equation $c^2=a^2+b^2$.

2) The length of the transverse axis is $2a$.

3) The length of the conjugate axis is $2b$.

4) The eccentricity $e=\frac{c}{a}>1$.

Example: Find the vertices , foci and asymptotes of the hyperbola given by the equations and sketch the graph of each equation

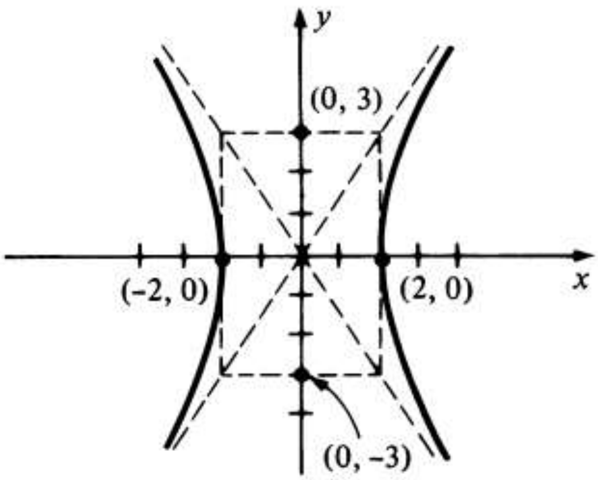
$$1)\ \frac{x^2}{4} - \frac{y^2}{9} = 1 \quad , \quad 2)\ 4x^2 - y^2 = 16$$

Solution:

1) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

$a^2 = 4 \ \Rightarrow a = 2 \quad , \quad b^2 = 9 \Rightarrow b = 3$

Since $c^2 = a^2 + b^2$ then $c^2 = 4 + 9 = 13 \ \Rightarrow c = \sqrt{13}$



Vertices : $(\pm a, 0) = (\pm 2,0)$

Foci : $(\pm c, 0) = \left(\pm\sqrt{13}, 0\right)$

Asymptotes: $\quad y = \pm\frac{b}{a}x \ \Rightarrow \ y = \pm\frac{3}{2}x$

2) Exercise .

Exercises

In exercises(1-2) find the vertex, focus and directrix of the parabola with the given equation and sketch the graph

1) $x^2 = -12y$ (Ans: $V(0,0)$, $F(0,-3)$, $y = 3$)
2) $3y^2 = -3x$ (Ans: $V(0,0)$, $F\left(-\frac{3}{8},0\right)$, $x = \frac{3}{8}$)

In exercises (3-4) find an equation for the parabola that satisfies the given condition

3) Focus $(2,0)$, directrix $x = -2$ (Ans: $y^2 = 8x$)
4) Vertex at the origin, symmetric to the y −axis , and passing through the point $(2,-3)$ (Ans: $3x^2 = -4y$)

In exercises (5-6) given the ellipse equations, find the eccentricity , the coordinates of the vertices , the foci . the equations of the directrices and sketch its graph

5) $16x^2 + 25y^2 = 400$ (Hint: $a = 5$, $b = 4$, $c = 3$)
6) $3x^2 + 2y^2 = 6$ (Hint: $a = \sqrt{3}$, $b = \sqrt{2}$, $c = 1$)

In exercises (7-9) find an equation for the ellipse satisfying the given conditions

7) $V_{1,2}(\pm 8,0)$, $F_{1,2}(\pm 5,0)$ (Ans: $\frac{x^2}{64} + \frac{y^2}{39} = 1$)
8) $V_{1,2}(0,\pm 5)$, length of minor axis 3 (Ans: $\frac{4x^2}{9} + \frac{y^2}{25} = 1$)
9) $V_{1,2}(0,\pm 6)$, passing through $(3,2)$ (Ans: $\frac{8x^2}{81} + \frac{y^2}{36} = 1$)

In exercises (10-11) find the vertices , foci , asymptotes of the hyperbola given by the equation and sketch its graph

10) $x^2 - y^2 = 1$ (Hint: $a = 1$, $b = 1$, $c = \sqrt{2}$)
11) $9x^2 - 16y^2 = 1$ (Hint: $a = 1/3$, $b = 1/4$, $c = 5/12$)

In exercises (12-14) find an equation for the hyperbola satisfying the given conditions

12) $F_{1,2}(\pm 5,0)$, $V_{1,2}(\pm 3,0)$ (Ans: $\frac{x^2}{9} - \frac{y^2}{16} = 1$)
13) $F_{1,2}(0,\pm 5)$, length of conjugate axis 4 (Ans: $\frac{y^2}{21} - \frac{x^2}{4} = 1$)
14) $V_{1,2}(\pm 3,0)$, equations of asymptotes $y = \pm 2x$ (Ans: $\frac{x^2}{9} - \frac{y^2}{36} = 1$)

CHAPTER 2 : LIMITS AND CONTINUITY FUNCTIONS

The Limits

Introduction: The notation of a limit is fundamental concept of calculus in interval University. You will learn how to evaluate limits and how they are used in the basic problems of calculus:

1) the tangent line problem (The derivative)

2) the area problem (Integral)

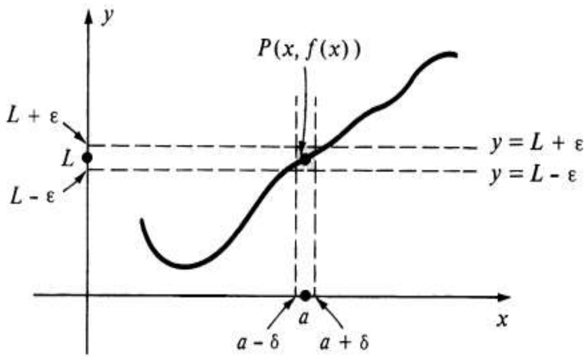
Definition of Limit:

Let f be a function that is defined on an open interval containing $'a'$, except possibly at $'a'$ itself, and let $'L'$ be a real number. The statement

$$\lim_{x\to a} f(x) = L$$

means that $\forall \varepsilon > 0$ و$\exists \delta > 0$ ،such that

$0 < |x-a| < \delta \; then \; |f(x)-L| < \varepsilon$



$\lim_{x\to a} f(x) = L$

Example: Prove that $\lim_{x\to 4}\frac{1}{2}(3x-1) = \frac{11}{2}$ by the definition.

Solution:

Let $f(x) = \frac{1}{2}(3x-1)$, $a = 4$ and $L = \frac{11}{2}$

As $|f(x)-L| < \varepsilon \iff \left|\frac{1}{2}(3x-1) - \frac{11}{2}\right| < \varepsilon$

$$\iff \left|\frac{3}{2}x - \frac{12}{2}\right| < \varepsilon$$

$$\iff \frac{3}{2}|x-4| < \varepsilon$$

$$\iff |x-4| < \frac{2\varepsilon}{3}$$

Thus , we can take $\delta = \frac{2\varepsilon}{3}$, then if $0 < |x-4| < \delta$, the last inequality in the list is true and consequently so is the first. Hence by Definition $\lim_{x\to 4}\frac{1}{2}(3x-1) = \frac{11}{2}$.

Techniques for finding Limits

Basic Limits: Let $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, then

$$1)\ \lim_{x\to a} b = b \quad , \quad 2)\ \lim_{x\to a} x = a \quad , \quad 3)\ \lim_{x\to a} x^n = a^n$$

$4)\ \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ provided either n is an odd or n is even and $a \geq 0$.

Example: Find $1)\ \lim_{x\to 4} 5$, $2)\ \lim_{x\to -3} \sqrt[5]{x}$, $3)\ \lim_{x\to 4} \sqrt{x}$

Solution:

1) $\lim_{x\to 4} 5 = 5$

2) $\lim_{x\to -3} \sqrt[5]{x} = \sqrt[5]{-3}$

3) $\lim_{x\to 4} \sqrt{x} = \sqrt{4} = 2$

Note: If $f(x)$ is a trigonometric function then $\lim_{x\to a} f(x) = f(a)$ provided $f(a)$ is defined.

Example: find $1)\ \lim_{x\to \pi} \sin x$, $2)\ \lim_{x\to \frac{\pi}{4}} \tan x$

Solution:

1) $\lim_{x\to \pi} \sin x = \sin \pi = 0$, 2) $\lim_{x\to \frac{\pi}{4}} \tan x = \tan \frac{\pi}{4} = 1$

Properties of the Limits

Let $a, b \in \mathbb{R}$, and $n \in \mathbb{Z}^+$ and $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = K$

1) $\lim_{x\to a}[bf(x)] = bL$

2) $\lim_{x\to a}[f(x) \pm g(x)] = L \pm K$

3) $\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot K$

4) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{K}$ provided $K \neq 0$

$$5)\ \lim_{x\to a}[f(x)]^n = L^n$$

Limits of Polynomial and Rational functions

1- If p is a polynomial function and $a \in \mathbb{R}$ then

$$\lim_{x\to a} p(x) = p(a)$$

2- If $r(x)$ is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and $a \in \mathbb{R}$, then

$$\lim_{x\to a} r(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ , provided } q(a) \neq 0.$$

Example: Find each limit (if they exist)

$$1)\ \lim_{x\to -1}(x^2+x-6) \quad , \quad 2)\ \lim_{x\to 2}\frac{x^2+x-6}{x+3}$$

Solution:

1) $\lim_{x\to -1}(x^2+x-6) = (-1)^2+(-1)-6$

2) $\lim_{x\to 2}\frac{x^2+x-6}{x+3} = \frac{(2)^2+(2)-6}{(2)+3} = \frac{0}{5} = 0$

Theorem

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

Example: Find the following Limits (if they exist)

$$1)\ \lim_{x\to 3^+}\frac{|x-3|}{x-3} \quad , \quad 2)\ \lim_{x\to 3^-}\frac{|x-3|}{x-3} \quad , \quad 3)\ \lim_{x\to 3}\frac{|x-3|}{x-3} \quad , \qquad 4)\ \lim_{x\to 2}\begin{cases} 2x+1 & , \ x<2 \\ x^2+1 & , \ x>2 \end{cases}$$

Solution:

Since $|x-3| = \begin{cases} x-3 & , \ x\geq 3 \\ x-3 & , \ x<3 \end{cases}$

Then $\frac{|x-3|}{x-3} = \begin{cases} 1 & , \ x\geq 3 \\ -1 & , \ x<3 \end{cases}$

1) $\lim_{x\to 3^+}\frac{|x-3|}{x-3} = \lim_{x\to 3^+} 1 = 1$

2) $\lim_{x\to 3^-}\frac{|x-3|}{x-3} = \lim_{x\to 3^-}(-1) = -1$

3) $\lim_{x\to 3}\frac{|x-3|}{x-3}$ does not exist ….why?

4) $\lim_{x\to 2}\begin{cases} 2x+1 & , \ x<2 \\ x^2+1 & , \ x>2 \end{cases}$ Exercise (Ans:5)

Indeterminate Limit

Let $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, the situation $\lim_{x\to a} \frac{f(x)}{g(x)}$ called indeterminate Limit (zero over zero) , In this case it takes some work on our part to find the limit.

Example: Evaluate the following limits (if they exist)

1) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$
2) $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$
3) $\lim_{x\to 3} \frac{x^3-27}{x-3}$
4) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

Solution:

1) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \frac{0}{0}$

Thus $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1} \frac{x+2}{x} = 3$

2) $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \frac{0}{0}$

thus $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$ $= \lim\limits_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$

$$= \lim_{x\to 0} \frac{x^2+100-100}{x^2\sqrt{x^2+100}+10}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2\sqrt{x^2+100}+10} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}$$

3) $\lim_{x\to 3} \frac{x^3-27}{x-3} = \frac{0}{0}$

thus $\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3}(x^2+3x+9) = 27$

4) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ Exercise (Ans: $\frac{1}{4}$)

Limits of trigonometric function

Theorem:

$$1)\ \lim_{x\to 0}\frac{\sin x}{x}=1 \quad , \quad 2)\ \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

Remarks:

1) $\lim_{x\to 0}\frac{\sin^n bx}{x^n}=b^n$ 2) $\lim_{x\to 0}\frac{x^n}{\sin^n bx}=\frac{1}{b^n}$

3) $\lim_{x\to 0}\frac{\tan^n bx}{x^n}=b^n$ 4) $\lim_{x\to 0}\frac{x^n}{\tan^n bx}=\frac{1}{b^n}$

Example: Find the following Limits (if they exist)

1) $\lim_{x\to 0}\frac{4x}{\tan 3x+\sin 2x}$ 2) $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$

3) $\lim_{x\to 1}\frac{\sin(3x-3)}{1-x^3}$ 4) $\lim_{x\to 1}\frac{\sin^2(\pi x)}{1-2x+x^2}$

Solution:

1) $\lim_{x\to 0}\frac{4x}{\tan 3x+\sin 2x}=\frac{0}{0}$

thus $\lim_{x\to 0}\frac{4x}{\tan 3x+\sin 2x}=\lim_{x\to 0}\frac{\frac{4x}{x}}{\frac{\tan 3x}{x}+\frac{\sin 2x}{x}}=\frac{\lim\limits_{x\to 0}4}{\lim\limits_{x\to 0}\frac{\tan 3x}{x}+\lim\limits_{x\to 0}\frac{\sin 2x}{x}}=\frac{4}{3+2}=\frac{4}{5}$

2) $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\frac{0}{0}$

thus $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x\sin x}\quad \left(\because\ 1-\cos x=2\sin^2\left(\frac{x}{2}\right)\right)$

$$=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x\sin x}=\lim_{x\to 0}\frac{\frac{2\sin^2\left(\frac{1}{2}x\right)}{x^2}}{\frac{x\sin x}{x^2}}=\frac{\lim\limits_{x\to 0}\frac{2\sin^2\left(\frac{1}{2}x\right)}{x^2}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}=\frac{2\left(\frac{1}{2}\right)^2}{1}=\frac{1}{2}$$

3) $\lim_{x\to 1}\frac{\sin(3x-3)}{1-x^3}=\frac{0}{0}$

thus $\lim_{x\to 1}\frac{\sin(3x-3)}{1-x^3}=\lim_{x\to 1}\frac{\sin\big(3(x-1)\big)}{(1-x)(1+x+x^2)}$

$$=\lim_{x\to 1}\frac{\sin\big(3(x-1)\big)}{-(x-1)}\lim_{x\to 1}\frac{1}{(1+x+x^2)}=-3\left(\frac{1}{3}\right)=-1$$

4) $\lim_{x\to 1}\frac{\sin^2(\pi x)}{1-2x+x^2}$ Exercise (Ans: π^2)

Exercises

1) *Using the definition of the limit to prove that*

$$a)\ \lim_{x\to 2} 3x-1=5 \quad b)\ \lim_{x\to 3}\frac{2}{5}x+\frac{1}{2}=\frac{17}{10} \qquad c)\ \lim_{x\to 1}\frac{2x^2-2}{x-1}=4$$

2) *Prove that*

$a)\ \lim_{x\to 3}\sqrt{2x-6}$ *does not exist*

$b)\ \lim_{x\to 5}\dfrac{2x-10}{|x-5|}$ *does not exist*

$c)\ \lim_{x\to 1} f(x)$ *does not exist if* $f(x)=\begin{cases} x+3 & if\ \ x<1 \\ \lfloor x\rfloor & if\ x>1\end{cases}$

In exercises(3 – 27) find the following limits (if they exist)

3) $\lim_{x\to 1}\frac{x^2+2x+3}{x-6}$ $\quad (\ ans:\ -\frac{6}{5}\ )$

4) $\lim_{x\to 2}\frac{2x+4}{x-7}$ $\quad (\ ans:\ -\frac{8}{5}\ )$

5) $\lim_{x\to 6}\sqrt{2x-5}$ $\quad (\ ans:\ \sqrt{7}\ )$

6) $\lim_{x\to 1}\frac{x^3-1}{x-1}$ $\quad (\ ans:\ 3\ )$

7) $\lim_{x\to 2}\frac{x^2+3x-10}{3x^2-5x-2}$ $\quad (\ ans: 1\ )$

8) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$ $\quad (\ ans:\ \frac{1}{2}\ )$

9) $\lim_{x\to 1}\begin{cases} 2x+1 & if\ \ x>1 \\ x^2-1 & if\ x<1\end{cases}$ $\quad (\ ans: does\ not\ exist\ )$

10) $\lim_{x\to 0}\frac{4x}{|x+1|-|x-1|}$ $\quad (\ ans:\ 2\ )$

11) $\lim_{x\to 4}\begin{cases} 6 & if\ \ x=4 \\ \frac{4-x}{5-\sqrt{x^2+9}} & if\ x\neq 4\end{cases}$ $\quad (\ ans:\ \frac{10}{8}\ )$

12) $\lim_{x\to 2}\frac{x^3+x^2-2x-8}{6x-12}$ $\quad \left(\ ans:\ \frac{7}{3}\right)$

13) $\lim_{x\to 2^+}(x-\lfloor x\rfloor)$ $\quad(\ ans:\ 0\ )$

14) $\lim_{x\to 1}(|x+1|+x^2+7)$ $\quad(\ ans:\ 10\ )$

15) $\lim_{x\to 0}\frac{\sin 6x}{\cos 4x}$ $\quad(\ ans:\ 0\ )$

16) $\lim_{x\to 0}\frac{x^3}{tan^3 2x}$ $\quad\left(ans:\frac{1}{8}\right)$

17) $\lim_{x\to 0}\frac{\tan x-\sin x}{x\cos x}$ $\quad(ans: 0\ )$

18) $\lim_{x\to 0}\frac{sin^2 x\cos x}{1-\cos x}$ $\quad(ans: 2\ )$

19) $\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}-x\right)}{x}$ $\quad(ans: 1)$

20) $\lim_{x\to 0}\frac{\cos 2x-1}{2x^2}$ $\quad(\ ans:\ -1\ )$

21) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)$ $\quad(\ ans:\ 0\ )$

22) $\lim_{x\to 1}\frac{\tan(\pi x)}{x^2-x}$ $\quad(ans:\ \pi\ )$

23) $\lim_{x\to 1}\frac{\cos\left(\frac{\pi x}{2}\right)}{x^2-1}$ $\quad\left(ans:\ -\frac{\pi}{4}\right)$

24) $\lim_{x\to 0}\frac{1-2x^2-2\cos x+cos^2 x}{x^2}$ $\quad(\ ans:\ -2\ )$

25) $\lim_{x\to -2}\frac{sin^2(\pi x)}{x^2+4x+4}$ $\quad(ans: \pi^2\ )$

26) $\lim_{x\to 0}\frac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$ $\quad(ans\ : 1\ )$

27) $\lim_{x\to 0}\frac{1-\sqrt{\cos x}}{x^2}$ $\quad(ans:\frac{1}{4})$

Continuity

Def: A function f is continuity at a number $'a'$ if the following three conditions are satisfied

1) f is defined on an open interval containing $'a'$ ($f(a)$ exist)

2) $\lim_{x\to a} f(x)$ exist , 3) $f(a) = \lim_{x\to a} f(x)$

Example: If $f(x) = 2x + 1$ discuss the continuity of $f(x)$ at $x = 1$.

Solution:

Since $f(1) = 2(1) + 1 = 3$

Also $\lim_{x\to 1}(2x + 1) = 3$, then $\lim_{x\to a} f(x) = f(a)$

Thus f is continuous at $= 1$.

Notes:

1) The polynomial function is continuous $\forall x \in \mathbb{R}$.
2) $f(x) = \sin x$ and $f(x) = \cos x$ are continuous $\forall x \in \mathbb{R}$.

Example: If $f(x) = \begin{cases} x + 3 & if \;\; x < 2 \\ x^2 + 1 & if \;\; x \geq 2 \end{cases}$ discuss the continuity of $f(x)$ at $= 2$.

Solution:

$$f(2) = 2^2 + 1 = 5 \ldots\ldots\ldots (*)$$

As $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x + 3) = 5$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^2 + 1) = 5$

$\Longrightarrow \lim\limits_{x\to 2} f(x) = 5$

$\lim_{x\to 2} f(x) = f(2)$ by $(*)$

Then f is continuous at $= 2$.

Exercise: Find the value of the constant c s.t. the function $f(x)$ is continuous at $x = \frac{\pi}{4}$,

$$f(x) = \begin{cases} \frac{1-\sin 2x}{\cos^2 2x} & , \;\; x \neq \frac{\pi}{4} \\ c \sin x & , \;\; x = \frac{\pi}{4} \end{cases}$$

Infinite Limit and Limit of infinite

Theorem: Let $\lim_{x\to a} f(x) \neq 0$ and $\lim_{x\to a} g(x) \neq 0$

To find $\lim_{x\to a} \frac{f(x)}{g(x)}$, we find

$$1)\ \lim_{x\to a^+} \frac{f(x)}{g(x)} \quad , \quad 2)\ \lim_{x\to a^-} \frac{f(x)}{g(x)}$$

Remarks:

1) $\lim_{x\to a^-} \frac{1}{(x-a)^n} = \infty$ also $\lim_{x\to a^+} \frac{1}{(x-a)^n} = \infty$ if n is even positive integer .
2) $\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty$ also $\lim_{x\to a^+} \frac{1}{(x-a)^n} = \infty$ if n is odd positive integer .

Example: Find the following limits (if they exist)

$$1)\ \lim_{x\to 3} \frac{2x^2}{9-x^2} \qquad , \qquad 2)\ \lim_{x\to 1} \frac{5x^2}{(x-1)^2} \qquad , \qquad 3)\ \lim_{x\to 1} \frac{1+x}{1-x^2}$$

Solution:

1) Since $\lim_{x\to 3}(9-x^2) = 0$

   Then, we will found $\lim_{x\to 3^-} \frac{2x^2}{9-x^2} = \infty$

   Also $\lim_{x\to 3^+} \frac{2x^2}{9-x^2} = -\infty$

   Thus $\lim_{x\to 3} \frac{2x^2}{9-x^2}$ does not exist.

2) Since $\lim_{x\to 1}(x-1)^2 = 0$, then we will found

$$\lim_{x\to 1^-} \frac{5x^2}{(x-1)^2} = \infty \quad , \quad \lim_{x\to 1^+} \frac{5x^2}{(x-1)^2} = \infty$$

   Thus $\lim_{x\to 1} \frac{5x^2}{(x-1)^2} = \infty$

3) Exercise (Ans : $\lim_{x\to 1} \frac{1+x}{1-x^2}$ does not exist)

Theorem: $\lim_{x\to\pm\infty} \frac{c}{x^k} = 0$ s.t. c is a constant and $\in \mathbb{Z}^+$.

Remark: Let $P_1(x), P_2(x)$ are two polynomials s.t.

$$P_1(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_0, a_1, .., a_{n-1}, a_n \in \mathbb{R}, n \in \mathbb{Z}^+, a_n \neq 0$$

$$P_2(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 : b_0, b_1, .., b_{m-1}, b_m \in \mathbb{R}, m \in \mathbb{Z}^+, b_m \neq 0$$

Then

1) $\lim_{x\to\pm\infty} P_1(x) = \lim_{x\to\pm\infty} a_n x^n$
2) $\lim_{x\to\pm\infty} \frac{P_1(x)}{P_2(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$

Example: Find the following limits (if they exist)

$$1)\ \lim_{x\to-\infty}(3x^3 - 10x^2 + 1) \quad , \quad 2)\ \lim_{x\to\infty}\frac{3}{x^2+5} \quad , \quad 3)\ \lim_{x\to-\infty}\frac{x^2-3x+7}{x^3+4x-9}$$

$$4)\ \lim_{x\to\infty}\frac{7x^2+x+11}{4-x} \quad , \quad 5)\ \lim_{x\to-\infty}\frac{x+3}{\sqrt{16x^2+5x}}$$

Solution:

1) $\lim_{x\to-\infty}(3x^3 - 10x^2 + 1) = \lim_{x\to-\infty} 3x^3 = -\infty$.
2) $\lim_{x\to\infty} \frac{3}{x^2+5} = \lim_{x\to\infty} \frac{3}{x^2} = 0$.
3) $\lim_{x\to-\infty} \frac{x^2-3x+7}{x^3+4x-9} = \lim_{x\to-\infty} \frac{x^2}{x^3} = \lim_{x\to-\infty} \frac{1}{x} = 0$.
4) $\lim_{x\to\infty} \frac{7x^2+x+11}{4-x} = \lim_{x\to\infty} \frac{7x^2}{-x} = \lim_{x\to\infty} -7x = -\infty$.
5) $\lim_{x\to-\infty} \frac{x+3}{\sqrt{16x^2+5x}} = \lim_{x\to-\infty} \frac{x+3}{\sqrt{x^2\left(16+\frac{5}{x}\right)}} = \lim_{x\to-\infty} \frac{x+3}{|x|\sqrt{16+\frac{5}{x}}}$

$$= \lim_{x\to-\infty} \frac{x}{-x\sqrt{16}} = \lim_{x\to-\infty} \frac{1}{-4} = -\frac{1}{4}$$

Note: The following forms also represent indeterminate forms

$$1)\ \frac{\pm\infty}{\pm\infty} \quad , \quad 2)\ 0\cdot(\pm\infty) \quad , \quad 3)\ \infty - \infty$$

Exercise

In exercises(1 – 3) discuss continuity of the following functions at $x = a$.

1) $f(x) = \begin{cases} x - x^2 & if\ \ x < -2 \\ 10 & if\ x = -2 \\ -5x & if\ x > -2 \end{cases}$ $\quad a = -2 \quad$ $(\,ans : not\ continuous)$

2) $f(x) = \begin{cases} \frac{3x+\sin x}{\tan 3x} & if\ \ x \neq 0 \\ \frac{5}{3} & if\ x = 0 \end{cases}$ $\quad a = 0 \quad$ $(ans :\ continuous)$

3) $f(x) = \begin{cases} \frac{1-\cos 2x}{1-\cos x} & if\ \ x \neq 0 \\ 4 & if\ x = 0 \end{cases}$ $\quad a = 0 \quad$ $(ans :\ continuous)$

In exercises (4 – 7) find the values of the constant $\boldsymbol{K}$ so that the function $f(x)$ is continuous at a

4) $f(x) = \begin{cases} x^2 + x + \boldsymbol{K} & if\ \ x < -1 \\ x^3 & if\ \ x \geq -1 \end{cases}$ $\quad a = -1 \quad$ $(\,ans\ :\ -1\,)$

5) $f(x) = \begin{cases} \frac{1-\cos 2\boldsymbol{K}x}{x\cos x} & if\ \ x \neq 0 \\ \boldsymbol{K} + 6 & if\ x = 0 \end{cases}$ $\quad a = 0 \quad$ $\left(ans :\ -\frac{3}{2}\ \ ,\ 2\ \right)$

6) $f(x) = \begin{cases} \frac{\sqrt{7x+2}-\sqrt{6x+4}}{x-2} & if\ \ x \neq 2 \\ \boldsymbol{K} & if\ x = 2 \end{cases}$ $\quad a = 2 \quad$ $(ans : 8\ \ )$

7) $f(x) = \begin{cases} \frac{1-\sin x}{(2x-\pi)^2} & ,\ x \neq \frac{\pi}{2} \\ K^3 & ,\ \ x = \frac{\pi}{2} \end{cases}$ $\quad$ $(\,ans : \frac{1}{2}\ \ )$

In exercises (8 – 19) find the following limits (if they exist)

8) $\lim_{x\to 4} \frac{5}{x-4}$ $\quad$ $(\,ans : does\ not\ exist\,)$

9) $\lim_{x\to -8} \frac{3x}{(x+8)^2}$ $\quad$ $(\,ans :\ -\infty\ \ )$

10) $\lim_{x\to 3} \frac{1}{x(x-3)^2}$ $\quad$ $(\,ans :\ \ \ \infty\ \ )$

11) $\lim_{x\to 1} \frac{4x^2}{x^2-4x+3}$ $\quad$ $(\,ans : does\ not\ exist\,)$

12) $\lim_{x\to \infty} \frac{5x^2-3x+1}{2x^2+4x-7}$ $\quad$ $\left(ans : \frac{5}{2}\ \right)$

13) $\lim_{x\to-\infty}\frac{4-7x}{2+3x}$ $\left(ans:\ -\frac{7}{3} \right)$

14) $\lim_{x\to-\infty}\frac{-3x^3-5x+7}{x^2-1}$ $(ans:\ \infty\ )$

15) $\lim_{x\to-\infty}|x-3|+x^3$ $(ans:\ -\infty)$

16) $\lim_{x\to\infty}\frac{x-3}{x^2+4}$ $(ans: 0\ )$

17) $\lim_{x\to\infty}\frac{\sqrt{16x^2-3}}{x+4}$ $(ans: 4\ )$

18) $\lim_{x\to\infty}\sqrt{x^2+5x}-x$ $(ans: 5\ )$

19) $\lim_{x\to 2}(\frac{4}{x}+\frac{3}{x-2})$ $(ans: does\ not\ exist\ )$

The Derivative

Tangent lines

Def: The slope of the tangent line to the graph of a function f at $(a, f(a))$ is

$$m_a = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{provided the limit exist}$$

Note: The tangent line of the curve $y = f(x)$ at the point $(a, f(a))$ is the line passes through the point $(a, f(a))$ with the slope m_a. It's equation

$$y = m_a(x-a) + f(a)$$

Example: 1) Find the slope of the function $f(x) = x^2$ at the point $(-2,4)$.

2)Find the equation of the tangent line at $(-2,4)$ to the graph of $f(x) = x^2$.

Solution:

1) The slope of the tangent line at $(-2,4)$ is m_{-2} s.t.

$$m_{-2} = \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0} \frac{(-2+h)^2-(-2)^2}{h}$$

$$= \lim_{h\to 0} \frac{4-4h+h^2-4}{h}$$

$$= \lim_{h\to 0} \frac{-4h+h^2}{h} = \lim_{h\to 0}(-4+h) = -4$$

2) The equation of the tangent line at $(-2,4)$ is

$$y = m_{-2}\big(x-(-2)\big) + f(-2)$$

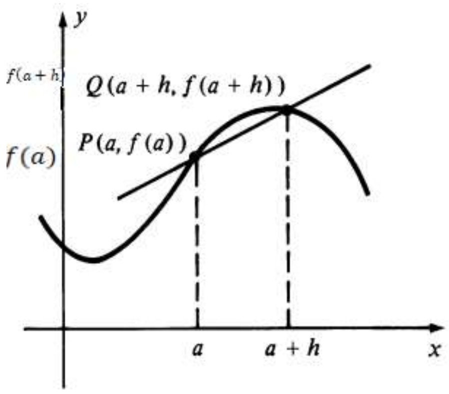
$$= -4(x+2) + 4$$

$\Rightarrow y = -4x - 4$

The derivative of the function at a point $-a-$

Def: The derivative of the function at the point $-a-$ denoted by $f'(a)$ s.t.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{Provided the Limit exist}$$



Note: Now we can write the equation of the tangent line in the form

$$y = f'(a)(x-a) + f(a)$$

Example: If $f(x) = 3x^2 - 12x + 8$ find 1) $f'(x)$, 2) $f'(3)$

3) The point on the graph at which the tangent line is a horizontal.

Solution:

1) Since $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Then $f'(x)$
$$= \lim_{h\to 0} \frac{3(x+h)^2 - 12(x+h) + 8 - 3x^2 + 12x - 8}{h}$$
$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 12x - 12h + 8 - 3x^2 + 12x - 8}{h}$$
$$= \lim_{h\to 0} \frac{6xh + 3h^2 - 12h}{h}$$
$$= \lim_{h\to 0} \frac{h(6x + 3h - 12)}{h}$$
$$= \lim_{h\to 0} (6x + 3h - 12)$$
$$= 6x - 12$$

2) $f'(3) = 6(3) - 12 = 6$
3) If the tangent line is horizontal , then $f'(x) = 0 \implies 6x - 12 = 0 \implies x = 2$ then the tangent line is horizontal at $(2, f(2)) = (2, -4)$.

Remark: we denoted of $f'(x)$ by

1) $\frac{dy}{dx}$, 2) $\frac{df}{dx}$, 3) $D_x y$, 4) y' and $f'(x)$ at $x = a$ is denoted by
1) $f'(a)$, 2) $\left.\frac{df}{dx}\right|_{x=a}$, 3) $\left.\frac{dy}{dx}\right|_{x=a}$, 4) $y'(a)$.

Techniques of differentiation

Rules for finding derivative

Let $f(x)$, $g(x)$ and $h(x)$ are functions differentiable , n any real number and c is a constant value , then

1) If $(x) = c \implies f'(x) = 0$.
2) If $f(x) = x^n \implies f'(x) = n\, x^{n-1}$.
3) If $f(x) = c\, g(x) \implies f'(x) = c\, g'(x)$.
4) If $f(x) = g(x) \pm h(x) \implies f'(x) = g'(x) \pm h'(x)$.
5) If $f(x) = g(x) \cdot h(x) \implies f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$.
6) If $f(x) = \frac{g(x)}{h(x)}$ provided $h(x) \neq 0$

$$\implies f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2} .$$

Example: Find the derivative of the following functions

1) $f(x) = 2x^3 + x^2 - 2$, 2) $f(x) = (3x^2 - 5)(2x^4 - x)$, 3) $f(x) = \frac{5x-5}{x^2+7}$.

Solution:

1) $f'(x) = 6x^2 + 2x$.
2) $f'(x) = (3x^2 - 5)'(2x^4 - x) + (3x^2 - 5)(2x^4 - x)'$

$$= 6x \cdot (2x^4 - x) + (3x^2 - 5)(8x^3 - 1)$$

$$= 12x^5 - 6x^2 + 24x^5 - 3x^2 - 40x^3 + 5$$

$$= 36x^5 - 40x^3 - 9x^2 + 5$$

3) $$f'(x) = \frac{(5x-5)'\,(x^2+7) - (5x-5)(x^2+7)'}{(x^2+7)^2}$$

$$= \frac{5 \cdot\, (x^2+7) - (5x-5) \cdot 2x}{(x^2+7)^2}$$

$$= \frac{5x^2 + 35 - 10x^2 + 10x}{(x^2+7)^2} = \frac{-5x^2 - 10x + 35}{(x^2+7)^2}$$

Exercise: Find an equation of the tangent line to the graph of $y = 2x^3 + 4x^2 - 5x - 3$ at the point $(0, -3)$. (Ans: $y = -5x - 5$)

Chain Rule

Theorem: If $y = f(u), u = g(x)$ and the derivatives $\frac{dy}{dx}$ and $\frac{du}{dx}$ both exist , then the composite function defined by $y = f\big(g(x)\big)$ has a derivative given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'\big(g(x)\big) \cdot g'(x)$$

Example: Find y' for each of the following

1) $y = (x^3 + 8)^5$, 2) $y = \dfrac{1}{(4x^2 + 6x - 7)^3}$, 3) $y = \sqrt{x^3 + x + 1}$

4) If $u = x^2 + 1$ and $y = u^2$

Solution:

1) Let $u = x^3 + 8 \Longrightarrow y = u^5$

Since $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

and $\frac{dy}{du} = 5u^4$, $\frac{du}{dx} = 3x^2$

then $\frac{dy}{dx} = 5u^4 \cdot 3x^2 = 15x^2(x^3 + 8)^4$

2) Let $u = 4x^2 + 6x - 7 \Longrightarrow y = \frac{1}{u^3} = u^{-3}$

Since $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

and $\frac{dy}{du} = -3u^{-4}$, $\frac{du}{dx} = 8x + 6$

then $\frac{dy}{dx} = -3u^{-4} \cdot (8x + 6) = -3(8x + 6)(4x^2 + 6x - 7)^{-4}$

$$= \frac{-3(8x + 6)}{(4x^2 + 6x - 7)^4}$$

3) Let $u = x^3 + x + 1 \Longrightarrow y = \sqrt{u} = u^{\frac{1}{2}}$

Since $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

and $\frac{dy}{du} = \frac{1}{2}u^{\left(\frac{1}{2}-1\right)} = \frac{1}{2}u^{-\frac{1}{2}}$, $\frac{du}{dx} = 3x + 1$

then $\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot (3x + 1) = \frac{1}{2}\,\frac{3x+1}{u^{\frac{1}{2}}} = \frac{1}{2}\,\frac{3x+1}{\sqrt{u}} = \frac{3x+1}{2\sqrt{x^3+x+1}}$

4) Exercise (Ans : $6x \cdot (x^2 + 1)$)

Notes:

1) If $f(x) = [g(x)]^n \Longrightarrow f'(x) = n[g(x)]^{n-1} \cdot g'(x)$
2) If $f(x) = \sqrt{g(x)} \Longrightarrow f'(x) = \frac{g'(x)}{2\sqrt{g(x)}}$

Derivative of trigonometric functions

Theorem:

1) $\frac{d}{dx}(\sin x) = \cos x$ 2) $\frac{d}{dx}(\cos x) = -\sin x$

3) $\frac{d}{dx}(\tan x) = \sec^2 x$ 4) $\frac{d}{dx}(\cot x) = -\csc^2 x$

5) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$ 6) $\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$

Example: Find $\frac{dy}{dx}$ for each of the following

$$1)\ y = 3x + \cot x \quad , \quad 2)\ y = \sec^6 x \quad , \quad 3)\ y = \frac{\cot x}{1 + \csc x}$$

Solution:

1) $\frac{dy}{dx} = 3 - \csc^2 x$

2) $\frac{dy}{dx} = 6\sec^5 x \cdot (\sec x)' = 6\sec^5 x \cdot (\sec x \cdot \tan x) = 6\sec^6 x \cdot \tan x$

3) $\frac{dy}{dx} = \frac{(\cot x)'(1+\csc x)-(\cot x)(1+\csc x)'}{[1+\csc x]^2} = \frac{-\csc^2 x(1+\csc x)-(\cot x)(-\csc x \cot x)}{[1+\csc x]^2}$

$$= \frac{-\csc^2 x - \csc^3 x + \csc x \cot^2 x}{[1+\csc x]^2}$$

Remark: If $u = g(x)$ is a differentiable function . then by chain rule and the above theorem , we get the following

1) $\frac{d}{dx}(\sin(u)) = u' \cos(u)$ 2) $\frac{d}{dx}(\cos(u)) = -u' \sin(u)$

3) $\frac{d}{dx}(\tan(u)) = u' \ \sec^2(u)$ 4) $\frac{d}{dx}(\cot(u)) = -u' \ \csc^2(u)$

5) $\frac{d}{dx}(\sec(u)) = u' \sec(u)\tan(u)$ 6) $\frac{d}{dx}(\csc(u)) = -u' \ \csc(u)\cot(u)$

Example: Find the derivative of the following functions

$$1)\ f(x) = \sec^2(x^3 - 5) \quad , \quad 2)\ f(x) = \tan(\sin x) \quad , \quad 3)\ f(x) = 3\tan(\sqrt{x})$$

Solution:

1) $f'(x) = 2\sec(x^3 - 5)\,(\sec(x^3 - 5))'$

$$= 2\sec(x^3 - 5)\sec(x^3 - 5)\tan(x^3 - 5)\,(3x^2)$$

$$= 6x^2\sec^2(x^3 - 5)\tan(x^3 - 5)$$

2) $f'(x) = \sec^2(\sin x) \cdot \cos x = \cos x \cdot \sec^2(\sin x)$

3) Exercise (Ans: $f'(x) = \frac{3\sec^2(\sqrt{x})}{2\sqrt{x}}$)

Implicit Differentiation

Example: Use implicit differentiation to find $\frac{dy}{dx}$

$$1)\ x^2y+3y^2=6 \quad , \quad 2)\ x+\sin y = xy$$

Solution:

1) $\frac{d}{dx}(x^2y)+\frac{d}{dx}(3y^2)=\frac{d}{dx}(6)$

$$\Rightarrow y\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(y)+\frac{d}{dx}(3y^2)=\frac{d}{dx}(6)$$

$$\Rightarrow 2xy+x^2\frac{dy}{dx}+6y\frac{dy}{dx}=0$$

$$\Rightarrow (x^2+6y)\frac{dy}{dx}=-2xy \Rightarrow \frac{dy}{dx}=\frac{-2xy}{x^2+6y}$$

2) $\frac{d}{dx}(x)+\frac{d}{dx}(\sin y)=\frac{d}{dx}(xy)$

$$\Rightarrow 1+\cos y\frac{dy}{dx}=y\frac{d}{dx}(x)+x\frac{d}{dx}(y)$$

$$\Rightarrow (\cos y - x)\frac{dy}{dx}=y-1$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-1}{\cos y - x}$$

Remark: It is possible to solve the previous example using the following rule

$$y'=\frac{-\left.\frac{df(x,y)}{dx}\right|_{y\ constant}}{\left.\frac{df(x,y)}{dy}\right|_{x\ constant}}$$

Provided we make $(x,y)=0$.

Example: Find the equations of the tangent and normal line to the curve

$x^2 - xy + y^2 = 7$ at the point $(-1,2)$

Solution:

Since $y' = \dfrac{-\frac{df(x,y)}{dx}\Big|_{y\ constant}}{\frac{df(x,y)}{dy}\Big|_{x\ constant}}$ and $x^2 - xy + y^2 - 7 = 0$

then $y' = \frac{y-2x}{2y-x} \implies y'(-1,2) = \frac{2-2(-1)}{2(2)-(-1)} = \frac{4}{5}$

since the equation the tangent line is $y = m(x-a) + f(a)$

then $y = \frac{4}{5}(x+1) + 2 \implies 4x - 5y + 14 = 0$

since the equation of the normal line is $y = \frac{-1}{m}(x-a) + f(a)$

then $y = \frac{-5}{4}(x+1) + 2 \implies 5x + 4y - 3 = 0$

Exercises

In exercises(1 – 4) find the slope of the tangent line at a on the graph of f by the definition .

1) $f(x) = 2 - x^2$, $a = 2$ (ans : -4)

2) $f(x) = \sqrt{x} + 1$, $a = 9$ (ans : $\frac{1}{6}$)

3) $f(x) = x + \frac{9}{x}$, $a = 3$ (ans : 0)

4) $f(x) = \frac{1}{2+x}$, $a = 1$ (ans : $-\frac{1}{9}$)

In exercises(5 – 9) differentiate the following functions by the definition of the derivstive .

5) $f(x) = 37$ (ans : 0)

6) $f(x) = 9x - 2$ (ans: 9)

7) $f(x) = 2 + 8x - 5x^2$ (ans : $8 - 10x$)

8) $f(x) = \frac{1}{x+2}$ (ans : $-(x+2)^{-2}$)

9) $f(x) = \sqrt{3x+1}$ (ans : $\frac{3}{2\sqrt{3x+1}}$)

10) Find the equations of all lines having slope -1 that are tangent to the curve $y = \frac{1}{x-1}$

(ans : $y = -x + 3$, $y = -x - 1$)

11) Show that the tangent line to the curve $y = x^3$ at (1 , 1) passes through the point (2 , 4)

12) Let $f(x) = \begin{cases} x^2 & , \ x \geq 0 \\ 2x & , \ x < 0 \end{cases}$

i) Show that f is continuous at $x = 0$

ii) Does the function f is differentiable at $x = 0$

In exercises(13 – 16) differentiate the following functions

13) $f(x) = (x^3 - 7)(2x^2 + 3)$ (ans : $10x^4 + 9x^2 - 28x$)

14) $f(x) = \frac{(2x^3 - 7x^2 + 4x + 3)}{x^2}$ (ans : $2 - \frac{4}{x^2} - \frac{6}{x^3}$)

15) $f(x) = \frac{(x^3-1)}{(x^3+1)}$ $\quad$ ($ans : \frac{6x^2}{(x^3+1)^2}$)

16) Find an equation of the tangent line to the graph of $y = \frac{5}{1+x^2}$ at each point

$$a)\ p(0,5) \qquad b)\ p(1\,,\frac{5}{2}) \qquad c)\ p(-2\,,1)$$

$(ans \quad a) y = 5 \qquad b)\ 5x + 2y - 10 = 0 \qquad c)\ 4x - 5y + 13 = 0\ )$

17)Find the coordinates of all points on the graph of $y = x^3 + 2x^2 - 4x +$ at which the tangent line is horizonta $\quad$ ($ans: \frac{2}{3}\ , -2$)

18) Suppose the functions $F(x)$ and $G(x)$ satisfy the following properties:

$$F(3) = 2, \quad G(3) = 4, \quad G(0) = 3$$
$$F'(3) = -1, \quad G'(3) = 0, \quad G'(0) = 0$$

(a) If $S(x) = \dfrac{F(x)}{G(x)}$, find $S'(3)$. Simplify your answer.

(b) If $T(x) = F(G(x))$, find $T'(0)$. Simplify your answer.

(c) If $U(x) = \ln(F(x))$, find $U'(3)$. Simplify your answer.

$(ans : \quad a) -\frac{1}{4} \qquad b)\ 0 \qquad c) \frac{-1}{2}\ )$

19) Given $F(x) = f^2(g(x))$, $g(1) = 2$, $g'(1) = 3$, $f(2) = 4$, and $f'(2) = 5$, find $F'(1)$.

($ans :\ 120$)

In exercises(20 – 28) differentiate the following functions

20) $f(x) = (x^2 - 3x + 8)^3$ $\quad$ ($ans : (6x - 9)(x^2 - 3x + 8)^2$)

21)$f(x) = (17x - 5)^{1000}$ $\quad$ ($ans : 17000(17x - 5)^{999}$)

22) $(x) = [(2x + 1)^{10} + 1]^{10}$ $\quad$ ($ans : 200(2x + 1)^9[(2x + 1)^9 + 1]^9$)

23) $(x) = \sin(x^2 + 2)$ $\quad$ ($ans : 2x\cos(x^2 + 2)$)

24) $f(x) = x^3 \cos(\frac{1}{x})$ $\quad$ ($ans : x\sin\left(\frac{1}{x}\right) + 3x^2\cos\left(\frac{1}{x}\right)$)

25) $f(x) = \frac{\cos 4x}{1-\sin 4x}$ $\quad$ ($ans : \frac{-4(1-\sin 4x)\sin 4x + 4\cos^2 4x}{(1-\sin 4x)^2}$)

26) $f(x) = \frac{\csc x}{\sqrt{x}}$ ($ans: -\frac{\csc x \cot x}{\sqrt{x}} - \frac{\csc x}{2x^{\frac{3}{2}}}$)

27) $f(x) = x^2 \sec\left(\frac{1}{x}\right)$ ($ans: 2x \sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right)$)

28) $f(x) = (\frac{1+\cos x}{\sin x})^{-1}$ ($ans: \frac{1}{1+\cos x}$)

In exercises (29 – 34) use implicit differentiable to find $\frac{dy}{dx}$

29) $3x - 2y + 4 = 2x^2 + 3y - 7x$ ($ans: \frac{dy}{dx} = \frac{10-4x}{5}$)

30) $2x^3 + x^2y + y^3 = 1$ ($ans: \frac{dy}{dx} = -\frac{6x^2+2xy}{x+3y}$)

31) $5x^2 + xy - 4y^2 = 0$ ($ans: \frac{dy}{dx} = \frac{10x-y}{x+8y}$)

32) $\sqrt{x} + \sqrt{y} = 1$ ($ans: \frac{dy}{dx} = 1 - x^{-\frac{1}{2}}$)

33) $y^2 = x \cos y$ ($ans\ \frac{dy}{dx} = \frac{\cos y}{2y+x\sin y}$)

34) $y = \sin(3x + 4y)$ ($ans: \frac{dy}{dx} = \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}$)

23) Find an equation of the tangent line to the curve $\sin x + \cos y = 1\ at\ \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

$$(ans: y = \frac{\pi}{2})$$

24) Suppose that y is a differentiable function of x satisfying

$y = xg(x^2 + y^2) - 13\ where$ g is a function such that $g(5) = 7, g'(5) = -2$

$and\ g''(5) = 12\ \ find\ \frac{dy}{dx}\ \ at\ the\ point\ (2, 1)$

$$(\ ans: -1\ )$$

CHAPTER 4 : APPLICATIONS OF THE DERIVATIVE

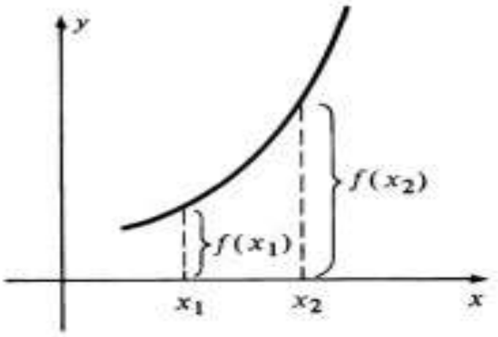
LOCAL EXTERMA OF FUNCTIONS

Def: Let a function f be defined on an interval I and let x_1, x_2 denote numbers in I

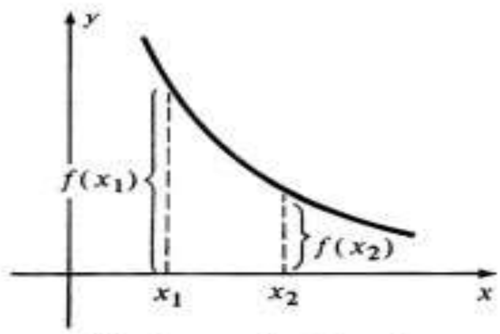
(i) f is **increasing** on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(ii) f is **decreasing** on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(iii) f is **constant** on I if $f(x_1) = f(x_2)$ for every x_1 and x_2 .



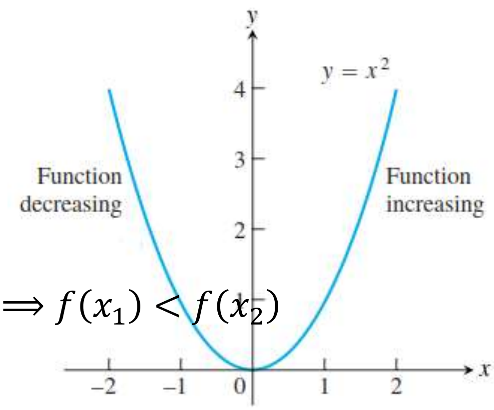
(i) Increasing function



(ii) Decreasing function

Example: Prove that $f(x) = x^2$ is increasing on the interval $[0, \infty)$ and decreasing on the interval $(-\infty, 0]$

Solution:



Let $x_1, x_2 \in [0, \infty) \implies 0 \le x_1 < x_2$

$\implies {x_1}^2 < {x_2}^2$

$\implies f(x_1) < f(x_2)$

Then f is increasing on the interval $[0, \infty)$.

Let $x_1, x_2 \in (-\infty, 0] \implies x_2 > x_1 \ge 0$

$\implies {x_1}^2 > {x_2}^2$

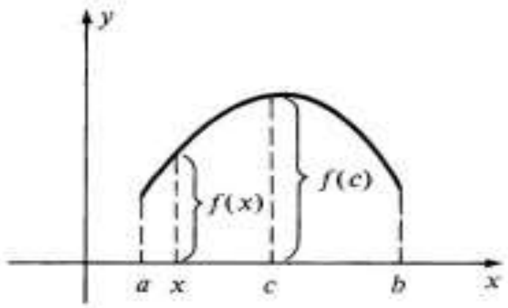
$\implies f(x_1) > f(x_2)$

Then f is decreasing on the interval $(-\infty, 0]$

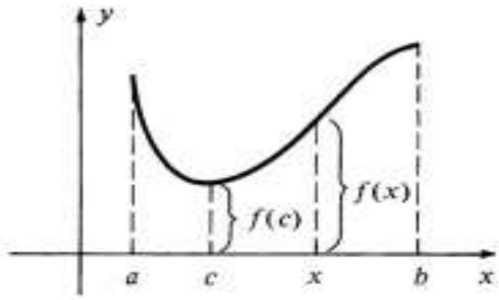
Def: Let a function f be defined on an interval I and let c be a number in I

(i) $f(c)$ is the **maximum value** of f on I if $f(x) \le f(c)$ for every x in I

(ii) $f(c)$ is the **minimum value** of f on I if $f(x) \ge f(c)$ for every x in I



(i) Maximum value $f(c)$



(ii) Minimum value $f(c)$

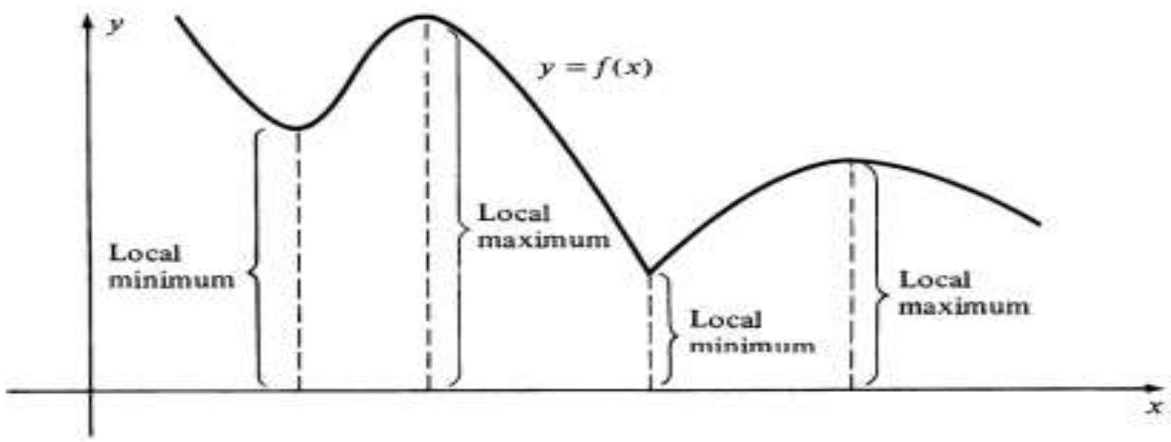
Theorem: If a function f is continuous on a closed interval $[a, b]$, then f takes on a minimum value or a maximum value at least once in $[a, b]$.

The extrema also called the **absolute minimum** and **absolute maximum values** for f on an interval.

Def: Let c be a number in the domain of the function f.

(i) $f(c)$ is a **local maximum** of f if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).

(ii) $f(c)$ is a **local minimum** of f if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

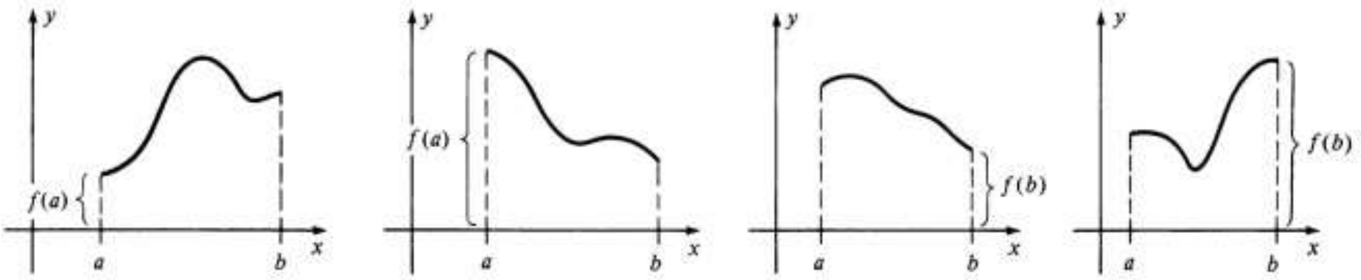


Theorem: If a function f has a local extremum at a number c in an open interval, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Corollary: If $f'(c)$ exists and $f'(c) \neq 0$, then $f(c)$ is not a local extremum of the function f.

Theorem: If a function f is continuous on a closed interval $[a, b]$ and has its maximum or minimum value at a number c in the open interval (a, b), then either $f'(c) = 0$ or $f'(c)$ does not exist.

Def: A number c in the domain of a function f is a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.



(i) Absolute minimum $f(a)$ (ii) Absolute maximum $f(a)$ (iii) Absolute minimum $f(b)$ (iv) Absolute maximum $f(b)$

Guidelines for finding the Absolute Extrema of a Continuous Function f on a Closed Interval $[a, b]$

1. Find all the critical numbers of f.
2. Calculate $f(c)$ for each critical number .
3. Calculate $f(a)$ and $f(b)$.
4. The absolute maximum and minimum of f on $[a, b]$ are the largest and smallest of the functional values calculated in 2 and 3.

Example: If $(x) = x^3 - 12x$, find the absolute maximum and minimum values of f on the closed interval $[-3,5]$, Sketch the graph of f.

Solution: the endpoints extremum are -3 and 5

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$$

Put $f'(x) = 0 \Longrightarrow 3(x + 2)(x - 2) = 0 \Longrightarrow (x + 2)(x - 2) = 0$

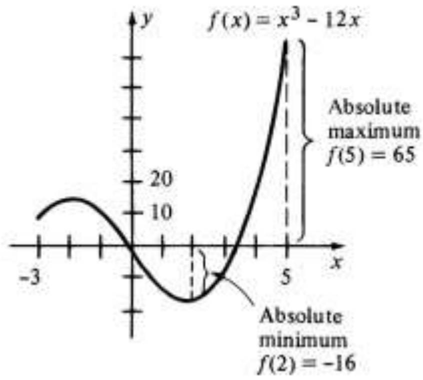
$\Longrightarrow x = \pm 2 \in [-3,5]$, then the critical numbers is -2 , 2

$$f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$$

$$f(2) = (2)^3 - 12(2) = 8 - 24 = -16$$

$$f(-3) = (-3)^3 - 12(-3) = -27 + 36 = 9$$

$$f(5) = (5)^3 - 12(5) = 125 - 60 = 65$$



Then the absolute maximum is $f(5) = 65$ and the absolute minimum is $f(2) = -16$

$f(-2)$ is a local maximum and $f(2)$ is a local minimum

Example: Find the critical numbers of f if $f(x) = (x+5)^2\ \sqrt[3]{x-4}$.

Solution: $f'(x) = (x+5)^2\,\frac{1}{3}(x-4)^{-2/3} + 2(x+5)(x-4)^{1/3}$

$$= \frac{(x+5)^2}{3(x-4)^{2/3}} + 2(x+5)(x-4)^{1/3}$$

$$= \frac{(x+5)^2 + 6(x+5)(x-4)}{3(x-4)^{2/3}}$$

$$= \frac{(x+5)[x+5+6(x-4)]}{3(x-4)^{2/3}} = \frac{(x+5)(7x-19)}{3(x-4)^{2/3}}$$

Consequently, $f'(x) = 0$ if $= -5$ *or* $x = \frac{19}{7}$. The derivative $f'(x)$ does not exist at $= 4$. Thus f has three critical numbers , namely -5 , $\frac{19}{7}$ and 4.

Example: If $(x) = 2\sin x + \cos 2x$, find the critical numbers of f, that are in the interval $[0,2\pi]$.

Solution:

$$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x = 2\cos x\,(1 - 2\sin x)$$

The derivative exists for all x, and $f'(x) = 0$ if either $\sin x = \frac{1}{2}$ or $\cos x = 0$. Hence the critical numbers of f in the interval $[0,2\pi]$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{\pi}{2}$ *and* $\frac{3\pi}{2}$.

Exercises

In Exercises 1-**2** find the absolute maximum and minimum of f on the indicated closed interval.

1 $f(x) = 5 - 6x^2 - 2x^3; [-3, 1]$

2 $f(x) = 1 - x^{2/3}; [-1, 8]$

Find the critical numbers of the functions defined in Exercises **3-6**

3 $f(x) = 4x^2 - 3x + 2$

4 $s(t) = 2t^3 + t^2 - 20t + 4$

5 $F(w) = w^4 - 32w$

6 $f(z) = \sqrt{z^2 - 16}$

Answers

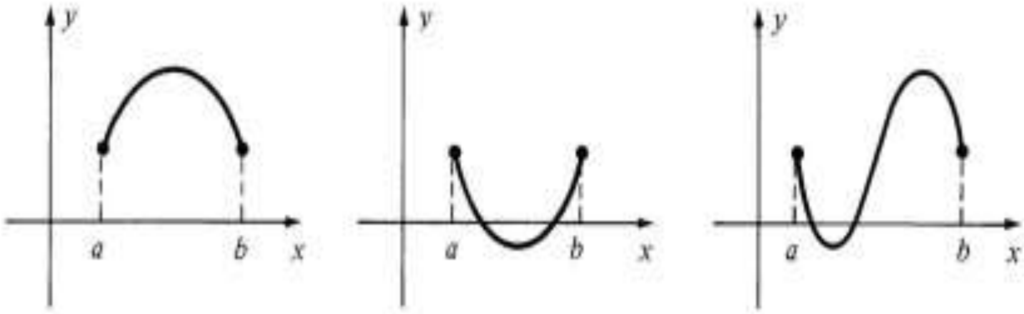
1 $5; -3$ 2 $1; -3$ **3** $\frac{3}{8}$ 4 $\frac{5}{3}$ and -2 **5** 2 **6** 4 and -4 (not 0)

Rolle's Theorem and the Mean Value Theorem

Theorem : Rolle's Theorem

If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for at least one number c in (a, b).

Some typical graphs of functions of this type



Corollary: If a function f is continuous on a closed interval $[a, b]$ and if $f(a) = f(b)$, then f has at least one critical number in the open interval (a, b).

Example: Prove that the function f defined by $f(x) = 4x^2 - 20x + 29$ satisfies the hypotheses of the Rolle's Theorem on the interval $[1,4]$ and find all numbers c in the interval $(1,4)$ such that $f'(c) = 0$.

Solution:

Since f is a polynomial function it is continuous and differentiable for all real numbers. In particular, It is continuous on $[1,4]$ and differentiable on $(1,4)$.

$f(1) = 4(1)^2 - 20(1) + 29 = 13 \quad and \quad f(4) = 4(4)^2 - 20(4) + 29 = 13$

Then $f(1) = f(4)$. Hence f satisfies the hypotheses of the Rolle's Theorem on the interval $[1,4]$

According to Rolle's Theorem , there exists a number c in $(1,4)$ such that $f'(c) = 0$.

Put $f'(x) = 0 \Longrightarrow 8x - 20 = 0 \Longrightarrow x = \frac{5}{2} \in (1,4)$ Hence $c = \frac{5}{2} \in (1,4)$

$\left[f'\left(\frac{5}{2}\right) = 8\left(\frac{5}{2}\right) - 20 = 20 - 20 = 0\right]$

<u>Theorem : The Mean Value Theorem</u>

If a function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad [or\ f(b) - f(a) = f'(c)(b - a)\ ].$$

Example: Prove that the function f defined by $f(x) = x^3 - 8x - 5$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1,4]$ and find all numbers c in the interval $(1,4)$ that satisfies the conclusion of the theorem.

Solution:

Since f is a polynomial function it is continuous and differentiable for all real numbers. In particular, It is continuous on $[1,4]$ and differentiable on $(1,4)$.

Hence f satisfies the hypotheses of the Mean Value Theorem on the interval $[1,4]$

According to Mean Value Theorem , there exists a number c in $(1,4)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Since $f'(x) = 3x^2 - 8 \Longrightarrow f'(c) = 3c^2 - 8$, $f(1) = -12$, $f(4) = 27$

This is equivalent to $3c^2 - 8 = \frac{27-(-12)}{4-1} \Longrightarrow 3c^2 - 8 = \frac{39}{3} \Longrightarrow 3c^2 - 8 = 13$

$\Longrightarrow c^2 = \frac{21}{3} \Longrightarrow c^2 = 7 \Longrightarrow c = \pm\sqrt{7}$

Since $-\sqrt{7} \notin (1,4)$, then $c = \sqrt{7} \in (1,4)$

Exercises

In Exercises **1-4** show that f satisfies the hypotheses of Rolle's Theorem on the indicated interval $[a, b]$ and find all numbers c in (a, b) such that $f'(c) = 0$.

1 $f(x) = 3x^2 - 12x + 11, [0, 4]$

2 $f(x) = 5 - 12x - 2x^2, [-7, 1]$

3 $f(x) = x^4 + 4x^2 + 1, [-3, 3]$

4 $f(x) = x^3 - x, [-1, 1]$

In Exercises **5-6** determine whether the function f satisfies the hypotheses of the Mean Value Theorem on the indicated interval $[a, b]$ and if so, find all numbers c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

5 $f(x) = x^3 + 1, [-2, 4]$

6 $f(x) = 5x^2 - 3x + 1, [1, 3]$

Answers

1 $c = 2$ 2 $c = -3$ 3 $c = 0$ 4 $c = -1/\sqrt{3}$ and $c = 1/\sqrt{3}$ 5 $c = 2$ 6 $c = 1.9$

The First Derivative Test

Theorem: Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b)

(i) If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.

(ii) If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

Example: If $f(x) = x^3 + x^2 - 5x - 5$, find the intervals on which f is increasing and the intervals on which f is decreasing**,** sketch the graph of f **.**

Solution:

$$f'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1)$$

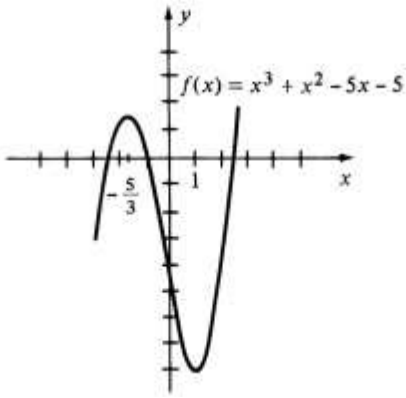
By theorem it is sufficient to find the intervals in which $f'(x) > 0$ and those in which $f'(x) < 0$. The factored form of $f'(x)$ and the critical numbers $\frac{-5}{3}$ and 1 suggest that we consider the open intervals $\left(-\infty, \frac{-5}{3}\right), \left(\frac{-5}{3}, 1\right)$ $, and$ $(1, \infty)$

Interval	k	Test value $f'(k)$	Sign of $f'(x)$	Variation of f
$\left(-\infty, \frac{-5}{3}\right)$	-2	3	$+$	Increasing on $\left(-\infty, \frac{-5}{3}\right]$
$\left(\frac{-5}{3}, 1\right)$	0	-5	$-$	Decreasing on $\left[\frac{-5}{3}, 1\right]$
$(1, \infty)$	2	11	$+$	Increasing on $[1, \infty)$

$$f(x) = x^3 + x^2 - 5x - 5 = x^2(x + 1) - 5(x + 1)$$

$$= (x^2 - 5)(x + 1)$$

$$f\left(\frac{-5}{3}\right) = \frac{40}{27} \quad , \quad f(1) = -8$$



Hence the $x -$ intercepts of the graph are $\sqrt{5}, -\sqrt{5}$ and -1,
The y –intercept is $f(0) = -5$

The First Derivative Test: Suppose c is a critical number of a function f and (a, b) is an open interval containing c. Suppose further that f is continuous on $[a, b]$ and differentiable on (a, b) , expect possibly at c.

(i) If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$ then $f(c)$ is a local maximum of f.

(ii) If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$ then $f(c)$ is a local minimum of f.

(iii) If $f'(x) > 0$ or $f'(x) < 0$ for all x in (a, b) except $= c$, then $f(c)$ is not a local extremum of f.

Example: Find the local extrema of f if $f(x) = x^3 + x^2 - 5x - 5$.

Solution:

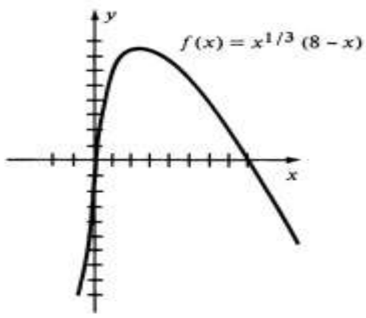
The critical numbers of f are $\frac{-5}{3}$ and 1 . We see from the table in previous example that the sign of $f'(x)$ changes from positive to negative as x increases through $\frac{-5}{3}$.

Hence, by the First Derivative Test, f has a local maximum at $\frac{-5}{3}$. This maximum value is $f\left(\frac{-5}{3}\right) = \frac{40}{27}$. Similarly, a local minimum occurs at 1 since the sign of $f'(x)$ changes from negative to positive as x increases through 1.This minimum value is $f(1) = -8$.

Example: Find the local maxima and minima of f if $f(x) = x^{\frac{1}{3}}(8 - x)$.

Solution:

$$f'(x) = x^{\frac{1}{3}}(-1) + (8 - x)\frac{1}{3}x^{\frac{-2}{3}} = \frac{-3x+8-x}{3x^{2/3}} = \frac{4(2-x)}{3x^{2/3}}$$



The critical numbers of f are 0 and 2

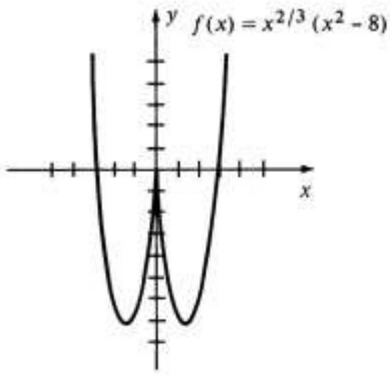
Interval	k	Test value $f'(k)$	Sign of $f'(x)$	Variation of f
$(-\infty, 0)$	-1	4	$+$	Increasing on $(-\infty, 0]$
$(0,2)$	1	$4/3$	$+$	Increasing on $[0,2]$
$(2, \infty)$	8	-2	$-$	Decreasing on $[2, \infty)$

By the First Derivative Test, f has a local maximum at 2 since the sign of $f'(x)$ changes from $+$ to $-$ as x increases through 2. This maximum value is $f(2) = 6\sqrt[3]{2} \approx 7.6$. The function does not have an extremum at 0 since the sign of $f'(x)$ does not changes as x increases through 0 .

Example: If $f(x) = x^{2/3}(x^2 - 8)$

1) Find the local extrema of , sketch the graph of f.
2) Find the absolute maximum and minimum values of f in each of the following intervals:

   $a)$ $\left[-1,\frac{1}{2}\right]$, $b)$ $[-1,3]$, $c)$ $[-3,-2]$



Solution:

1) $f'(x) = x^{2/3}(2x) + \frac{2}{3}x^{-1/3}(x^2 - 8) = \frac{6x^2+2(x^2-8)}{3x^{1/3}} = \frac{8(x^2-2)}{3x^{1/3}}$

The critical numbers of f are $-\sqrt{2}$,0 and $\sqrt{2}$

Interval	k	Test value $f'(k)$	Sign of $f'(x)$	Variation of f
$(-\infty,-\sqrt{2})$	-8	$-248/3$	$-$	Decreasing on $(-\infty,-\sqrt{2}]$
$(-\sqrt{2},0)$	-1	$8/3$	$+$	Increasing on $[-\sqrt{2},0]$
$(0,\sqrt{2})$	1	$-8/3$	$-$	Decreasing on $[0,\sqrt{2}]$
$(\sqrt{2},\infty)$	8	$248/3$	$+$	Increasing on $[\sqrt{2},\infty)$

By the First Derivative Test, f has a local minimum at $-\sqrt{2}$ and $\sqrt{2}$ and a local maximum at 0. The corresponding functional values are $f(0) = 0$ and

$f(\sqrt{2}) = -6\sqrt[3]{2} = f(-\sqrt{2})$

2)

Interval	Minimum	Maximum
$\left[-1,\frac{1}{2}\right]$	$f(-1) = -7$	$f(0) = 0$
$[-1,3]$	$f(\sqrt{2}) = -6\sqrt[3]{2}$	$f(3) = \sqrt[3]{9}$
$[-3,-2]$	$f(-2) = -4\sqrt[3]{4}$	$f(-3) = \sqrt[3]{9}$

Exercises

In Exercises 1- 8 find the local extrema of f. Describe the intervals in which f is increasing or decreasing,

1 $f(x) = 5 - 7x - 4x^2$

2 $f(x) = 2x^3 + x^2 - 20x + 1$

3 $f(x) = x^4 - 8x^2 + 1$

4 $f(x) = x^{4/3} + 4x^{1/3}$

5 $f(x) = x^2\sqrt[3]{x^2 - 4}$

6 $f(x) = x^{2/3}(x - 7)^2 + 2$

7 $f(x) = x^3 + (3/x)$

8 $f(x) = 10x^3(x - 1)^2$

In Exercises 9-11 find the local extrema of f on the interval $[0, 2\pi]$, and determine where f is increasing or decreasing on $[0, 2\pi]$.

9 $f(x) = \cos x + \sin x$

10 $f(x) = \frac{1}{2}x - \sin x$

11 $f(x) = 2\cos x + \sin 2x$

Answers

1 Max: $f(-\frac{7}{8}) = \frac{129}{16}$; increasing on $(-\infty, -\frac{7}{8}]$; decreasing on $[-\frac{7}{8}, \infty)$

2 Max: $f(-2) = 29$; min: $f(\frac{5}{3}) = -\frac{548}{27}$; increasing on $(-\infty, -2]$ and $[\frac{5}{3}, \infty)$; decreasing on $[-2, \frac{5}{3}]$

3 Max: $f(0) = 1$; min; $f(-2) = -15$; min: $f(2) = -15$; increasing on $[-2, 0]$ and $[2, \infty)$; decreasing on $(-\infty, -2]$ and $[0, 2]$

4 Min: $f(-1) = -3$; increasing on $[-1, \infty)$; decreasing on $(-\infty, -1]$

5 Max: $f(0) = 0$; min: $f(-\sqrt{3}) = f(\sqrt{3}) = -3$; increasing on $[-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$; decreasing on $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$

6 Max: $f(\frac{7}{4}) \approx 42$; min: $f(0) = 2$; min: $f(7) = 2$; increasing on $[0, \frac{7}{4}]$ and $[7, \infty)$; decreasing on $(-\infty, 0]$ and $[\frac{7}{4}, 7]$

7 Max: $f(-1) = -4$; min: $f(1) = 4$; increasing on $(-\infty, -1]$ and $[1, \infty)$; decreasing on $[-1, 0)$ and $(0, 1]$.

8 Max: $f(\frac{3}{5}) \approx 0.346$; min: $f(1) = 0$; increasing on $(-\infty, \frac{3}{5}]$ and $[1, \infty)$; decreasing on $[\frac{3}{5}, 1]$

9 Max: $f(\pi/4) = \sqrt{2}$; min: $f(5\pi/4) = -\sqrt{2}$; increasing on $[0, \pi/4]$ and $[5\pi/4, 2\pi]$; decreasing on $[\pi/4, 5\pi/4]$

10 Min: $f(\pi/3) = (\pi - 3\sqrt{3})/6$; max: $f(5\pi/3) = (5\pi + 3\sqrt{3})/6$; decreasing on $[0, \pi/3]$ and $[5\pi/3, 2\pi]$; increasing on $[\pi/3, 5\pi/3]$

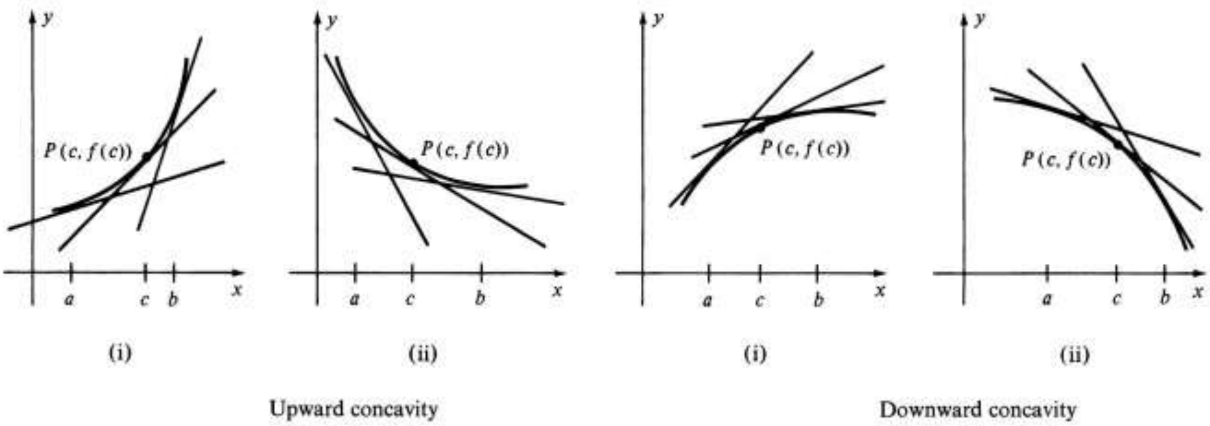
11 Max: $f(\pi/6) = 3\sqrt{3}/2$; min: $f(5\pi/6) = -3\sqrt{3}/2$; increasing on $[0, \pi/6]$ and $[5\pi/6, 2\pi]$; decreasing on $[\pi/6, 5\pi/6]$

Concavity and the Second Derivative Test

Def: Let f be a function that is differentiable at a number c.

(i) The graph of f is **concave upward** at the point $P\big(c, f(c)\big)$ if there exists an open interval (a, b) containing c such that on (a, b) the graph of f is above the tangent line through P.

(ii) The graph of f is **concave downward** at the point $P\big(c, f(c)\big)$ if there exists an open interval (a, b) containing c such that on (a, b) the graph of f is below the tangent line through P.



Test for Concavity: Suppose a function f is differentiable on an open interval containing c and $f''(c)$ exists.

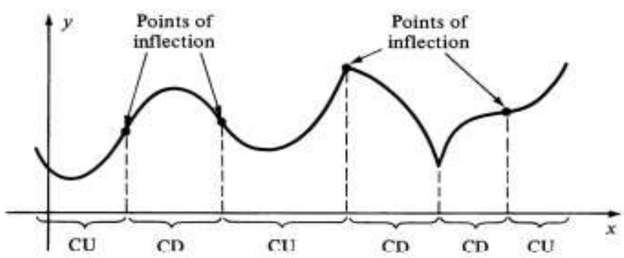
(i) If $f''(c) > 0$ the graph is concave upward at $P\big(c, f(c)\big)$.

(ii) If $f''(c) < 0$ the graph is concave downward at $P\big(c, f(c)\big)$.

Def: A point $P\big(k, f(k)\big)$ on the graph of a function f is a **point of inflection** if there exists an open interval (a, b) containing k such that one of the following statements holds.

(i) $f''(x) > 0$ if $a < x < k$ and $f''(x) < 0$ if $k < x < b$; or

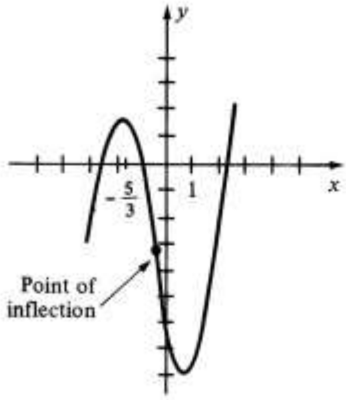
(ii) $f''(x) < 0$ if $a < x < k$ and $f''(x) > 0$ if $k < x < b$



Example: if $f(x) = x^3 + x^2 - 5x - 5$ determine intervals on which the graph of f is concave upward and intervals on which the graph of f is concave downward.

Solution:

$$f''(x) = 6x + 2 = 2(3x + 1)$$



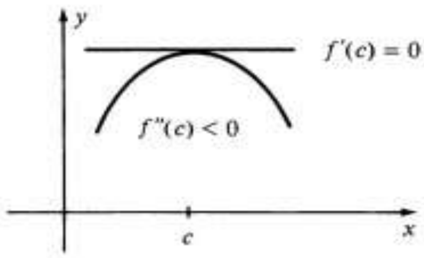
Hence $f''(x) < 0$ if $3x + 1 < 0$, that is, if $< -\frac{1}{3}$. it follows from the Test for Concavity that the graph is concave downward on the infinite interval $\left(-\infty, -\frac{1}{3}\right)$. Similarly $f''(x) > 0$ if $> -\frac{1}{3}$. and, therefore the graph is concave upward on $\left(-\frac{1}{3}, \infty\right)$.

The point $P\left(-\frac{1}{3}, \frac{88}{27}\right)$ at which $f''(x)$ changes sign (and the concavity changes from downward to upward) is a point of inflection.

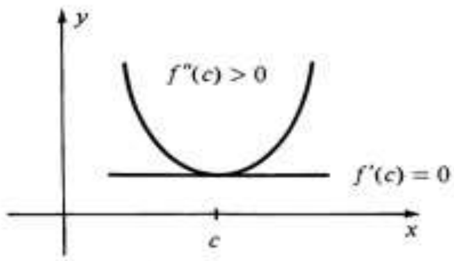
<u>The Second Derivative Test</u>: Suppose a function f is differentiable on an open interval containing c and $f'(c) = 0$.

(i) If $f''(c) < 0$, then f has a local maximum at c.

(ii) If $f''(c) > 0$, , then f has a local minimum at c.

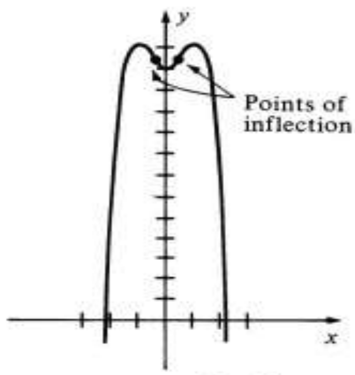


(i) Local maximum



(ii) Local minimum

Example: If $(x) = 12 + 2x^2 - x^4$, use the Second Derivative Test to find local maxima and minima of f. Discuss concavity, find the points of inflection, and sketch the graph of f.



$f(x) = 12 + 2x^2 - x^4$

Solution:

$$f'(x) = 4x - 4x^3 = 4x(1 - x^2)$$

$$f''(x) = 4 - 12x^2 = 4(1 - 3x^2)$$

The expression for $f'(x)$ is used to find the critical numbers $0, 1, and -1$. The values of f'' at these numbers are

$$f''(0) = 4 > 0 \quad , \quad f''(1) = -8 < 0 \quad , \quad f''(-1) = -8 < 0$$

Hence, by the Second Derivative Test there is a local minimum at 0 and local maxima at $1, and - 1$. The corresponding functional values are $f(0) = 12\ ,\ and$

$f(1) = 13 = f(-1)$

Critical number c	$f''(c)$	Sign of $f''(c)$	Conclusion
-1	-8	$-$	Local max: $f(-1) = 13$
0	4	$+$	Local min: $f(0) = 12$
1	-8	$-$	Local max: $f(1) = 13$

To locate the possible points of inflection we solve the equation $f''(x) = 0$, that is , $4(1 - 3x^2) = 0$. Evidently, the solution are $-\sqrt{3}/3$ and $\sqrt{3}/3$

Interval	k	Test value $f''(k)$	Sign of $f''(x)$	Concavity
$(-\infty, -\sqrt{3}/3)$	-1	-8	$-$	Downward
$(-\sqrt{3}/3, \sqrt{3}/3)$	0	4	$+$	Upward
$(\sqrt{3}/3, \infty)$	1	-8	$-$	Downward

The points $(\pm\sqrt{3}/3, 113/9)$ are points of inflection

Example: If $(x) = 2\sin x + \cos 2x$, find local extrema of f on the interval $[0,2\pi]$.

Solution:

$$f'(x) = 2\cos x - 2\sin 2x$$

$$f''(x) = -2\sin x - 4\cos 2x$$

The critical numbers of f in the interval $[0,2\pi]$ are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}\ and\ \frac{3\pi}{2}$.

$$f''\left(\frac{\pi}{6}\right) = -3\ , \quad f''\left(\frac{5\pi}{6}\right) = -3\ , \quad f''\left(\frac{\pi}{2}\right) = 2\ , \quad f''\left(\frac{3\pi}{2}\right) = 6$$

The local maxima are $\frac{\pi}{6}\ and\ \frac{5\pi}{6}$, and the local minima are $\frac{\pi}{2}\ and\ \frac{3\pi}{2}$.

Exercises

In Exercises 1-9 use the Second Derivative Test (whenever applicable) to find the local extrema of f. Discuss concavity, find x-coordinates of points of inflection, and sketch the graph of f.

1 $f(x) = x^3 - 2x^2 + x + 1$

2 $f(x) = 3x^4 - 4x^3 + 6$

3 $f(x) = 2x^6 - 6x^4$

4 $f(x) = (x^2 - 1)^2$

5 $f(x) = \sqrt[5]{x} - 1$

6 $f(x) = x^2 - (27/x^2)$

7 $f(x) = x/(x^2 + 1)$

8 $f(x) = \sqrt[3]{x^2}(3x + 10)$

9 $f(x) = 8x^{1/3} + x^{4/3}$

Answers

In Exercises 1-9 the notations CU and CD mean that the graph is concave upward or downward, respectively, in the interval that follows. PI denotes point(s) of inflection.

1 Max: $f(\frac{1}{3}) = \frac{31}{27}$; min: $f(1) = 1$. CD on $(-\infty, \frac{2}{3})$; CU on $(\frac{2}{3}, \infty)$; x-coordinate of PI is $\frac{2}{3}$.

2 Min: $f(1) = 5$; CU on $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$; CD on $(0, \frac{2}{3})$; x-coordinates of PI are 0 and $\frac{2}{3}$.

3 Max: $f(0) = 0$ (by first derivative test); min: $f(-\sqrt{2}) = f(\sqrt{2}) = -8$; CU on $(-\infty, -\sqrt{\frac{6}{5}})$ and $(\sqrt{\frac{6}{5}}, \infty)$; CD on $(-\sqrt{\frac{6}{5}}, \sqrt{\frac{6}{5}})$; x-coordinates of PI are $\pm\sqrt{\frac{6}{5}}$.

4 Max: $f(0) = 1$; min: $f(-1) = f(1) = 0$. CU on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$; CD on $(-1/\sqrt{3}, 1/\sqrt{3})$; x-coordinates of PI are $\pm 1/\sqrt{3}$.

5 No local extrema; CU on $(-\infty, 0)$; CD on $(0, \infty)$; PI $(0, -1)$

6 No max or min. CU on $(-\infty, -3)$ and $(3, \infty)$; CD on $(-3, 0)$ and $(0, 3)$; x-coordinates of PI are ± 3.

7 Min: $f(-1) = -\frac{1}{2}$; max: $f(1) = \frac{1}{2}$. CU on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$; CD on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$; x-coordinates of PI are $0, \pm\sqrt{3}$.

8 Max: $f(-\frac{4}{3}) \approx 7.27$; min: $f(0) = 0$; CD on $(-\infty, 0)$ and $(0, \frac{2}{3})$; CU on $(\frac{2}{3}, \infty)$; PI is $(\frac{2}{3}, 10\sqrt[3]{12}/3)$.

9 Min: $f(-2) \approx -7.55$; CU on $(-\infty, 0)$ and $(4, \infty)$; CD on $(0, 4)$; x-coordinates of PI are 0 and 4.

Def: Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Example: If $(x) = \frac{2x^2-5}{3x^2+x+2}$, determine the horizontal asymptote.

Solution:

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2-5}{3x^2+x+2} = \frac{2}{3}$, Also $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2-5}{3x^2+x+2} = \frac{2}{3}$

Then the line $y = \frac{2}{3}$ is a horizontal asymptote for the graph of f.

Example: Find the horizontal asymptote of $f(x) = \frac{4x}{x^2+9}$.

Solution:

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x}{x^2+9} = 0$, Also $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x}{x^2+9} = 0$

Then the line $y = 0$ (the x –axis) is a horizontal asymptote for the graph of f.

Def: Vertical Asymptote

A line $x = a$ is a **Vertical l asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

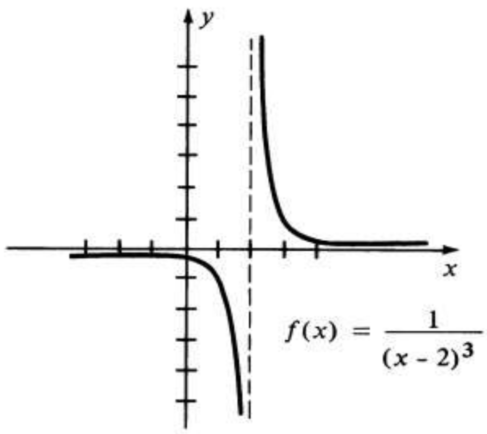
Example: Find the vertical asymptote of $f(x) = \frac{1}{(x-2)^3}$.

Solution:

$$\lim_{x\to 2^+} \frac{1}{(x-2)^3} = \infty$$

$$\lim_{x\to 2^-} \frac{1}{(x-2)^3} = -\infty$$

The line $x = 2$ is a vertical asymptote



Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of , and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptote.
7. Plot key points, such as the intercepts and the points found in Steps 3-5, and sketch the curve.

Example: Sketch the graph $(x) = \frac{2x^2}{9-x^2}$. (using the graphing strategy)

Solution:

1. $D_f = \mathbb{R}/\{-3,3\}$

   $f(-x) = \frac{2(-x)^2}{9-(-x)^2} = \frac{2x^2}{9-x^2} = f(x)$ the graph is symmetric with respect to the $y-$ axis.

2. $f'(x) = \frac{4x(9-x^2)-2x^2(-2x)}{[9-x^2]^2} = \frac{36x-4x^3+4x^3}{[9-x^2]^2} = \frac{36x}{[9-x^2]^2}$

$$f''(x) = \frac{36(9-x^2)^2 - (36x)(2)(9-x^2)}{[9-x^2]^4} = \frac{(9-x^2)[36(9-x^2)+144x^2]}{[9-x^2]^4}$$

$$= \frac{324+108x^2}{[9-x^2]^3} = \frac{108(x^2+3)}{[9-x^2]^3}$$

3. The critical points of f is 0 , -3 and 3 are not the critical numbers because $-3,3 \notin D_f$

Since $f''(0) = \frac{324}{729} = \frac{4}{9} > 0$, then the local minima at 0

4.

Interval	k	Test value $f'(k)$	Sign of $f'(x)$	Variation of f
$(-\infty, 0)$	-1	$-9/16$	$-$	Decreasing on $(-\infty, 0]$
$(0, \infty)$	1	$9/16$	$+$	Increasing on $[0, \infty)$

5. Since $f''(0) = \frac{4}{9} > 0$, then the graph is concave upward at $P(0, \frac{4}{9})$

   There is no inflection points because $-3,3 \notin D_f$

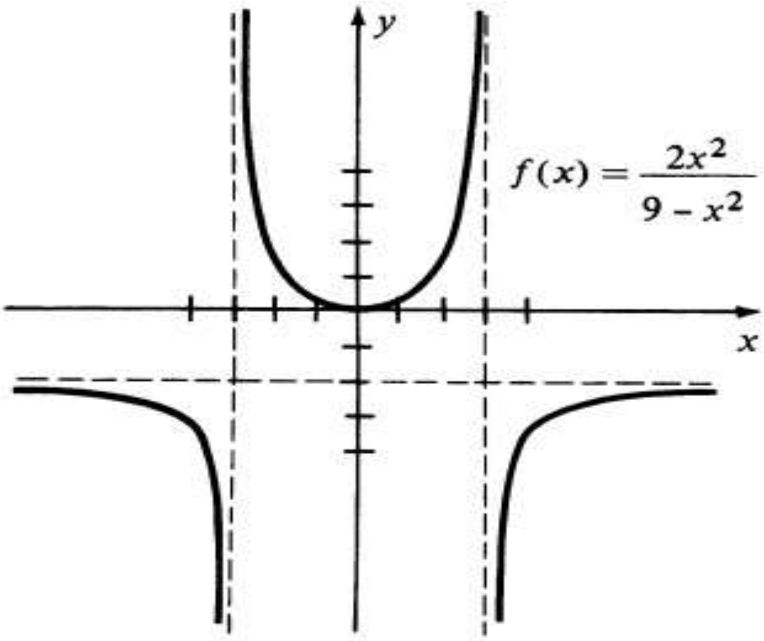
6. Since $\lim_{x\to\pm\infty}\frac{2x^2}{9-x^2}=-2$, then the line $y=-2$ is a horizontal asymptote.

   Since $\lim_{x\to 3^-}\frac{2x^2}{9-x^2}=\infty$, $\lim_{x\to 3^+}\frac{2x^2}{9-x^2}=-\infty$

   $\lim_{x\to -3^-}\frac{2x^2}{9-x^2}=-\infty$, $\lim_{x\to -3^+}\frac{2x^2}{9-x^2}=\infty$

   Then the lines $x=3$ and $x=-3$ are a vertical asymptotes.

7.



Exercises:

In exercises (1- 3) Sketch the following graph. (using the graphing strategy)

1) $f(x)=\frac{x^2-2x+4}{x-2}$
2) $f(x)=\frac{x^2}{x^2-x-2}$
3) $f(x)=\frac{x^2-9}{2x-4}$