Recitation 4/7

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Projection Matrix

Let $A \in \mathbb{R}^{m\times n}$, matrix with linearly independent columns. Then, $P = A(A^TA)^{-1}A^T$ is a projection matrix onto the column space of A.

In other words, for $b \in \text{col}(A)$, $Pb = b$, and for $b \notin \text{col}(A)$, Pb is a vector in $\text{col}(A)$ with the property of $||Pb - b||$ being minimum.

Easy to obtain P with QR decomposition - $P = QQ^T$ when $A = QR$.

Determinant

Determinants are scalar real number, only defined on **square matrices**

1. Cofactor expansion formula - Most Linear Algebra Textbooks
2. Product of Pivots(Strang)
3. Product of Singular values is equal to absolute value of determinant

An important properties of determinant
- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$

Problems

1.Consider $\mathbb{R}^3$ space.
(a) We have xy-plane and a vector $v = (3, 2, 1)$. Draw a picture and figure out the projected vector of v onto xy-plane without computation.

(b) xy-plane is a span of two vectors. What are those vectors?

(c) Let A be a matrix with two columns obtained in problem (b). What is the QR decomposition of A?

(d) Use formula $P = QQ^T$, compute P and Pv. Does it agree with your result in (a)?

(e) A column space of matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{pmatrix}$ is also xy-plane. Compute $P = B(B^TB)^{-1}B^T$ and compare it with previous results.

2. (a)Compute the determinant of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ using cofactor expansion formula.

(b) Write down the cofactor expansion of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$.(Don't compute)

3. True or false. Find a counterexample or explain why

(a) $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & \\ & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is a rank-r SVD of A. Since the singular values are 3, 2, 0, $\det A = 0$.

(b) A determinant of square orthogonal matrix is 1.

(c) A determinant of projection matrix is 1.

(d) A determinant of diagonal matrix is product of diagonal entries.

(e) A determinant of square matrix with nonzero nullspace is always zero.

(f) A matrix $A \in \mathbb{R}^{n\times n}$ has $(n^2 - n)$ zero entries and n nonzero entries. The determinant is zero unless it is a diagonal matrix.

(g) n-by-n matrix with more than $n^2 - n$ zero entries always has determinant zero.

(h) For non-square matrix A, determinant of $A^T A$ equals determinant of AA^T.

(i) Matrix A has only ones and zeros in it. Its determinant is always one or zero.

(j) A determinant of doubly stochastic matrix(rows and columns have sum one) is always one.