

Recitation 4/14

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Volumes, Matrix Calculus

A region in \mathbb{R}^n is transformed into another region in \mathbb{R}^n under the left multiplication of $A \in \mathbb{R}^{n \times n}$. (i.e., all the vectors inside the region is multiplied on the left by A)

$\det(A)$ is the scaling factor of volumes between two regions.

2 by 2 SVD explanation

Matrix Calculus - Remember $d(AB) = (dA)B + A(dB)$

Eigenvalues

For a square matrix $A \in \mathbb{R}^{n \times n}$, we call λ an **eigenvalue** of A if

$$\text{There exists a vector } x \in \mathbb{R}^n \text{ such that } Ax = \lambda x$$

Moreover, we call x an **eigenvector** of A .

In other words, if a vector is multiplied by A and it remains the scalar multiple of itself, we call such vector an eigenvector, and the scalar factor becomes eigenvalue.

Problems

1. (a) Assume A has QR decomposition $A = QR$. Express dA in terms of Q, R, dQ, dR .

(b) Assume A has SVD $A = U\Sigma V$. Express dA in terms of SVD matrices.

(c) Let A be an orthogonal matrix. Prove that $A^T dA$ is skew-symmetric. Note that $(dA)^T = d(A^T)$.

(d) Let $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and fix A (so $dA = 0$). What is the relationship between dx and dy ?

(e) From (d), Assume x_1, x_2, x_3 are unrelated ($\frac{dx_i}{dx_j} = 0$ for $i \neq j$). Find a derivative $\frac{dy_i}{dx_j}$ in terms of A_{ij} by using simple division.

2. A tilted square with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$ is transformed by left multiplication of $A = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$. Given that the image is still a quadrilateral (and the vertices are the images of original four vertices), Compute the volume (area) of the image. Compare it with the determinant of A .

3. (a) What is the volume of square with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 0), (1, 1, 0)$ in \mathbb{R}^3 ?

(b) Let $A \in \mathbb{R}^{n \times n}$ be a singular matrix, so that the column space has dimension less than n . Explain why determinant is zero in terms of volumes.

4. Compute eigenvalues and corresponding eigenvectors of these matrices. (Multiply $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and let it $\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$, then solve it for λ)

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 0 \\ -3 & -1 \end{pmatrix}$.

True or False. Explain or give counterexample.

- (a) If x is an eigenvector of A then $2x$ is also an eigenvector of A .
- (b) If λ is an eigenvalue of A then $-\lambda$ is an eigenvalue of $-A$.
- (c) If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2 .
- (d) Assume A is not a symmetric matrix. Then A and A^T cannot have same eigenvalues.
- (e) A doubly stochastic matrix always have eigenvalue one.
- (f) Eigenvalue of real matrix is always a real number.
- (g) At least one eigenvalue of complex matrix is a complex number.
- (h) Let x, y be two vectors which are not colinear. They can be both eigenvectors of a same eigenvalue λ of A .