Practice Problems

1. Say A is a 3×3 real matrix with eigenvalues $\lambda_1 = -1, \lambda_2 = -3+4i, \lambda_3 = -3-4i$, with corresponding eigenvectors x_1, x_2, x_3.

a) What are the trace and determinant of $2A$?

b) Two eigenvectors of A are $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$. What is x_3?

Solution. a) The eigenvalues of $2A$ are $2\lambda_1, 2\lambda_2, 2\lambda_3$. So the trace is

$$2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 2(-1 + (-3+4i) + (-3-4i)) = 2(-7) = -14$$

and the determinant is

$$2\lambda_1 \cdot 2\lambda_2 \cdot 2\lambda_3 = 8(-1)(-3+4i)(-3-4i) = -200.$$

b) Since A is real, its complex eignevectors must come in complex-conjugate pairs. So

$$x_3 = \overline{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}.$$

2. Using the same matrix A as in 1, which of the following has unbounded magnitude (i.e. magnitude blowing up) as $n \to \infty$ or $t \to \infty$? Assume y is chosen at random.

a) $A^n y$ as $n \to \infty$

b) $A^{-n} y$ as $n \to \infty$

c) The solution of $\frac{dx}{dt} = Ax$ as $t \to \infty$ for the initial condition $x(0) = y$.

d) The solution of $\frac{dx}{dt} = -Ax$ as $t \to \infty$ for the initial condition $x(0) = y$.

Solution. Notice that the eigenvalues of A satisfy $|\lambda| \geq 1$ and $\mathrm{Re}[\lambda] < 0$. This is enough information for us to answer the first 4 parts of this question.

a) This has unbounded magnitude. If we write $y = c_1 x_1 + c_2 x_2 + c_3 x_3$, then

$$A^n y = c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2 + \lambda_3^n c_3 x_3$$

and λ_2^n, λ_3^n become larger and larger in magnitude as $n \to \infty$ (you can see this by writing those eigenvalues in polar form). Since y was chosen at random, c_2, c_3 are likely nonzero.

b) The magnitude of this vector will stay bounded as $n \to \infty$ (though it may not converge to any vector in particular). Remember the eigenvalues of A^{-1} are $1/\lambda_i$ and $|1/\lambda_i| \leq 1$. So writing

$$A^{-n} y = c_1 \lambda_1^{-n} x_1 + c_2 \lambda_2^{-n} x_2 + \lambda_3^{-n} c_3 x_3$$

we see that the second and last term will decay as $n \to \infty$ (e.g. by writing those eigenvalues in polar form). The first term will always have the same magnitude.

c) The solution to this equation is

$$x(t) = e^{At}y = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2 + c_3 e^{\lambda_3 t}x_3$$

and it has bounded magnitude as $t \to \infty$. This is because $\mathrm{Re}[\lambda_j] < 0$ for all eigenvalues, so $e^{\lambda_j t}$ always approaches zero as $t \to \infty$ (you can see this by writing $\lambda_j = a + ib$).

d) The eigenvalues of $-A$ are $-\lambda_j$, so they all have positive real parts. This means that the solution

$$x(t) = e^{-At}y = c_1 e^{-\lambda_1 t}x_1 + c_2 e^{-\lambda_2 t}x_2 + c_3 e^{-\lambda_3 t}x_3$$

will have unbounded magnitude as $t \to \infty$, since each term has magnitude which blows up.

3. Using the same matrix A as in 1, write down the exact solution $x(t)$ to $\frac{dx}{dt} = Ax$ for the initial condition $x(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Solution. As in part c) above, the general solution to $\frac{dx}{dt} = Ax$ is

$$x(t) = e^{At}x(0) = c_1 e^{-1t}x_1 + c_2 e^{(-3+4i)t}x_2 + c_3 e^{(-3-4i)t}x_3$$

where c_1, c_2, c_3 are some constants depending on $x(0)$. Because the initial conditions are real, we expect $c_2 = \overline{c_3}$.

Setting $t = 0$, we get

$$x(0) = c_1 x_1 + c_2 x_2 + c_3 x_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Eyeballing, we see that this is true if $c_1 = c_2 = c_3 = 1$, so the exact solution is

$$x(t) = e^{-1t}x_1 + e^{(-3+4i)t}x_2 + e^{(-3-4i)t}x_3.$$

4. Use the series formula for e^{At} to show that

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Use this to conclude that $x(t) = e^{At}x(0)$ satisfies $\frac{dx}{dt} = Ax$.

Solution. The series formula for e^{At} is

$$e^{At} = 1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

(Note: t is a scalar here! So $A^n t^n$ is well-defined). We want to take the derivative with respect to t; that is, we treat A as a constant and t as a variable. Taking derivatives "term by term" in the series gives

$$\begin{aligned}\frac{d}{dt}e^{At} &= A + 2\frac{A^2 t}{2!} + 3\frac{A^3 t^2}{3!} + \dots \\ &= A(1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots) \\ &= Ae^{At}.\end{aligned}$$

Now we compute

$$\begin{aligned}\frac{d}{dt}(e^{At}x(0)) &= x(0)\left(\frac{d}{dt}(e^{At})\right) \\ &= x(0)Ae^{At} \\ &= A(e^{At}x(0))\end{aligned}$$

which shows the property we were supposed to show.