Greek Mathematics

We start by looking at four great theorems of antiquity. Ancient Greek mathematics is justly considered one of the first flowerings of human intellectual achievement. Between 500 and 200 BC Pythagoras, Eudoxus, Euclid, Archimedes and their contemporaries revolutionized our ideas of number and space and laid the foundations for science and mathematics as we know it. Nearly two thousand years passed before the next mathematical revolution took place.

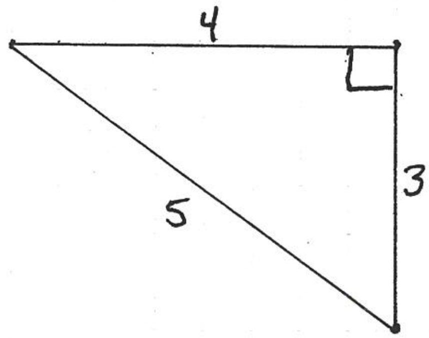
Though we all learn the basic facts of geometry and number theory discovered by the Greeks, few of us are ever told why these results are so important, so startling, so beautiful. Like magicians the Greeks started with nothing and conjoured up an edifice of ideas that has stood for two thousand five hundred years.

There are four classical theorems, each of which changed our way of thinking about mathematics, each of which resonates throughout history.

The Pythagorean Theorem

"The square on the hypotenuse is the sum of the squares on the other two sides"

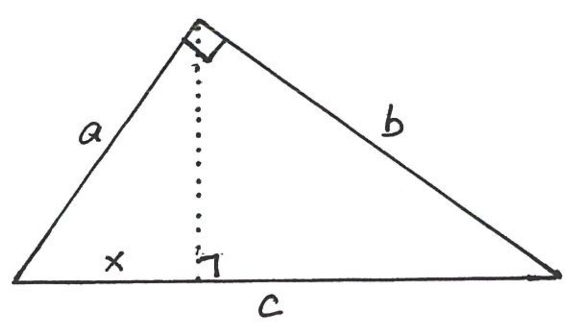
A thousand years before Pythagoras, Egyptian engineers noticed that a triangle whose sides are in proportion 3:4:5 could be used to construct right angles:



but nobody seems to have considered this a special case of a general phenomenon. Indeed, this appears to be a happy accident since the obvious pattern corresponding to 3:4:5 would be 6:7:8, which does not give a right triangle. Even if someone noticed that 5:12:13 is also a right triangle, is it obvious the pattern is $a^2+b^2=c^2$ instead of $a^2=b+c$? It is only in hindsight that we think the relationship between right triangles and squares on the sides is obvious.

Even if you guess the correct pattern, how could you possibly know that the pattern applies to ALL right triangles? You certainly cannot draw every right triangle and measure them all. Furthermore, it is a staggering leap of faith to believe there could be such order and harmony in a universe which seems to be dominated by frivolous gods. The discovery of mathematical truth independent of the vagaries of time or the "real" world is the greatest contribution of the Pythagoreans. Their fundamental principle was that proof should replace observation or experiment, and their first great achievement was the theorem that bears their name.

With modern notation the proof of the Pythagorean theorem is quite simple. Consider the right triangle below:



The dotted line is the altitude onto the hypotenuse c.
The three right triangles thus created are similar, so

$$\frac{c}{b} = \frac{b}{c-x} \quad \text{and} \quad \frac{c}{a} = \frac{a}{x}$$

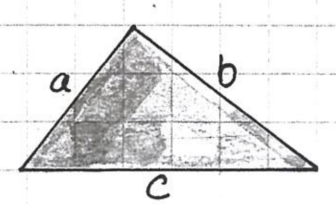
Cross multiplying the first equation gives $c(c-x) = b^2$, hence $c^2 - cx = b^2$, and $c^2 = b^2 + cx$. The second equation gives $cx = a^2$, so $c^2 = b^2 + a^2$ and we're done.

This was not how the Pythagoreans proved their theorem, since they had none of the algebraic tools we used above. The Pythagorean proof was much more geometric in nature, perhaps along the lines of the proof in Euclid's "Elements". There are hundreds of proofs of this theorem, but it is not important which was used by the Pythagoreans; the amazing part of the story is that they concieved of such a thing, that they brought order and logic to a chaotic world.

Of course the Pythagorean theorem is fundamental to all of mathematics. The simple formula $a^2+b^2=c^2$ gives the key link between geometry and algebra, though not until 2100 years after Pythagoras would Descartes show that geometry and algebra may be unified using his system of coordinates, graphs, and equations, all based on the Pythagorean theorem.

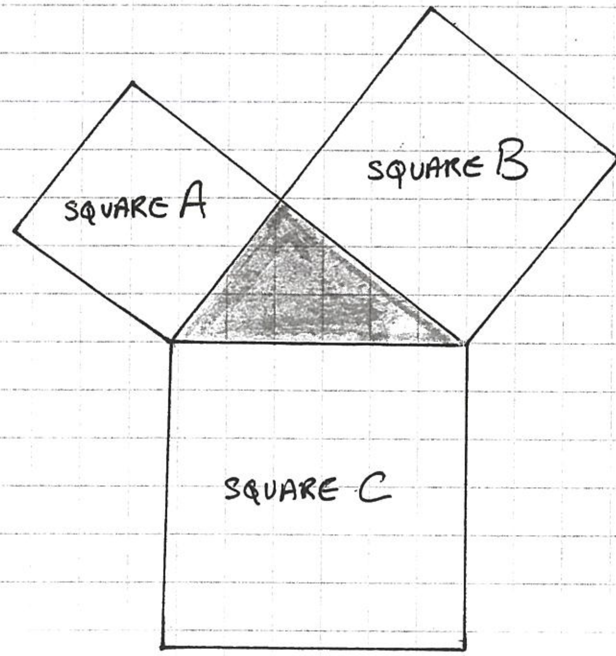
Addendum I: Euclid's Proof

Since geometry is woefully neglected in the schools today, let me add a few words about the Greek interpretation of this theorem. We tend to think about adding squares of numbers, the length of one side squared plus the length of the other side squared, etc. But the Greeks understood the addition of squares to mean the addition of two geometric squares. In other words, we tend to illustrate the Pythagorean theorem like this:



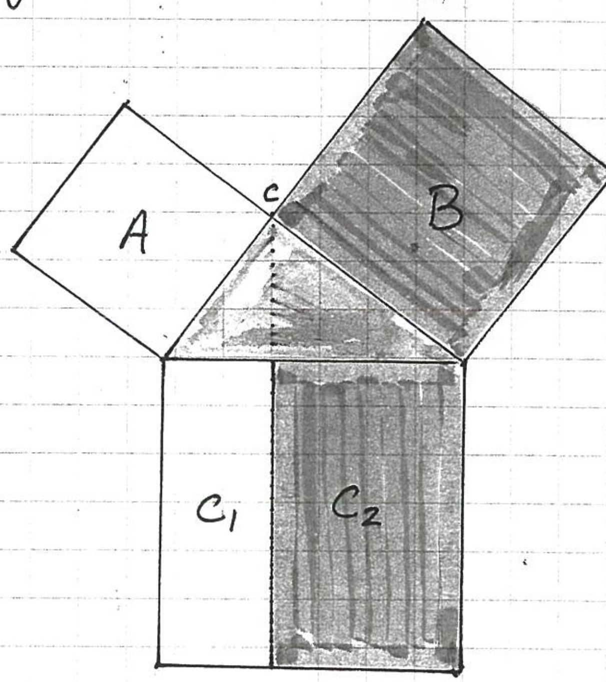
$$a^2+b^2=c^2$$

But the Greeks would actually put squares on each side of the triangle and then say the sum of the two smaller squares is the square on the hypoteneuse:



$$A + B = C$$

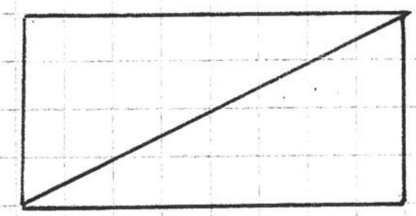
Euclid, in a flourish of ingenuity proved this without the aid of algebra. First, draw an altitude from vertex c as shown. This divides the square on the hypoenuse into two rectangles, marked C_1 and C_2. Euclid shows $A = C_1$ and $B = C_2$, so $A + B = C_1 + C_2 = C$.

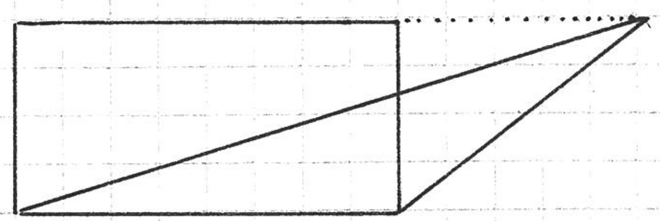


We will need two well-known preliminary results from high-school geometry:

If a triangle and a rectangle have the same base and height, the rectangle is twice the triangle

This deserves a couple of pictures

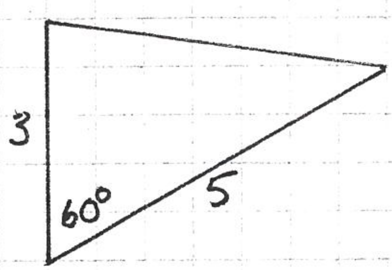


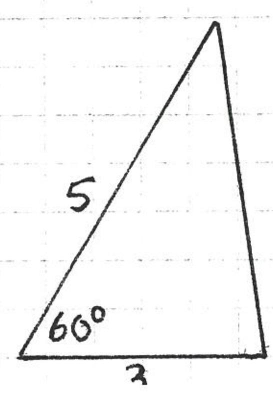


On the left it is obvious the rectangle is twice the triangle, and on the right the triangle has the same base and height, so the same is true.

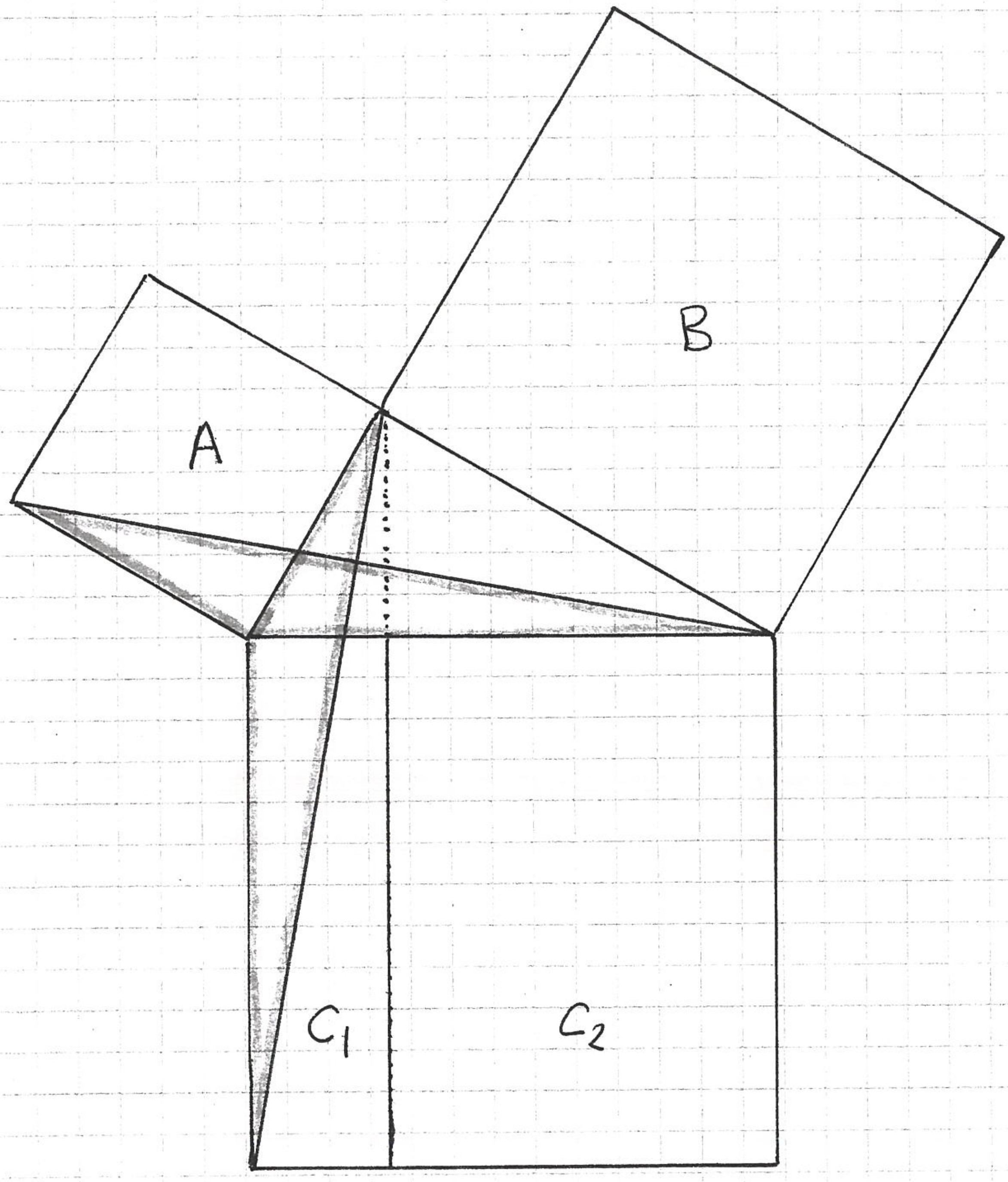
If two triangles share two sides and the included angle, then the triangles are the same

This is known as "Side-Angle-Side". It says these two triangles are the same



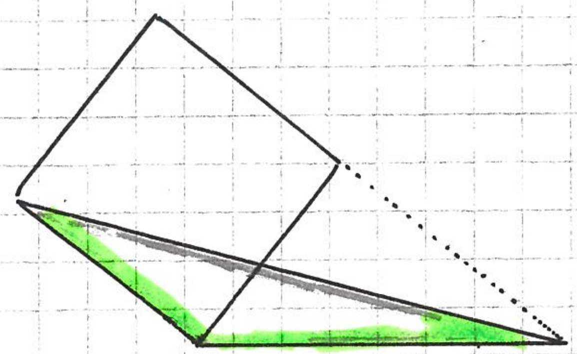


Euclid's Proof

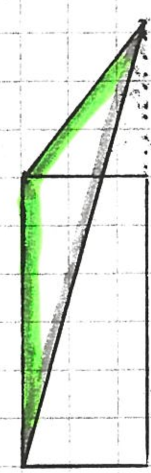


We will show Square A = Rectangle C_1. That Square B = Rectangle C_2 will follow by symmetry.

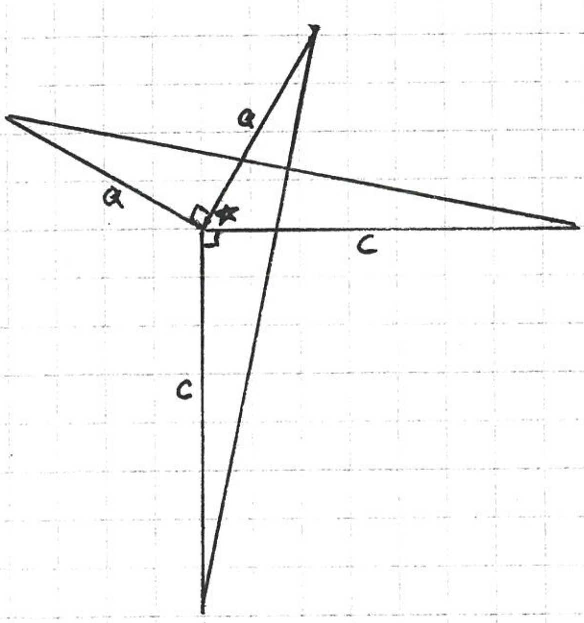
We have added the two shaded triangles to our previous picture. Look at the top-left and you see



so the square is twice the triangle



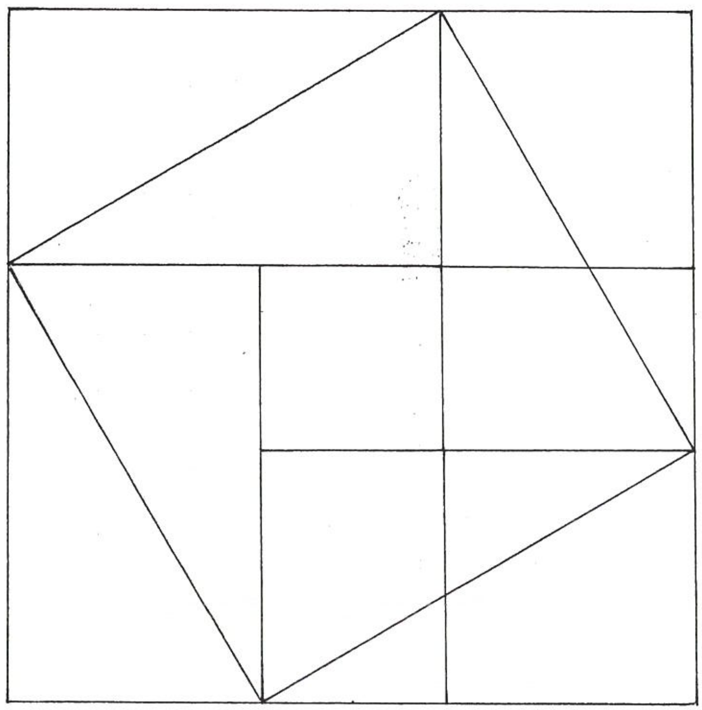
Look at the lower-left and you see so the rectangle is twice that triangle. Now to show the square equals the rectangle we need only show the two triangles are the same. But both triangles have one side c, the hypotenuse of the right triangle, and they also have a side a of the square A. Finally,



the included angle of both triangles is a right angle plus the angle * indicated on the left. Hence, by Side-Angle-Side the triangles are equal, and we're done.

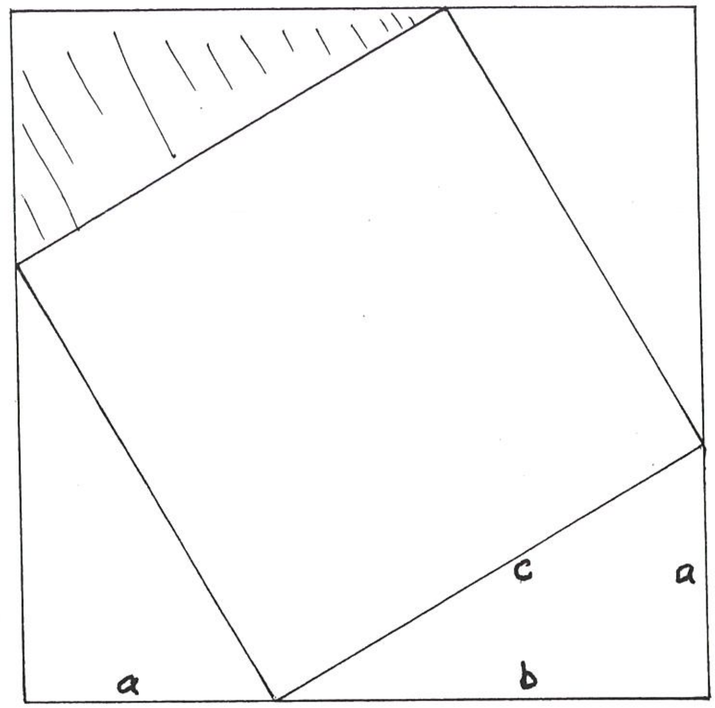
Addendum II : Other Proofs

There are hundreds of proofs of the Pythagorean Theorem, but most are based on the diagram below



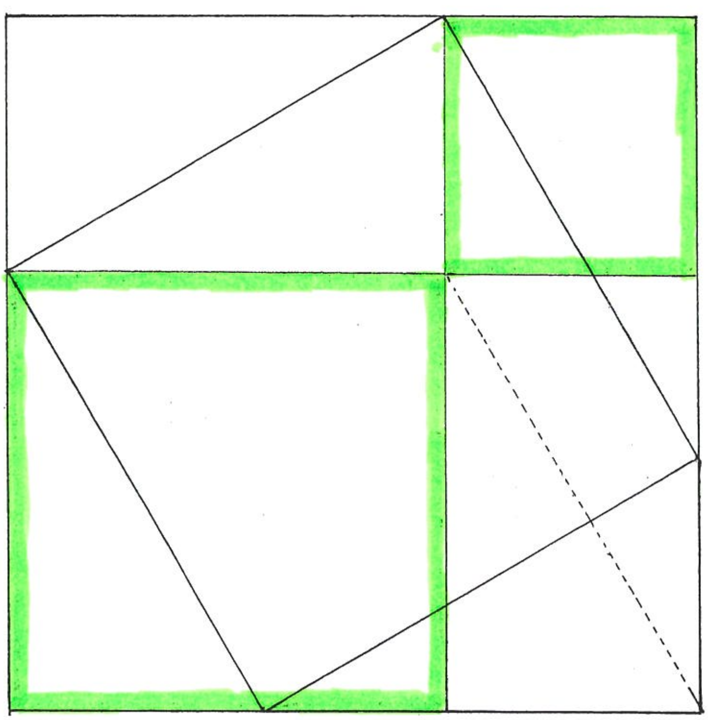
The diagram contains several copies of a right triangle and the various squares on its sides, and it was used in ancient China in the next three proofs. The fourth proof below was given in India during the middle ages, but it probably dates to 300-500 BC.

China I : The diagram here is known as the "Chinese square". We see that the outside square has side $a+b$, area $(a+b)^2$. On the other hand it is built from one square with area c^2 and four triangles each with area $\frac{1}{2}ab$. Hence



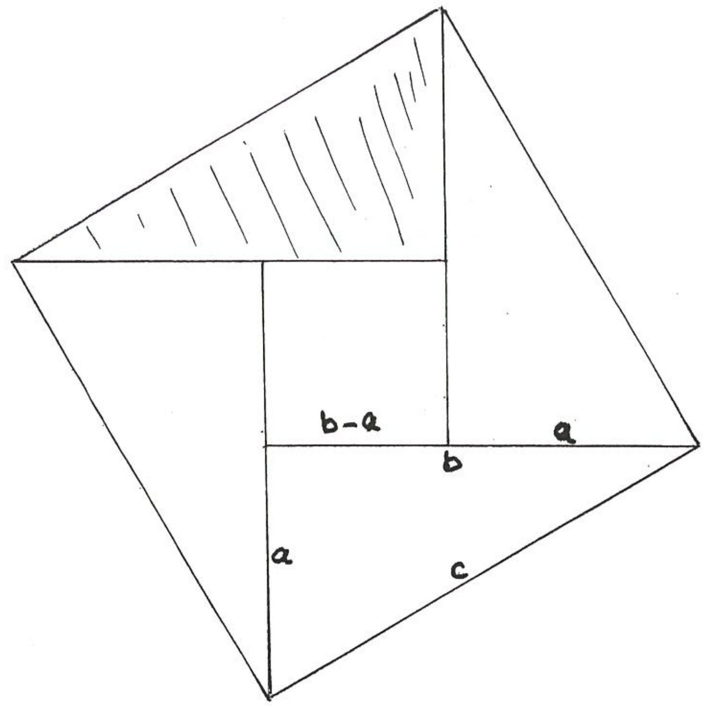
$$(a+b)^2 = c^2 + 4\cdot\tfrac{1}{2}ab \quad \text{so} \quad a^2+2ab+b^2 = c^2+2ab$$

and $a^2+b^2=c^2$. But this uses algebra and is counter to the ancient geometric tradition. Here is the same argument given geometrically:



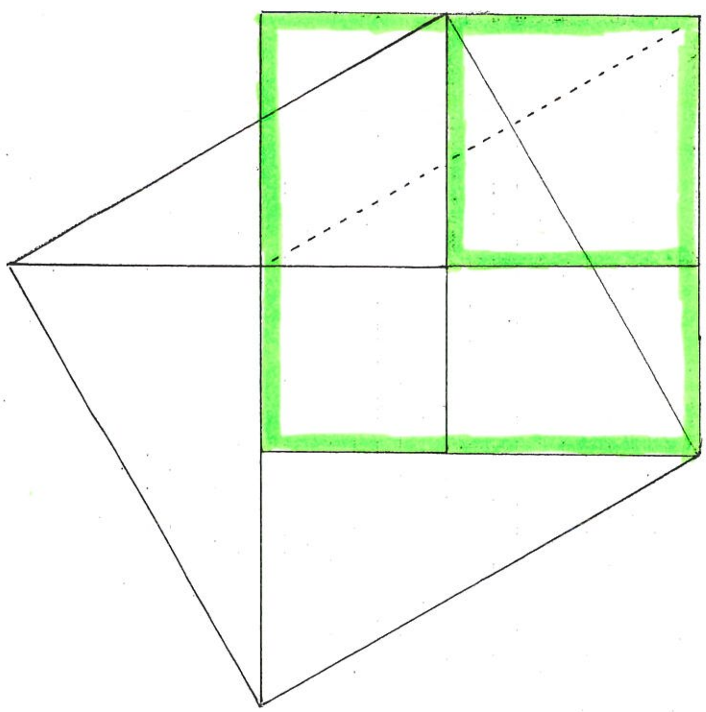
The outside square is the diagonal square plus four triangles. But we can also say the outside square is the two inner squares plus four triangles. Hence the diagonal square equals the two inner squares.

China II : The diagram here is the basis for another algebraic proof of our theorem :



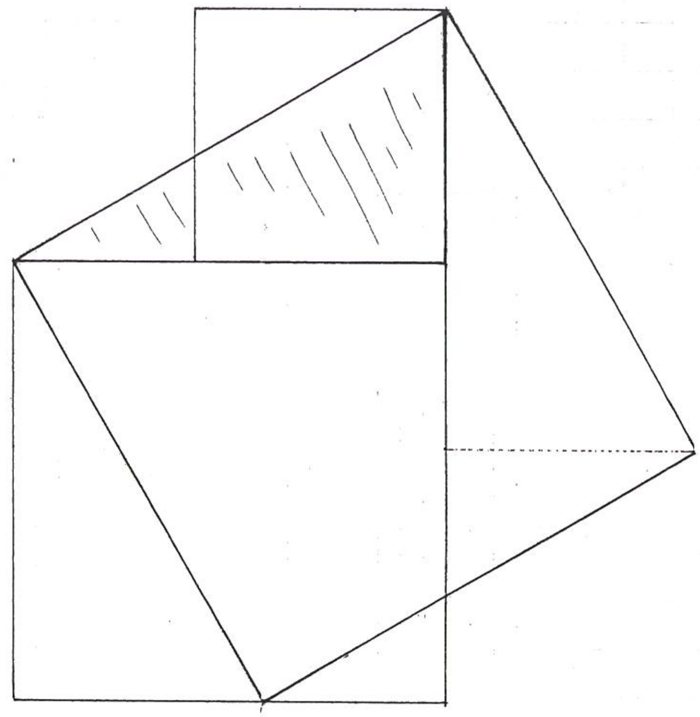
$$c^2 = (b-a)^2 + 4 \cdot \frac{1}{2} a b$$
$$= a^2 - 2ab + b^2 + 2ab$$
$$= a^2 + b^2$$

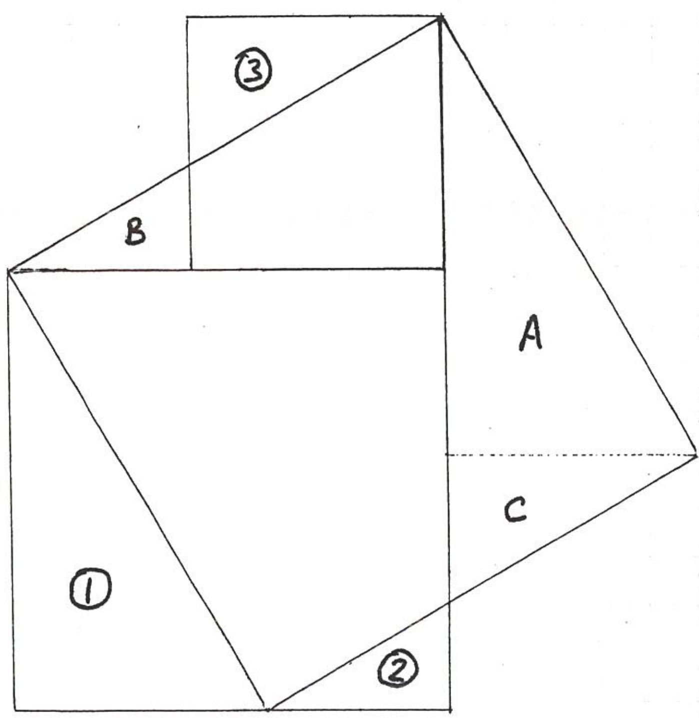
Here is the same argument given geometrically :



The small square in the middle can be looked at two ways : It is the diagonal square less four triangles, or it is the two marked squares less the "same" four triangles. Hence the diagonal square equals the sum of the two marked squares.

China III : This version is said to go back to the Han dynasty, but the only written record is from the middle ages :

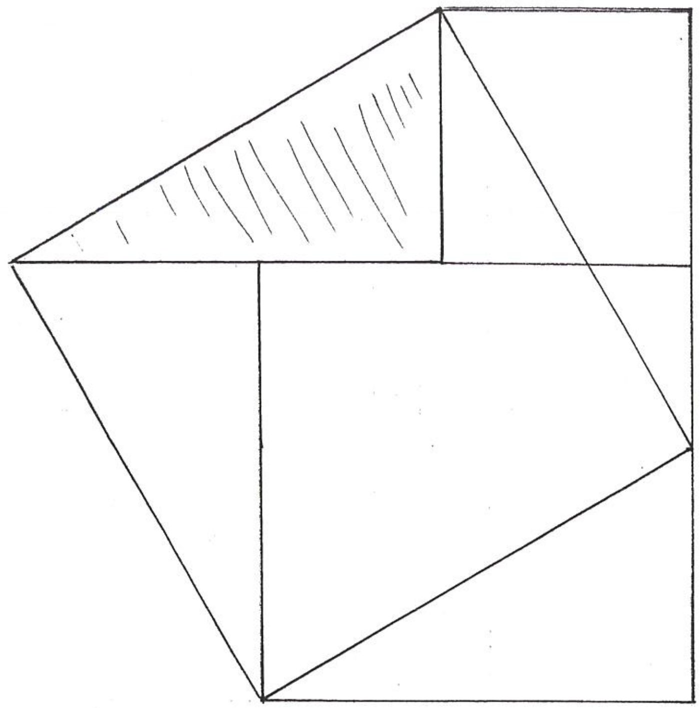


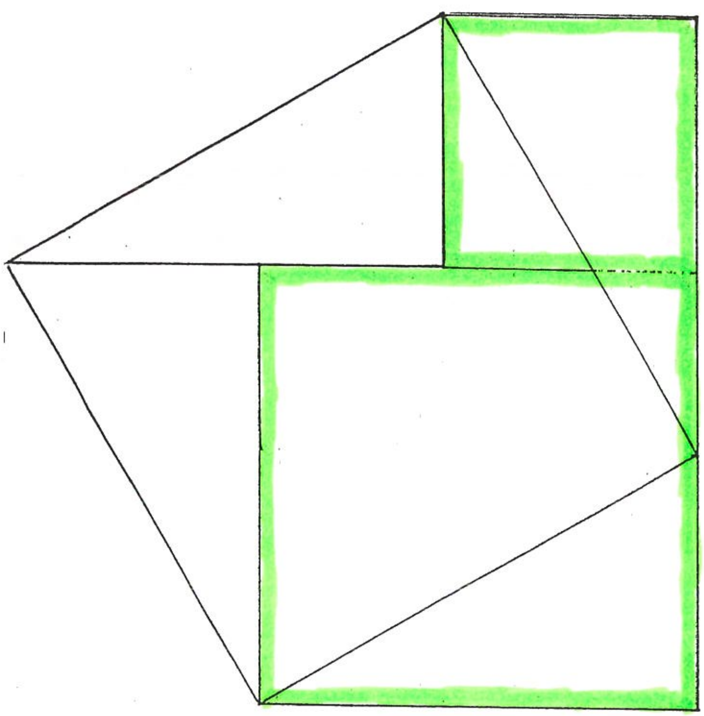


The diagonal square and the other squares overlap. The diagonal square is the overlap plus triangles A, B, and C. The other squares are the overlap plus triangles 1, 2, and 3. But clearly A = 1, B = 2 and C = 3, so we're done.

The Indian diagram :

Some writers claim this predates all the others, but there is much debate and no written records.

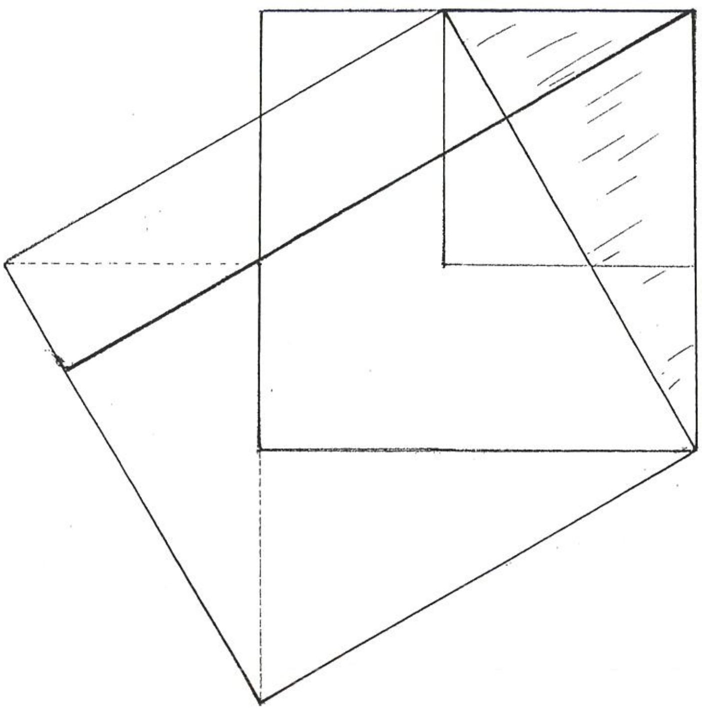




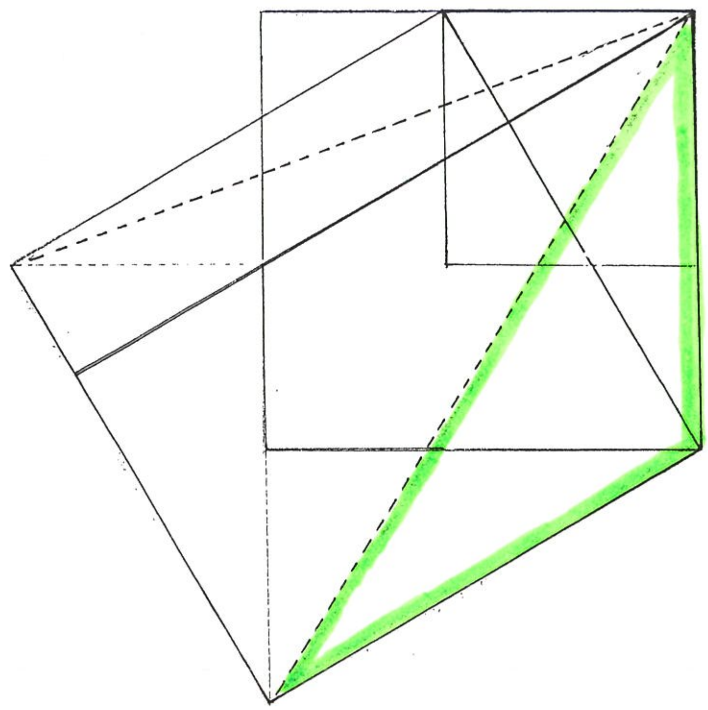
Again we have the diagonal square overlapping the other two squares, and they differ by the "same" two triangles.

Just for completeness, here are Euclid's diagram and Euclid's proof folded over onto the Chinese square:

Euclid's diagram



Euclid's proof



Euclid's proof looks much more complicated than the others, but we cheated when we presented the other "proofs"; we left out all justification that the squares were in fact squares and that all those triangles were the "same". Remembering that its proof is what makes the Pythagorean Theorem so important, these are egregious omissions

Addendum III : The Wizard of Oz

In the classic film "The Wizard of Oz" the scarecrow journeys to Oz in search of a brain. After finally receiving his diploma the straw man points a finger at his head and says

> The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. Oh joy, rapture, I've got a brain!

Aside from the misquote the implication is clear. If you understand the Pythagorean Theorem you have a brain. If not, your head is made of straw.