Euler, e, and i

Leonhard Euler was one of the greatest mathematicians of all time. He lived in the eighteenth century when the full power of calculus was developed by Newton, Leibniz, the three Bernoullis, Euler, Lagrange, and others. It was the century of total scientific optimism, when all the laws of physics could be explicated by differential equations, every equation had a solution, new functions could be created as needed, and new magic numbers seemed to drop out of the sky.

Anyone who has studied calculus has seen "Euler's number" e. Euler was not the first to encounter e (it comes up in the study of compound interest), but he was the first to recognize its seminal position in the mathematical universe.

Euler started by expanding the expression $\left(1+\frac{x}{n}\right)^n$ using the binomial theorem:

$$\left(1+\frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{n}\right)^3 + \cdots$$

$$= 1 + x + \left(\frac{n-1}{n}\right)\frac{x^2}{2!} + \frac{(n-1)}{n}\left(\frac{n-2}{n}\right)\frac{x^3}{3!} + \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\left(\frac{n-3}{n}\right)\frac{x^4}{4!} + \cdots$$

$$= 1 + x + \left(1+\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{x^4}{4!} \cdots$$

Now Euler argued that "when n is an infinite number" the quantities $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots$ all vanish and we get

$$\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Call this function $E(x)$. We have two ways of looking at this function

① $E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

② $E(x) = \left(1+\frac{x}{n}\right)^n$ where n is an infinite number

Look at ① above and calculate the derivative $E'(x)$.

$$E'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \dots$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = E(x)$$

Hence $E(x)$ is its own derivative. Further from ①

we see that $E(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

and this is the number Euler called e.

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Now look at ② above and substitute $n = mx$.
Because n is an infinite number so is m, and
we get.

$$E(x) = \left(1 + \frac{x}{n}\right)^n = \left(1 + \frac{x}{mx}\right)^{mx} = \left(1 + \frac{1}{m}\right)^{mx} = \left[\left(1 + \frac{1}{m}\right)^m\right]^x$$

The expression $\left(1 + \frac{1}{m}\right)^m$ in the brackets is $E(1)$

from equation ②, so we get $E(x) = e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \left(1+\frac{1}{n}\right)^n \text{ where } n \text{ is infinite}$$

A modern mathematics student imbued with limits and epsilons and deltas might cringe at the phrase "n is an infinite number", but in the eighteenth century it was common to talk of infinite numbers and infinitesimally small numbers as if they really existed. Before you sneer at the ignorance of Euler and his contemporaries remember that it was they who formulated and solved all the fundamental problems of differential calculus, integral calculus, multidimensional calculus, ordinary differential equations, partial differential equations, complex analysis, physics, etc.

Next Euler looked at the expression $e^{i\theta}$ where $i = \sqrt{-1}$. Once again it is best not to ask too many questions about this definition. I can imagine Euler saying something like "Hey, what's your problem? It works." We got

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

Now because $i = \sqrt{-1}$ we know $i^2 = -1$ and so $i^3 = -i$ and $i^4 = 1$ and $i^5 = i$, etc. Hence

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \cdots$$

Now differentiate this term by term

$$\left(e^{i\theta}\right)' = i - \theta - i\frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{i\theta^4}{4!} - \cdots = i e^{i\theta}$$

(as we expected from the chain rule). Since $e^0 = 1$ we can say

$y = e^{i\theta}$ is a solution of the differential equation $y' = iy$ with condition $y(0) = 1$

Now look at the function $z(\theta) = \cos\theta + i\sin\theta$. Simple differentiation gives

$$z'(\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = iz$$

Because $z(0)=1$, we see that $y = z(\theta)$ is also a solution of the differential equation $y' = iy$ with condition $y(0)=1$. Since the differential equation completely determines the power-series of a function we know that $e^{i\theta} = z(\theta)$, that is

$$\star \quad \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

There are several important consequences of this unexpected link between the exponential function and the trig functions. First, setting $\theta = \pi$ yields

$$e^{\pi i} = -1 \quad \text{OR} \quad e^{\pi i} + 1 = 0$$

In a recent survey of mathematicians this equation was voted the most "beautiful" formula in mathematics.

It concisely expresses a previously unknown relationship amongst the five most important numbers in mathematics.

Another corollary of Euler's formula ☆ comes from substituting $-\theta$ in the place of θ yielding

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Adding this to ☆ gives $\dfrac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$

while subtracting gives $\dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$

These two formulas say that the trig functions can be built from e^x. In fact all of the basic formulas of trigonometry may be derived from these equations. Eulers exponential function truely is the most fundamental function of the mathematical universe.

Footnote: It has been said that THE LORD gave to Moses three tablets with HER laws written upon them. The Hebrew children understood the first two tablets easily enough, but the third tablet had the formula

$$e^{\pi i} + 1 = 0$$

written out as a poem in Hebrew (since mathematical notation had not yet been codified). Not understanding its significance but frightened by its power, the Hebrews buried the tablet in the desert. For thirty centuries rabbinical scolars, priests, and imams tried to decifer its meaning. I sometimes wonder if THE LORD simply took pity on mankind and whispered in Euler's ear.