Math 340 – Jan. 4

Prof. Sergey Norin

Topics:
- Graph Theory: study of networks
- Probability
- Enumeration

Prerequisites:
Math 240 or 235
- Methods of proofs:
  - by contradiction
  - by induction
- Basic notions of graph theory
- Basics of counting (eg. product rule and binomial theorem)
- Series

6 assignment
- best 5 count to grade

Exams are closed book

OH. Wed. after class

—

Discrete Probability

Monty Hall Problem

A host offers a choice of three doors, behind one is a car and behind the others are goats. You pick door A. The host opens another door (say C) behind which is a goat. We are offered a choice to switch to door B.

What is the correct answer?
1. Stick with A
→ 2. Switch to B
3. Doesn't matter.

—

Enumeration

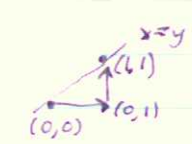
Catalan numbers:

$c_0, c_1, c_2, \ldots, c_n, \ldots$

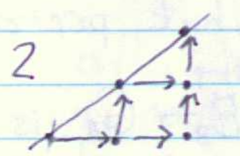
c_n – number of walks on integer lattice from $(0,0)$ to (n,n) going "up" or "to the right" by one unit each time, not going over the $x=y$ line

$c_0 = 1$

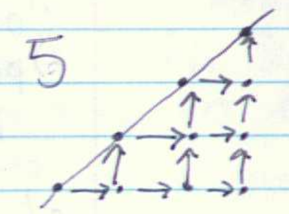
$c_1 = 1$



$c_2 = 2$



$c_3 = 5$

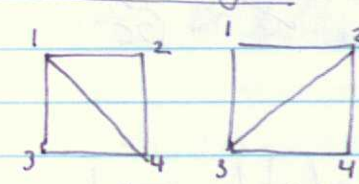


Triangulations of a regular n+2-gon

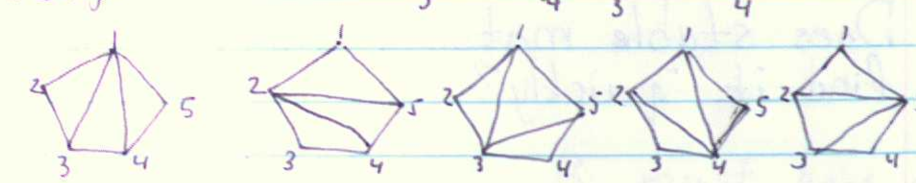
$c_1 = 1$

1 triangle

$c_2 = 2$



$c_3 = 5$



$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

by Taylor series

$$c(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

Stable Marriage Problem

n boys & n girls

- our goal is a <u>matching</u> (ie. each boy is paired up with exactly one girl)
- each boy and girl order girls (or respectively boys) in their order of preference

$B_A : G_A > G_B > G_C$ $\quad$ $G_A : B_C > B_B > B_A$

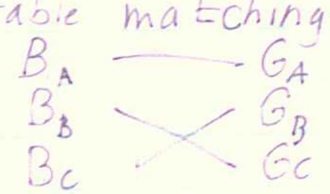
$B_B : G_C > G_A > G_B$ $\quad$ $G_B : B_B > B_C > B_A$

$B_C : G_B > G_C > G_A$ $\quad$ $G_C : B_A > B_B > B_C$

- a matching is <u>stable</u> if there does not exist a pair of boy and girl who are not matched to each other, but prefer each other to their respective partners

ie. $B_B - G_A$ is not stable

stable matching:



Does stable matching always exist & can we find it "quickly"?

<u>Boy Proposal Algorithm:</u> (BPA)

1. At each step, choose an arbitrary boy not currently matched ("engaged") to anyone
2. He proposes to the girl he likes most who hasn't rejected him yet
3. The girl he proposes to accepts the proposal if she is not yet engaged, or if she prefers the proposing boy to her current partner with whom she breaks up
4. Algorithm stops when all boys are engaged.

<u>Observation:</u>

1. Once a girl gets engaged, she remains engaged, and her partners only improve
2. Boys' partners can only get worst

<u>Theorem:</u>

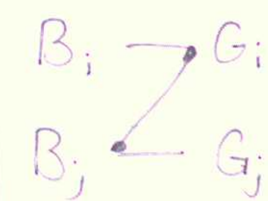
Boy Proposal Algorithm (BPA) always terminates in a stable match.

Proof:

The algorithm always <u>terminates.</u> There is at most n^2 steps since no boy can propose to a girl twice. Moreover, when it terminates all the boys will be matched. Indeed, if a boy asked all the girls in his list then all the girls have been asked, so they are all engaged and thus all the boys are too.

The resulting matching is <u>stable</u>.

Suppose not, namely there are G_i & B_j who prefer each other current partners B_i & G_j respectively



$G_i : B_j > B_i$

$B_j : G_i > G_j$

This is a contradiction as B_j must have proposed to G_i and therefore G_i will end up with a boy she likes at least as much as B_j, so not B_i. ∎

Also known as the Gale-Shapley Algorithm, 1962.

Nobel Prize in economics in 2012.

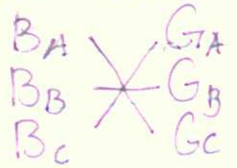
Does the choice of a boy proposing matter?

Does it benefit boys (and/or girls) to misrepresent their preferences?

(No difference)

There exists potentially several stable matchings.

– here every girl gets her 1st preference



We say that girl G and boy B are <u>valid</u> partners if there is some stable matching such that G & B are matched.

For boy B, let $G^+(B)$ denote the valid partner of B that he likes most.

<u>Theorem:</u>
In BPA, every boy B is matched to $G^+(B)$.

Math 340 - Jan. 6

Recall:

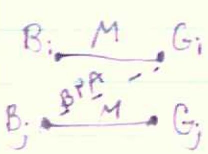
- Stable Marriage Problem
  - proved algorithm gives a stable matching (BPA)
  - a boy B is a valid partner for girl G if they are matched in some stable matching

Lemma:

In BPA a girl never rejects or breaks up with a valid partner.

Proof:

Suppose otherwise, namely, that some girl G_i rejects (or breaks up) with a valid partner B_i and choose the first such moment.
Let B_j be the boy G_i rejects B_i for.
There is a stable matching M such that G_i & B_i are paired. Let G_j be the partner of B_j in M.



$G_i : B_j > B_i$

Since M is stable $\Rightarrow B_j : G_j > G_i$, then B_j would have proposed to G_j first so he would be engaged to her. Thus we have a contradiction. □

For a boy B, let $G^+(B)$ denote the valid partner of B he likes most

Theorem:

In BPA, each boy B is matched to $G^+(B)$

Proof:

Every boy ends up with a valid partner and the first valid partner B proposes to is $G^+(B)$ and by the lemma she will never break up this engagement. □

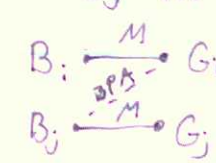
For girl G, let $B^-(G)$ denote the valid partner she likes least.

Theorem:

In BPA, each girl ends up with $B^-(G)$.

Proof:

Suppose otherwise, namely, some girl G_i ends up with $B_j \neq B^-(G)$. But in some stable matching M, G_i & $B^-(G_i) = B_i$ are matched and let B_j be matched to G_j in M.



$G_i : B_j > B_i$ ← because B_i valid partner for G_i

$B_j : G_i > G_j$

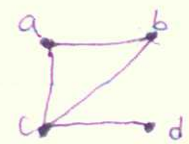
By the previous theorem, $G_i = G^+(B_j)$ and as G_j is a valid partner to B_j, he preferes G_i. Thus we have a contradiction. □

Basics of Graph Theory

A *graph* G is a pair $(V(G), E(G))$ where $V(G)$ is a finite set and $E(G)$ is a set of pairs of elements of $V(G)$.

Elements of $V(G)$ are called *vertices* and elements of $E(G)$ are edges.

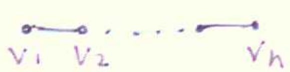
$V(G) = \{a, b, c, d\}$
$E(G) = \{ab, ac, bc, cd\}$



Standard graphs:

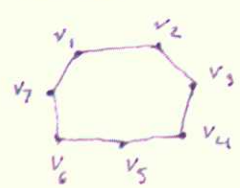
K_n - *complete graph*: n vertices and every two vertices are joined by an edge

Elements of an edge are its ends.
Two vertices are *adjacent* if they are ends of some edge.
An edge is *incident* to its ends.
A *path* P_n on n vertices (with ends v_1 and v_n)



$V(P_n) = \{v_1, \dots, v_n\}$
$E(P_n) = \{v_1v_2, \dots, v_{n-1}v_n\}$

A *cycle* C_n $(n \geq 3)$



A *subgraph* $H \subseteq G$ is a graph H s.t.
$V(H) \subseteq V(G)$ and
$E(H) \subseteq E(G)$

A path (or cycle) is a subgraph of G.

A *degree* of a vertex v of G, denoted $\deg_G(v)$ or $\deg(v)$, is the number of edges containing or more specifically the number of vertices *adjacent* to v (called neighbours of v).

A graph obtained from G by *deleting an edge* $e \in E(G)$, denoted $G \setminus e$, is a subgraph of G with $V(G \setminus e) = V(G)$, $E(G \setminus e) = E(G) - \{e\}$.

Ex. $G \setminus cd$

Deleting a vertex $v \in V(G)$, denoted $G \setminus v$ with $V(G \setminus v) = V(G) - \{v\}$, $E(G \setminus v)$ - set of all edges of G not incident to v.

A graph G is *connected* if every pair $u, v \in V(G)$ there is a path in G with ends u and v.
$G \setminus cd$ is not connected.

A connected component of G is a *maximal subgraph* of G which is connected

The components of G, denoted $\text{comp}(G)$, is the number of components of G.
$\text{comp}(G \setminus cd) = 2$.

A *forest* is a graph with no cycles.

A *tree* is a connected forest.

Lemma:
If F is a forest then $|E(F)| - |V(F)| + \text{comp}(F) = 0$.
In particular, if F is a tree
$$|E(F)| = |V(F)| - 1.$$

No proof given in this course (as well for other lemmas)

Lemma: Handshaking Lemma
In any graph G,
$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

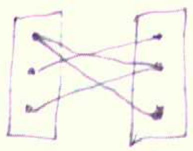
A leaf is a vertex of degree one.

Lemma:
If T is a tree with ≥ 2 vertices then T has ≥ 2 leaves.

Lemma:
In a tree there is exactly one path joining any pairs of vertices. Equivalently, for any $u, v \in V(T)$ there is a unique path in T with ends u and v.

A bipartition of G is a partition (A, B) of $V(G)$ ($A \cap B = \emptyset$, $A \cup B = V(G)$) such that every edge has one end in A and another in B.

A graph is bipartite if it admits a bipartition.



A B

Theorem:
A graph G is bipartite iff it does not contain an odd cycle (a cycle with an odd number of vertices)

A proper vertex k-colouring of a graph G is a map $\varphi: V(G) \to \{1, 2, \ldots, k\}$ such that if u and v are adjacent that $\varphi(u) \neq \varphi(v)$.

The chromatic number of G, $\chi(G)$, is the minimum k such that there exists a proper vertex k-colouring of G.

$\chi(G) \leq 1 \iff E(G) = \emptyset$ (edgeless)
$\chi(G) \leq 2 \iff G$ is bipartite

Theorem:
$\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

For $K_n \Rightarrow n$ vertices, $\deg(v) = n - 1 \Rightarrow \chi(K_n) \leq n$.

Math 340 - Jan. 11

Matchings

Ex. Dorm room assignments
- assign people to rooms with capacity 2

Graph vertices - people
edges - pairs of suitable roommates

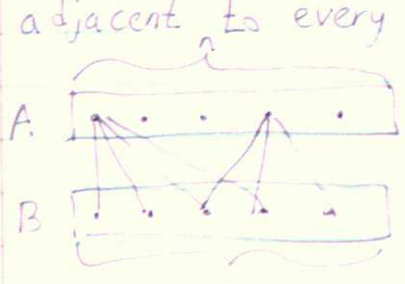
A matching M in a graph G is a collection of edges no two of which share an end.

A vertex is covered by M if it is an end of some edge in M, uncovered otherwise.

A matching is perfect if all the vertices are covered.

Ex. For which n does K_n have a perfect matching?
If and only if n is even. Since it is perfect there will be $\frac{n}{2}$ edges in the matching.

Ex. Complete bipartite graph $K_{n,n}$ is a graph with bipartite (A, B) such that $|A| = |B| = n$ and every vertex of A is adjacent to every vertex of B.



How many perfect matchings are there in $K_{n,n}$?
$n!$

1 vertex has n options, 2nd has $n-1$ options, ...

Listing the matches of vertices A gives an ordering of B, there are $n!$ orderings.

Ex. Count perfect matchings in K_{2n}.

$$(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot \ldots \cdot 1$$

By induction:
Induction step: there are $(2n-1)$ choices of a match for vertex one
By induction hypothesis
$(2n-3)(2n-5)\ldots 1$ matchings of the remaining $2n-2$ vertices

Another method:
There are $\binom{2n}{n}$ ways to partition vertices of K_{2n} into disjoint sets A, B, $|A| = |B| = n$. There are $n!$ perfect matchings joining vertices of A to vertices of B.
$\binom{2n}{n}n!$, but every perfect match is counted 2^n times.
So there are $\binom{2n}{n}n!/2^n$ perfect matchings.

Thus $$\frac{\binom{2n}{n}n!}{2^n} = (2n-1)(2n-3)\ldots 1$$

Our goal is to find maximum matching in a graph G: matching with maximum # of edges.

Maximal matching in G is a matching M which is not a subset of a larger matching.

Maximal and maximum are not always equal.

Let M be a matching.
A path P is <u>M-alternating</u> if the edges of P alternate between edges in M and not in M.

A M-alternating path is <u>M-augmenting</u> if it has ≥ 1 edge and the ends of it are not covered by M.

Observation:
If there exists an M-augmenting path P in G then $E(P) \,\Delta\, M$ is a matching M' of G such that $|M'| = |M| + 1$.
symmetric difference ↑

Note: $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$

<u>Theorem</u>: Berge
Let M be a matching in a graph G. Then M is maximum if and only if G contains no M-augmenting path.

Proof:
($\Rightarrow$): if G contains an M-augmenting path, then M is not maximum (by observation above)

($\Leftarrow$): Suppose M is not maximum, our goal is to show that G contains an M-augmenting path.

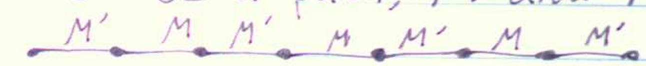
Let M' be a matching in G with $|M'| > |M|$

Consider the graph with the edge set of $M \Delta M'$

The graph with the edge set of $M \Delta M'$ has maximum degree ≤ 2.

Every connected component of this graph is a path or a cycle. If it is a cycle it is an even cycle alternating between edges of M and M'.

$|M'| > |M|$.
Some component of $M \Delta M'$ contains more edges of M' than M.
It must be a path, M and M'-alternating.

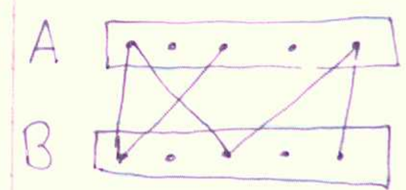


In fact starting and ending with an edge of M', which is <u>M-augmenting</u>.

$\square$

We will restrict our attention to bipartite graphs.

<u>Bipartite Graphs</u>

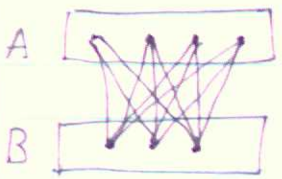


A - job candidates

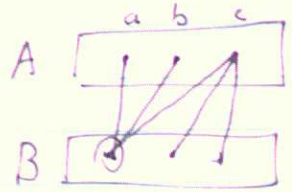
B - positions

Let us try to understand when it is possible to find positions for all job candidates.
A matching in the bipartite graph with bipartition (A, B) which covers A. (Every vertex in A is an end of a matching edge).

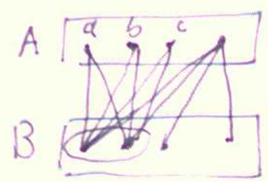
Examples where no such matchings exist.



- no enough jobs



- both a,b want the same job



- only 2 jobs for a,b,c

For $S \subseteq V(G)$, let $N(S)$ (neighbourhood) denote the vertices in G which are adjacent to at least one vertex in S.

<u>Theorem</u>: Hall

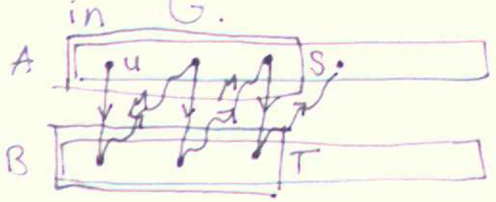
Let G be a bipartite graph with bipartition (A,B) then there is a matching in G covering A if and only if $(*)\ |N(S)| \geq |S|$ for every $S \subseteq A$.

$(*)$ is known as Hall's condition for S.

Proof:

$(\Rightarrow)$: If $|S| > |N(S)|$ then S cannot be covered by a matching because every vertex in S must be matched to a vertex in $N(S)$.

$(\Leftarrow)$: Suppose $(*)$ holds for every $S \subseteq A$.
Consider the maximum matching M in G.
Suppose some $u \in A$ is uncovered.
By the choice of M, there is no M-augmenting path in G.



Let S and T be subsets of A and B reachable by M-alternating paths from u.

- every vertex in T is matched to a vertex in S by an edge of M

- $|S| > |T|$ because M matches T into $S \setminus \{u\}$

- $N(S) \subseteq T$ as any edge from S to $B \setminus T$ is not in M can be used to extend an alternating path from u.

$|N(S)| \leq |T| < |S|$

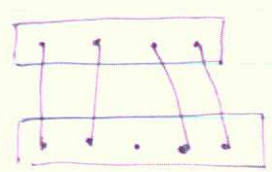
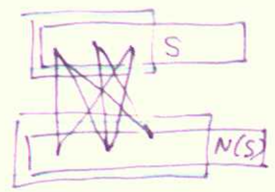
So $(*)$ is violated for S. Contradiction. $\square$

Math 340 - Jan. 13

Theorem: Hall!

Let G be a bipartite graph with bipartition (A, B). Then G has a matching covering A iff

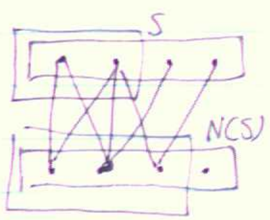
$|N(S)| \geq |S|$ for every $S \subseteq A$ $\leftarrow$ Hall's condition

A  or A 

B

Perfect matching is a matching covering all vertices.
A graph G is **k-regular** if every vertex of G has degree k.

Theorem:
If a bipartite graph G is k-regular for some $k > 0$. Then G has a perfect matching.

A 

B

$|A| = |B|$

- every edge has exactly one end in A

$$k|A| = \sum_{v \in A} deg(v) = |E(G)|$$

Symmetrically, $|E(G)| = \sum_{v \in B} deg(v) = k|B|$

$$\Rightarrow |A| = |B|$$

Proof:
It is enough to find a matching covering A, by Hall's theorem we need to show

$|N(S)| \geq |S| \quad \forall S \subseteq A$

Counting edges between S and $N(S)$, as before.
Counting ends in S, we have $k|S|$ such edges, and counting ends in $N(S)$, there are $\leq k|N(S)|$.
So $k|N(S)| \geq k|S| \Rightarrow |N(S)| \geq |S|$.

$\square$

k-edge colouring of G is a function
$c: E(G) \to \{1, 2, \ldots, k\}$
such that $c(e) \neq c(f)$ if $e, f \in E(G)$ share an end.

Ex. $K_{3,3}$ 

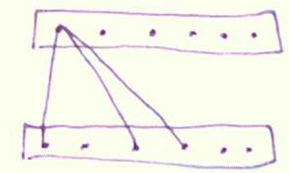
Edge-chromatic number $\chi'(G)$ is the minimum k such that G has a k-edge colouring

$\chi'(K_{3,3}) = 3$

In general, $\chi'(G) \geq \Delta(G) \leftarrow$ maximum degree of G.
because all edges incident to some fixed vertex must receive different colours. (equality doesn't always hold.)

K_3 △ $\chi'(K_3) = 3$
$\Delta(K_3) = 2$

Applications

Ex. A  tutors

B students

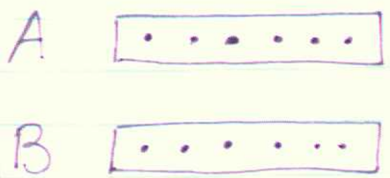
Edges represent pairs tutor student, we are interested in scheduling one-on-one tutoring sessions.

What is the minimum number of time slots (colours) necessary? (Modelled by edge colouring).

Theorem:
Let G be a k-regular bipartite graph then $\chi'(G) = k$.

Proof:

Assume $k \geq 1$.



$n = |A| = |B|$ as shown in previous proof

$|E(G)| = kn$

Note: k-edge colouring corresponds to partitioning $E(G)$ into k matchings $M_1, M_2, \dots, M_k$

$|M_i| \leq n$

By the above, the theorem is equivalent to showing that edges of such graph can be partitioned into k perfect matchings.

Proof by induction on k.
By previous theorem there exists a perfect matching M_k of G.

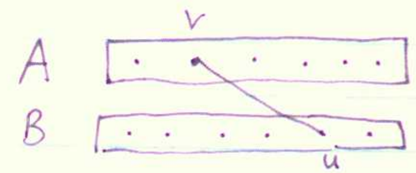
Base case: $k = 0$ $\quad$ 0-colouring is valid.

Delete the edges M_k from G to obtain G'.
G' is $(k-1)$-regular, so by the induction hypothesis the edges of G' can be coloured using colours $\{1, 2, \dots, k-1\}$. Use colour k on edges of M_k to obtain a colouring of G with k colours.

$\square$

Lemma:
If G is a bipartite graph with $\Delta(G) \leq k$ then G is a subgraph of some k-regular bipartite graph.

Proof:
Let (A, B) be a bipartition of G, by adding vertices to A and B if necessary, we assume that $|A| = |B| \geq k$



If G is not k-regular, we'd like to add edges.

Let G' be a graph with $V(G') = V(G)$ such that $\Delta(G') \leq k$ and subject to that $|E(G')|$ is as large as possible.

We want to show that G' is k-regular.

If not, then $\deg(v) < k$ for some $v \in V(G')$ say $v \in A$
Counting, as before, $|E(G')| < k|A| = k|B|$.
Therefore there exists $u \in B$ $\deg(u) < k$.
Add an edge joining u and v to G'.
So the resulting graph has max. degree almost k, more edges than G', contradicting the choice of G', so G' is k-regular.

Issue:
- v-u may already have a connecting edge.
- can resolve through careful choice of joining edges.

$\square$

Theorem:
If G is a bipartite graph then $\chi'(G) = \Delta(G)$

Proof:
$\chi'(G) \geq \Delta(G)$ as mentioned earlier.
Let $\Delta(G) = k$.
By the result before there exists a k-regular bipartite graph G' s.t. G is a subgraph of G' and $\chi'(G') = k$.

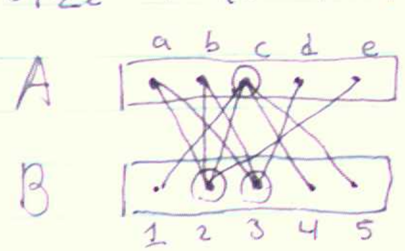
Every k-edge colouring of G' produces a k-edge colouring of G, so $\chi'(G) \leq k = \Delta(G)$.

□

Theorem: Vizing
If G is a simple graph (no parallel edges) then $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G)+1$
If $P \neq NP$, then there is no efficient $\longrightarrow \Delta$ way to determine whether $\chi'(G) = 3$ or $\chi'(G) = 4$.

What can we say about maximum size of a matching in a bipartite graph G?

How can we certify that G has no matching of size k (with k edges)?



What is the largest size of a matching?
a2 b3 c4

A vertex cover in a graph G is $X \subseteq V(G)$ such that every edge has ≥ 1 end in X.

If G has a vertex cover of size k then it has no matching of size $>k$.

For bipartite graphs the converse holds
If there is no matching of size $>k$ then there is a vertex cover of size $\leq k$.