Solutions

1 (Lecture recap—skip if you feel like it)**.** A function f is called "linear" if $f(x+y) = f(x) + f(y)$ for all x and y and $f(rx) = rf(x)$ for any scalar r.

a) Is $f(x) = mx + b$ linear? What about $f(x) = x^2$? (In both cases, f is a function on the real numbers)

b) Show that $f(x) = Ax$ is linear for any 2×2 matrix A. (Here, x is any 2×1 vector.)

c) Show that $f(X) = AX$ is linear for any 2×2 matrix A. (Here, X is any 2×2 matrix.)

Solution.

a) For the first function, we check

$$\begin{aligned} f(x+y) &= mx + b + my + b \\ &= m(x+y) + 2b \end{aligned}$$

while $f(x+y) = m(x+y) + b$. These are equal only if $b = 0$, so f is not linear if $b \neq 0$. If $b = 0$, then $f(rx) = rx = rf(x)$, so f is linear.

For the second function, $f(rx) = r^2x^2 \neq rf(x)$. So f is not linear.

b) Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Then

$$Ax = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}, Ay = \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix}, A(x+y) = \begin{bmatrix} a(x_1+y_1) + b(x_2+y_2) \\ c(x_1+y_1) + d(x_2+y_2) \end{bmatrix}$$

and

$$A(rx) = \begin{bmatrix} arx_1 + brx_2 \\ crx_1 + drx_2 \end{bmatrix} = r \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = rAx.$$

So f is linear.

c) Write $X = [x\ y]$ and $V = [u\ v]$; that is, x, y, u, v are 2-component column vectors. Then

$$AX + AV = [Ax\ Ay] + [Au\ Av] = [Ax + Au\ Ay + Av]$$

and

$$A(X+V) = [A(x+u)\ A(y+v)].$$

Using b), we know that these are equal. Also

$$A(rX) = [Arx\ Ary] = [rAx\ rAy] = rAx,$$

again using b).

2. Say x, y, z are 4-component column vectors. The equation

$$x(y+z) = xy + xz = yx + zx$$

is nonsense (why?) but is a few symbols away from being true. Decorate with transposes to make it a true equation.

Solution. The equation is nonsense because we can't multipl two 4-component vectors. One way to decorate with transposes to get a true equation is

$$x^T(y+z) = x^T y + x^T z = y^T x + z^T x.$$

3. Say P is the 4×4 linear operation that reverses the order, i.e.

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

What does P do to the 4×4 identity matrix I? How can you use this to write down P?

More generally, if you know how a linear operation A behaves on a vector of variables, how can you write down the matrix for A?

Solution.

$$P \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Remember that the identity matrix satisfies $IP = PI = P$, so the matrix PI above is exactly P. In general, you can compute the matrix for any linear operation by writing down how it acts on the columns (or rows) of I.

4. Find the LU factorization of

$$A = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

What 3 conditions on a, b, c guarantee that $A = LU$ has 3 pivots?

Solution. We start by using Gaussian elimination to find U. Subtracting the first row from the second and third gives

$$\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix}.$$

Subtracting the second row from the third gives

$$\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} = U.$$

To compute L, we remember that the elimination matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

so $E_2E_1A = U$. The elimination matrices are nonsingular and square, so $A = E_1^{-1}E_2^{-1}U$. We have

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

So $L = E_1^{-1}E_2^{-1}$, which is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(You should learn a faster way to compute L in one of the next lectures.)

To guarantee A has 3 pivots, we need $a \neq 0$, $a \neq b$ and $b \neq c$.

5. Consider the matrices

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and set $A = UB^{-1}L$. Without inverting any matrices, compute the second column of A^{-1}.

Solution. The second column x of A^{-1} is

$$x = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

That is, x is the vector so that $Ax = [0\ 1\ 0]^T$. Since $A = UB^{-1}L$, we are actually trying to solve

$$UB^{-1}Lx = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We'll do this in a couple of steps. Let $y = B^{-1}Lx$, so we need to solve

$$Uy = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since U is upper triangular, we can do this just by substitution:

$$\begin{aligned} y_3 = 0 &\Longrightarrow\ y_3 = 0 \\ y_2 + 2y_3 = 1 &\Longrightarrow\ y_2 = 1 \\ y_1 + y_2 + y_3 = 0 &\Longrightarrow\ y_1 = -1. \end{aligned}$$

Now we want to solve $B^{-1}Lx = y$ for x. This is the same as solving $Lx = By$ for x. Remember, we know y, so the right hand side is just a vector. So we have

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

We can finish using substitution again:

$$\begin{aligned} x_1 = 1 \quad &\implies x_1 = 1 \\ -x_1 + x_2 = -1 &\implies x_2 = 0 \\ -2x_1 + x_2 + x_3 = -1 &\implies x_3 = 1. \end{aligned}$$