Topic: four subspaces associated to a matrix.

(1) Let A be an $n \times n$ invertible matrix. Then $\mathrm{col}(A) = \mathbb{R}^n$ and $\mathrm{null}(A) = \{0\}$.

(2) Let A and B be matrices such that A is invertible and AB makes sense. Then

$$\begin{aligned} \mathrm{null}(AB) &= \mathrm{null}(B) \\ \mathrm{col}(AB) &= \{Ax \text{ for } x \in \mathrm{col}(B)\} \\ \mathrm{rank}(AB) &= \mathrm{rank}(B). \end{aligned}$$

(3) Let A and B be matrices such that A is invertible and BA makes sense. Then

$$\begin{aligned} \mathrm{col}(BA) &= \mathrm{col}(B) \\ \mathrm{null}(BA) &= \{A^{-1}x \text{ for } x \in \mathrm{null}(B)\} \\ \mathrm{rank}(BA) &= \mathrm{rank}(B). \end{aligned}$$

(4) Let A be an $n \times n$ invertible matrix. Let $r \leq n$, and let B be the $n \times r$ matrix built from the first r columns of A. Then

$$B = A \left(\begin{array}{c} \mathrm{Id}_{r\times r} \\ \hline 0_{(n-r)\times r} \end{array} \right).$$

Use (2) to deduce $\mathrm{null}(B) = 0$ and $\mathrm{rank}(B) = r$.

(5) Let A be an $n \times n$ invertible matrix. Let $r \leq n$, and let B be the $r \times n$ matrix built from the first r rows of A. Then

$$B = \left(\begin{array}{c|c} \mathrm{Id}_{r\times r} & 0_{(n-r)\times r} \end{array} \right) A.$$

Use (3) to deduce $\mathrm{col}(B) = \mathbb{R}^n$ and $\mathrm{null}(B)$ is the span of the last $(n-r)$ columns of A^{-1}.

(6) Let A be an $n \times m$ matrix, and fix an integer $r \leq n, m$. Assume that $A = EFG$ where E is an $n \times r$ matrix arising from the construction in (4), F is an invertible $r \times r$ matrix, and G is an $r \times m$ matrix arising from the construction in (5). Then

$$\begin{aligned} \mathrm{col}(A) &= \mathrm{col}(E) \\ \mathrm{rank}(A) &= r \\ \mathrm{null}(A) &= \mathrm{null}(G). \end{aligned}$$

(7) Suppose P is a square matrix such that $P^2 = P$. Then $b \in \mathrm{col}(P)$ if and only if $Pb = b$.

(8) Let Q be an orthogonal matrix. Then $b \in \mathrm{col}(Q)$ if and only if $QQ^\top b = b$.

(9) Let $A = QR$ be a QR decomposition (where R is invertible). Then $b \in \mathrm{col}(A)$ if and only if $QQ^\top b = b$.

(10) [1] Let $A = U\Sigma V^\top$ be a rank-r SVD. Then $b \in \mathrm{col}(A)$ if and only if $UU^\top b = b$.

(11) Determine the column space, null space, and rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(12) Determine the column space, null space, and rank of the matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 6 & 8 \\ 0 & 0 & 7 \end{pmatrix}.$$

(13) Consider the following full SVD of a matrix:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Write down the rank-r SVD for this matrix, and determine its column space, null space, and rank.

(14) Any $n \times m$ matrix can be expressed as the sum of (at most) $\min(m, n)$ rank-1 matrices.

[1] Thanks to Sungwoo for this problem idea.