Midterm Preparation Questions

October 16, 2017

The exam will be *closed-book*. The questions listed below are more or less of the same kind that could appear on the midterm exam. However, *more* questions will be asked in the exam, also regarding topics *not* appearing here. In general, you should know the material that was covered in class, in the homework assignments, quizzes, and handouts.

On pages 3 and 4 are listed the rules for the Natural Deduction calculus and the Axiomatic calculus. If needed, these will be provided in the exam.

Be very careful in your choice of words, since incorrect use of terminology will result in a deduction of points in the midterm.

1. *Sets*

   You should be able to know the basics of set theory and of infinite cardinalities. Here some examples:

   (a) What is the powerset of the empty-set?
   (b) What is Cantor's Theorem?
   (c) How can you show that $|\mathbb{R}| > |\mathbb{N}|$?
   (d) How can you show that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$?

2. *Diagonalization*

   Prove that $S = \{f \mid f : \mathbb{N} \supset \mathbb{N}\}$ (the set all functions from $\mathbb{N}$ to $\mathbb{N}$) is not countably infinite (denumerable). In other words, prove that the cardinality of S is not $\aleph_0$.

3. *Terminology*

   You should be able to give correct, one-sentence definitions of all terms that were introduced in this course. For example:

   (a) What does it mean for a function to be *injective*?
   (b) What is a *derivation* in an axiomatic calculus?
   (c) What is a *formula* in first-order logic?
   (d) What does it mean to say that first-order logic is *sound*?
   (e) What does it mean to say that a theory is *complete*?

4. *Propositional and first-order logic*

   In addition to the proof on the handouts, here are some theorems that you should be able to prove.

   Prove the following using the ND calculus (see rules at the end of the handout).

   (a) $\vdash \sim(A \vee B) \supset (\sim A \wedge \sim B)$
   (b) $\sim A \wedge \sim B \vdash \sim(A \vee B)$
   (c) $\vdash (\sim B \supset \sim A) \supset (A \supset B)$
   (d) $A \supset B \vdash \sim B \supset \sim A$
   (e) $\vdash (\sim B \supset \sim A) \supset (A \supset B)$
   (f) $\vdash \sim\sim A \supset A$
   (g) $P, \sim P \vdash Q$
   (h) $P \vee Q, \sim P \vdash Q$
   (i) $\exists y : \forall x : A(x,y) \vdash \forall x : \exists y : A(x,y)$

5. *Formalizations*

   Consider the following first-order language $\mathcal{L}$ with the unary predicate symbols O and P, the binary predicate symbols L and G, the binary function symbol f, and the constants t, s, and n.

   We interpret it with the positive integers (i. e., 1, 2, 3, ...) as the universe of discourse and the following mapping:

$L(x,y)$	$\Longrightarrow$ x is less than y	$G(x,y)$	$\Longrightarrow$ x is greater than y
$E(x)$	$\Longrightarrow$ x is even	$O(x)$	$\Longrightarrow$ x is odd
$P(x)$	$\Longrightarrow$ x is prime	$f(x,y)$	$\Longrightarrow$ the product of x and y
t	$\Longrightarrow$ 2	s	$\Longrightarrow$ 7
e	$\Longrightarrow$ 11		

   (a) Translate each of the following statements into *two* English sentences.

   First into an English sentence that is very close to the formal expression, and second, into a sentence that is as colloquial as possible, but which still has the same meaning.

   i. $\forall x : \big(E(x) \supset \exists y : \big(O(y) \wedge G(x,y)\big)\big)$
   ii. $\exists x : \forall y : \sim(x = y) \supset L(x,y)$
   iii. $\forall y : \forall z : \big(E(y) \vee E(z)\big) \supset E(f(y,z))$
   iv. $\exists x : \forall y : (E(x) \wedge P(x) \wedge E(y) \wedge P(y)) \supset (x = y)$

   (b) What are the truth values of (i)–(iv) under the above interpretation?

   (c) Translate the following statements into formal sentences in $\mathcal{L}$.

   v. If a number is even and a prime, then that number is 2.
   vi. The product of any two odd positive integers is odd.
   vii. There is at least one prime number that is greater than 7 and less than 11.

Natural Deduction calculus

$\sim$–Intro $\quad \dfrac{\begin{matrix}[\mathsf{A}]\\ \vdots \\ \bot\end{matrix}}{\sim\mathsf{A}}$

$\sim$–Elim $\quad \dfrac{\mathsf{A} \quad \sim\mathsf{A}}{\bot}$

$\wedge$–Intro $\quad \dfrac{\mathsf{A} \quad \mathsf{B}}{\mathsf{A} \wedge \mathsf{B}}$

$\wedge$–Elim $\quad \dfrac{\mathsf{A} \wedge \mathsf{B}}{\mathsf{A}} \quad \dfrac{\mathsf{A} \wedge \mathsf{B}}{\mathsf{B}}$

$\vee$–Intro $\quad \dfrac{\mathsf{A}}{\mathsf{A} \vee \mathsf{B}} \quad \dfrac{\mathsf{B}}{\mathsf{A} \vee \mathsf{B}}$

$\vee$–Elim $\quad \dfrac{\mathsf{A} \vee \mathsf{B} \quad \begin{matrix}[\mathsf{A}]\\ \vdots \\ \mathsf{C}\end{matrix} \quad \begin{matrix}[\mathsf{B}]\\ \vdots \\ \mathsf{C}\end{matrix}}{\mathsf{C}}$

$\supset$–Intro $\quad \dfrac{\begin{matrix}[\mathsf{A}]\\ \vdots \\ \mathsf{B}\end{matrix}}{\mathsf{A} \supset \mathsf{B}}$

$\supset$–Elim $\quad \dfrac{\mathsf{A} \quad \mathsf{A} \supset \mathsf{B}}{\mathsf{B}}$

ex falso quodlibet $\quad \dfrac{\bot}{\mathsf{A}}$

RAA $\quad \dfrac{\begin{matrix}[\sim\mathsf{A}]\\ \vdots \\ \bot\end{matrix}}{\mathsf{A}}$

In the following a and x are variables, and t is a term:

$\dfrac{\mathsf{A}\,(\mathsf{a}/\mathsf{x})}{\forall \mathsf{x}\ \mathsf{A}}\ \forall \mathit{Intro} \qquad \dfrac{\forall \mathsf{x}\ \mathsf{A}}{\mathsf{A}\,(\mathsf{t}/\mathsf{x})}\ \forall \mathit{Elim}$

$\dfrac{\mathsf{A}\,(\mathsf{t}/\mathsf{x})}{\exists \mathsf{x}\ \mathsf{A}}\ \exists \mathit{Intro} \qquad \dfrac{\exists \mathsf{x}\ \mathsf{A} \quad \begin{matrix}[\,\mathsf{A}\,(\mathsf{a}/\mathsf{x})\,]\\ \vdots \\ \mathsf{C}\end{matrix}}{\mathsf{C}}\ \exists \mathit{Elim}$

Restrictions:

$\forall$*Intro* and $\exists$*Elim*: The (free) variable a (called an 'Eigenvariable', sometimes also a 'parameter') must be free for x in A.

$\forall$ *Intro*: a may not occur free in $\mathsf{A}(\mathsf{x})$, nor in any hypothesis on which the derivation depends, i. e., in any uncancelled hypothesis in the derivation of $\mathsf{A}(\mathsf{a}/\mathsf{x})$.

$\exists$ *Elim*: the (free) variable a may not occur free in $\mathsf{A}(\mathsf{x})$, nor in C, nor in any hypothesis on which the derivation depends (i. e., uncancelled hypotheses in the derivations of $\exists \mathsf{x} : \mathsf{A}$ and C).

$\forall$ *Elim* and $\exists$ *Intro*: The term t must be free for x in A.

Axiomatic Calculus

I. Propositional logic

Inference rule: From A and $\mathsf{A} \supset \mathsf{B}$ infer B (*modus ponens*).

Substitution rule: Any *wff* can be substituted for the metalogical variables A, B, and C (of course, the same metalogical variable must be substituted by the same *wff*).

Axiom schemata:

1. $\sim\mathsf{A} \supset (\mathsf{A} \supset \mathsf{B})$
2. $\mathsf{B} \supset (\mathsf{A} \supset \mathsf{B})$
3. $(\mathsf{A} \supset \mathsf{B}) \supset ((\sim\mathsf{A} \supset \mathsf{B}) \supset \mathsf{B})$
4. $(\mathsf{A} \supset (\mathsf{B} \supset \mathsf{C})) \supset ((\mathsf{A} \supset \mathsf{B}) \supset (\mathsf{A} \supset \mathsf{C}))$
5. $\mathsf{A} \supset (\mathsf{B} \supset (\mathsf{A} \wedge \mathsf{B}))$
6. $(\mathsf{A} \wedge \mathsf{B}) \supset \mathsf{A}$
7. $(\mathsf{A} \wedge \mathsf{B}) \supset \mathsf{B}$
8. $\mathsf{A} \supset (\mathsf{A} \vee \mathsf{B})$
9. $\mathsf{B} \supset (\mathsf{A} \vee \mathsf{B})$
10. $((\mathsf{A} \vee \mathsf{B}) \wedge \sim\mathsf{A}) \supset \mathsf{B}$

II. First-order logic

- **Logical axioms.** For all variables x and y, terms t, and all well-formed formulae A and B (we write $\mathsf{A}(\mathsf{x})$ to indicate that x is a free variable in A; if we write A, it may or may not contain free variables):

  CP. All axiom schemata of propositional logic.

  L1. (*Substitution*) $\big(\forall \mathsf{x} : \mathsf{A}(\mathsf{x})\big) \supset \mathsf{A}(\mathsf{t})$
  where $\mathsf{A}(\mathsf{t})$ arises by substituting t for all the occurrences of x in A for which t is free for x (i. e., $\mathsf{A}(\mathsf{t}) = \mathsf{A}(\mathsf{x})(\mathsf{t}/\mathsf{x})$).

  L2. ($\forall$*-Distribution*) $\big(\forall \mathsf{x} : (\mathsf{A} \supset \mathsf{B})\big) \supset (\mathsf{A} \supset \forall \mathsf{x} : \mathsf{B})$
  if A contains no free occurrence of x.

- **Axioms for equality.**

  E1. $(\mathsf{x} = \mathsf{x})$

  E2. $(\mathsf{x} = \mathsf{y}) \supset \big(\mathsf{A}(\mathsf{x}) \supset \mathsf{A}(\mathsf{y})\big)$
  where $\mathsf{A}(\mathsf{y})$ arises from $\mathsf{A}(\mathsf{x})$ by replacing some, but not necessarily all, free occurrences of x by y, and y is free for the occurrences of x which it replaces.

- **Inference rules.**

  Modus ponens: From A and $\mathsf{A} \supset \mathsf{B}$ infer B.

  Generalization: From A infer $\forall \mathsf{x} : \mathsf{A}$.

  Substitution rule: Any *wff* can be substituted for the metalogical variables A and B (of course, the same metalogical variable must be substituted by the same *wff*).

