3411. Prove that each of the two given systems of vectors S and S' is a basis. Find the matrix of the change of the base S to S'.

a) $S = ((1, 2, 1), (2, 3, 3), (3, 8, 2))$,
$S' = ((3, 5, 8), (5, 14, 13), (1, 9, 2))$;

b) $S = ((1, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1,), (1, 3, 2, 3))$,
$S' = ((1, 0, 3, 3), (-2, -3, -5, -4), (2, 2, 5, 4), (-2, -3, -4, -4))$.

3412. Prove that in the space $\mathbb{R}[x]_n$ of polynomials of degree $\leq n$ with real coefficients the systems

$$\{1, x, \dots, x^n\} \text{ and } \{1, x-a, (x-a)^2, \dots, (x-a)^n\} \quad (a \in \mathbb{R})$$

are bases. Find the coordinates of the polynomial $f(x) = a_0 + a_1x + \dots + a_nx^n$ in these bases and the matrix of change from the first basis to the second one.

3413. What happens with the matrix of the change from one basis to another if

a) we interchange two vectors of the first base;

b) we interchange two vectors of the second base;

c) we write the vectors of both bases in inverse order?

3414. Prove that the following systems of vectors are linearly independent and complete them to a basis of the space of rows

a) $a_1 = (2, 2, 7, -1)$, $a_2 = (3, -1, 2, 4)$, $a_3 = (1, 1, 3, 1)$;

b) $a_1 = (2, 3, -4, -1)$, $a_2 = (1, -2, 1, 3)$;

c) $a_1 = (4, 3, -1, 1, 1)$, $a_2 = (2, 1, -3, 2, -5)$,
$a_3 = (1, -3, 0, 1, -2)$, $a_4 = (1, 5, 2, -2, 6)$;

d) $a_1 = (2, 3, 5, -4, 1)$, $a_2 = (1, -1, 2, 3, 5)$.

35 Subspaces

3501. Find out whether the following sets of vectors form a subspace of appropriate vector spaces:

a) vectors of the plane with the origin O whose ends belong to one of two given lines which are intersecting at the point O;

b) vectors of the plane with the origin O whose ends belong to a given line;

c) vectors of the plane with the origin O whose ends do not belong to given line;

d) vectors of the coordinate plane whose ends belong to the first quadrant;

e) vectors of the space $\mathbb{R}^n$ with integer coordinates;

f) vectors of an arithmetic space F^n, where F is a field, which are solutions of a given system of linear equations;

g) vectors of a linear space which are linear combinations of the given vectors $a_1, \dots, a_k$;

h) bounded sequences of complex numbers;

i) convergent sequences of real numbers;

j) sequences of real numbers with the fixed limit a;

k) sequences $u(n)$ of elements of a field F satisfying the recurrence equation

$$u(n+k) = f(n) + a_0u(n) + a_1u(n+1) + \dots + a_{k-1}u(n+k-1),$$

where $(f(n))$ is a fixed sequence of elements of F, k is a fixed natural number, and $a_i \in F$;

l) polynomials of even degree with coefficients in a field F;

m) polynomials with coefficients in a field F which do not contain even powers of the variable x;

n) elements of the space 2^M (see Exercise 409) of even cardinalities;

o) elements of 2^M of odd cardinalities.

3502. Prove that the following sets of vectors in a space F^n, where F is a field, form subspaces. Find their bases and dimensions:

a) vectors in which the first and last coordinate coincide;

b) vectors in which the coordinates with even indices are equal to 0;

c) vectors in which the coordinates with even indices are equal;

d) vectors of the form $(\alpha, \beta, \alpha, \beta, \dots)$;

e) vectors which are solutions of a homogeneous system of equations.