MIT 18.06 Final Exam, Spring 2022
Johnson

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/14
2	/10
3	/17
4	/17
5	/10
6	/16
7	/16
total	/100

Problem 0 (∞ points): Honor code

Copy the following statement with your signature into your solutions:

> *I have completed this exam **closed-book/closed-notes** entirely on my **own.***
>
> [your signature]

Problem 1 (4+4+6 points):

The matrix A is given by

$$A = LUL^{-1}U^{-1}$$

for

$$L = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & 3 & 1 & \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 2 & 0 & 1 & 1 \\ & -1 & 0 & -1 \\ & & -2 & 1 \\ & & & 1 \end{pmatrix}.$$

(a) Write an expression for A^{-1} in terms of L, U, L^{-1}, and/or U^{-1} (but you **don't** need to actually multiply or invert the terms!).

(b) What is the determinant of A?

(c) Solve $PAx = b$ for x, where P is the 4×4 permutation that swaps the 1st and 4th elements of a vector, and $b = \begin{pmatrix} -5 \\ 4 \\ 11 \\ -3 \end{pmatrix}$. (You can get partial credit by just outlining a reasonable sequence of steps here that doesn't involve a lot of unnecessary calculation.)

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Problem 2 (4+6 points):

(a) If a and x are vectors in $\mathbb{R}^n$, then $aa^T x$ can be computed using either **left-to-right** as $(aa^T)x$ or **right-to-left** as $a(a^T x)$, where the parentheses indicate the order of operations. Roughly count the number of arithmetic operations (additions and multiplications) in these two approaches: say whether each approach scales proportional to n, n^2, n^3, etcetera.

(b) A is an $n \times n$ real matrix and x is an n-component real vector. Indicate which of the following **must be equal** to one another:

$$\text{trace}(Axx^T), \quad \text{trace}(xAx^T), \quad \text{trace}(x^T Ax), \quad x^T Ax,$$

$$\text{trace}(x^T xA), \quad xx^T A, \quad \text{trace}(xx^T A), \quad \text{determinant}(xx^T A).$$

For the expressions that are equal, indicate how you would evaluate this quantity in a cost (in arithmetic operations) proportional to n^2.

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Problem 3 (4+4+4+5 points):

You have a 4×3 matrix $A = \begin{pmatrix} q_1 & 2q_2 & 3q_1 + 4q_2 \end{pmatrix}$, where we have expressed the three columns of A in terms of the orthonormal vectors

$$q_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad q_2 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) What is the rank of A?

(b) Give a basis for $N(A)$.

(c) You are asked to calculate the projection matrix P onto $C(A)$. Your friend Harvey Ard suggests applying the formula $P = A(A^T A)^{-1} A^T$ he memorized in linear algebra. Explain why this won't work here, and give an even simpler (correct) formula for P in terms of the quantities above. (You need not evaluate P numerically, just write a formula in terms of products of quantities defined above.)

(d) Find the *closest* vector to $x = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in $N(A^T)$.

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Problem 4 (3+4+4+6 points):

The nullspace $N(A)$ of the real matrix A is spanned by the vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Give as much true information as possible about the size (the number of rows and columns) of A.

(b) Give an eigenvector and eigenvalue of the matrix $B = (3I - A^T A)(3I + A^T A)^{-1}$.

(c) Aside from the eigenvalue identified in the previous part, all *other* eigenvalues λ of B must be (**circle/copy all that apply**): purely real, purely imaginary, zero, negative real part, positive real part, $|\lambda| < 1$, $|\lambda| > 1$, $|\lambda| \leq 1$, $|\lambda| \geq 1$.

(d) Give a good approximate formula for of $B^n \begin{pmatrix} 0 \\ -1 \\ 0 \\ 8 \end{pmatrix}$ for large n. (Give an explicit numerical vector, possibly including simple functions of n like 2^n or n^3... no other abstract symbolic formulas.)

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Problem 5 (10 points):

Describe (give an explicit numerical result with as few unknowns as possible) all possible linear combinations of the vectors

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \; a_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \; a_3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \; a_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

that give the vector $x = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$.

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Problem 6 (8+8 points):

Professor May Trix is trying to construct an 18.06 homework question in which $\frac{dx}{dt} = Ax$ has the solution

$$x(t) = v_1 \cos(2t) + v_2 e^{-t} + v_3 \sin(2t)$$

for some constant vectors v_1, v_2, v_3, and some initial condition $x(0)$. Help May construct A, v_1, v_2, v_3, and $x(0)$:

(a) Write down a numerical formula for a possible **real** matrix A such that A is as **small in size as possible** and where A contains **no zero entries**. Your formula can be left as a product of some matrices and/or matrix inverses — you *don't* need to multiply them out or invert any matrices, but you should give possible numeric values for all of the matrices in your formula. (You *don't* need to explicitly check that your A has no zero entries as long as zero entries seem unlikely. e.g. the inverse of a matrix with no special structure probably has no zero entries.)

(Note that there are many possible answers here, but they will all have certain things in common.)

(b) Using the numbers that you chose from the formula in your previous part, give possible corresponding (numeric) values for $x(0)$, v_1, v_2, and v_3.

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Problem 7 (8+8 points):

Suppose that we have a sequence of m data points (x_i, y_i) coming from a physics experiment that we want to fit to a line $cx + d$, where the coefficients c and d are chosen to minimize the sum of the squares of the errors. But, because some of the data points have more measurement error than others, we don't weight the errors equally in minimizing the error. In particular, suppose that we want to minimize:

$$E = \text{weighted error} = w_1(cx_1+d-y_1)^2+w_2(cx_2+d-y_2)^2+\cdots = \sum_{i=1}^{m} w_i(cx_i+d-y_i)^2.$$

where $w_1, w_2, \ldots, w_m > 0$ are some positive weights associated with each data point (more uncertain data points have smaller weight).

(a) To **convert this into a standard least-squares** problem, show that we can rewrite E in the form $E = \|Mu-v\|^2$ for some matrix M, an unknown vector u, and a known vector v — give explicit expressions for M, u, and v in terms of the points (x_i, y_i), the weights w_i, and the unknowns c and d.

(b) More generally, suppose that we are minimizing $E = (Ax-b)^T W(Ax-b)$ over $x \in \mathbb{R}^n$ where A is an $m \times n$ real matrix, b is an m-component real vector, and W is an $m \times m$ real-symmetric *positive-definite* "weight" matrix. Using the **properties of positive-definite matrices** from class, show that we can rewrite E as a **standard least-squares problem** $E = \|Mx - v\|^2$ for some matrix M and vector v: that is, explain how M and v could be related to A, W, and b.

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