MIT 18.06 Practice Exam 1, Spring 2023
Strang and Horning

Your name: ______________________________
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Student ID: ______________________________

Recitation: ______________________________

Problem 1 (10+4+4+10+6=34 points):

Record your answers in the alloted spaces. You may use the rest of this page and the following for your calculations.

(a) Compute a new factorization of the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & 5 & 2 \\ 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix},$$

by adding linearly independent rows of A (in order from top to bottom) to the factor R_{new} and choosing the columns of C_{new} so that $A = C_{new}R_{new}$:

$$C_{new} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad R_{new} = \begin{pmatrix} 3 & 2 & 1 & 5 & 2 \\ 2 & 2 & 0 & 4 & 0 \end{pmatrix}$$

(b) Put an X next to the correct answer. The column space of A is

(i) a line ________

(ii) a plane ___ X ___

(iii) the whole 3D space ________

(iv) none of the above ________

Solution: The column space of A is the span of the two columns in C_{new}, so the column space is a plane.

(c) Put an X next to the correct answer. The row space of A is

(i) a line ________

(ii) a plane ___ X ___

(iii) a 3D subspace ________

(iv) none of the above ________

Solution: The row space of A is the span of the two rows in R_{new}, so the row space is a plane.

(d) Use $A = C_{new}R_{new}$ to compute Ax for the vector $x = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ in two steps:

$$R_{new}x = \begin{pmatrix} 13 \\ 8 \end{pmatrix} \qquad Ax = C_{new}(R_{new}x) = \begin{pmatrix} 13 \\ 8 \\ 5 \end{pmatrix}$$

(e) If we multiply the "dot-product" way, $y = R_{new}x$ requires ___2___ dot product(s) bewteen 5×1 vectors and $Ax = C_{new}y$ requires 3 dot product(s) between ___ 2 ___$\times 1$ vectors.

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Problem 2 (16+4+4+12=36 points):

Record your answers in the alloted spaces. You may use the rest of this page and the following for your calculations.

(a) Compute the factorization $A = LU$ of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix}.$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: To eliminate the subdiagonal entries in the first column of A, we subtract multiples of 2, 4, and 8 of the first row, respectively. These multipliers land in the first column of L, in the location of the zeros they were used to create. So far we have,

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & & 1 & 0 \\ 8 & & & 1 \end{pmatrix} \qquad E_{41}E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 4 & -6 & 1 \end{pmatrix}.$$

We use the $(2,2)$ pivot in $E_{41}E_{31}E_{21}A$ to introduce zeros in the last two entries of the second column with multipliers 2 and 4, respectively. We now have

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & & 1 \end{pmatrix} \qquad E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -3 \end{pmatrix}.$$

Finally, we use the $(3,3)$ pivot in $E_{42}E_{32}E_{41}E_{31}E_{21}A$ to introduce zeros in the last entry of the third column with multiplier 2. We conclude that

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \qquad U = E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Put an X next to the correct answer. The matrix A is

(i) invertible __ X __

(ii) not invertible ______

Solution: Both L and U are triangular matrices with nonzero diagonal entires, therefore, they are invertible. Therefore, $A = LU$ is also invertible with $A^{-1} = U^{-1}L^{-1}$.

(c) The rank of A is __ 4 __.

Solution: Since A is invertible its columns are linearly independent. It has 4 columns so rank(A) = 4.

(d) Use $A = LU$ to solve $Ax = b$ for the vector $b = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}^T$.

$$x = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Solution: We need to solve $Ax = (LU)x = b$. First, we solve the lower triangular system $Ly = b$ with forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We can calculate that $y_1 = 1$, $y_2 = 1 - 2y_1 = -1$, $y_3 = 0 - 4y_1 - 2y_2 = -4 + 2 = -2$, and $y_4 = 1 - 8y_1 - 4y_2 - 2y_3 = 1 - 8 + 4 + 4 = 1$. We then solve the upper triangular system $Ux = y$ with back substitution:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

We can calculate that $x_4 = 1$, $x_3 = -2 + 2x_4 = 0$, $x_2 = -1 + 2x_3 - x_4 = -1 + 0 - 1 = -2$, and $x_1 = 1 - x_3 = 1 - 0 = 1$. We conclude that $x = \begin{pmatrix} 1 & -2 & 0 & 1 \end{pmatrix}^T$.

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Problem 3 (6+6+10+8=30 points):

Record your answers in the alloted spaces. You may use the rest of this page and the following for your calculations. Consider the matrix $A = LPU$ given by:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U}.$$

(a) The matrices L (True), P (True), and U (True) are invertible. (Write True or False next to each).

Solution: The matrices L and U are triangular with nonzero diagonals, therefore, they are invertible. The matrix P has linearly independent columns (a permutation matrix exchanging rows 2 and 3), therefore, it is invertible.

(b) Write A^{-1} in terms of L^{-1}, P^{-1}, and U^{-1} (without computing any numbers):

$$A^{-1} = U^{-1}P^{-1}L^{-1}$$

Solution: Since the individual matrices are invertible, the inverse of the product $A = LPU$ is the product of the inverses in reverse order.

(c) Consider the linear system $Ax = b$. What right-hand-side vector b should one choose so that the system has solution $x =$ (**first** column of A^{-1})?

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution: Since A is invertible, we have unique solution $x = A^{-1}b$. The entries in b tell us how to combine the columns of A^{-1} to get x. If we want the first column of A^{-1}only, we need to take $b_1 = 1$ and all other entries of b should be zero.

(d) Compute x, the first column of A^{-1}:

$$x = \begin{pmatrix} 1 \\ -1/3 \\ 0 \\ 0 \end{pmatrix}.$$

Solution: We need to solve $Ax = (LPU)x = b$. We first solve the lower triangular system $Lz = b$, then the system $Py = z$, and finally, the upper triangular system $Ux = y$. The two triangular systems are best solved using forward substitution and backsubstitution, as in problem 2 above. To solve $Py = z$,we only need to exchange rows 2 and 3 on both sides to get $y_1 = z_1, y_2 = z_3, y_3 = z_2$,and $y_4 = z_4$.

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