(1) [a]

There are $\underline{5!}$ ways of ranking the contestants.

Both Jerry and Elaine get awarded if they are both placed in the top three positions. This can be done in

$\underline{{}^3P_2 = 6}$ ways, and for each of these 6 ways, the other 3 contestants can be ranked in

$\underline{3! = 6}$ ways.

Hence, the number of ways of ranking the contestants so that both Jerry and Elaine get awarded is $6 \times 6 = 36$.

Hence, required probability is

$$\frac{36}{5!} = \frac{36}{120} = \underline{\frac{3}{10}}.$$

(1) [b]

Let A = {Kramer and Newman do not get any awards}, and

$B = \{$George gets first prize$\}$.

Then $|A|$ = Number of ways in which Kramer and Newman can hold the fourth and the fifth positions

$$= 2! \times 3! = 12.$$

$$|A \cap B| = 2 \times 2! = 4$$

Hence $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

$$= \frac{|A \cap B| / 5!}{|A| / 5!}$$

$$= \frac{4}{12} = \frac{1}{3}. \quad \blacksquare$$

(2) [a] There are $(2n)!$ possible arrangements of the $2n$ people.

Now $\{X_n = n\} = \{$all couples sit together$\}$.

Think of each couple as a unit.

Then there are $\underline{n!}$ arrangements of these units.

Now each couple have $\underline{2! = 2}$ possible arrangements among themselves.

Hence, number of arrangements in which all couples sit together is $\underline{n! \times 2^n}$. Hence

$$P(X_n = n) = \frac{n! \times 2^n}{(2n)!}.$$

[This is similar to Q.1 in Assignment 1, where you were asked to find the probability that all three girls stand together, and ~~one~~ one way of solving the problem was to think of the three girls as a single unit. See solutions to Assignment 1.)

(2) [b]

Sol^{n} 1: For $1 \le i \le n$, let

$$Y_i = \mathbb{1}\{i\text{-th couple sit together}\}.$$

Then $EY_i = P(i\text{-th couple sit together})$,

and $X_n = Y_1 + \cdots + Y_n$.

$\Rightarrow EX_n = EY_1 + \cdots + EY_n$.

Again, think of the i-th couple as a single unit. Then this unit and the $(2n-2)$ other people can be arranged in $\underline{(2n-1)!}$ ways, and the i-th couple have $\underline{2! = 2}$ possible arrangements among themselves. Hence

P(i-th couple sit together)

$$= \frac{2 \times (2n-1)!}{(2n)!} = \frac{1}{n}$$

$$\Rightarrow E X_n = \underbrace{\frac{1}{n} + \dots + \frac{1}{n}}_{n \text{ times}} = 1.$$

<u>Sol^n 2:</u> $\rightarrow$ For $1 \le i \le 2n-1$, let

$Y_i = 1\{$ i-th and $(i+1)$-st position is occupied by a couple $\}$.

You can choose one among n couples in <u>n many ways</u>,

and then fill up i-th and $(i+1)$-st position with that couple in <u>$2! = 2$</u>

ways, and fill up the remaining $(2n-2)$ places in $(2n-2)!$ ways.

Hence

$$P(Y_i = 1) = \frac{n \times 2 \times (2n-2)!}{(2n)!}$$

$$= 1/(2n-1).$$

$\Rightarrow EY_i = P(Y_i = 1) = 1/(2n-1)$

for $1 \leq i \leq 2n-1$.

Since

$X_n = Y_1 + Y_2 + \dots + Y_{2n-1}$,

$EX_n = EY_1 + \dots + EY_{2n-1}$

$$= \left(\frac{1}{2n-1}\right) \times (2n-1) = 1. \blacksquare$$

(3) [a] Let Y denote the number of tosses required to get the first head. Then

$Y \sim Geo(p)$, where

p is the probability of landing heads in a single toss. Then

$100 = E(Y) = 1/p$

$\Rightarrow p = 1/100$.

(Finding p in this way was done in class. See the problems solved on 02/07/2017.)

Then $X \sim Bin(300, p)$.

$\Rightarrow EX^2 = V(X) + (EX)^2$

$= 300p(1-p) + (300p)^2$

$$= 3 \times \frac{99}{100} + 9$$

$$= 9 + \frac{297}{100}$$

$$= \frac{1197}{100}. \quad \blacksquare$$

(Computing EX^2 from the expression for the variance and EX was explained in class. See the problems solved on 02/02/2017, where $E(3X^2+X+2)$ was computed.)

(3)(b) $P(X \geq 2)$

$$= 1 - P(X=0) - P(X=1).$$

The distrib^{n} of X can be approximated by a $Poi(\lambda)$ distribution, where

$\lambda = 300 \times \frac{1}{100} = 3.$

Hence,

$P(X \geq 2) \approx 1 - e^{-\lambda} - e^{-\lambda} \cdot \lambda$

$= 1 - 4 \cdot e^{-3}.$

(A similar problem on Poisson approximation was solved in class on 02/07/2017.).

[4] (This is Exercise 3.197 of the textbook.)

[a] The probability p that a single 1-cm^3 sample has one or more bacteria colonies, is given by

$p = 1 - e^{-2}.$

Let Y_4 = number of samples (out of the four selected) that contain one or more bacteria colonies.

Then $Y_4 \sim Bin(4, p)$. So

$P(Y_4 \geq 2)$

$= 1 - P(Y_4 = 0) - P(Y_4 = 1)$

$= 1 - (1-p)^4 - 4p(1-p)^3$

$= 1 - e^{-8} - 4e^{-6}(1 - e^{-2})$

$= 1 - 4e^{-6} + 3e^{-8}$.

(4) [b] Let Y_n be the analogue of Y_4 defined above. Then

$P(Y_n = 0) = (1-p)^n = e^{-2n}$

Hence, the minimum ~~number~~ number required

~~of~~ of samples is the <u>smallest n</u> for which

$$e^{-2n} < 1-0.999 = 0.001$$

$$\Leftrightarrow \quad -2n < \log_e 10^{-3} = -3 \log_e 10$$

$$\Leftrightarrow \quad n > \frac{3}{2} \log_e 10 .$$

Hence the answer is $\left\lceil \frac{3}{2} \log_e 10 \right\rceil$

(5) [a] Jim ~~wi~~ loses each game with probability $1-p$. Hence, if Y denotes the number of games Jim loses, then

$$Y \sim Bin(7, 1-p) \text{, and}$$

$$X = 100 - 10Y .$$

Hence

$$EX = 100 - 10EY$$

$$= 100 - 10(1-p) \cdot 7$$

$$= 30 + 70p. \quad \blacksquare$$

(b) $V(X) = 10^2 V(Y)$

$$= 100 \times 7(1-p) \cdot p$$

$$= 700p(1-p). \quad \blacksquare$$