3905. Find the images and the kernels of the linear operators from Exercise 3901.

3906. Prove that the operator of differentiation

a) is singular on the space of polynomials of degree $\leq n$;

b) is nonsingular on the space of functions with the bases $(\cos t, \sin t)$.

3907. Prove that any subspace of a vector space is:

a) the kernel of some linear operator;

b) the image of some linear operator.

3908. Prove that two linear operators of rank 1 having the same kernels and images are commuting.

3909. Let $\mathcal{A}$ be a F-linear operator on a subspace L of a space V different from V. Prove that there exist infinitely many linear operators on V, whose restriction to L coincides with $\mathcal{A}$, provided the field F is infinite.

3910. Let $\mathcal{A}$ be a linear operator on a space V, and L be a subspace of V. Prove that

a) the image $\mathcal{A}(L)$ and the preimage $\mathcal{A}^{-1}(L)$ are subspaces of V;

b) if $\mathcal{A}$ is nonsingular and V is finite-dimensional then

$$\dim \mathcal{A}(L) = \dim \mathcal{A}^{-1}(L) = \dim L.$$

3911. Let $\mathcal{A}$ be a linear operator on a space V, L be a subspace of V, and $L \cap \operatorname{Ker}\mathcal{A} = 0$. Prove that any linearly independent system of vectors in L maps by $\mathcal{A}$ to a linearly independent system.

3912. Prove, for linear operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$, the Frobenious inequality

$$\operatorname{rk}\mathcal{BA} + \operatorname{rk}\mathcal{AC} \leq \operatorname{rk}\mathcal{A} + \operatorname{rk}\mathcal{BAC}.$$

3913. A linear operator $\mathcal{A}$ is a *pseudoreflection*, if $\operatorname{rk}(\mathcal{A} - \mathcal{E}) = 1$. Prove that any linear operator on a n-dimensional space is a product of at most n pseudoreflections.

3914. Prove that the set of operators $\mathcal{X}$, such that $\mathcal{AX} = 0$ for a linear operator $\mathcal{A}$ on a n-dimensional space, is a vector space. Find its dimension.

3915. Find the matrix of the operator:

a) $(x_1, x_2, x_3) \mapsto (x_1, x_1 + 2x_2, x_2 + 3x_3)$ on the space $\mathbb{R}^3$ with a basis of unit vectors;

b) of the rotation of the plane through an angle α with an arbitrary orthonormal basis;

c) of the rotation of the three-dimensional space through an angle $2\pi/3$ around the line which is given in a rectangular system of coordinates by the equations $x_1 = x_2 = x_3$ with the basis of unit vectors of the coordinate axes;

d) of the projection of the three-dimensional space with the basis (e_1, e_2, e_3) to the axis of the vector e_2 in parallel with the coordinate plane of the vectors e_1 and e_3;

e) $x \mapsto (x, a)a$ on an Euclidean space with the orthonormal basis (e_1, e_2, e_3) if $a = e_1 - 2e_3$;

f) $X \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X$ on the space $\mathbf{M}_2(\mathbb{R})$ with a basis of matrix units;

g) $X \mapsto X \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the space $\mathbf{M}_2(\mathbb{R})$ with a basis of matrix units;

h) $X \mapsto {}^tX$ on the space $\mathbf{M}_2(\mathbb{R})$ with a basis of matrix units;

i) $X \mapsto AXB$ (A, B are fixed matrices in the space $\mathbf{M}_2(\mathbb{R})$ with a basis of matrix units;

j) $X \mapsto AX + XB$ (A, B are fixed matrices) on the space $\mathbf{M}_2(\mathbb{R})$ with a basis of matrix units;

k) of the differentiation on the space $\mathbb{R}[x]_n$ with the basis $(1, x, \ldots, x^n)$;

l) of the differentiation on the space $\mathbb{R}[x]_n$ with the basis $(x^n, x^{n-1}, \ldots, 1)$;

m) of the differentiation on the space $\mathbb{R}[x]_n$ with the basis

$$\left(1, x-1, \frac{(x-1)^2}{2}, \ldots, \frac{(x-1)^n)}{n!}\right).$$

3916. Prove that the space $\mathbb{R}^3$ has a unique linear operator which maps the vectors $(1,1,1)$, $(0,1,0)$, $(1,0,2)$ to the vectors $(1,1,1)$, $(0,1,0)$, $(1,0,1)$, respectively. Find its matrix with a basis of unit vectors.

3917. Let a vector space V be a direct sum of subspaces L_1 and L_2 with bases $(a_1, \ldots, a_k)$ and $(a_{k+1}, \ldots, a_n)$, respectively. Prove that the projection onto L_1 in parallel with L_2 is a linear operator and find its matrix with the basis $(a_1, \ldots, a_n)$.

3918. Find the general form of matrices of linear operators on the n-dimensional space with a basis $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$, which map the given independent vectors $a_1, \ldots, a_k$ $(k < n)$ to the given vectors $b_1, \ldots, b_k$.