**MATH 251, homework 5, due date Monday Feb 9.**

**Problem 1.** Find a homogenous system of linear equations whose space of solutions is $\mathrm{Im}T$ for the following $T\colon \mathbf{R}^3 \to \mathbf{R}^4$:

$$T(x_1, x_2, x_3) = (3x_1 + 4x_2 + 2x_3, x_1 + 2x_2, 2x_1 + x_2 + 3x_3, -x_1 + 5x_2 - 7x_3).$$

**Problem 2.** Let $a_0 = 1, a_1 = 1$, and for $n \geq 2$ let $a_n = a_{n-1} + a_{n-2}$ be the Fibonacci sequence. Show that for $n \geq 1$, the Fibonacci number $a_n$ is the determinant of

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

the $n$ by $n$ matrix with all entries zero except for 1 on the main diagonal and the diagonal above it and $-1$ on the diagonal below it.

**Problem 3.** Prove that the determinant of the $n$ by $n$ matrix $(d_{ij})$, where $d_{ij}$ is the greatest common divisor of the numbers $i$ and $j$, is equal to $\varphi(1)\varphi(2)\cdots\varphi(n)$.

Here $\varphi(k)$ is the *Euler function*, i.e. the number of numbers $1 \leq i \leq k$ such that the greatest common divisor of $i$ and $k$ is 1. This function satisfies the *Gauss formula*

$$\sum_{d|n} \varphi(d) = n.$$

Hint: For $1 \leq i, j \leq n$ define $p_{ij}$ to be equal to 1 if $j$ divides $i$ and 0 otherwise. Prove

$$d_{ij} = \sum_{k=1}^{n} p_{ik} p_{jk} \varphi(k),$$

and write the matrix $(d_{ij})$ as the sum of $n^n$ matrices.