إحصاء واحتمالات

هندسة إلكترونية وحاسوب

المستوى الثالث

إعـــداد :

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Introduction

Ex: A computer passwords consists of a letter of the alphabet followed by 3 or 4 digits find

1- The number of passwords in which no digits repeated

(26x10x9x8)+(26x10x9x8x7)=1499760

2- The total number of passwords consists created

(25x10x10x10)+(26x10x10x10x10)=286000

Note: we use the sum rule when the events cannot occurring in the same time

Ex: Find the number of : The numbers between 0 to 9

(i) 2- Digit even numbers

9x5=45

(ii) 2- Digit odd numbers

9x5=45

(iii) 2- Digit odd number with distend digits

8x5=40

(iv) 2- Digit even number with distend digits

(9x1)+(8x4)=41

Fractionation

Suppose that M be the positive integer M!=M(M-1)........x2x1

Note: 0!=1

Permutation

$P(n,r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$ $r \le n$, r and $n \in Z^{+}$

${}^{6}P_{2} = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6x5x4!}{4!} = 30$

Ex: Find the value of n

a- ${}^{n}P_{2} = 56$

${}^{n}P_{2}=56 \rightarrow n(n-1)=56 \rightarrow n2-n-56=0 \rightarrow (n-8)(n+7)=0$

$n-8=0 \rightarrow n=8$, $n+7=0 \rightarrow n=-7$ impossible

b- ${}^{n-2}P_{3} = 24$

${}^{n-2}P_{3}=24 \rightarrow {}^{n-2}P_{3}={}^{4}P_{3} \rightarrow n-2=4 \rightarrow n=6$

Ex: How many ways to set 3 boys in rows of 10 chair

${}^{10}P_{3}=10x9x8=720$

Ex: How many ~~[illegible]~~ ways can six men and six women seated in rows if :

1- Any person may sit next another

$^{12}P_{12}=12!=479001600$

2- Men and women must accept alternate

$^6P_6 \times {}^6P_6 \times {}^2P_2 = 6! \times 6! \times 2! = 1036800$

3- men sit together and also women sit together

$2! \times {}^6P_6 \times {}^6P_6 = 2! \times 6! \times 6! = 1036800$

Note: The number of the circle permutation of n! different object taken by (n-1)!

Theory: The number of distinct permutation that can be formed a contain of n – object in which first object appear k1,times the second object k2 times $\frac{n!}{K1!\,K2!..Kn!}$

Ex: Find the number of the different word that can be formed the letter of MISSISSIPPI

Solution: $\frac{11}{4! \times 4! \times 2! \times 1!} = \frac{11}{1152}$

Combination

We use the combination to counting problems where the order does unimportant

$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Ex: $^6C2 = \frac{6!}{2!(6-2)!} = \frac{6 \times 5 \times 4!}{2! \times 4!} = 15$

Ex: How many ways to make a committee contain at least 5 students from a group of 7 students

$^7C_5 + {}^7C_6 + {}^7C_7 = \frac{7!}{5! \times 2!} + \frac{7!}{6! \times 1!} + \frac{7!}{7! \times 1!} = 29$

Ex: Prove that $\binom{n}{r} \div \binom{n}{r-1} = \frac{n-r+1}{r}$

$\frac{n!}{r!(n-r)!} \times \frac{(r-1)! \times (n-r)!}{n!} = \frac{(r-1)!\,(n-r+1)\,(n-r)!}{r(r-1)!\,(n-r)!} = \frac{n-r+1}{r}$

Note: $^nC_r = {}^nC_{n-r}$

Ex: $^8C_3 = {}^8C_5$, $^{12}C_2 = {}^{12}C_{10}$

Note: If $^nC_{r1} = {}^nC_{r2}$ $r_1+r_2=n$ or $r_1=r_2$

Ex: How many ways can students choose 8 of 10 question to answer on exam and how many ways if they must choose 2 at least from the first three question

a- $^{10}C_8 = \frac{10!}{8! \times 2!} = 45$

b- $({}^3C_2 \times {}^7C_6) + ({}^3C_3 \times {}^7C_5) = \frac{3!}{2!} \times \frac{7!}{6!} + \frac{3!}{3!} \times \frac{7!}{5! \times 2!} = 42$

Sample Space and Events

Defn: An experiment is any operation whose out comes cannot prediction with certainty

Defn: The sample Space of an experiment is the set of all possible out comes for the experiment

Out comes is called a sample point

Example 1 : we roll a single die one time

Then the experiment is the roll of the die.

The sample space {1,2,3,4,5,6}.

Example 2 : Consider we toss two coins

Then $S = \{HH, HT, TH, TT\}$

Defn: An event is a sub set of a sample space in example 1 $S = \{1,2,3,4,5,6\}$

Then each set of the following is an event $A=\{1\}$, $B=\{1,3,5\}$, $C=\{2,4,6\}$

The empty set $\emptyset$ and **S** are events

$\emptyset$ is called impossible event

S is called the sure event

Defn: Two events A and B are said to be mutually – exclusive if they are disjoint $(A\cap B) = \emptyset$

Axioms of probability Let S be a sample space. Let $\mathcal{E}$ be the class of events. And Let P be areal valued function defined on $\mathcal{E}$ .Then P is called a probability function and P(A) is called the probability of the event A if the following axioms holds :

1 For any event A $0 \le P(A) \le 1$

2 $P(S) = 1$

3 $P(A_1 \cup A_2 \cup \ldots\ldots\ldots\ldots \cup A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$

for any i,j $A_i \cap A\, A_j = \emptyset$

Theorem 1 $P(\emptyset) = 0$

Proof since $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$

Then $P(S \cup \emptyset) = P(S) \rightarrow P(S) + P(\emptyset) = P(S)$

$1 + P(\emptyset) = 1 \rightarrow P(\emptyset) = 0$

Theorem 2

If $A^{\backslash}$ is the complement of A then $P(A^{\backslash}) = 1 - P(A)$

Proof since $S = A \cup A^{\backslash}$

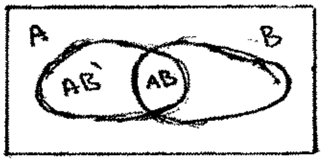
Then $P(S) = P(A \cup A^{\backslash}) \rightarrow P(S) = P(A) + P(A^{\backslash})$ since $A \cap A^{\backslash} = \emptyset$

$1 = P(A) + P(A^{\backslash}) \rightarrow P(A^{\backslash}) = 1 - P(A)$

Theorem 3 $P(A \cap B^{\backslash}) = P(A) - P(AB)$

Proof $A = AB^{\backslash} \cup AB \rightarrow P(A) = P(AB^{\backslash} \cup AB) \rightarrow P(A) = P(AB^{\backslash}) + P(AB)$

Then $P(AB^{\backslash}) = P(A) - P(AB)$ since $AB^{\backslash} \cap AB = \emptyset$

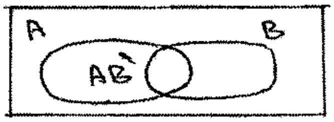


Theorem 4

If A and B be two events then $P(A \cup B) = P(A) + P(B) - P(AB)$

Proof $A \cup B = AB^{\backslash} + B \rightarrow P(A \cup B) = P(AB^{\backslash}) + P(B) \rightarrow P(A \cup B) = P(A) - P(AB) + P(B)$

Then $P(A \cup B) = P(A) + P(B) - P(AB)$



Defn: (Finite Probability Space)

Let S be a finite space $S = \{a_1, a_2, \ldots\ldots a_n\}$ a finite probability space is obtained by assigning to each point a_i S areal number P_i called a probability of a_i satisfy the following

1- $P_i \geq 0$

2- $P_1 + P_2 + \ldots\ldots P_n = 1$

The probability P(A) is defined to be the sum of the probability of the point in A

Example Let 3 coins tossed and the number of heads observed then $S = \{0,1,2,3\}$

(0 All tail , 1 one head , 2 two heads , 3 all heads)

The probability space $P(0) = \frac{1}{8}$, $P(1) = \frac{3}{8}$, $P(2) = \frac{3}{8}$, $P(3) = \frac{1}{8}$

Let A be the event that at least one head appears

B be the event that all heads or all tails

Then $A = \{1,2,3\}$, $B = \{0,3\}$

$P(A) = P(1) + P(2) + P(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$

$P(B) = P(0) + P(3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Defn: (Sample Space with equally likely)

$$P(A) = \frac{\text{number of element in } A}{\text{number of elements in } S} = \frac{n(A)}{n(S)}$$

Or $P(A) = \frac{\text{number of ways that event A occur}}{\text{number of ways that sample space S occur}}$

Example 1 Let a card be selected at random from an ordinary pack of 52 cards

Let A={the card is a spade} B={the card is a face card}

Solution: since a sample space is equality likely

$P(A) = \frac{\text{the number of points in A}}{\text{the number of points in S}} = \frac{13}{52}$

$P(B) = \frac{\text{the number of the face card}}{\text{number of cards}} = \frac{12}{52}$

$P(A \cap B) = \frac{\text{number of the spade in face card}}{\text{number of cards}} = \frac{3}{52}$

Example 2 Let we choose integer number X such $1 \le X \le 60$ find the probability of

A)X is even B)X is a multiple of 7 C) $X=n^2$ for some integer n

Solution: $P(A)\ \frac{30}{60} = \frac{1}{2}$

since the multiple of 7 are {7,14,21,28,35,42,49,56} then $P(B) = \frac{8}{60}$

since $X=n^2$, then we have the set {1,4,9,16,25,36,49} then $P(C) = \frac{7}{60}$

Example 3 Three light bulbs are chosen at random from 15 bulbs of which 5 are defective, find the probability P that : (i) None is defective (ii) Exactly one is defective

(iii) At least one is defective

Solution: There are $\binom{15}{3} = 455$ ways to select 3 bulbs From is

(i) $\frac{\binom{10}{3}\binom{5}{0}}{\binom{15}{3}} = \frac{120}{455} = \frac{24}{91}$

(ii) $\frac{\binom{10}{2}\binom{5}{1}}{\binom{15}{3}} = \frac{45 \times 5}{455} = \frac{255}{455}$

(iii) the event at least one is defective is the complement of the event none are defective

$P = 1 - \frac{24}{91} = \frac{67}{91}$

Another solution $\frac{\binom{10}{2}\binom{5}{1}}{\binom{15}{3}} + \frac{\binom{10}{1}\binom{5}{2}}{\binom{15}{3}} + \frac{\binom{10}{0}\binom{5}{3}}{\binom{15}{3}} = \frac{67}{91}$

Conditional Probability

The conditional probability of A , given that B has occurred is $P(A\backslash B)=\frac{P(A\cap B)}{P(B)}$ if $P(B)>0$

Example 1 We roll a pair of fair dice one time and given that the two numbers that occur are not the same. Compute the probability that the sum is 7 , the sum is 4 ?

Solution: $S=\{(X_1,X_2) : X_1=1,2,\ldots\ldots,6 \quad , \quad X_2=1,2,\ldots\ldots,6\}$

A= two numbers are different = 30 elements

B= sum is 7 = {16,25,34,52,61}

C= sum is 4 = {13,22,31}

Then $P(A)=\frac{30}{36}=\frac{5}{6}$, $P(B)=\frac{6}{36}=\frac{1}{6}$, $P(C)=\frac{3}{36}=\frac{1}{12}$

$P(A\cap B)=\frac{6}{36}=\frac{1}{6}$, $P(A\cap C)=\frac{2}{36}=\frac{1}{18}$, $P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{1}{5}$

$P(C\backslash A)=\frac{P(A\cap C)}{P(A)}=\frac{1}{18}\times\frac{12}{1}=\frac{2}{3}$

Example 2 A pair dice is thrown, find the probability P that the sum is 10 or greater of

(i) 5 appear on the first dice

(ii) 5 appear on at least one dice

Solution: The point of the sample space = 36

Let E be the event that the sum is 10 or greater E= {(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)}

Then $P(E)=\frac{6}{36}=\frac{1}{6}$

Let A be the event 5 appear in the first dice A= {(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)}

Then $P(A)=\frac{1}{6}$

Let B be the event 5 appear on at least one dice

Then $P(B)=\frac{11}{36}$, $P(E\cap A)=\frac{2}{36}=\frac{1}{18}$, $P(E\cap B)=\frac{3}{36}=\frac{1}{12}$

Now $P(E\backslash A)=\frac{1}{18}\times\frac{6}{1}=\frac{1}{3}$, $P(E\backslash B)=\frac{3}{36}\times\frac{36}{11}=\frac{3}{11}$

Example Let $P(A\cup B)=\frac{4}{10}$, $P(A\cap B)=\frac{1}{10}$, $P(B)=\frac{2}{10}$

Find $P(A\backslash B)$, $P(B\backslash A)$, $P(B\backslash A^{\backslash})$, $P(A^{\backslash}\backslash B^{\backslash})$

Solution: $P(A)=P(A\cup B)-P(B)+P(A\cap B)=\frac{4}{10}-\frac{2}{10}+\frac{1}{10}=\frac{3}{10}$

$P(A\backslash B)=\frac{P(A\cap B)}{P(B)}=\frac{1}{10}\times\frac{10}{2}=\frac{1}{2}$, $P(B\backslash A)=\frac{P(A\cap B)}{P(A)}=\frac{1}{10}\times\frac{10}{3}=\frac{1}{3}$

$$P(B\backslash A') = \frac{P(A'\cap B)}{P(A')} = \frac{P(B)-P(AB)}{1-P(A)} = \frac{\frac{2}{10}-\frac{1}{10}}{\frac{7}{10}} = \frac{1}{10} \times \frac{10}{7} = \frac{1}{7}$$

$$P(A'\backslash B') = \frac{P(A'\cap B')}{P(B')} = \frac{P(A\cup B)'}{1-P(B)} = \frac{1-P(A\cup B)}{1-P(B)} = \frac{6}{10} \times \frac{10}{8} = \frac{3}{4}$$

<u>Example</u> Find $P(A\backslash B)$ if $A\subset B$, A,B are mutually exclusive

<u>Solution</u>: If $A\subset B$ then $A\cap B=A$ Then $P(B\backslash A)=\frac{P(A)}{P(A)}=1$

If A,B mutually exclusive Then $A\cap B=\emptyset$ $P(B\backslash A)=\frac{0}{P(A)}=0$

<u>Example</u> In a certain college , 25% of students failed math , 15% of students failed computer and 10% of the students failed both math and computer , a student is selected at random.

(i) If he failed computer , what is the probability that he failed math

(ii) If he failed math , what is the probability that he failed computer

(iii) What is the probability that he failed math or computer

<u>Solution</u>: Let A be the event the students failed math $P(A)=\frac{25}{100}$

Let B be the event the students failed computer $P(B)=\frac{15}{100}$

Then $P(A\cap B)=10/100$

(i) $P(A\backslash B)=\frac{P(A\cap B)}{P(B)} = \frac{10}{100} \times \frac{100}{15} = \frac{2}{3}$

(ii) $P(B\backslash A)=\frac{P(A\cap B)}{P(A)} = \frac{10}{100} \times \frac{100}{25} = \frac{2}{5}$

(iii) $P(A\cup B)=P(A)+P(B)-P(AB)=\frac{25}{100}+\frac{15}{100}-\frac{10}{100}=\frac{3}{10}$

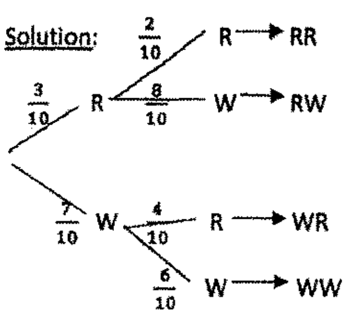
Since we have $P(A\backslash B)=\frac{P(A\cap B)}{P(B)}$ implies that $P(A\cap B)=P(B)\ P(A\backslash B)$.......1

And $P(B\backslash A)=\frac{P(A\cap B)}{P(A)}$ implies that $P(A\cap B)=P(A)\ P(B\backslash A)$.......2

<u>Example 1</u> Box contain 3 red balls and 7 white balls if we choose at random a ball from the box and we put another ball from another color and then we choose a second ball from the box , find the probability of :

1-The second ball is red 2-Two balls is white 3-Two balls from the same color

<u>Solution:</u>



Suppose that

A- be the event of the second ball is red

B- be the event of two balls are white

C- be the event of two balls from the same color

$P(A)=\frac{3}{10}\times\frac{2}{10}+\frac{7}{10}\times\frac{4}{10}=\frac{17}{50}$

$P(B)=\frac{7}{10}\times\frac{6}{10}=\frac{21}{50}$

$P(C)=\frac{3}{10}\times\frac{2}{10}+\frac{7}{10}\times\frac{6}{10}=\frac{12}{25}$

<u>Example 2</u> A class has 10 boys and 5 girls, three students are selected at random, one after the other, find the probability that

(i) The first two are boys and third is girl

(ii) The first and third are boys and second is a girl

(iii) The first and third are of the same sex and the second is of the opposite sex

<u>Solution</u>: (i) $\frac{10}{15}\times\frac{9}{14}\times\frac{5}{13}=\frac{45}{273}$

(ii) $\frac{10}{15}\times\frac{5}{14}\times\frac{9}{13}=\frac{45}{273}$

(iii) $\frac{10}{15}\times\frac{5}{14}\times\frac{9}{13}+\frac{5}{15}\times\frac{10}{15}\times\frac{4}{13}=\frac{65}{273}$

<u>Boy's theorem</u> Suppose that we are given K events $A_1, A_2, \ldots\ldots A_K$ such that

1- $\bigcup_{i=1}^{k} A_i = S$ 2- $A_i \cap A_j = \emptyset$, $i\neq j$

then for any event $B \subset S$ $P(A_i\backslash B)=\frac{P(Ai)\ P(B\backslash Aj)}{\sum_{i+1}^{k}\ P(Ai)\ P(B\backslash Ai)}$ $i=1,2,\ldots\ldots,K$

<u>Proof</u> We can write $B=B\cap S$ $B=B\cap(\bigcup_{i=1}^{k} A_i) = \bigcup_{i=1}^{k}(B\cap A_i)$

Then $P(B)=P(\bigcup_{i=1}^{k}(B\cap A_i))$ we have $(B\cap Ai)\cap(B\cap Aj)=\emptyset$ $\forall i\neq j$

Then $P(B)=\sum_{i+1}^{k}\ P(B\cap A_i)=\sum_{i+1}^{k}\ P(A_i)\ P(B\backslash A_i)$.........

Then $P(A_i\backslash B)=\frac{P(B.Aj)}{P(B)}=\frac{P(Aj)\ P(B\backslash Aj)}{\sum_{i+1}^{k}\ P(Ai)\ P(B\backslash Ai)}$

<u>Example</u> Three machines A,B and C produce respectively 50%, 30%and 20% of items of a factory, the percentages of defective out put of these machines are 3%, 4%, 5%, suppose the item is selected at random, what is the probability that the item is defective

<u>Solution</u>: Let X be an event that an item is defective

Then $P(X)=P(A\cap X)+P(B\cap X)+P(C\cap X)=P(A)\ P(X\backslash A) + P(B)\ P(X\backslash B) + P(C)\ P(X\backslash C)$

$=(0.50)(0.03)+(0.30)(0.04)+(0.20)(0.05)=0.37$

Example In above example suppose the item is selected random and found it defective, find the probability that the item was probability by A

$$P(A\backslash X)=\frac{P(A)\,P(X\backslash A)}{\sum P(Ai)\,P(X\backslash Ai)}=\frac{P(A)\,P(X\backslash A)}{P(A)\,P(X\backslash A)+P(B)\,P(X\backslash B)+P(C)\,P(X\backslash C)}$$

$$=\frac{(0.50)(0.03)}{(0.50)(0.03)+(0.30)(0.04)+(0.20)(0.05)}=\frac{53}{112}$$

Defn An events A and B are independent if $P(A\cap B)=P(A)\ P(B)$

Example Let a fair coin be tossed three times, consider the following events

A={first toss is head}, B={second toss is head}, C={exactly two heads are toss in ....}

Solution: S={HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}

A={HHH,HHT,HTH, HTT} $P(A)=\frac{4}{8}=\frac{1}{2}$, B={HHH,HHT,THH,THT} $P(B)=\frac{4}{8}=\frac{1}{2}$

C={HHT,THH} $P(C)=\frac{2}{8}=\frac{1}{4}$, A.B={HHH,HHT} $P(A\cap B)=\frac{2}{8}=\frac{1}{4}$

A.C={HHT} $P(A\cap C)=\frac{1}{8}$, B.C={HHT,THH} $P(B\cap C)=\frac{2}{8}=\frac{1}{4}$

$P(A)\ P(B)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}=P(A\cap B)$ independent

$P(A)\ P(C)=\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}=P(A\cap C)$ independent

$P(B)\ P(C)=\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}=P(B\cap S)$ not independent

Example If $P(A\cup B)=P(A)\ P(B^{\backslash})+P(B)$ Proof that A and B are independent

Proof by theorem $P(A\cup B)=P(A)\ [1-P(B)]+P(B)-P(AB)$ 1

We have $P(A\cup B)=P(A)\ [1-P(B)]+P(B)$

$P(A\cup B)=P(A)+P(B)-P(A)\ P(B)$ 2

From 1 and 2 $P(AB)=P(A)\ P(B)$

Ex 1

1- Compute each of the following

${}^{n+1}P_{r+1}$, ${}^{7}p2$, ${}^{n}P_{n-1}$, ${}^{n+1}P_{n-1}$, ${}^{n}C_{n}$, ${}^{n}C_{n-1}$, ${}^{100}C_{2}$, $\binom{n}{n-2}$, $\binom{n+1}{n-1}$

2- Proof the following statements

a- ${}^{n+1}P_{r+1} = (n+1)\,{}^{n}P_{r}$ b- ${}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1} = {}^{n}P_{r}$ c- $n\ {}^{n-1}P_{n-1} = {}^{n}P_{n}$

d- ${}^{n+1}C_{r} = {}^{n}C_{r-1} + {}^{n}C_{r}$ e- ${}^{n}C_{r+1} + 2\,{}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+2}C_{r+1}$ f- $\dfrac{\binom{n}{1}\binom{2n}{2}}{\binom{3n}{3}} = {}^{2n}P_{2} / {}^{3n-1}P_{2}$

3- Find the value of n in each of the following

a- ${}^{n}C_{2} = 435$ b- $5\,{}^{n}C_{6} = 12\,{}^{n}C_{4}$ c- $2\,{}^{n}C_{4} = 35\,{}^{n/2}C_{3}$

4- Find the value of r in each of following

a- ${}^{28}C_{r+4} = {}^{28}C_{r-2}$ b- ${}^{n}P_{r} = 120\,{}^{n}C_{n-r}$

5- How many 3 digit numbers can be formed by using the 6 digit {2,3,4,5,6,8}

a- Repetitions of digit are allowed

b- Repetitions are not allowed

c- The number is to be odd and repetitions are not allowed

d- The number is to be even and repetitions are not allowed

e- The number is to be a multiple of 5 and repetition are not allowed

f- The number must contain the digit 5 and repetition are not allowed

6- A book shelf to be used to display six new books suppose that there are eight computer science books and five math –books from which to choose. If we decide to show four computer science books and two math-books and are required to keep the books in each subject to ... there , how many different displays are possible

7- Find the number of subsets of each possible size of a set contain four elements

8- How many 3-digit numbers can be formed by using the 7 digits {1,2,3,4,5,6,7} and these numbers must be greater than 400 ?

9- How many distinct permutation can be formed from all the letters of each words Good

10- A committee of three people is to be chosen from four married couples

a- How many different committees are there ?

b- If the committee contain one man and two women ?

c- How many committee are there such that no two committee numbers are married to each other ?

11- How many different arrangements of the letters in the word ((B O U G H T)) can be formed if the vowels must be kept next each other ?

12- In how many ways can seven people be seated in a circle

13- In class there are 10 boys and 8 girls and we want to form a committee of 7 person

a- In how many ways can a committee selected

b- If the chair from the boys

c- If there are a chair and writter

d- If there are a chair and writter from girls

14- How many different 8-card hands with 5 red cards and 3 black cards can be dealt from a deck of 52 cards

15- If n fair coins are tossed and the results recorded how many

a- Record sequences are possible

b- Sequences contain exactly three tails , assuming $n \geq 3$?

c- Sequences contain exactly K heads , assuming $n \geq K$?

16- The box contains 15 balls , 8 of which are red and 7 are black , in how many ways can 5 balls be chosen of that : a- all 5 are red b- 2 red and 3 black

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Ex 2

1- Given S={1.2.3} , A{1} , B={2} , C={3} , $P(A)=\frac{1}{3}$, $P(B)=\frac{1}{3}$

Find : (i) $P(C)$ (ii) $P(A \cup B)$ (iii) $P(A')$ (iv) $P(A' \cap B')$ (v) $P(A' \cup B')$

2- Given an experiment such that $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{2}$ and $P(A \cup B)=\frac{2}{3}$

Compute $P(A')$, $P(B')$, $P(A \cap B)$, $P(A' \cap B')$, $P(A' \cup B)$

3- For any events A,B and C proof that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

4- An experiment consists A selecting a digit from the digit 0 to 9 in such away that each digit has same chance of being selected as any other . we name the digit selected A . these lines of code are then executed :

If A<2 Then B=12 , Else B=17 If B=12 Then C=A-1 , Else C=0

a- Construct a tree to illustrate the ways in which values can be a singed to each variable A,B and C

b- Find the sample space generated by the free

c- Find the probability that : (i) A is an even (ii) C is negative (iii) C=0 (iv) $C \le 1$

5- If A and B two events probability :

(i) $P(A) + P(B) \le 1 + P(AB)$ (ii) If $P(A) \le P(B)$ THEN $P(A') \ge P(B')$

6- Given an experiment such that : $P(B)=\frac{1}{4}$, $P(A \cup B)=\frac{1}{3}$, Find $P(A)$ if :

(i) $A \cap B = \emptyset$ (ii) $B \subset A$

7- A coin is tossed 4 times , find the probability P that :

(i) first tossed is tail (ii) second tossed is head (iii) at fast 3 heads appear

(iv) exactly two tails

8- If 10 girls in class , 3 have blue eyes , if two of the girls are chosen at random , what is the probability that :

(i) both have blue eyes (ii) neither has blue eyes (iii) at least one has blue eyes

9- Show that the conditional probability satisfies the axiom of probability. Note P(E)

10- If A and B are independent events prove that :

(i) A, B' are independent (ii) A', B' are independent

11- We are given 3 boxes as follows :

Box A contain 3 red balls and 5 white balls

Box B contain 2 red ball and 1 white ball

Box C contain 2 red balls and 3 white balls

If a box is selected ass random and select from if a ball which becomes red

What is the probability that it come from A.

12- If $P(A)=\frac{1}{2}$, $P(B\backslash A)=\frac{2}{3}$, Find $P(AB')$

13- If $P(A)=\frac{6}{10}$, $P(B)=\frac{3}{10}$, Find $P(AUB)$ in the following cases :

(i) A,B are independent (ii) B c A (iii) A and B are mutually exclusive

14- Prove that $P(A'\backslash B) = 1-P(A\backslash B)$

15- If A and B are independent and $P(A)=\frac{1}{2}$, $P(AUB)=\frac{2}{3}$

Find: (i) $P(B)$ (ii) $P(A\backslash B)$ (iii) $P(B'\backslash A)$

16- Two digits are selected at random from the digit 1-9

(i) If the sum is odd , what is the probability that 2 is one of numbers selected

(ii) If 2 is one of the digits selected , what is the probability that the sum is odd

17- If $P(A'UB) = 1-P(A).P(B')$ Proof that A.B are independent

Random Variable and probability Distribution

Def: A random variable X is areal – valued function of the elements of sample space S

Example 1 When a coin is tossed three times so that sample space S={HHH,HHT,HTH,THH,HTT,THT,TTH,TTT}

If we define X=number of head produced

Thus the value if X=are {0,1,2,3}

Hence X is a random variable whose domain is a sample space and its range is the set {0,1,2,3}

If Y= the absolute value of the difference between the number of H and the number of T

Then Y={1,3}

Def: A random variable X is called discrete if it is range R_X is discrete (has finite or countably in finite number of elements) set of real numbers

A random variable X is called Continuous if its range is an interval or union of intervals on the real line

In example 1 the random variables X and Y are discrete

Example 2 A pair of a fair dice is rolled one times.

Let X be the sum of the two numbers that occur

Then $S=\{(x_1,x_2): x_1 = 1,2,\ldots\ldots,6,\ x_2 = 1,2,\ldots.,6\}$

We define $X((x_1,x_2))=x_1+x_2$

Then the range of X is $R_X = \{2,3,4,\ldots\ldots.,12\}$ Then X is a discrete random variable

Example 3 We select 1 student at random from those registered of the university of HADRAMOUT

Let Y be the weight of the selected student

We assume that there are 300 students registered and they are numbers from 1-300 .

Then $S = \{1,2,\ldots\ldots,300\}$

The random variable Y is defined to be

Y(w)= weight of the student w , for $w \in S$

If we assume no student weight less than 30kg or more than 100kg

Thus the range of Y $R_Y = \{x / 30 \le x \le 100\}$

Thus Y is a continous random variable

Discrete Distribution Function

In example 1 we can write a table listing the values of the random variable and the probability of each value as follows:

X	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

In example 2

The random variables X and the corresponding probabilities are shown as follow:

Ex:

P(X=6)={(5,1),(1,5),(2,4),(4,2),(3,3)= $\frac{5}{36}$

N	P(X=N)
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Def: A table listing all possible values that a random variable (discrete) can take together with the associated probabilities is called a probability function or probability distribution

The probability function for X is a function denoted by $R_x(x)$ or P(X=x)

The probability function for a discrete variable X is a function $P_x(x)$ satisfies the following conditions:

1- $P_x(x) \geq 0$ 2- $\sum_x P_x(X)=1$

Def: The distribution function for a random variable X denoted by $F_x(t)$ is a function of a real variable t such that $F_x(t)=P(X\leq t)=\sum_x P_x(x)$ $-\infty<t<\infty$

Example Suppose that the probability function of a random variable X is $P_X(x)=\frac{x}{10}$,x=1,2,3,4

List the value that could be included in the probability distribution and the value that could listed in the distribution function to the value of X

Solution: The probability distribution

$P_X(1)=\frac{1}{10}$, $P_X(2)=\frac{2}{10}$, $P_X(3)=\frac{3}{10}$, $P_X(4)=\frac{4}{10}$

The distribution function

$F_X(1)=P(X\leq1)=\sum_{x\leq1} P_X(x)=P_X(1)=\frac{1}{10}$

$F_X(2)=P(X\leq2)=\sum_{x\leq2} P_X(x)=P_X(x)=P_X(1)+P_X(2)=\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$

$F_X(3)=P(X\leq3)=\sum_{x\leq3} P_X(x)=P_X(x)= P_X(1)+P_X(2)+P_X(3)=\frac{1}{10}+\frac{2}{10}+\frac{3}{10}=\frac{6}{10}$

$F_X(4)=P(X\leq4)=\sum_{x\leq4} P_X(x)=P_X(x)= P_X(1)+P_X(2)+P_X(3)+P_X(4)=\frac{1}{10}+\frac{2}{10}+\frac{3}{10}+\frac{4}{10}=\frac{10}{10}=1$

Example Given that the distribution function of X is

t	0	1	2	3	4	5
$F_X(t)$	0.0459	0.2415	0.5747	0.8585	0.9794	1

Find: $P(X\leq2)$, $P(X>2)$, $P(X=2)$, $P(1\leq X\leq3)$

Solution: From the table we get $P(X\leq2)=0.5747$

Since we have $P(X\leq2)+ P(X>2)=1$

$P(X>2)=1- P(X\leq2)=1-0.5747=0.4253$

Since $P(X\leq2)= P(X\leq2)+P(X\leq1)$

$P(X=2)= P(X\leq2)- P(X\leq1)=0.5747-0.2415=0.3332$

$P(1\leq X\leq3)= P(X=1)+ P(X=2)+ P(X=3)+ P(X=0)-P(X=0)= P(X\leq3)- P(X\leq0)=0.8126$

Def: If X is a discrete random variable with the probability distribution $P_X((x)$ then the expected value X ((some times called the expectation of X or the mean)) is

$$M = E(x)= \sum_x X P_X(x)$$

Example Let a pair of fair dice is tossed and Let X assign to each element in S the maximum of its numbers, Find the expected value of X

Solution: The sample space $S=\{(x_1,x_2)\ \ x1 = 1,2,\ldots,6\ ,\ x2 = 1,2,\ldots,6\}$

The random variable X(a,b) = max(a,b)

$P(X=1)=P(\{(1,1)\})=\frac{1}{36}$

$P(X=2)=P(\{(2,1),(1,2),(2,2)\})=\frac{3}{36}$

$P(X=3)=P(\{(3,1),(1,3),(3,2),(2,3),(3,3)\})=\frac{5}{36}$

$P(X=4)=P(\{(4,1),(1,4),(4,2),(2,4),(4,3),(3,4),(4,4)\})=\frac{7}{36}$

$P(X=5)=P(\{(5,1),(1,5),(5,2),(2,5),(5,3),(3,5),(5,4),(4,5),(5,5)\})=\frac{9}{36}$

$P(X=6)=P(\{(6,1),(1,6),(6,2),(2,6),(6,3),(3,6),(6,4),(4,6),(6,5),(5,6),(6,6)\})=\frac{11}{36}$

Therefore we have the following table

X_i	1	2	3	4	5	6
$P_X(X_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Then the expectation or the mean of X

$$M=E(x)=\sum_{i=1}^{6} X_i P_X(X_i) = X_1P_X(X_1)+X_2P_X(X_2)+X_3P_X(X_3)+X_4P_X(X_4)+X_5P_X(X_5)+X_6P_X(X_6)$$

$$=(1)(\tfrac{1}{36})+(2)(\tfrac{3}{36})+(3)(\tfrac{5}{36})+(4)(\tfrac{7}{36})+(5)(\tfrac{9}{36})+(6)(\tfrac{11}{36})=\frac{161}{36}$$

<u>Example</u> Repeat the previous example such that the random variable Y which assign to each point (a,b) in S the sum of its numbers

<u>Solution</u>: $S=\{(X_1,X_2)\ /\ X_1=1,2,\ldots\ldots,6\ ,\ X_2=1,2,\ldots\ldots,6\}$ random variable $Y=a+b$

Therefore we have the following table

X_i	2	3	4	5	6	7	8	9	10	11	12
$P_X(X_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Then the mean of Y $M=E(Y)=\sum_{i=1}^{12} X_i P_X(X_i)=\frac{252}{36}=7$

<u>Def</u>: The mathematical expectation for a function of a random variable X , H(X) is a discrete $E(H(x))=\sum_x H(x)\,P_X(x)$

Where $P_X(x)$ is the probability distribution for X

Properties the expectation

1- Let X be a random variable and let a,b are real numbers then $E(ax+b)=aE(x)+b$

2- If H(x) and G(x) are function of the random variable X then $E(H(x)+G(x))=E(H(x))+E(G(x))$

<u>Proof</u> Let X be a random variable and its probability distribution is $P_X(x)$

Then $E(ax+b)\,P_X(x)=\sum(ax\,P_X(x)+bx\,P_X(x))$

$$=\sum_x ax\,P_X(x)+\sum_x bx\,P_X(x)=a\sum_x xP_X(x)+b\sum_x P_X(x)$$

$$=aE(x)+b \qquad (\sum_x P_X(x)=1)$$

2 Let f(x) be function of a random variable X such that $f(x)=H(x)+G(x)$

Then $E(f(x)) = \sum f(x)\, P_X(x) = \sum (H(x)+G(x))\, P_X(x) = \sum H(x)\, P(x) + \sum G(x)\, P_X(x)$
$= E(H(x)) + E(G(x))$

Variance and Standard Deviation

Def: Let X be a random variable with probability distribution $P_X(x)$. The variance of X , denoted by $\sigma_x^2 = \sum_x (x-M)^2 P_X(x) = E((x-M)^2)$ where M is the mean of X ,

The square root of the variance is called the standard deviation and denoted by σ_x

Example: Find the variance and standard deviation for a random variable Y taken in example 2

Solution: , $\sigma_y^2 = E(y-M_y)^2 = \sum (y-M_y)^2 P_y(y)$ and $M = \sum_y y\, P_y(y) = \frac{252}{36} = 7$

$$\sigma_y^2 = \frac{105 \times 2}{36} = \frac{35}{6} \rightarrow \sigma_y = \sqrt{\frac{35}{6}}$$

Y	$P_y(y)$	$yP_y(y)$	$y-M_y$	$(y-M_y)^2$	$(y-M_y)^2 P_y(y)$
2	$\frac{1}{36}$	$\frac{2}{36}$	-5	25	$\frac{25}{36}$
3	$\frac{2}{36}$	$\frac{6}{36}$	-4	16	$\frac{32}{36}$
4	$\frac{3}{36}$	$\frac{12}{36}$	-3	9	$\frac{27}{36}$
5	$\frac{4}{36}$	$\frac{20}{36}$	-2	4	$\frac{16}{36}$
6	$\frac{5}{36}$	$\frac{30}{36}$	-1	1	$\frac{5}{36}$
7	$\frac{6}{36}$	$\frac{42}{36}$	0	0	0
8	$\frac{5}{36}$	$\frac{40}{36}$	1	1	$\frac{5}{36}$
9	$\frac{4}{36}$	$\frac{36}{36}$	2	4	$\frac{16}{36}$
10	$\frac{3}{36}$	$\frac{30}{36}$	3	9	$\frac{27}{36}$
11	$\frac{2}{36}$	$\frac{22}{36}$	4	16	$\frac{32}{36}$
12	$\frac{1}{36}$	$\frac{12}{36}$	5	25	$\frac{25}{36}$
		$\frac{252}{36}$			$\sum = \frac{2(105)}{36}$

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Theorem $\sigma_x^2 = E(x^2) - M^2$

Proof $\sigma_x^2 = \sum_x (X-M)^2 P_X(x) = \sum_x (X-2xM+M^2)\, P_X(x)$

$= \sum_x X^2 P_X(x) - M \sum_x X\, PX(x) + M^2 P_X(x)$

$= E(x^2) - 2M\, E(x) + M^2 = E(x^2) - 2M^2 + M^2 = E(x^2) - M^2$

Theorem For any random variable X , Let Y =ax+b Where a and b are constants,
Then $\sigma_x^2 = a^2 \sigma_x^2$

Where σ_x^2 is the variance of X

σ_{y^2} Is the variance of y

Proof $\sigma_{x^2} = E(y^2)-(M_y)^2 = E(ax+b)^2(M_y)^2 = \sum(ax+b)^2 P_X(x) - (a\,E(x)+b)^2$

$= \sum(a^2x^2+2ab+b^2)\,P_x(x) - (a^2\,E^2(x)+2ab\,E(x)+b^2)$

$= a^2\sum x^2 P_X(x)+2ab\sum P_X(x)+b^2\sum P_X(x)-a^2E^2(x)-2ab\,E(x)-b^2$

$= a^2\,E(x^2)+2ab\,E(x)+b^2-a^2\,E^2(x)-2ab\,E(x)-b^2$

$= a^2\,E(x^2)-a^2\,E^2(x)=a^2(E(x^2)-(M_x)^2) = a^2\sigma_{x^2}$

Joint distribution

Let X and y a random variables on a sample space **S** with respective images sets

$X(S)=\{x_1,x_2,\ldots,x_n\}$

$Y(S)=\{y_1,y_2\ldots y_m\}$

Thus $X(S)\,Y(X) = \{(x_1,y_1),(x_1,y2),\ldots,(x_1,y_m)$

$(x_2,y_1),(x_2,y_2),\ldots,(x_2,y_m$

$\vdots \quad \vdots \quad \vdots$

$(x_n,y_1),(x_n,y_2),\ldots,(x_n,y_m)\}$

$=\{(x,y) \quad x \in X(S)=y \in Y(s)\}$

The function on X(S) Y(S) define by $h(x_i,y_j)=P(X=x_i\,,\,Y=y_j)$ is called the joint distribution or joint probability function of X and Y and is usually given in the form of table

X \Y	y_1	y_2	…………	y_m	sum
X_1	$h(x_1,y_1)$	$h(x_1,y_2)$	…………	$h(x_1,y_m)$	$F(x_1)$
X_2	$h(x_2,y1)$	$h(x_2,y_2)$	…………	$h(x_2,y_m)$	$F(x_2)$
:	:	:		:	:
:	:	:		:	:
x_n	$h(x_n,y_1)$	$h(x_n,y_2)$		$h(x_n,y_m)$	$F(x_n)$
	$Y(y_1)$	$Y(y2)$		$Y(y_m)$	1

Note $P_y(y_j)=g(y_j)$ $P_X(x_i)=f(x_i)$

The joint distribution function h satisfies the following

1- $h(x_i,y_j)\geq 0$ 2- $(\sum_{i=1}^{n}\ )(\sum_{j=1}^{m}\ )\,h(x_i,y_j)=1$

Example 1: Let X and Y be a random variable such that

$X(a,b)=\max(a,b)$ and $Y(a,b)=a+b$

When tossing a pair of a fair dice

Find the joint distribution of X and Y

X\Y	2	3	4	5	6	7	8	9	10	11	12	sum
1	$\frac{1}{36}$	0	0	0	0	0	0	0	0	0	0	$\frac{1}{36}$
2	0	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	0	0	$\frac{3}{36}$
3	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	$\frac{5}{36}$
4	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	$\frac{7}{36}$
5	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	$\frac{9}{36}$
6	0	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{11}{36}$
Sum	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

<u>Example 2:</u> Let X and Y be two random variables such that

X represent the number of heads occur

Y the absolute value between the number of head and tail occur when we tossing a coin three times

Y\X	0	1	2	3
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$

<u>Def</u>: Let X and Y are a random variables with the above joint distribution function $h(x_i,y_j)$ then the con variance of X and Y denoted by Cov(x,y)and define by

$Cov(x,y)=E(x,y)-M_x\, M_y$ where $E(x,y)=\sum x_i\, y_j\, h(x_i,y_j)$

<u>Example</u>: Find the con variance of X and Y in example 1

<u>Solution</u>: $Cov(x,y)=E(x,y)-M_x\, M_y$

$E(x,y)=(\sum_{i\,j}\ )X_i\, Y_j\, h(x_i,y_j)$

$=(1x2x\frac{1}{36})+(2x3x\frac{2}{36})+(2x4x\frac{1}{36})+\ldots+(6x12x\frac{1}{36})=\frac{1232}{36}=34.2$

$(M_x=4.47\quad M_y=7)$

$Cov(x,y)=E(x,y)-M_x\, M_y=34.2-(4.47)(7)=2.9$

<u>Example 2</u> H.w

Def: The correlation of X and Y denoted by P(x,y), Pxy is define by $P(x,y)=\frac{Cov(x,y)}{\sigma x\ \sigma y}$

Example: Find the correlation P(x,y) using the pervious example

Solution: $P(x,y)=\frac{Cov(x,y)}{\sigma x\ \sigma y}$

$Cov(x,y)=E(xy)-M_x M_y$

$E(xy)=\sum X_i Y_j h(x_i,y_j)=34.2$ $M_x=4.47$ $M_y=7$

Then the correlation $P(x,y)=\frac{Cov(x,y)}{\sigma x\ \sigma y}=P(x,y)=\frac{2.9}{(1.4)(2.4)}=0.86$

Notes: 1- $P(x,y)=P(y,x)$ 2- $P(x,x)=1$ 3- $1\geq P\geq -1$

4- P=0 that mean X and Y not correlation

Def: A finite number of random variable x,y...z on a sample space S are said to be independent $P(X=x_i, Y=y_j, \ldots, Z=z_k)=P(X=x_i)P(Y=y_j)\ldots\ldots P(Z=z_k)$

Joint distribution of X and Y $h(x_i,y_j)=P(X=x_i, Y=y_j)=P(X=x_i, Y=y_i)=F(x_i)\ g(y_j)$

Theorem Let X and Y are independent random variable , then

1 $E(x.y)=E(x)\ E(y)$ 2 $Cov(x,y)=0$ 3 $\sigma_{x+y}^{2}=\sigma_x^2+\sigma_y^2$

Def: The k-th moment of a random variable X , denoted by m_k , is the expected value of X to k.th power , k=1,2,.......

That is $m_k=E(x^k)=\sum x^k P_X(x)$

Example: Find the first five moments of X if X has the following probability distribution

X_i	-2	1	3
$P_X(x_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Solution: First moment $m_1=E(x)=\sum_{i=1}^{3} X_i P_X(x)$

$=x_1 P_X(x_1)+x_2 P_X(x_2)+x_3 P_X(x_3)=-2(\frac{1}{2})+1(\frac{1}{4})+3(\frac{1}{4})=0$

$m_2=E(x^2)=\sum_{i=1}^{3} X_i^2 P_X(x_i)$

$=(-2)^2(\frac{1}{2})+(1)^2(\frac{1}{4})+(3)^2(\frac{1}{4})=2+\frac{1}{4}+\frac{9}{4}=\frac{18}{4}=\frac{9}{2}$

$m_3=E(x^3)=\sum_{i=1}^{3} X_i^3 P_x(x_i)$

$=(-2)^3(\frac{1}{2})+(1)^3(\frac{1}{4})+(3)^3(\frac{1}{4})=-4+\frac{1}{4}+\frac{27}{4}=\frac{12}{4}=3$

$m_4=\sum_{i=1}^{3} X_i^4 P_x(x_i)=\ldots\ldots$

$m_5=\sum_{i=1}^{3} X_i^5 P_x(x_i)=\ldots\ldots$

Ex 3

* Find the following joint distribution of X and Y

X\Y	-2	-1	4	5	Sum
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
Sum	0.3	0.3	0.1	0.3	1

Find: 1- E(x) and E(y) 2- cov(x,y) 3- σ_x , σ_y and P(x,y)

* If $P_x(x) = \frac{1}{4}$ x = 2,4,8,16 = 0 other wise

Compute 1- E(x) 2- $E(\frac{1}{x})$ 3- $E(x^2)$ 4- $E(2^{x/2})$ 5- σX^2 and σY

* A fair coin is tossed three times. Let X denoted 1 or 0 according as a head or tail occurs on the first tossed and Let Y denoted the number of heads which occur

Determine: 1- The probability function of X and Y

2- Expected value of X and Y

3- The joint distribution h of X and Y

4- Correlation coefficient P. P(x,y)

* Find the con variance and the correlation coefficient of two random variable X and Y if :

E(x) = 2 , E(y) = 3 , $E(y^2) = 16$, E(xy) = 10 , $E(x^2) = 9$

* The correlation coefficient of two random variable X and Y is $-\frac{1}{4}$ while their variances are 3 and 5 Find the con variance

* Let X and Y be independent random variable with :

E(x) = 3 , $E(x^2) = 25$, E(y) = 10 and $E(y^2) = 164$

Find : 1- E(3x+y-8) 2- E(2x-3y+7) 3- Var X 4- σX 5- Var Y 6- σY

7- Var[3x+y-8] 8- Var[2x-3y+7] 9- E((y-10)/8) and Var[(y-10)/8)

10- E((x-3)/4) and Var[(x-3)/4]

* Suppose X be a random variable with probability function as following :

$P_x(x) = 0.1$ at X = 0 , $P_x(x) = 0.9$ at X = 1 , $P_x(x) = 0$ other wise

Find the first five moments of X

1- box contains 4 balls numbered 1,2,3,4 respectively. Let Y be the number that occurs if 1 ball is drawn at random from the box. What is the probability function for Y

2- Suppose that the probability function of a random variable X is

X	1	2	3	4	5	6	7	8
P(x)	0.02	0.03	0.05	0.2	0.4	0.2	0.07	?

Find: a- P(8)   b- Find the table for the distribution function F   c- Find P(x≤4) , P(x<4)

3- Suppose that the distribution function of a random variable X is

t	0	1	2	3	4	5
$F_x(t)$	0.13	0.27	0.53	0.84	0.92	1

Find: a- P(x≤3) , P(x=3) , P(x=2 or x=3)   b- The probability distribution of X

4- The probability function a random variable X is given by

$$P(x)=\begin{cases} 2P & X=1 \\ P & X=2 \\ 4P & X=3 \\ 0 & \text{other wise} \end{cases}$$

Where P is constant

Find: $P(0\le X<3)$, $P(X>1)$

5- Suppose that the distribution function of a random variable X is

$$F_x(t)=\begin{cases} 0 & t<3 \\ =\frac{1}{3} & 3\le t<4 \\ =\frac{1}{2} & 4\le t<5 \\ =\frac{2}{3} & 5\le t<6 \\ =1 & t\ge 6 \end{cases}$$

Find: a- The probability function for X , $P(3<X\le5)$

b- Graph of the above distribution function

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Statistics

Statistics is concerned with scientific methods for collecting, organizing, summarizing, presenting and analysis data, as well as drawing void conclusion on the basic of such analysis

Defn: Raw data are collected data which have not been organized numerically

Defn: An array is an arrangement raw numerical data is ascending or descending order, the difference between the largest and the smallest numbers is called the range of data

Example: Arrange the numbers {17,45,38,27,6,48,11,57,34,22} in an arrange

Solution: In ascending order 6,11,17,22,27,34,38,45,48,57

In descending order 57,48,45,38,34,27,22,17,11,6

The range is 57-6=51

Frequency distribution

A tabular arrangement of data by classes together with the corresponding class-frequency is called a frequency distribution

The following table shows a frequency distribution of masses of 100 students

Masses	No of students
50-52	5
53-55	18
56-58	24
59-61	27
62-64	8

The first class consists of masses from 50-52 and the corresponding class frequency is 5

The symbol 50-52 is called the class interval and the end numbers 50 and 52 are called class limits. The smaller number 50 is called lower class limit and the larger number 52 is called the upper class limit.

Data organized as in the frequency distribution are called grouped data

Class boundaries

The masses are recorded to the nearest kg then the class 50-52 includes all measurements from 49.5 to 52.5 kg , thus the numbers 49.5 and 52.5 are called class boundaries or true class limits and so 49.5 is the lower boundaries class limits 52.5 is the upper boundaries class limit.

The class interval (size of class interval) $C = 52.5-49.5 = 3$

Histogram and Frequency polygons

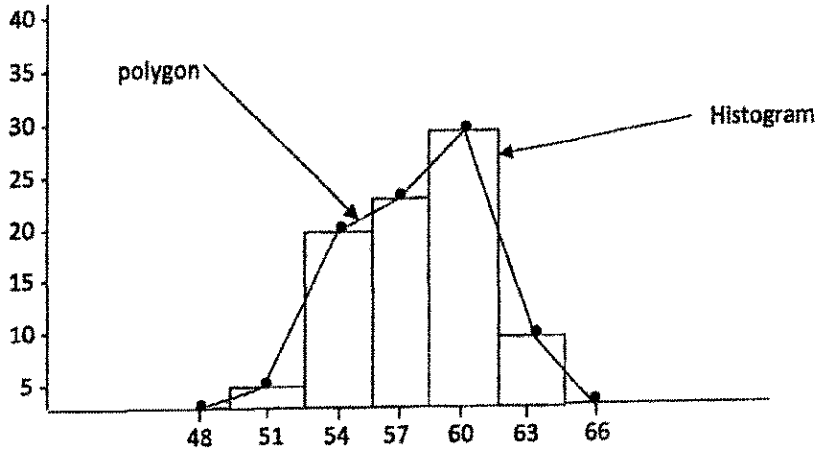
Histogram and frequency polygons are two graphical representation of frequency distribution

1- A histogram or frequency histogram consists of a set of rectangles having:

a- Basis on a horizontal axis centration class marks and the lengths equal to the class interval size

b- Area represent to the class frequency

2- A frequency polygon is a line graph of class frequency which can be obtained by connected midpoints of the tops of the rectangles in the histogram and frequency polygons of the previous example of the students



Measure of central tendency mean, median and mode

An average is a value which representation of a set of data , such values tend to a center of a set of n numbers $x_1, x_2, \dots x_n$ denoted by x , is defined by $\bar{x} = \frac{X1+X2+\cdots+Xn}{n} = \frac{\sum_{i=1}^{n} Xi}{n}$

Example: Find the mean of the numbers 8,3,5,10,12

Solution: $\bar{x} = \frac{8+3+5+10+12}{5} = \frac{38}{5} = 7.6$

Defn: If the numbers $x_1, x_2, \dots, x_k$ occur with frequency $F_1, F_2, \dots, F_k$, then the mean is

$$\frac{F1\,X1+F2\,X2+\cdots+Fk\,Xk}{F1+F2+\cdots+Fk} = \frac{\sum Fi\,Xi}{\sum Fi}$$

Example: If we put the numbers in a frequency distribution without classes

Xi	37	38	39	42	43	44	45	46	48
Fi	3	3	1	2	2	1	1	1	1
Fi Xi	111	114	39	84	86	44	45	46	48

$\bar{x} = \frac{\sum Fi\,Xi}{\sum Fi} = \frac{617}{15} = 41.15$

Now , If we put the above data in the frequency distribution with classes

Example:

Masses	No of students
50-52	5
53-55	18
56-58	42
59-61	27
62-64	8

Solution:

Mass (kg)	Class mark	Frequency	Fi Xi
50-52	51	5	255
53-55	54	18	972
56-58	57	42	2394
59-61	60	27	1620
62-64	63	8	504
		100	5745

$\bar{X} = \frac{\sum Fi\,Xi}{\sum Fi} = \frac{5745}{100} = 75.45$

The median

The median of numbers arranged in an arrange is middle value or the arithmetic of the two middle values

Example: The median of 3,4,5,6,8,8,10 is 6

The median of 5,5,7,9,11,12,15,18 is $\frac{9+11}{2} = 10$

For grouped data , the median is given by Median = $L_1 + (\frac{\frac{N}{2}-(\sum F)}{F\ median})\ C$

Where L_1: Lower class boundary of median class

N: Total number of frequencies

($\sum$F): Sum of frequencies of all classes lower than the median class

F median: Frequency of median class

C: Size of median class interval

Example: Find the median of following frequency distribution

Class	25-29	30-34	35-39	40-44	45-49
Frequency	7	19	14	7	13

Solution:

Class	F	$\sum$F
24-29	7	7
30-44	19	26
35-39	14	40
40-44	7	47
45-49	13	60

Median = $L + (\frac{\frac{N}{2}-(\sum F)}{F\ median})\ C$

first we find the median class $\frac{N}{2} = \frac{60}{2} = 30$

The median class 35-39 , L=34.5 , $\sum$F=26 , F median = 14 , C = 39.5-34.5 = 5

Then median = $34.5 + (\frac{30-26}{14})\ 5 = 34.5 + (\frac{4}{14})5 = 34.5 + \frac{20}{14} \approx 35.93$

The mode

The mode of a set of numbers is that value which occurs with the greatest frequency

The mode may not exist, and if it does exist it may not be unique

Example: 1) The set 2,2,5,7,9,9,9,10 has mode 9

2) The set 3,5,7,0,12,15 has no mode

3) The set 1,3,4,4,4,5,7,7,7 has two modes 4 and 7

For a frequency distribution the mode can be obtained from the formula:

Mode = $L + (\frac{\Delta 1}{\Delta 1 + \Delta 2})\ C$

Where L: Lower class boundary and model class

Δ_1: Frequency of model class boundary of model class – frequency of next lower class

Δ_2: Frequency of model class – frequency of the next higher class

C: Size of model class interval

Example: Find the mode of the following frequency distribution

Class	Frequency
50-59.99	8
60-69.99	10
70-79.99	16
80-89.99	14
90-99.99	10
100-109.99	5
110-119.99	2

Solution: The mode is given by mode $= L_1 + (\frac{\Delta 1}{\Delta 1+\Delta 2})\, C$

First the mode class 70-79.99 , $L_1=69.995$, $\Delta_1=16-10=6$, $\Delta_2=16-14=2$

$C=79.995-69.995=10$

Mode $= 69.995 + (\frac{6}{6+2})10 = +69.995 + \frac{60}{8} = 69.995 + 7.5 = 77.5$

Measure of dispersions

The degree to which numerical data tend to spread about an average value is called dispersions of data

1) Range The rang of a set of numbers is the difference between the largest and the smallest numbers in the set

2) The mean deviation M.D $= \frac{\sum |xi-\bar{x}|}{N}$ where x is the mean

Example: Find the mean deviation of the 6,7,10,8,5,4,9,7

Solution: $\bar{x} = \frac{\sum xi}{N} = \frac{56}{8} = 7$

The mean deviation M.D $= \frac{\sum |xi-\bar{x}|}{N} =$

$$\frac{|6-7|+|7-7|+|10-7|+|8-7|+|5-7|+|4-7|+|9-7|+|7-7|}{8} = \frac{1+3+1+2+3+2}{8} = 1.5$$

Note: For the frequency M.D$= \frac{\sum Fi|xi-\bar{x}|}{\sum Fi}$

Example:

Xi	3	7	11	15	19
Fi	2	5	9	7	2

Solution:

Xi	Fi	Fi Xi	(Xi-X̄)	Fi(Xi-X̄)
3	2	6	8.32	16.64
7	5	35	4.32	21.60
11	9	99	0.32	2.88
15	7	105	3.68	25.76
19	2	38	7.68	15.36

The arithmetic mean $\bar{x} = \frac{\sum Fi\,Xi}{\sum Fi} = \frac{283}{25} = 11.32$ M.D $= \frac{82.42}{25} = 3.24$

3) The variance

The variance of a set of data $x_1, x_2, \ldots, x_n$ denoted by S^2 is define by $S^2 = \frac{\sum (Xi-\bar{X})^{\wedge}2}{N}$

In frequency $S^2 = \frac{\sum Fi(Xi-\bar{X})^{\wedge}2}{\sum Fi}$

Defn: The standard deviation is the position square root of variance

Example: Find the variance and standard deviation 5,7,10,12,6

Solution:

Xi	Xi-$\bar{X}$	(Xi-$\bar{X}$)^2
5	-3	9
7	-1	1
10	2	4
12	4	16
6	-2	4

Variance $= \frac{\sum (Xi-\bar{X})^{\wedge}2}{N} = \frac{34}{5} = 6.4$

The stand deviation $S = \sqrt{6.8} = 2.61$

Example: Find the variance and standard deviation

Mass(kg)	No of students
60-62	5
63-65	18
66-68	42
69-71	27
72--74	6

Solution:

Class	Fi	Xi	Fi Xi	Xi-$\bar{X}$	(Xi-$\bar{X}$)^2	Fi(Xi-$\bar{X}$)^2
60-62	5	61	305	7	49	245
63-65	18	64	1152	3	9	162
66-68	42	67	2814	0	0	0
69-71	27	70	1890	3	9	243
72-74	6	73	438	6	36	216
						866

$\bar{x} = \frac{\sum FiXi}{\sum Fi} = \frac{6572}{98} = 67$

Therefore $S^2 = \frac{\sum Fi(Xi-\bar{X})^{\wedge}2}{\sum Fi} = \frac{866}{98} = 8.84$

Then $S = \sqrt{S^2} = \sqrt{8.84} = 2.97$

The Binomial distribution

The binomial is an experiment which has two outcomes called success and failure

We consider expected and independent trials of an experiment

Let P be the probability of success so that $q = 1 - P$ is the probability of failure

Consider a series of experiments which have the following properties:

1- The result of each experiment can be classified in to one of two outcomes say success and failure

2- The probability P of a success is the same of each experiment

3- Each experiment is independent of all others

4- The series consists of a fixed number of experiments

Defn: Let X be the total number of success in n repeated independent experiment with probability P of success on a given trail

X called the binomial random variable with parameters P and n. and the probability distribution for X is called the binomial distribution

This probability function is defined as:

Theorem If X is binomial random variable with parameters n and P then the binomial distribution $P(x=n) = P_X(x) = \binom{n}{x} P^X q^{n-X}$ thus $b(x;n,P) = \binom{n}{x} P^X q^{n-X}$

Example If we tossed a coin 6 times, what is the probability of:

1- Exactly two heads 2- No heads 3- At least 4 heads 4- At least one head

Solution For each time the sample space {h,t}

Then , the probability of h $P = \frac{1}{2}$ the probability T $q = 1 - \frac{1}{2} = \frac{1}{2}$

Then X is the number of head

1- $b(2;6,\frac{1}{2}) = \binom{6}{2}(\frac{1}{2})^2 (\frac{1}{2})^4 = \frac{6x5}{2x1} (\frac{1}{2})^6 = \frac{15}{64}$

2- $P(X=0) = (0,6,\frac{1}{2}) = \binom{6}{0} (\frac{1}{2})^0 (\frac{1}{2})^6 = \frac{1}{64}$

3- $P(X\geq4) = P(X=0) + P(X=5) + P(X=6) = \binom{6}{4}(\frac{1}{2})^4 (\frac{1}{2})^2 + \binom{6}{5}(\frac{1}{2})^5 (\frac{1}{2}) + \binom{6}{6}(\frac{1}{2})^6 (\frac{1}{2})^0 = \frac{45}{8}$

4- $P(X\geq1) = 1 - P(X=0) = 1 - \frac{1}{64} = \frac{63}{64}$

Example If a die is rolled four times

1- What is the probability that exactly two 6 ' S

2- What is the probability of obtaining two or fewer 6 ' S

Solution For each roll the sample space $S = \{1,2,3,4,5,6\}$

The probability of getting $6 = \frac{1}{6}$ The probability of no 6 is $q = 1 - \frac{1}{6} = \frac{5}{6}$

Then X is the number of 6 that occur

1- $b(2;4,\frac{1}{6}) = \binom{4}{2}(\frac{1}{6})^2(\frac{5}{6})^2 = \frac{25}{12}$

2- $P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=b(0;4,\frac{1}{6})+b(1;4,\frac{1}{6})+b(2;4,\frac{1}{6})=\frac{625}{1296}+\frac{560}{1296}+\frac{150}{1296}=\frac{1275}{1296}$

Theorem Let X be a random variable with the binomial distribution $b(x;n,P)$, Then

1- $E(x) = nP$ 2- $\sigma X^2 = nP\, q$, Where the standard deviation $\sigma X = \sqrt{nP\, q}$

Proof 1- since $b(x;n,P) = \binom{n}{x}P^x q^{n-x}$

by the definition $E(x)= \sum_{x=0}^{n} X\, Px(X)= \sum_{x=0}^{n} X\, b(x;n.P) = \sum_{x=0}^{n} X\binom{n}{x}P^X q^{n-x}$

$=\sum_{x=1}^{n} X \frac{n!}{X!(n-x)!} P^x q^{n-x} = nP\sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} P^{x-1} q^{n-x}$

Put $y=x-1$ where $x=1 \rightarrow y=0$, $x=n \rightarrow y=n-1$

$= nP\sum_{y=0}^{n-1} \frac{n!}{y!(n-1-y)!} P^y q^{n-1-y} = nP\sum_{x=0}^{n} \binom{n-1}{Y} P^y q^{(n-1)-y} = nP(P+q)^{n-1} = nP(1)^{n-1} = nP$

2- Home work

Example Determine the expected number of days in a family with 8 children, assuming the sex distribution to be equally probable

What is the probability that the expected number of bays goes occur?

Solution we have $n=8$, $P=\frac{1}{2}$, $q=\frac{1}{2}$

$E(x)=np = 8x\frac{1}{2} =4$

The probability of the expected number of bays $b(4;8,\frac{1}{2})= \binom{8}{4}(\frac{1}{2})^4(\frac{1}{2})^4 = 0.27$

Poisson distribution

A model often used for counting probabilities associated with number of success occurring within a time interval of given length or within a region of space of a given size is the Poisson probability model

The Poisson distribution is defined as follows $P(x;\lambda)=P(x=u)=\frac{e^{-\lambda}\lambda^{X}}{X!}$ $X=0,1,2,\ldots$

Where $\lambda>0$ and some constant $e=2.7182$

Which is the probability of exactly X success occurring in a time interval of a given length (region of space of a given size) and λ is the overage number of success occurring in a time interval of a given length (or region of space of a given size)

Example If a person receives five calls on the overage during a day, what is the probability that he will receive fewer that five calls tomorrow? Exactly Five calls?

Solution The overage number of calls per a day $\lambda=5$

Letting X be the number of calls person will receive tomorrow

$P(X\leq 4) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= P(0;5)+P(1;5)+P(2;5)+P(3;5)+P(4;5)= \sum_{x=0}^{4} P(X;5) = \sum_{x=0}^{4} \frac{e^{-5}\, 5^X}{X!}$

$= e^{-5} [\, 1+5+\frac{25}{2}+\frac{125}{6}+\frac{625}{24}\,] = 0.44049$

$P(X=5) = P(5;5) = \frac{e^{-5}\, 5^5}{5!} = 0.17547$

Theorem Let X be a random variable with Poisson distribution $P(X;\lambda)$ Then

1- $E(x) = \lambda$ 2- $\sigma_X^2 = \lambda$

Proof $E(x) = (\sum_{x=0}^{\infty} X\, P(X;\lambda)) = (\sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^X}{X!}) = (\sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^X}{(X-1)!}) = \lambda\, (\sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{X-1}}{(X-1)!})$

Let $y=x-1$ as $x=1$ $y=0$, $x=\infty$ $y=\infty$

$= \lambda\, (\sum_{y=1}^{\infty} \frac{e^{-\lambda} \lambda^X}{(y)!}) = \lambda\, (\sum_{x=0}^{\infty} P(y;\lambda)) = \lambda \,.\, 1 = \lambda$

2- Home work

Problem Show that the Poisson distribution $P(X;\lambda)$ is a probability distribution

That $\sum_{x=0}^{\infty} P(X;\lambda) = 1$

Proof $\sum_{x=0}^{\infty} P(X;\lambda) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^X}{X!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^X}{X!} = e^{-\lambda} e^{\lambda} = e^{-\lambda+\lambda} = e^0 = 1$