Practice Problems

1. True or False (give a good reason if true/counterexample or reason if false)

1. If the zero vector is in the column space of a matrix A, then the columns of A are linearly dependent.
   Solution: False; $A = I$ is a counterexample. The zero vector is in the column span of every matrix, because the zero vector is in every subspace.

2. If the columns of a matrix are dependent, so are the rows.
   Solution: False. A counterexample is any matrix with more columns than rows, but full row rank, e.g.
   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

3. The column space of a 2×2 matrix is the same as its row space.
   Solution: False. Consider
   $$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$
   Then $C(A) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$, but $R(A) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$.

4. The column space of a 2×2 matrix has the same dimension as its row space.
   Solution: True. The dimensions of both spaces are the rank of A.

5. The columns of a matrix are a basis for the column space.
   Solution: False. The columns will always span the column space, but they may not be linearly independent. A counterexample is any matrix with a column of all 0's, or any matrix with more columns than rows.

6. A and A^T have the same number of pivots.
   Solution: True. The number of (nonzero) pivots is the rank of A, which is equal to the rank of A^T.

7. A and A^T have the same left nullspace.
   Solution: False. The left nullspace of A is $N(A^T)$. The left nullspace of A^T is $N(A)$. These are usually not equal; for example, if A is 2×3, then $N(A)$ is a subspace of $\mathbb{R}^3$ and $N(A^T)$ is a subspace of $\mathbb{R}^2$.

8. If the row space equals the column space then $A^T = A$.
   Solution: False. A counterexample is any invertible matrix which is not symmetric, like
   $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$
   .

9. If $A^T = -A$, then the row space of A equals the column space.
   Solution: True. $C(A) = C(-A) = C(A^T) = R(A)$.

2. If w_1, w_2, w_3 are independent vectors in $\mathbb{R}^3$, show that the differences

$$\begin{aligned} v_1 &= w_2 - w_3 \\ v_2 &= w_1 - w_3 \\ v_3 &= w_1 - w_2. \end{aligned}$$

are *dependent*. Find the matrix A so that

$$\begin{bmatrix} \\ w_1 & w_2 & w_3 \\ \\ \end{bmatrix} A = \begin{bmatrix} \\ v_1 & v_2 & v_3 \\ \\ \end{bmatrix}.$$

Which matrices above are singular?

Solution. To show that v_1, v_2, v_3 are dependent, we need to find a linear relation that they satisfy. Playing around, you can see that

$$v_1 - v_2 + v_3 = (w_2 - w_3) - (w_1 - w_3) + (w_1 - w_2) = 0.$$

The matrix A is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

The matrix $(v_1\ v_2\ v_3)$ is singular, and so is A (if A weren't singular, then it would be impossible for $(v_1\ v_2\ v_3)$ to be singular).

3. Construct $A = uv^T + wz^T$ whose column space has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and whose row space has basis $(1, 0), (1, 1)$. Write A as a 3×2 matrix times a 2×2 matrix.

Solution. From the problem, we know A should be 3×2. Let's try to find A such that the columns are an invertible linear combination of $u = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, and $w = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and the first two rows are $(1, 0)$ and $(1, 1)$.

Inspection (or other techniques for solving linear equations) will show you that this is possible by

$$\left(0 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \frac{1}{2} & \frac{7}{2} \end{pmatrix} = A.$$

And so we see that

$$\begin{aligned} A &= \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

So we can take

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad z = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

4. If a subspace S is contained in a subspace V, prove that $S^\perp$ contains $V^\perp$.

Solution. Suppose $w \in V^\perp$. Then for all $v \in V$, $w \cdot v = 0$. Since S is contained in V, this means that for all $s \in S$, we have $w \cdot s = 0$. So by the definition of $S^\perp$, w is in $S^\perp$. That is, for every vector $w \in V^\perp$, we have shown w is also in $S^\perp$. This means $V^\perp \subset S^\perp$.