Practice Problems

1. The "tribonacci numbers" are the sequence defined by $T_1 = 1$, $T_2 = 1$, $T_3 = 2$ and the recurrence

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}.$$

a) Write this 3-term recurrence as a square matrix R acting on the last _____ numbers in this sequence.

b) Find a formula for T_n involving only T_1, T_2, T_3 and R.

c) The eigenvalues of this matrix are roughly $-0.42 \pm 0.61i$ and 1.84. What does this tell you about the behavior of the Tribonacci numbers for large n?

Solution. a) The recurrence has 3 terms, so the matrix R should have 3 columns (we need to input 3 numbers). Since R is square, it will also output 3 numbers. Imitating the process from lecture, we are actually looking for the matrix R satisfying

$$R \begin{pmatrix} T_{n-1} \\ T_{n-2} \\ T_{n-3} \end{pmatrix} = \begin{pmatrix} T_n \\ T_{n-1} \\ T_{n-2} \end{pmatrix}.$$

Using the recurrence, it's not too hard to see that

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note: you might get a slightly different matrix depending on how you choose to order your vector of T_i's.

b) We can obtain T_n by applying R some number of times to the vector $v = [T_3\ T_2\ T_1] = [2\ 1\ 1]$ (technically this will result in a vector, and T_n will be a coordinate of this vector). Notice that T_4 is $e_1^T(Rv)$ and $T_5 = e_1^T(R^2 v)$. So $T_n = e_1^T(R^{n-3}v)$.

c) Since T_n is a coordinate of $R^{n-3}[2\ 1\ 1]$, we will write $[2\ 1\ 1]$ in an eigenbasis for R and then analyze what happens when we multiply repeatedly by R. Say that v_1 is the eigenvector of R corresponding to the real eigenvalue $\lambda_1 = 1.84$ and v_2, v_3 correspond to the complex eigenvalues λ_2, λ_3. Say also that we've found the expansion of $[T_3\ T_2\ T_1] = [2\ 1\ 1]$ in the eigenvectors, and we get

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = c_1 v_1 + c_2 v_2 + c_3 v_3.$$

Now, remember that $R^j v_r = (\lambda_r)^j v_r$. So

$$R^j \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = c_1(1.84)^j v_1 + c_2(-0.42 + 0.61i)^j v_2 + c_3(-0.42 - 0.61i)^j v_3.$$

We know that the coefficient of v_1 is getting larger and larger as j grows. What about the coefficients of v_2, v_3? The best thing to do here is to think about $-0.42 \pm 0.61i$ in polar form. That is, $-0.42+0.61i = re^{i\theta}$ where $r = \sqrt{(-0.42)^2+0.61^2}$. It's easier to understand multiplication in polar form: $(re^{i\theta})^j = r^j e^{i\theta j}$. Notice that $r<1$ here (it's about 0.74) so r^j is getting closer and closer to 0 as j increases. This means that $(re^{i\theta})^j$ is getting closer and closer to 0 as j increases, since the $e^{i\theta j}$ part always has modulus 1. We just thought about the coefficent of v_2, but the coefficient of v_3 behaves in the same way. In summary, the coefficients of v_2, v_3 are shrinking.

So as j gets very large, the vector $[T_j\ T_{j-1}\ T_{j-2}]$ gets very close to parallel with v_1, the first eigenvector. This means that the ratios between T_j, T_{j-1}, T_{j-2} approach the ratios between the corresponding entries of v_1.

2. Consider the matrix

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

from lecture, which has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, and corresponding eigenvectors $x_1 = [1\ 1]$ and $x_2 = [1\ 2]$.

a) What do we get if we take the vector $x = [3\ 4] = 2x_1 + x_2$ and multiply 100 times by A^{-1}?

b) What happens if we take x and multiply many times by $(2A-5I)^{-1}$? Does it converge to a particular vector?

c) More generally, if we have an arbitrary matrix A with all eigenvalues distinct, and we multiply a vector x repeatedly by A^{-1}, it typically approaches what eigenvector? When might this fail to happen?

Solution. a) We know that the eigenvectors of A, A^{-1} are the same, and the eigenvalues are reciprocal. So the eigenvalues of A^{-1} are $\lambda_1 = 1/2$ and $\lambda_2 = 1/3$, and the corresponding eigenvectors are x_1, x_2 respectively. We also know that $A^{-n}x_i = (\lambda_i)^{-n}x_i$. So

$$A^{-100}x = 2(1/2)^{100}x_1 + (1/3)^{100}x_2 = (1/2^{99})x_1 + (1/3^{100})x_2.$$

b) We can figure this out without ever computing $(2A-5I)^{-1}$. This is because the eigenvectors of $A, 2A, 2A-5I, (2A-5I)^{-1}$ are all the same. So we just need to figure out how the eigenvalues of A and $(2A-5I)$ are related.

Scaling A scales the eigenvalues by the same amount, so the eigenvalues of $2A$ are $4, 6$. Translating $2A$ by $-5I$ adds -5 to each eigenvalue, so the eigenvalues of $2A-5I$ are $-1, 1$. So the eigenvalues of $(2A-5I)^{-1}$ are $-1, 1$ (since $-1 = 1/-1$ and $1 = 1/1$). This means

$$(2A-5I)^{-n}x = 2(-1)^n x_1 + (1)^n x_2 = 2(-1)^n x_1 + x_2.$$

So $(2A-5I)^{-n}x$ does not converge to a particular vector! It vacillates between $-2x_1 + x_2$ and $2x_1 + x_2$ depending on if n is even or odd.

c) Say the eigenvalues of A are $\lambda_1, \ldots, \lambda_r$ and the basis of eigenvectors is $v_1, \ldots, v_r$. Say we're interested in some vector x, which can be expressed in the basis of eigenvectors as

$$x = c_1v_1 + c_2v_2 + \cdots + c_rv_r.$$

As we've seen above,

$$A^{-n}x = c_1(1/\lambda_1)^n v_1 + \cdots + c_r(1/\lambda_r)^n v_r.$$

Usually, there will be one eigenvalue, say λ_1 with smallest modulus, so $1/\lambda_1$ has larger modulus than all the other eigenvalues of A^{-1}. The vector $A^{-n}x$ tends to approach an eigenvector with eigenvalue $1/\lambda_1$, since the coefficent $c_1(1/\lambda_1)^n$ of v_1 is much much larger in modulus than the coefficients on the other eigenvectors.

There is an exception to this, though—when another eigenvalue, say λ_2, has the same modulus as λ_1. Then two coefficients are comparable in modulus, and $A^{-n}x$ does not approach an eigenvector.