Solution set for Assignment 1

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Exercise 1

Theorem. *For any natural number a, a^2 is even if and only if a is even.*

Proof. To prove the claim we need to show both directions of the biconditional 'iff':

1. If a^2 is even, then is a is even;
2. If a is even, then a^2 is even.

Case 1: We prove the first conditional by a *proof by contradiction*.

Assume, for the sake of contradiction, that under the condition that a^2 is even, a is not even.

If a is not even, then it is odd. By the definition, we can write a as $a = 2b + 1$. But then,

$$a^2 = (2b + 1)^2 = 4b^2 + 4b + 1 = 2(2b^2 + 2b) + 1$$

Thus, by definition, a^2 is also odd. This contradicts our assumption that a^2 is even. Thefore, the assumption is false, and a is even.

Alternatively, we could also have proved the contrapositive of the claim, that is, that if a is odd (i. e., not even), then a^2 is also odd.

Case 2: It is left to prove that if a is even then so is a^2. Suppose a is even, then, by definition, there exists a natural number b such that $a = 2b$. But then, $a^2 = (2b)^2 = 4b^2 = 2(2b^2)$. Thus, a^2 is also even, by definition.

□

Exercise 2

- **My descendants**
  1. *Base clause:* All my children are my descendants.
  2. *Inductive clause:* If x is my descendant, then x's children are my descendants.
  3. *Final clause:* No one else is my descendant.
- **Palindromes** (composed of symbols from the set Σ)
  1. *Base clause:* For all $x \in \Sigma$, x is a palindrome.

2. *Inductive clause:* If P is a palindrome and $x \in \Sigma$, then xPx is a palindrome.
3. *Final clause:* Nothing else is a palindrome.

- **Binary Trees**

1. *Base clause:* The **empty tree** or **null tree** is a binary tree.
2. *Inductive clause:* If T_1 and T_2 are binary trees and r is a node, then the graph obtained by adding an edge directed from r to the roots of T_1 and T_2 is a binary tree.
3. *Final clause:* Nothing else is a binary tree.

Exercise 3

Claim: For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} 2i = n(n+1)$$

Proof. (By mathematical induction on n.)

- *Base case.*
  Let $n = 1$, then

$$\sum_{i=1}^{n} 2i = \sum_{i=1}^{1} 2i = 2 = 1(1+1) = n(n+1)$$

- *Induction step.*
  Induction Hypothesis (IH): Suppose the claim holds for some m, i.e.

$$\sum_{i=1}^{m} 2i = m(m+1).$$

  To show: The claim holds for $m+1$.

$$\begin{aligned} \sum_{i=1}^{m+1} 2i &= \sum_{i=1}^{m} 2i + 2(m+1) \\ &= m(m+1) + 2(m+1) && \text{by IH} \\ &= (m+1)(m+2) \end{aligned}$$

- *Conclusion.* The claim holds for all natural numbers.

□

Exercise 4

The domain of well-formed *geq*-strings is recursively defined as follows:

1. (Base clause) Any string of the form '`-geq-`' is a well-formed *geq*-string.
2. (Inductive clause 1) If 'x `geq` x' is a well-formed *geq*-string, so is 'x`-geq` x`-`', where x is composed of hyphens only.
3. (Inductive clause 2) If 'x `geq` y' is a well-formed *geq*-string, so is 'x`-geq` y', where x and y are composed of hyphens only.
4. (Final clause) Nothing else is a well-formed *geq*-string.

(a) Three well-formed *geq*-strings: '`--geq-`', '`----geq-`', '`--geq--`'.
Three strings that are not well-formed *geq*-strings: '`-* geq -*`', 'x`-geq-`', '`-geq--`'.

(b) **Claim:** If 'x `geq` y' is a well-formed *geq*-string, then the number of hyphens in x is greater than or equal to the number of hyphens in y.

Proof. (By mathematical induction on the structure of 'x `geq` y'.)

- *Base case.*
  Let 'x `geq` y' be of the form '`-geq-`'. Then the number of hyphens in x (= '`-`') is clearly equal to the number of hyphens in y (= '`-`').
- *Induction step 1.*
  Induction Hypothesis (IH): Suppose the claim holds for some 'x `geq` x', i. e., the number of hyphens in x, say m, is greater than or equal to the number of hyphens in x, which is also m.
  To show: The claim holds for 'x`-` `geq` x`-`'.
  But the number of hyphens in 'x`-`' is $m + 1$, which is equal to the number of hyphens in 'x`-`'.
- *Induction step 2.*
  Induction Hypothesis (IH): Suppose the claim holds for some 'x `geq` y', i. e., the number of hyphens in x, say m, is greater than or equal to the number of hyphens in y, say n.
  To show: The claim holds for 'x`-` `geq` y'.
  But the number of hyphens in 'x`-`' is $m + 1$, which is, by induction hypothesis, greater than or equal to $n + 1$, which is greater than n, the number of hyphens in y.
- *Conclusion.* The claim holds for all well-formed *geq*-strings.

□

(c) Yes, you can obtain such a *geq*-string by taking '`- geq -`' and applying the Inductive Clause 1 $(a - 1)$-many times.

(d) Yes, you can obtain such a *geq*-string by taking '`- geq -`', applying the Inductive Clause 1 $(b - 1)$-many times, and then applying the Inductive Clause 2 $(a - b)$ times.

Exercise 5

Consider the sets $A = \{b, c, d, e\}$ and $B = \{a, b, d, f\}$.

a. F_1 is a total bijective function from A to B.

b. F_2 is a total function from A to B. It is neither injective nor surjective.

c. F_3 is not a function from A to B, since a, for instance, is not an element of A.

d. F_4 is not a function, since b is mapped to more than one element.

e. F_5 is a not a total function. It is a partial function from A to B. It is injective.

Exercise 6

(a) Recall that a function f from A to B is said to be injective or 1-1 if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$.

Claim: The function $f(x) = x + 7$ is injective on the natural numbers.

Proof. Suppose $f(x) = f(y)$ for some x, $y \in \mathbb{N}$. Then, it follows:

$$\begin{array}{rcll} x+7 & = & y+7 & \text{by definition of } f \\ x & = & y & \text{by subtracting 7 from both sides,} \end{array}$$

from which we can conclude that f is injective. □

(b) $f : \mathbb{N} \to \mathbb{N}$ is surjective, if for all $y \in \mathbb{N}$ there exists an $x \in \mathbb{N}$ such that $f(x) = y$. Suppose, for the sake of contradiction, that f is surjective and pick $y = 2$. Then by definition, there must exits $x \in \mathbb{N}$ such that $f(x) = 2$. That is,

$$\begin{array}{rcll} x+7 & = & 2 & \text{by definition of } f \\ x & = & -5 & \text{by subtracting 7 from both sides.} \end{array}$$

But $-5 \notin \mathbb{N}$ (-5 is not a natural number, it is an integer). Therefore, there is no $x \in \mathbb{N}$ such that $f(x) = 2$, which contradicts the assumption that f is surjective. Conclusion: f is not surjective.

Exercise 7

Claim: The power set of the natural numbers, $\mathcal{P}(\mathbb{N})$, is not denumerable.

Proof. (By contradiction, using diagonalization.)

Assume, for the sake of contradiction, that $\mathcal{P}(\mathbb{N})$ is denumerable, i. e., there exist a bijection f from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. Using such a bijection, we can construct the following table:

	1	2	3	...
1	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	...
2	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	...
3	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

where, for all $i, j \in \mathbb{N}$, a_{ij} is defined as follows

$$a_{i,j} := \begin{cases} 0, & \text{if } j \notin f(i) \\ 1, & \text{if } j \in f(i) \end{cases}$$

Thus, if f maps i to the subset of the natural numbers A_i (that is, $f(i) = A_i$), then A_i contains the natural number j if and only if $a_{i,j} = 1$. In other words, $a_{i,j} = 0$ indicates that the number j *is not* an element of the set $f(i)$, and $a_{i,j} = 1$ indicates that the number j *is* an element of the set $f(i)$. — Note that nothing depends on the choice of 0 and 1 for the values of $a_{i,j}$; we could just as well have chosen the words '*no*' and '*yes*' instead. Make sure you understand why this is the case.

Example. Suppose the bijection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is defined, on the first three natural numbers, as follows

$$\begin{aligned} f(1) &= \{2, 3, 108\} \\ f(2) &= \{x \in \mathbb{N} \mid \exists y \in \mathbb{N} \,.\, x = 2y + 1\} \\ f(3) &= \emptyset \end{aligned}$$

That is, f maps 1 to the set of three elements $\{2, 3, 108\}$, 2 to the set of odd numbers and 3 to the empty set. Using the definition given for the $a_{i,j}$'s, we get:

$$a_{1,1} = 0, a_{1,2} = 1, a_{1,3} = 1, ..., a_{1,107} = 0, a_{1,108} = 1, a_{1,109} = 0, ...$$
$$a_{2,1} = 1, a_{2,2} = 0, a_{2,3} = 1, a_{1,4} = 0, ...$$
$$a_{3,1} = 0, a_{3,2} = 0, a_{3,3} = 0, a_{3,4} = 0, ...$$

Thus, the table described above would look as follows:

	1	2	3	4	...
1	0	1	1	0	...
2	1	0	1	0	...
3	0	0	0	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now, we construct a set B as follows:

Let $i \in \mathbb{N}$,

1. If $a_{i,i} = 1$, then $i \notin B$. (i. e., if i *is* an element of the set $f(i)$, then i *is not* an element of B);
2. If $a_{i,i} = 0$, then $i \in B$ (i. e., if i *is not* an element of the set $f(i)$, then i *is* an element of B).

Clearly B is a set containing only natural numbers, i. e., $B \subseteq \mathbb{N}$.

(Note: We could have constructed the set B in a simpler way, but defining it as

$$B = \{x \in \mathbb{N} \mid x \notin f(x)\},$$

analogously to the proof of Cantor's Theorem, presented in class. However, then the use of the 'diagonal' would be much less apparent. Make sure you understand why both definitions work!)

Thus, $B \in \mathcal{P}(\mathbb{N})$ and, by assumption, B must be in the range of f. That is, there exists $n \in \mathbb{N}$ such that $f(n) = B$.

Now, there are two possibilities: either $n \in B$ or $n \notin B$.

- **Case 1**: $n \in B$. By assumption $B = f(n)$, thus $n \in f(n)$.

  But then, by definition of B, it must be that $n \notin B$. $\lightning$
- **Case 2**: $n \notin B$. By assumption $B = f(n)$, thus $n \notin f(n)$.

  But then, by definition of B, it must be that $n \in B$. $\lightning$

Both cases lead to a contradiction. Thus, we can conclude that there is no bijection f from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$, i. e., that is $\mathcal{P}(\mathbb{N})$ is not denumerable. $\square$

Exercise 8

a) The cardinality of $A \cup \{a\}$ is still $\aleph_0$.

   To prove this we need to find a bijection from $\mathbb{N}$ to $A \cup \{a\}$.

   Since A is countably infinite, there must be a bijection f from $\mathbb{N}$ to A. Now, define a bijection g from $\mathbb{N}$ to $A \cup \{a\}$ as follows: $g(0) = a$ and $g(n) = f(n - 1)$ for all $n \in \mathbb{N}$.

b) The cardinality of $A \cup B$ is still $\aleph_0$.

   To prove this we need to find a bijection from $\mathbb{N}$ to $A \cup B$.

   Let A and B be two sets of countably infinite sets, whose elements are indexed by natural numbers, i. e., $A = \{a_i \mid i \in \mathbb{N}\}$ and $B = \{b_i \mid i \in \mathbb{N}\}$. Now consider the function $f : \mathbb{N} \to A \cup B$ such that $f(2i) = a_i$ and $f(2i + 1) = b_i$. The function f is clearly injective and, if $|A| = |B| = \aleph_0$, f is also surjective.

Exercise 9

(a) It is the mathematical study of formal mathematical theories.

(b) In our case, the object language is the (formal) language in which we make formal proofs. The meta-language is the language in which we reason *about* the object-language (it consists of formal and non-formal elements).

(d) *Principia Mathematica* is a three-volume work by Bertrand Russell and Alfred North Whitehead, published in 1910–1913. It includes a formal presentation of arithmetic.

(e) State three English adjectives that are autological and three that are heterological.

Autological:	Forgettable	Unique	Finite
Heterological:	Long	Colorful	Warm

Now, is adjective heterological itself heterological, or not?