Practice Problems

1. When does the least square solution $\hat{x}$ minimizing $||b - Ax||$ give an exact solution to $Ax = b$?

Solution. The vector $\hat{x}$ gives an exact solution to $Ax = b$ exactly if $A\hat{x} = b$. We would like to understand when this happens.

Notice that $A\hat{x}$ is in the column space of A, so if b is not in $C(A)$ then $\hat{x}$ can't give an exact solution. On the other hand, if b is in the column space, we claim that $\hat{x}$ is always an exact solution. Indeed, if b is in the column space, then we can find an x so that $Ax = b$. So the minimum value of $||b - Ax||$ is 0. Since $\hat{x}$ minimizes this distance, we must have that $||b - A\hat{x}|| = 0$. In other words, $A\hat{x} = b$ so $\hat{x}$ is an exact solution as claimed.

2. We want to project onto the plane $x - y - 2z = 0$. To do this, choose 2 vectors spanning that plane (the nullspace of what matrix?) and make them the columns of a matrix A so that $C(A)$ is the plane. Then compute the projection of the point

$$b = \begin{pmatrix} 0 \\ 6 \\ 12 \end{pmatrix}$$

onto this plane.

Solution. The plane $x - y - 2z = 0$ is the nullspace of

$$B = \begin{pmatrix} 1 & -1 & -2 \end{pmatrix}.$$

Since it's a small example, we can eyeball a basis of this plane; the vectors (1 1 0) and (2 0 1) work. So we have

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The projection p of b onto $C(A)$ is

$$p = A(A^TA)^{-1}A^Tb.$$

To avoid taking inverses, we use the strategy of setting $\hat{x} = (A^TA)^{-1}A^Tb$, solving the normal equations $A^TA\hat{x} = A^Tb$, and then computing $p = A\hat{x}$.

We compute:

$$A^TA = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \qquad A^Tb = \begin{pmatrix} 6 \\ 12 \end{pmatrix}.$$

So we have the equation

$$\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}.$$

Solving (however you like), we get $x_1 = 1$ and $x_2 = 2$. Now, we need to compute $A\hat{x}$:

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = p.$$

Alternate solution: we know that $b = p + e$, where p is the projection of b to the plane $x - y - 2z = 0$ and e is the projection of b to the orthogonal complement of that plane. So if we can compute e, we can compute $p = b - e$.

The orthogonal complement of the plane is the vector space S spanned by $v = (1 \ -1 \ -2)$. Using the formula from lecture, the projection of b onto S is

$$e = \frac{vv^T}{v^T v} b.$$

Now, $v^T v = 6$ and

$$vv^T = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$$

so

$$e = \frac{1}{6} \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 10 \end{pmatrix}$$

and

$$p = b - e = \begin{pmatrix} 0 \\ 6 \\ 12 \end{pmatrix} - \begin{pmatrix} -5 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

3. Say P is an $m \times m$ orthogonal-projection matrix onto an n-dimensional subspace. What is the rank of $A = (I - P)P$? What is the rank of $B = (I - P) - P$? Show that B is orthogonal. (Hint: helpful to compute B^2, and also to draw a picture of what B does.)

Solution. Let's say S is the subspace P projects onto.

For the first question, we multiply

$$A = (I - P)P = P - P^2 = P - P = 0.$$

So A has rank 0. This makes sense; $I - P$ is the projection onto $S^\perp$. So A first projects x to S, and then projects the resulting point of S to $S^\perp$. But if s is a vector in S, the closest point in $S^\perp$ will always be 0.

For the second question, it's easiest to picture the case where we're in the plane and S is 1-dimensional (so a line through the origin). In that case, B takes a vector and reflects it across the line through the origin perpendicular to S; that is B is reflection across $S^\perp$. This gives us the intuition that B is full rank, since reflection is an invertible transformation. We'll show this using the hint.

Taking the hint, we compute

$$B^2 = I - 4P + 4P^2 = I - 4P + 4P = I$$

where we are using the fact that $P^2 = P$. This means that B is invertible (in fact, B is its own inverse) and in particular is rank m.

We've already computed that $B^2 = I$, so $B^{-1} = B$. We'd like to show that B is *orthogonal*; that is, that $B^{-1} = B^T$. It's enough to show that $B = B^T$. Remember from class that for a projection matrix P, $P = P^T$. So we check

$$B^T = (I - 2P)^T = I - 2P^T = I - 2P = B$$

as desired.

4. If A is $m \times n$ with rank n, what is the complexity of finding the projection p onto $C(A)$ of a point b by

(1) forming the projection matrix P (using the formula from class) then doing the multiplication Pb

(2) forming the normal equations, solving them for $\hat{x}$ and then computing $p = A\hat{x}$.

Solution. For (1): The formula is

$$P = (A^T A)^{-1} A^T.$$

To compute A^T, you do roughly mn arithmetic operations (there are mn entries of A^T and to get a single entry, you do 1 operation, which is selecting a particular entry of A). Computing the product $A^T A$ uses mn^2 operations (there are n^2 entries of $A^T A$ and to get a single entry, you do about m operations). Inverting $A^T A$ uses n^3 operations (e.g. using Gaussian elimination). Doing the final multiplication uses mn^2 operations (there are mn entries of P and computing one of them takes about n operations). Since $m > n$, the overall computational complexity is on the order of mn^2.

For (2): Notice that the most expensive part of the above computation was actually multiplying the matrices $A^T A$, which took mn^2 operations. This will also be the most expensive part of this method. The normal equations are $A^T A\hat{x} = A^T b$. Computing $A^T b$ takes mn operations; solving the equations takes n^3 operations; multiplying $A\hat{x}$ in the end takes mn operations. So again, the overall computational complexity is on the order of mn^2.