Recitation 4/21

Sungwoo Jeong Tuesday 10AM, 11AM

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Eigenvalues, Continued

- Eigenvalues are values λ such that shifted matrix $A - \lambda I$ have nonempty nullspace.

$$\exists x \neq 0 \text{ such that } (A - \lambda I)x = 0$$

- Determinant of $A - \lambda I$ is a degree n polynomial of λ. n solutions of the polynomial are exactly the eigenvalues. (Why?)
- We have exactly n eigenvalues, counting multiplicities (i.e. for 3 by 3 matrix we can have three eigenvalues, 1, 1, 2)

Diagonalization

- Let $A \in \mathbb{R}^{n \times n}$ have eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding eigenvectors $x_1, \dots, x_n$. n equations $Ax_i = \lambda x_i$ can be simultaneously represented as,

$$AX = X\Lambda$$

where X is a matrix with i^{th} column x_i, Λ is a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$.

- **If X is invertible**(X has linearly independent eigenvectors), We can now express A as

$$A = X\Lambda X^{-1}$$

and this is called a **Diagonalization or Eigendecomposition** of A.

- Why Diagonalization is so powerful and important?

Problems

1. Compute eigenvalues of $A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 3 & 1 \\ 2 & 4 & -1 \end{pmatrix}$ using the polynomial $\det(A - \lambda I)$.

2. (a) Let $f_0, f_1, \dots$ be Fibonacci sequence with $f_0, f_1 = 0, 1$. Find 2 by 2 matrix A such that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = A \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$. Then, express $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$ in terms of A.

(b) Find eigenvectors, eigenvalues, and eigendecomposition of A.

(c) (Challenging) Express the eigendecomposition of A^{100}. With a small assumption(Regard a very small number as 0), prove that the ratio f_{101}/f_{100} is same as the largest eigenvalue.

(d) Find formula for f_n.

3. Think about another sequence $g_0, g_1, \ldots$ with relationship $g_{i+1} = 2g_i + g_{i-1}$ and $g_0, g_1 = 0, 1$. Find formula for g_n.

4. True or false. Prove or give counterexample.

(a) Diagonalizable matrices are invertible.

(b) Invertible matrices are diagonalizable.

(c) Non-diagonalizable matrices can be invertible.

(d) Non-invertible matirice can be diagonalizable.

(e) If A is diagonalizable then A^5 is diagonalizable.

(f) Squared singular values of A are eigenvalues of A^2

(g) Squared singular values of A are eigenvalues of $A^T A$

(h) Squared singular values of A are eigenvalues of AA^T