MIT 18.06 Practice Exam 2 Solutions, Spring 2023

Gilbert Strang and Andrew Horning

Your name: ______________________________
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Student ID: ______________________________

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Problem 1:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Consider the following 3×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$$

1(a) Use elementary row operations to reduce A to the echelon form R.

$$R = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Subtracting the first row from the second reduces A to the form

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}.$$

Subtracting the second row from the third completes the transformation: the first two columns are pivot columns and the remaining columns correspond to free variables.

1(b) Use the echelon form R to write down a basis for the column space and row space of A.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \qquad \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

Solution: The first two columns of R are the pivot columns, so the first two columns of A are a basis for the column space. The nonzero rows of R are a basis for the row space.

1(c) Use the echelon form R to write down a basis for the nullspace of A.

$$R = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Solution: The last three columns of R correspond to the free variables x_3, x_4, x_5. We can rewrite the equations from the first two rows of the nullspace equation $Rx = 0$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -2 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ -3 \end{pmatrix}.$$

Setting $x_3 = 1, x_4 = 0, x_5 = 0$ allows us to solve for the pivot variables: $x_1 = -1$ and $x_2 = -1$. Our first basis vector for the nullspace of R is $\begin{pmatrix} -1 & -1 & 1 & 0 & 0 \end{pmatrix}^T$. We proceed analogously for the remaining free variables. Set $x_3 = 0, x_4 = 1, x_5 = 0$ and solve for the pivot variables: $x_1 = -1$ and $x_2 = -2$. Our second basis vector is $\begin{pmatrix} -1 & -2 & 0 & 1 & 0 \end{pmatrix}^T$. Finally, set $x_3 = 0, x_4 = 0, x_5 = 1$ and solve for the pivot variables: $x_1 = -2$ and $x_2 = -3$. Our third basis vector is $\begin{pmatrix} -2 & -3 & 0 & 0 & 1 \end{pmatrix}^T$. Recall that R and A have the same row space and the same nullspace, so this basis for the nullspace of R is also a basis for the nullspace of A.

1(d) Write down the general solution to $Ax = b$, when $b = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}^T$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solution: The vector b is the sum of the first two (pivot) columns of A, so a particular solution is $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \end{pmatrix}^T$. Any combination of vectors in the nullspace of A is also a solution to $Ax = b$, so the general solution is a combination of the particular solution and combinations of the basis vectors for the nullspace of A.

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Problem 2:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Follow the steps in 2(a)-(c) to find the parabola $b = C + Dt + Et^2$ that is closest to the four points $(t_1, t_2, t_3, t_4)^T = (-1, 0, 1, 2)^T$ and $(b_1, b_2, b_3, b_4)^T = (0, -1, 0, 3)^T$.

2(a) Write down the 4×3 coefficient matrix A and right-hand side b associated with the 4 equations $b_k = C + Dt_k + Et_k^2$ (for $k = 1, 2, 3, 4$) for the 3 unknowns, C, D, and E.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 3 \end{pmatrix}$$

Solution: Each of the four equations gives us a row of the matrix A. The first equation is $b_1 = C + Dt_1 + Et_1^2$. The coefficients multiplying the unknowns are $\begin{pmatrix} 1 & t_1 & t_1^2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$, the first row of A. The first entry of the right hand side is $b_1 = 0$. The same process fills in the second row of the matrix A and the second entry of the right-hand side b, the third, and the fourth.

2(b) Compute the 3×3 matrix $M = A^T A$ and the 3×1 vector $c = A^T b$. Is the matrix M invertible? Why?

$$M = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{pmatrix}, \qquad c = \begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix}$$

Solution: The matrix $M = A^T A$ has the same nullspace as A and the nullspace of A is trivial because its columns are linearly independent. Consequently, the nullspace of $A^T A$ is trivial and its columns are linearly independent. A square matrix with linearly independent columns is invertible, so $M = A^T A$ is invertible.

2(c) Use elimination to solve the normal equations $Mx = c$ for the coefficients of the best fit parabola, $x = \begin{pmatrix} C & D & E \end{pmatrix}^T$.

$$x = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Solution: We can solve the normal equations by doing elimination on the augmented linear system

$$\begin{pmatrix} M & c \end{pmatrix} = \begin{pmatrix} 4 & 2 & 6 & 2 \\ 2 & 6 & 8 & 6 \\ 6 & 8 & 18 & 12 \end{pmatrix}.$$

We subtract the first row from twice the second row and three times the first row from twice the third row to get

$$\begin{pmatrix} 4 & 2 & 6 & 2 \\ 0 & 10 & 10 & 10 \\ 0 & 10 & 18 & 18 \end{pmatrix}.$$

We then subtract the second row from the third row to get

$$\begin{pmatrix} 4 & 2 & 6 & 2 \\ 0 & 10 & 10 & 10 \\ 0 & 0 & 8 & 8 \end{pmatrix}.$$

Backward substitution yields $x_3 = 1$, $x_2 = 0$, and $x_1 = -1$, so that $x = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$. The best fit parabala is $b = -1 + t^2$. Note that in this example, b happens to lie exactly in the column space of A! If we compute Ax, we get *exactly* b. This is not the typical situation when A is rectangular and has full column rank, but it is a reassuring demonstration that the least-squares routine returns an exact solution when it exists: the smallest possible value of $Ax - b$ is zero.

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Problem 3:

Record your final answer in the alloted spaces. You may use the remaining space for your calculations.

Given two column vectors $x = (1, 1, 0)^T$ and $y = (0, 1, 1)^T$, consider the following two 3×3 *oblique projection* matrices (I is the 3×3 identity matrix):

$$N = \frac{xy^T}{y^T x}, \qquad M = I - \frac{xy^T}{y^T x}.$$

3(a) What are the dimensions of the four fundamental subspaces of N? Write down one nonzero vector in each subspace.

Solution: The column space and row space of the rank 1 matrix N are one dimensional. Every column is a multiple of x and every row is a multiple of y^T. Therefore, the nullspace of N is the space of vectors orthogonal to y and the nullspace of N^T is the space of vectors orthogonal to x. Both of these spaces have dimension $3 - 1 = 2$ by the Counting Theorem. The vector $v = (0, -1, 1)^T$ has $v^T y = 0$ and is in the nullspace of N, i.e., it is orthogonal the row space of N. The vector $w = (1, -1, 0)^T$ has $w^T x = 0$ and is in the nullspace of N^T, i.e., it is orthogonal to the row space of N^T.

3(b) What are the dimensions of the four fundamental subspaces of M? Write down one nonzero vector in each subspace.

Solution: Vectors in the nullspace of M satisfy $Mv = v - \frac{xy^T}{y^T x} v = 0$ or, equivalently, $v = \frac{y^T v}{y^T x} x$. They are all multiples of x! Similarly, vectors in the nullspace of M^T are all multiples of y. So the nullspaces of M and M^T have dimension 1. By the Counting Theorem, the column and row spaces of M have dimension $3 - 1 = 2$. The column space is orthogonal to the nullspace of M^T so any vector orthogonal to y is in the column space, e.g., $(0, -1, 1)^T$. Similarly, the row space is orthogonal to the nullspace of M, so any vector orthogonal to x is in the row space, e.g., $(1, -1, 0)^T$.

3(c) Use the four fundamental subspaces to explain why NM and MN are the zero matrix.

Solution: From part (a) and (b), the rows of N are orthogonal to the columns of M and vice versa. Therefore, all the dot products in the matrix-matrix products NM and MN are zero. Try it!

3(d) Are either of N or M an orthogonal projection matrix? Why or why not? (Recall that an orthogonal projection matrix P satisfies $P^2 = P$ and $P^T = P$.)

Solution: No, neither is an orthogonal projection matrix. They satisfy $M^2 = M$ and $N^2 = N$, but neither is symmetric so $N^T \neq N$ and $M^T \neq M$. Such matrices are called oblique projection matrices. They project onto their column space (so that $M^2 = M$), but they do not project *orthogonally* onto the column space. Instead, they project at an *oblique* angle.

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