I. DEFINITION OF THE SYMBOLS

§ 1. The signs customarily employed in the general theory of magnitudes are of two kinds. The first consists of letters, of which each represents either a number left indeterminate or a function left indeterminate. This indeterminacy makes it possible to use letters to express the universal validity of propositions, as in

$$(a + b)c = ac + bc.$$

The other kind consists of signs such as $+$, $-$, $\sqrt{}$, 0, 1, and 2, of which each has its particular meaning.[4]

I adopt this basic idea of distinguishing two kinds of signs, which unfortunately is not strictly observed in the theory of magnitudes,[5] *in order to apply it in the more*

[4] ⟦Footnote by Jourdain (*1912*, p. 238):
Russell (*1908*) has expressed it: "A variable is a symbol which is to have one of a certain set of values, without its being decided which one. It does not have first one value of a set and then another; it has at all times *some* value of the set, where, so long as we do not replace the variable by a constant, the 'some' remains unspecified."

On the word "variable" Frege has supplied the note: "Would it not be well to omit this expression entirely, since it is hardly possible to define it properly? Russell's definition immediately raises the question what it means to say that 'a symbol has a value'. Is the relation of a sign to its significatum meant by this? In that case, however, we must insist that the sign be univocal, and the meaning (value) that the sign is to have must be determinate; then the variable would be a sign. But for him who does not subscribe to a formal theory a variable will not be a sign, any more than a number is. If, now, you write 'A variable is represented by a symbol that is to represent one of a certain set of values', the last defect is thereby removed; but what is the case then? The symbol represents, first, the variable and, second, a value taken from a certain supply without its being determined which. Accordingly, it seems better to leave the word 'symbol' out of the definition. The question as to what a variable is has to be answered independently of the question as to which symbol is to represent the variable. So we come to the definition: 'A variable is one of a certain set of values, without its being decided which one'. But the last addition does not yield any closer determination, and to belong to a certain set of values means, properly, to fall under a certain concept; for, after all, we can determine this set only by giving the properties that an object must have in order to belong to the set; that is, the set of values will be the extension of a concept. But, now, we can for every object specify a set of values to which it belongs, so that even the requirement that something is to be a value taken from a certain set does not determine anything. It is probably best to hold to the convention that Latin letters serve to confer generality of content on a theorem. And it is best not to use the expression 'variable' at all, since ultimately we cannot say either of a sign, or of what it expresses or denotes, that it is variable or that it is a variable, at least not in a sense that can be used in mathematics or logic. On the other hand, perhaps someone may insist that in '$(2 + x)(3 + x)$' the letter 'x' does not serve to confer generality of content on a proposition. But in the context of a proof such a formula will always occur as a part of a proposition, whether this proposition consists partly of words or exclusively of mathematical signs, and in such a context x will always serve to confer generality of content on a proposition. Now, it seems to me unfortunate to restrict to a particular set the values that are admissible for this letter. For we can always add the condition that a belong to this set, and then drop that condition. If an object $\varDelta$ does not belong to the set, the condition is simply not satisfied and, if we replace 'a' by '$\varDelta$' in the entire proposition, we obtain a true proposition. I would not say of a letter that it has a signification, a sense, a meaning, if it serves to confer generality of content on a proposition. We can replace the letter by the proper name '$\varDelta$' of an object $\varDelta$; but this $\varDelta$ cannot anyhow be regarded as the *meaning* of the letter; for it is not more closely allied with the letter than is any other object. Also, generality cannot be regarded as the meaning of the Latin letter; for it cannot be regarded as something independent, something that would be added to a content already complete in other respects. I would not, then, say 'terms whose meaning is indeterminate' or 'signs have variable meanings'. In this case signs have no denotations at all." [Frege, 1910.]⟧

[5] Consider 1, log, sin, lim.

comprehensive domain of pure thought in general. I therefore divide all signs that I use into *those by which we may understand different objects* and *those that have a completely determinate meaning.* The former are *letters* and they will serve chiefly to express *generality.* But, no matter how indeterminate the meaning of a letter, we must insist that throughout a given context the letter *retain* the meaning once given to it.

Judgment

§ 2. A judgment will always be expressed by means of the sign

$$\vdash\!\!\!-\!\!\!-,$$

which stands to the left of the sign, or the combination of signs, indicating the content of the judgment. If we *omit* the small vertical stroke at the left end of the horizontal one, the judgment will be transformed into a *mere combination of ideas* [*Vorstellungsverbindung*],[6] of which the writer does not state whether he acknowledges it to be true or not. For example, let

$$\vdash\!\!\!-\!\!\!- A$$

stand for ⟦bedeute⟧ the judgment "Opposite magnetic poles attract each other";[7] then

$$-\!\!\!-\!\!\!- A$$

will not express ⟦ausdrücken⟧ this judgment;[8] it is to produce in the reader merely the idea of the mutual attraction of opposite magnetic poles, say in order to derive consequences from it and to test by means of these whether the thought is correct. When the vertical stroke is omitted, we express ourselves *paraphrastically*, using the words "the circumstance that" or "the proposition that".[9]

Not every content becomes a judgment when $\vdash\!\!\!-\!\!\!-$ is written before its sign; for

[6] ⟦Footnote by Jourdain (*1912*, p. 242):
"For this word I now simply say 'Gedanke'. The word 'Vorstellungsinhalt' is used now in a psychological, now in a logical sense. Since this creates obscurities, I think it best not to use this word at all in logic. We must be able to express a thought without affirming that it is true. If we want to characterize a thought as false, we must first express it without affirming it, then negate it, and affirm as true the thought thus obtained. We cannot correctly express a hypothetical connection between thoughts at all if we cannot express thoughts without affirming them, for in the hypothetical connection neither the thought appearing as antecedent nor that appearing as consequent is affirmed." [Frege, 1910.]⟧

[7] I use Greek letters as abbreviations, and to each of these letters the reader should attach an appropriate meaning when I do not expressly give them a definition. ⟦The "A" that Frege is now using is a capital alpha.⟧

[8] ⟦Jourdain had originally translated "bedeuten" by "signify", and Frege wrote (see *Jourdain 1912*, p. 242):
"Here we must notice the words 'signify' and 'express'. The former seems to correspond to 'bezeichnen' or 'bedeuten', the latter to 'ausdrücken'. According to the way of speaking I adopted I say 'A proposition expresses a thought and signifies its truth value'. Of a judgment we cannot properly say either that it signifies or that it is expressed. We do, to be sure, have a thought in the judgment, and that can be expressed; but we have more, namely, the recognition of the truth of this thought."⟧

[9] ⟦Footnote by Jourdain (*1912*, p. 243):
"Instead of 'circumstance' and 'proposition' I would simply say 'thought'. Instead of 'beurtheilbarer Inhalt' we can also say 'Gedanke'." [Frege, 1910.]⟧

example, the idea "house" does not. We therefore distinguish contents that *can become a judgment* from those that *cannot*.[10]

The horizontal stroke that is part of the sign $\vdash$ *combines the signs that follow it into a totality, and the affirmation expressed by the vertical stroke at the left end of the horizontal one refers to this totality.* Let us call the horizontal stroke the *content stroke* and the vertical stroke the *judgment stroke*. The content stroke will in general serve to relate any sign to the totality of the signs that follow the stroke. *Whatever follows the content stroke must have a content that can become a judgment.*

§ 3. A distinction between *subject* and *predicate* does *not occur* in my way of representing a judgment. In order to justify this I remark that the contents of two judgments may differ in two ways: either the consequences derivable from the first, when it is combined with certain other judgments, always follow also from the second, when it is combined with these same judgments, [[and conversely,]] or this is not the case. The two propositions "The Greeks defeated the Persians at Plataea" and "The Persians were defeated by the Greeks at Plataea" differ in the first way. Even if one can detect a slight difference in meaning, the agreement outweighs it. Now I call that part of the content that is the *same* in both the *conceptual content*. Since *it alone* is of significance for our ideography, we need not introduce any distinction between propositions having the same conceptual content. If one says of the subject that it "is the concept with which the judgment is concerned", this is equally true of the object. We can therefore only say that the subject "is the concept with which the judgment is chiefly concerned". In ordinary language, the place of the subject in the sequence of words has the significance of a *distinguished* place, where we put that to which we wish especially to direct the attention of the listener (see also § 9). This may, for example, have the purpose of pointing out a certain relation of the given judgment to others and thereby making it easier for the listener to grasp the entire context. Now, all those peculiarities of ordinary language that result only from the interaction of speaker and listener—as when, for example, the speaker takes the expectations of the listener into account and seeks to put them on the right track even before the complete sentence is enunciated—have nothing that answers to them in my formula language, since in a judgment I consider only that which influences its *possible consequences*. Everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated; *nothing is left to guesswork*. In this I faithfully follow the example of the formula language of mathematics, a language to which one would do violence if he were to distinguish between subject and predicate in it. We can imagine a language in which the proposition "Archimedes perished at the capture of Syracuse" would be expressed thus: "The violent death of Archimedes at the capture of Syracuse is a fact". To be sure, one can distinguish between subject and predicate here, too, if one wishes to do so, but the subject contains the whole content, and the predicate serves only to turn the content into a judgment. *Such a*

[10] On the other hand, the circumstance that there are houses, or that there is a house (see § 12 [[footnote 15]]), is a content that can become a judgment. But the idea "house" is only a part of it. In the proposition "The house of Priam was made of wood" we could not put "circumstance that there is a house" in place of "house". For a different kind of example of a content that cannot become a judgment see the passage following formula (81).

[[In German Frege's distinction is between "beurtheilbare" and "unbeurtheilbare" contents. Jourdain uses the words "judicable" and "nonjudicable".]]

language would have only a single predicate for all judgments, namely, "is a fact". We see that there cannot be any question here of subject and predicate in the ordinary sense. *Our ideography is a language of this sort, and in it the sign* $\vdash$ *is the common predicate for all judgments.*

In the first draft of my formula language I allowed myself to be misled by the example of ordinary language into constructing judgments out of subject and predicate. But I soon became convinced that this was an obstacle to my specific goal and led only to useless prolixity.

§ 4. The remarks that follow are intended to explain the significance for our purposes of the distinctions that we introduce among judgments.

We distinguish between *universal* and *particular* judgments; this is really not a distinction between judgments but between contents. *We ought to say "a judgment with a universal content", "a judgment with a particular content".* For these properties hold of the content even when it is *not* advanced as a judgment but as a [[mere]] proposition (see § 2).

The same holds of negation. In an indirect proof we say, for example, "Suppose that the line segments AB and CD are not equal". Here the content, that the line segments AB and CD are not equal, contains a negation; but this content, though it can become a judgment, is nevertheless not advanced as a judgment. Hence the negation attaches to the content, whether this content becomes a judgment or not. I therefore regard it as more appropriate to consider negation as an adjunct of a *content that can become a judgment.*

The distinction between categoric, hypothetic, and disjunctive judgments seems to me to have only grammatical significance.[11]

The apodictic judgment differs from the assertory in that it suggests the existence of universal judgments from which the proposition can be inferred, while in the case of the assertory one such a suggestion is lacking. By saying that a proposition is necessary I give a hint about the grounds for my judgment. *But, since this does not affect the conceptual content of the judgment, the form of the apodictic judgment has no significance for us.*

If a proposition is advanced as possible, either the speaker is suspending judgment by suggesting that he knows no laws from which the negation of the proposition would follow or he says that the generalization of this negation is false. In the latter case we have what is usually called a *particular affirmative judgment* (see § 12). "It is possible that the earth will at some time collide with another heavenly body" is an instance of the first kind, and "A cold can result in death" of the second.

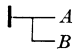
Conditionality

§ 5. If A and B stand for contents that can become judgments (§ 2), there are the following four possibilities:

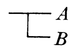
(1) A is affirmed and B is affirmed;
(2) A is affirmed and B is denied;
(3) A is denied and B is affirmed;
(4) A is denied and B is denied.

[11] The reason for this will be apparent from the entire book.

Now

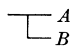


stands for the judgment that *the third of these possibilities does not take place, but one of the three others does*. Accordingly, if



is denied, this means that the third possibility takes place, hence that A is denied and B affirmed.

Of the cases in which

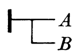


is affirmed we single out for comment the following three:

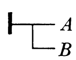
(1) A must be affirmed. Then the content of B is completely immaterial. For example, let $\vdash\!\!—A$ stand for $3 \times 7 = 21$ and B for the circumstance that the sun is shining. Then only the first two of the four cases mentioned are possible. There need not exist a causal connection between the two contents.

(2) B has to be denied. Then the content of A is immaterial. For example, let B stand for the circumstance that perpetual motion is possible and A for the circumstance that the world is infinite. Then only the second and fourth of the four cases are possible. There need not exist a causal connection between A and B.

(3) We can make the judgment

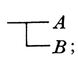


without knowing whether A and B are to be affirmed or denied. For example, let B stand for the circumstance that the moon is in quadrature with the sun and A for the circumstance that the moon appears as a semicircle. In that case we can translate

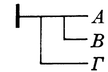


by means of the conjunction "if": "If the moon is in quadrature with the sun, the moon appears as a semicircle". The causal connection inherent in the word "if", however, is not expressed by our signs, even though only such a connection can provide the ground for a judgment of the kind under consideration. For causal connection is something general, and we have not yet come to express generality (see § 12).

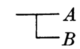
Let us call the vertical stroke connecting the two horizontal ones the *condition stroke*. The part of the upper horizontal stroke to the left of the condition stroke is the content stroke for the meaning, just explained, of the combination of signs



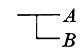
to it is affixed any sign that is intended to relate to the total content of the expression. The part of the horizontal stroke between A and the condition stroke is the content stroke of A. The horizontal stroke to the left of B is the content stroke of B. Accordingly, it is easy to see that



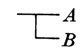
denies the case in which A is denied and B and Γ are affirmed. We must think of this as having been constructed from



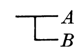
and Γ in the same way as



was constructed from A and B. We therefore first have the denial of the case in which

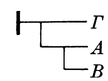


is denied and Γ is affirmed. But the denial of



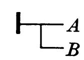
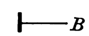
means that A is denied and B is affirmed. From this we obtain what was given above. If a causal connection is present, we can also say "A is the necessary consequence of B and Γ", or "If the circumstances B and Γ occur, then A also occurs".

It is no less easy to see that

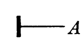


denies the case in which B is affirmed but A and Γ are denied.[12] If we assume that there exists a causal connection between A and B, we can translate the formula as "If A is a necessary consequence of B, one can infer that Γ takes place".

§ 6. The definition given in § 5 makes it apparent that from the two judgments

 and 

the new judgment



[12] ⟦There is an oversight here, already pointed out by Schröder (*1880*, p. 88).⟧

follows. Of the four cases enumerated above, the third is excluded by

$$\begin{array}{l}\vdash\!\!\top\!\!-\, A\\ \quad\llcorner\!-\, B\end{array}$$

and the second and fourth by

$$\vdash B,$$

so that only the first remains.

We could write this inference perhaps as follows:

$$\begin{array}{c}\begin{array}{l}\vdash\!\!\top\!\!-\, A\\ \quad\llcorner\!-\, B\end{array}\\ \\ \vdash B\\ \hline \vdash A.\end{array}$$

This would become awkward if long expressions were to take the places of A and B, since each of them would have to be written twice. That is why I use the following abbreviation. To every judgment occurring in the context of a proof I assign a number, which I write to the right of the judgment at its first occurrence. Now assume, for example, that the judgment

$$\begin{array}{l}\vdash\!\!\top\!\!-\, A\\ \quad\llcorner\!-\, B,\end{array}$$

or one containing it as a special case, has been assigned the number X. Then I write the inference as follows:

$$(\mathrm{X}):\ \begin{array}{c}\vdash B\\ \hline \vdash A.\end{array}$$

Here it is left to the reader to put the judgment

$$\begin{array}{l}\vdash\!\!\top\!\!-\, A\\ \quad\llcorner\!-\, B\end{array}$$

together for himself from $\vdash B$ and $\vdash A$ and to see whether it is the judgment X above.

If, for example, the judgment $\vdash B$ has been assigned the number XX, I also write the same inference as follows:

$$(\mathrm{XX})::\ \begin{array}{c}\begin{array}{l}\vdash\!\!\top\!\!-\, A\\ \quad\llcorner\!-\, B\end{array}\\ \hline \vdash A.\end{array}$$

Here the double colon indicates that $\vdash B$, which was only referred to by XX, would have to be formed, from the two judgments written down, in a way different from that above.

Furthermore if, say, the judgment $\vdash \Gamma$ had been assigned the number XXX, I would abbreviate the two judgments

$$\begin{array}{rc} & \begin{array}{l}\vdash\!\!\top\!\!\top\!\!-\, A\\ \quad\vert\;\llcorner\!-\, B\\ \quad\llcorner\!\!-\!\!-\, \Gamma\end{array}\\ (\mathrm{XXX})::\ & \hline \\ & \begin{array}{l}\vdash\!\!\top\!\!-\, A\\ \quad\llcorner\!-\, B\end{array}\\ (\mathrm{XX})::\ & \hline \\ & \vdash A\end{array}$$

still more thus:

$$\begin{array}{rc} & \begin{array}{l}\vdash\!\!\top\!\!\top\!\!-\, A\\ \quad\vert\;\llcorner\!-\, B\\ \quad\llcorner\!\!-\!\!-\, \Gamma\end{array}\\ (\mathrm{XX,\ XXX})::\ & \hline\hline \\ & \vdash A.\end{array}$$

Following Aristotle, we can enumerate quite a few modes of inference in logic; I employ only this one, at least in all cases in which a new judgment is derived from more than a single one. For, the truth contained in some other kind of inference can be stated in one judgment, of the form: if M holds and if N holds, then Λ holds also, or, in signs,

$$\begin{array}{l}\vdash\!\!\top\!\!\top\!\!-\, \Lambda\\ \quad\vert\;\llcorner\!-\, M\\ \quad\llcorner\!\!-\!\!-\, N.\end{array}$$

From this judgment, together with $\vdash N$ and $\vdash M$, there follows, as above, $\vdash \Lambda$. In this way an inference in accordance with any mode of inference can be reduced to our case. Since it is therefore possible to manage with a single mode of inference, it is a commandment of perspicuity to do so. Otherwise there would be no reason to stop at the Aristotelian modes of inference; instead, one could continue to add new ones indefinitely: from each of the judgments expressed in a formula in §§ 13–22 we could make a particular mode of inference. *With this restriction to a single mode of inference, however, we do not intend in any way to state a psychological proposition; we wish only to decide a question of form in the most expedient way.* Some of the judgments that take the place of Aristotelian kinds of inference will be listed in § 22 (formulas (59), (62), and 65)).

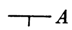
Negation

§ 7. If a short vertical stroke is attached below the content stroke, this will express the circumstance that *the content does not take place.* So, for example,

$$\vdash\!\!\!\top\!\!-\, A$$

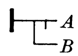
means "A does not take place". I call this short vertical stroke the *negation stroke.*

The part of the horizontal stroke to the right of the negation stroke is the content stroke of A; the part to the left of the negation stroke is the content stroke of the negation of A. If there is no judgment stroke, then here—as in any other place where the ideography is used—no judgment is made.



merely calls upon us to form the idea that A does not take place, without expressing whether this idea is true.

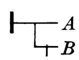
We now consider some cases in which the signs of conditionality and negation are combined.



means "The case in which B is to be affirmed and the negation of A to be denied does not take place"; in other words, "The possibility of affirming both A and B does not exist", or "A and B exclude each other". Thus only the following three cases remain:

A is affirmed and B is denied;
A is denied and B is affirmed;
A is denied and B is denied.

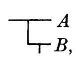
In view of the preceding it is easy to state what the significance of each of the three parts of the horizontal stroke to the left of A is.



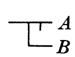
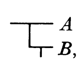
means "The case in which A is denied and the negation of B is affirmed does not obtain", or "A and B cannot both be denied". Only the following possibilities remain:

A is affirmed and B is affirmed;
A is affirmed and B is denied;
A is denied and B is affirmed;

A and B together exhaust all possibilities. Now the words "or" and "either—or" are used in two ways: "A or B" means, in the first place, just the same as



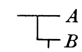
hence it means that no possibility other than A and B is thinkable. For example, if a mass of gas is heated, its volume or its pressure increases. In the second place, the expression "A or B" combines the meanings of both

 and 

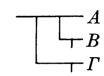
so that no third is possible besides A and B, and, moreover, that A and B exclude each other. Of the four possibilities, then, only the following two remain:

A is affirmed and B is denied;
A is denied and B is affirmed.

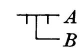
Of the two ways in which the expression "A or B" is used, the first, which does not exclude the coexistence of A and B, is the more important, and *we shall use the word "or" in this sense.* Perhaps it is appropriate to distinguish between "or" and "either—or" by stipulating that only the latter shall have the secondary meaning of mutual exclusion. We can then translate



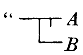
by "A or B". Similarly,



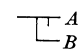
has the meaning of "A or B or Γ".



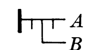
means

"  is denied",

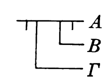
or "The case in which both A and B are affirmed occurs". The three possibilities that remained open for



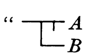
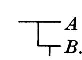
are, however, excluded. Accordingly, we can translate



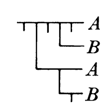
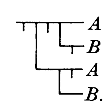
by "Both A and B are facts". It is also easy to see that



can be rendered by "A and B and Γ". If we want to represent in signs "Either A or B" with the secondary meaning of mutual exclusion, we must express

"  and "

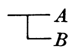
This yields

 or also 

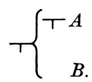
Instead of expressing the "and", as we did here, by means of the signs of conditionality and negation, we could on the other hand also represent conditionality by means of a sign for "and" and the sign of negation. We could introduce, say,

$$\left\{\begin{array}{l}\Gamma\\ \Delta\end{array}\right.$$

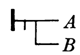
as a sign for the total content of Γ and Δ, and then render



by

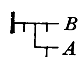


I chose the other way because I felt that it enables us to express inferences more simply. The distinction between "and" and "but" is of the kind that is not expressed in the present ideography. The speaker uses "but" when he wants to hint that what follows is different from what one might at first expect.

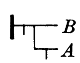


means "Of the four possibilities the third, namely, that A is denied and B is affirmed, occurs". We can therefore translate it as "B takes place and (but) A does not".

We can translate the combination of signs



by the same words.



means "The case in which both A and B are denied occurs". Hence we can translate it as "Neither A nor B is a fact". What has been said here about the words "or", "and", and "neither —nor" applies, of course, only when they connect contents that *can become judgments*.

Identity of content

§ 8. Identity of content differs from conditionality and negation in that it applies to names and not to contents. Whereas in other contexts signs are merely representatives of their content, so that every combination into which they enter expresses only a relation between their respective contents, they suddenly display their own selves when they are combined by means of the sign for identity of content; for it expresses the circumstance that two names have the same content. Hence the introduction of a sign for identity of content necessarily produces a bifurcation in the meaning of all