Practice Problems

1. Remember that a matrix Q is *unitary* if $Q^H Q = I$. A matrix is *orthogonal* if it is real and unitary; that is, if it is real and $Q^T Q = I$.

a) Find the flaw in this argument:

False Claim: all eigenvalues of an orthogonal matrix are ± 1. Indeed, if $Qx = \lambda x$,

$$\lambda^2 x^T x = (Qx)^T (Qx) = x^T (Q^T Q) x = x^T x,$$

therefore $\lambda^2 = 1$, so $\lambda = \pm 1$.

If you're stuck, think about what happens for a rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

b) Correct the proof to show

True Claim: all eigenvalues of a unitary matrix have magnitude 1 (e.g. $\lambda = e^{i\phi}$ for some ϕ).

c) Show that the eigenvectors for different eigenvalues of a unitary matrix are orthogonal.

d) Show that the determinant of any real unitary matrix (e.g., an orthogonal matrix) is ± 1 using eigenvalues. (Note: you already proved this on a previous pset in a different way.)

2. Here is a quick "proof" that the eigenvalues of **every** real matrix A are real:

$$\textbf{False Proof: } Ax = \lambda x \text{ gives } x^T A x = \lambda x^T x, \quad \text{so } \lambda = \frac{x^T A x}{x^T x} = \frac{\text{real}}{\text{real}}.$$

Find the flaw in this reasoning – a hidden assumption that is not justified. You can test those steps on the 90° rotation matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda = i, \; x = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

3. a) If S is a positive definite matrix, show that S^{-1} is also positive definite.

b) If S and T are positive definite, show that their sum $S + T$ is also positive definite. If $S = A^H A$ and $T = B^H B$ for full-column-rank matrices A and B, then can you write down a full column-rank matrix C so that $S + T = C^T C$?

4. Say A is a 3×3 real matrix. The matrix $B = A + A^T$ has eigenvalues $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 1$, with corresponding eigenvectors $x_1 = [1\ 2\ 1]$, $x_2 = [-2\ 1\ 0]$ and $x_3 = [1\ 2\ -5]$.

a) What is e^B? (It's fine to leave your answer as a product of several matrices, as long as each matrix is written down explicitly)

b) Let $C = (I - B)(I + B)^{-1}$. What are the eigenvalues and eigenvectors of C?

c) Give a good approximation for

$$y = C^{100} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

in terms of a single eigenvector.