Practice Problems

1. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

One eigenvalue of A is -1. The other eigenvalue is a double root of $\det(A - \lambda I)$. What is the other eigenvalue? (Hint: use the trace, don't do much calculation.)

Solution. We will use the fact that the trace of A is the sum of its eigenvalues. Say the mystery eigenvalue is λ. Then we have

$$\begin{aligned} 3 &= -1 + 2\lambda \\ 4 &= 2\lambda \\ \lambda &= 2. \end{aligned}$$

2. Suppose M is a positive Markov matrix (so one eigenvalue equals 1, all other eigenvalues have $|\lambda| < 1$). Why is M^∞ a rank-1 matrix?

Solution. We know that the eigenvalues of M^n are the nth powers of eigenvalues of M. So for n large, M^n has one eigenvalue equal to 1 and the remaining eigenvalues are extremely small numbers. This means that M^∞ has one eigenvalue equal to 1 and all other eigenvalues equal to zero. The number of nonzero eigenvalues is the rank of the matrix (the number of zero eigenvalues is the dimension of the nullspace).

3. $x^T Ay = \operatorname{tr}(AB)$ where B is what matrix? (Hint: use the cyclic property of the trace and recall that the trace of a 1x1 matrix a is a).

Solution. We would like to get a trace on the left hand side. The left hand side is a scalar, so is equal to its own trace. That is. $x^T Ay = \operatorname{tr}(x^T Ay)$. We also know that $\operatorname{tr}(BC) = \operatorname{tr}(CB)$ for any B, C. So $\operatorname{tr}(x^T Ay) = \operatorname{tr}(Ayx^T)$, and $B = yx^T$.

4. Suppose A is an $m \times n$ full column-rank matrix with thin SVD $U\Sigma V^T$ (so that V is square/unitary and U is $m \times n$). By inspection of $A^T A$ in comparison with the diagonalization formula, the eigenvectors of $A^T A$ are ___ and its eigenvalues are ___ .

Solution. First, we compute $A^T A$:

$$A^T A = V\Sigma U^T (U\Sigma V^T) = V\Sigma^2 V^T$$

since U has orthonormal columns so $U^T U = I$. The diagonalization formula is $B = X\Lambda X^{-1}$, where the columns of X are the eigenvectors of B and Λ is diagonal and the diagonal entries are eigenvalues of B. Comparing, we see that the columns of V are the eigenvectors of $A^T A$ and the eigenvalues are σ^2, the squares of the singular values.

5. Suppose A is $m \times m$, full rank, and we compute its QR factorization $A = QR$, e.g. by Gram–Schmidt. Claim: the matrix $B = RQ$ has the same eigenvalues as A. Why?

Solution. We start by relating B to A. Since $A = QR$, the matrix

$$RQ = Q^{-1}(QR)Q = Q^{-1}AQ.$$

This tells us that A and RQ are similar matrices, so they have the same eigenvalues.