Practice Problems

1. True, false, or neither (that is, sometimes true, sometimes false):

a) If v, w are eigenvectors of A, then so is $v + w$ and cv for c any scalar.

b) If $v \in N(A)$ is not the zero vector, then v is an eigenvector of A.

Solution. a) The vector cv will always be an eigenvector, with the same eigenvalue as v. If v, w are eigenvectors with the *same* eigenvalue, then $A(v + w) = Av + Aw = \lambda v + \lambda w = \lambda(v + w)$, so $v + w$ is another eigenvector (as long as it's nonzero). If v, w have *different* eigenvalues, then $v + w$ is *not* an eigenvector.

b) is true! Since v is in $N(A)$, $Av = 0 = 0 \cdot v$, so v is an eigenvector with eigenvalue 0.

2. Describe as many eigenvalues and corresponding eigenvectors as you can (without doing any serious calculation) for

a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$.

b) A projection matrix P onto some subspace S (pick a particular 2-dimensional subspace of $\mathbb{R}^3$ if you're confused).

c) The permutation matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

d) A rank one matrix uv^T (pick particular 3-component vectors u, v if you're confused).

Solution. a) I fixes every vector; that is $Ix = x$ for any x. So all vectors are eigenvectors of I, with eigenvalue 1. For the upper-triangular matrix, the easy eigenvector is [1 0 0], which has eigenvalue 4. The other eigenvectors are more difficult.

b) P fixs every vector in S, so all vectors in S are eigenvectors, with eigenvalue 1. On the other hand, P sends every vectors in $S^\perp$ to 0, so all vectors in $S^\perp$ are eigenvectors with eigenvalue 0.

c) M fixes the vector $v = [1\ 1\ 1]$, so v is an eigenvector with eigenvalue 1. The other eigenvalues/vectors are a little more complicated.

d) First, every vector in the nullspace is an eigenvector with eigenvalue 0. Now, say q is a vector which is *not* in the nullspace. Then $uv^t q = u(v^T q) = (v^T q)u$ since $v^T q$ is just a scalar. That is, $(uv^T)q$ is always a scalar multiple of u. So the only way for q to be an eigenvector is if q is a scalar multiple of u. The vector cu is an eigenvector with eigenvalue $c \cdot v^T u$.

3. For which angles θ does the rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

have real eigenvalues? (Hint: The roots of the quadratic equation ax^2+bx+c are $(-b\pm\sqrt{b^2-4ac})/2a$.)

Solution. Remember that the eigenvalues are the roots of the equation

$$\det(R - \lambda I) = 0.$$

In this situation,

$$R - \lambda I = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

and

$$\begin{aligned} \det(R - \lambda I) &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta \\ &= \lambda^2 - 2\lambda\cos\theta + 1. \end{aligned}$$

Using the quadratic formula, the roots of this equation are real exactly when $4\cos^2\theta - 4 > 0$ (this is the part inside the square root in the quadratic formula). Now $\cos^2\theta \le 1$, and $\cos^2\theta = 1$ only when $\theta = 0, \pi$. So we see that R has real eigenvalues only when it is the identity matrix or rotation by π.

4. Suppose that A, B, C are $m \times m$ matrices with eigenbases that you know. What do you know about the eigenvectors and eigenvalues of A^{2022}? A^{-1} (assuming A is invertible)? A^T? AB? $A+B$?

Solution.

- A^{2022}: Say $Av = \lambda v$. Then $A^{2022}v = \lambda^{2022}v$. That is, the eigenvectors are the same; the eigenvalues are 2022 powers.
- A^{-1}: I can manipulate the equation $Av = \lambda v$ to involve an A^{-1}, by multiplying both sides by A^{-1} on the left. I get $v = \lambda A^{-1}v$; dividing both sides by λ, I get $(1/\lambda)v = A^{-1}v$. So the eigenvectors are the same, but the eigenvalues are reciprocals.
- A^T: Can't say anything about the eigenvectors in general, but the eigenvalues will be the same. The eigenvalues of A are the roots of $\det(A - \lambda I)$ and the determinant of a matrix and its transpose are equal, so $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$.
- AB Can't say anything in general. If A and B happen to have an eigenvector, say v, in common, then it will be an eigenvector of AB (and the eigenvalues will multiply).
- $A+B$ Can't say anything in general. Again, if A and B happen to have an eigenvector, say v, in common, then v will be an eigenvector of $A+B$, and the eigenvalues will add.

5. Suppose that A is the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) What is the pattern when you multiply A repeatedly by some vector? After _______ multiplications, you get back the same vector, so

$$A^{\_\_} = \_\_\_\_\_\_\_\_\_\_\_\_.$$

b) What are eigenvalues and eigenvectors of A? Is this consistent with the previous part?

c) Write the vector $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the basis of the eigenvectors and give a formula for $A^n x$.

d) What are the eigenvectors and eigenvalues of $B = 2A + I$?

Solution. a) $A[a\ b] = [b\ \ -a]$. So after 4 multiplications, we get back the same vector. This means $A^4 = I$.

b) The matrix A is actually an example of a rotation matrix from Problem 3, with $\theta = -\pi/2$. So we know we should expect 2 complex eigenvalues. The characteristic equation is

$$\det(A - \lambda I) = \lambda^2 + 1$$

and the two roots of this equation are $\lambda_1 = i$ and $\lambda_2 = -i$. The two eigenvectors v_1, v_2 satisfy

$$Av_1 = iv_1 \qquad \text{and} \qquad Av_2 = -iv_2.$$

Say $v_1 = [a_1\ b_1]$ and $v_2 = [a_2\ b_2]$. Then these equations are

$$\begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} = \begin{pmatrix} ia_1 \\ ib_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} b_2 \\ -a_2 \end{pmatrix} = \begin{pmatrix} -ia_2 \\ -ib_2 \end{pmatrix}.$$

The first equations tell us that $b_1 = ia_1$ and $ib_1 = -a_1$ (these two equations actually contain the same information, since multiplying both sides of the first equation by i give the second). The simplest vector satisfying these conditions is $[1\ i]$.

The second equations tell us that $b_2 = -ia_2$ and $ib_2 = a_2$ (again, these equations contain the same information). The simplest vector satisfying these equations is $[i\ 1]$.

Remember that the eigenvalues of A^4 are exactly the eigenvalues of A raised to the 4th power. Since $A^4 = I$ and the only eigenvalue of I is 1, I expect to get 1 if I raise any eigenvalue of A to the 4th power. And indeed I do, so this checks out.

c) Writing $[1\ 0]$ in the basis of eigenvectors is the same as solving the equations

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Doing this gives

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{-i}{2} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Now

$$A^n x = A^n \left(\frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{-i}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} \right) = \frac{i^n}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{(-i)^{n+1}}{2} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

d) Say $Ax = \lambda x$. Then we see that $Bx = 2Ax + x = 2\lambda x + x = (2\lambda + 1)x$, so x is *also* an eigenvector of B with eigenvalue $2\lambda + 1$. Similarly, using the fact that $A = (1/2)(B - I)$ we can check that every eigenvector of B is also an eigenvector of A.

So the eigenvalues of B are $2\lambda_1 + 1 = 2i + 1$ and $2\lambda_2 + 1 = -2i + 1$. And the eigenvectors are v_1, v_2.