Practice Problems

1. Consider the singular value decomposition $A = U\Sigma V^T$ where

$$U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix} \qquad V = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}.$$

a) What orthonormal bases does this give for $C(A)$ and $C(A^T)$?

b) Write A as $\sigma_1 R_1 + \sigma_2 R_2$ where R_1 and R_2 are rank 1 matrices.

c) What is a good rank 1 approximation for A?

d) If you apply A to a unit circle in $\mathbb{R}^2$, what is the output? (A vague answer is fine.)

e) Why not choose $V = I$, which is another orthonormal basis for $C(A^T)$? What does A do to the columns of I?

2. Suppose A is square and upper triangular (with nonzero diagonal entries). If you perform Gram-Schmidt on the columns of A, what can you say about the square matrix Q whose columns are the Gram-Schmidt vectors?

3. a) Give a 4×3 matrix A with 3 different, nonzero columns such that blindly applying Gram-Schmidt to the columns of A will lead you to divide by zero.

b) What property of A causes Gram-Schmidt to fail?

c) To find an orthonormal basis for $C(A)$, you should instead apply Gram-Schmidt to what matrix B?