MIT 18.06 Exam 3, Spring 2022
Johnson

Your name: ____________________________________

Recitation: ____________________________________

problem	**score**
1	/30
2	/21
3	/24
4	/25
total	/100

Problem 0 (∞ points): Honor code

Copy the following statement with your signature into your solutions:

> *I have completed this exam **closed-book/closed-notes** entirely on my **own.***
>
> [your signature]

Problem 1 (30 points):

The matrix

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

has an eigenvalue $\lambda_1 = 1$ and corresponding eigenvector $x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(a) What is the other eigenvalue λ_2 and a corresponding eigenvector $x_2 = \begin{pmatrix} 1 \\ ?? \end{pmatrix}$?

(b) B is a 2×2 matrix such that $Bx_k = (1-\lambda_k+\lambda_k^2)x_k$ for the two eigenvectors ($k = 1, 2$). What is B?

(c) What is $A^{3/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

(blank page for your work if you need it)

Problem 2 (21 points):

A is a square matrix such that $N(A-I)$ is spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $N(A-5I)$ is spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(a) Without much calculation, you can tell that A **is** / **is not** (choose 1) Hermitian because __________________.

(b) What is A? You can leave your answer as a **product of matrices and/or matrix inverses** without multiplying/inverting them.

(c) What is e^{A+I}? You can leave your answer as a **product of matrices and/or matrix inverses** without multiplying/inverting them, but your answer should not have exponentials of matrices or infinite series.

(blank page for your work if you need it)

Problem 3 (24 points):

For each of the following, say whether it **must** be true, it **may** be true, or it **cannot** be true. No justification needed.

(a) If a matrix is diagonalizable, it **must/may/cannot** have orthogonal eigenvectors.

(b) M is a Markov matrix. If $M^n x$ converges to a steady state as $n \to \infty$ for *any* vector x, then M **must/may/cannot** be a positive Markov matrix (i.e. have all entries > 0).

(c) If a matrix A is *not* diagonalizable, then $\det(A - \lambda I)$ **must/may/cannot** have repeated roots.

(d) If $A^n x$ goes to zero as $n \to \infty$ for *some* x, then A**must/may/cannot** have an eigenvalue λ with $|\lambda| > 1$

(e) If $e^{At}x$ goes to zero as $t \to \infty$ for *every* x, then A**must/may/cannot** have an eigenvalue λ with $|\lambda| > 1$

(f) If A has an eigenvector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then it **must/may/cannot** have an eigenvector $\begin{pmatrix} -3 \\ -6 \\ -9 \end{pmatrix}$.

Problem 4 (25 points):

Suppose A is a real-symmetric matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 0$, and $\lambda_4 = 7$, with corresponding eigenvectors:

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \ x_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \ x_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Now, we construct a sequence of vectors $y_0, y_1, y_2, \ldots$ where each vector y_{k+1} in the sequence is computed from the previous vector y_k by solving

$$(A - 2I)y_{k+1} = y_k$$

for y_{k+1}. If $y_0 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, **give a good approximation** for y_{100}.

(blank page for your work if you need it)