# 4

# Incompleteness Everywhere

## 4.1 The Incompleteness Theorem Outside Mathematics

The incompleteness theorem is a theorem about the consistency and completeness of formal systems. As noted in the introductory chapter, "consistent," "inconsistent," "complete," "incomplete," and "system" are words used not only in a technical sense in logic, but in various senses in ordinary language, and so it is not surprising that the incompleteness theorem has been thought to have a great many applications outside mathematics.

A few examples:

- Religious people claim that all answers are found in the Bible or in whatever text they use. That means the Bible is a complete system, so Gödel seems to indicate it cannot be true. And the same may be said of any religion which claims, as they all do, a final set of answers.

- As Gödel demonstrated, all consistent formal systems are incomplete, and all complete formal systems are inconsistent. The U.S. Constitution is a formal system, after a fashion. The Founders made the choice of incompleteness over inconsistency, and the Judicial Branch exists to close that gap of incompleteness.

- Gödel demonstrated that any axiomatic system must be either incomplete or inconsistent, and inasmuch as Ayn Rand's philosophy of Objectivism claims to be a system of axioms and propositions, one of those two conditions must apply.

It will be noted that all of these misstate the incompleteness theorem by leaving out the essential condition that the system must be capable of formalizing a certain amount of arithmetic. There are many complete and consistent formal systems that do not satisfy this condition. If we remember to include the condition, supposed applications of the incompleteness theorem such as those illustrated will less readily suggest themselves, since neither the Bible, nor the constitution of the United States, nor again the philosophy of Ayn Rand is naturally thought of as a source of arithmetical theorems.

The incompleteness theorem is a mathematical theorem, dealing with formal systems such as the axiomatic theory PA of arithmetic and axiomatic set theory ZFC. Formal systems are characterized by a formal language, a set of axioms in that language, and a set of formal inference rules which together with the axioms determine the set of theorems of the system. The Bible is not a formal system. To spell this out: it has no formal language, but is a collection of texts in ordinary language, whether Latin, English, Japanese, Swahili, Greek, or some other language. It has no axioms, no rules of inference, and no theorems. Whether something follows from what is said in the Bible is not a mathematical question, but a question of judgment, interpretation, belief, opinion. Similarly for the Constitution and the philosophy of Ayn Rand. Deciding what does or does not follow from these texts is not a task for mathematicians or computers, but for theologians, believers, the Supreme Court, and just plain readers, who must often decide for themselves how to interpret the text and have no formal rules of inference to fall back on.

Thus, we need only ask the question "is the Bible (the Constitution, etc.) a formal system?" for the answer to be obvious. Of course it's not a formal system. It's nothing like a formal system. To be sure, if we set aside the mathematical notion of a formal system and use the words "formal" and "system" in an everyday sense, it can be said that the Constitution is a formal system "after a fashion." That is to say, from the point of view of ordinary language, the Constitution contains lots of formal language, and it looks rather systematic. But nobody would seriously claim that there is any such thing as the formally defined language, the axioms, and the rules of inference of the Constitution. So the incompleteness theorem does not apply to the Constitution, the Bible, the philosophy of Ayn Rand, and so on.

Without pretending that these various systems of thought, legislation, philosophy, and so on constitute formal systems, we could apply to them *analogues* of the formal notions of consistency and completeness. For the Bible to be complete in such a sense, analogous to that used in logic, would mean that every statement that makes sense in the context of Bible reading could be reasonably held to be decided by the Bible, in the sense that either that statement or its negation can be held to be explicitly or implicitly asserted in the Bible. If we raise the question whether the Bible is complete in this sense, the answer is again pretty obvious: it is not. For example, it makes good sense in the context of Bible reading to ask whether Moses sneezed on his fifth birthday. No information on this point can be found in the Bible. Hence, the Bible is incomplete. Similarly, the Constitution is incomplete, since it does not tell us whether or not wearing a polka-dot suit is allowed in Congress. Ayn Rand's philosophy of Objectivism is incomplete, since we cannot derive from it whether or not there is life on Mars, even though it makes sense in the context of Objectivism to ask whether there is life on Mars. We don't need Gödel to tell us that these "systems" are in this sense incomplete. Trivially, any doctrine, theory, or canon is incomplete in this analogical sense. Such trivial observations are presumably not at issue in the comments quoted. But there is no more substantial or interesting use to be made of the incompleteness theorem in discussing the Bible, the Constitution, Objectivism, etc.

Similar remarks apply to the following reflections by John Edwards in the electronic magazine *Ceteris Paribus*:

> We can view rules for living, whether they are cultural mores of the sort encoded in maxims, or laws, principles, and policies meant to dictate acceptable actions and procedures, as axioms in a logical system. Candidate actions can be thought of as propositions. A proposition is proved if the action it corresponds to can be shown to be allowed or legal or admissible within the system of rules; it is disproved if it can be shown to be forbidden, illegal, or inadmissible. In the light of Gödel's theorem, does it not seem likely that any system of laws must be either inconsistent or incomplete?

To say that we "can view" rules for living and so on as axioms in a logical system is unexceptionable since anything, broadly speaking, can be viewed as anything else. Furthermore, in the case of viewing a lot of things as "systems" with "axioms" and "theorems," it is demonstrably the case that many people find it natural and satisfying to view things this way. It is a different question whether any conclusions can be drawn, or any substantial claims supported, on the basis of such analogies and metaphors. In the quoted passage, the suggested conclusion is that a system of laws must be "inconsistent or incomplete." Given the accompanying explanation of what "inconsistent" and "incomplete" mean here, it is an easy observation that all systems of laws, rules of living, and so on, are both inconsistent and incomplete and will remain so. In other words, in the case of legal systems, there will always be actions and procedures about which the law has nothing to say, and there will always be actions and procedures on which conflicting legal viewpoints can be brought to bear. Hence the need for courts and legal decisions. References to Gödel's theorem can only add a rhetorical flourish to this simple observation.

## 4.2 "Human Thought" and the Incompleteness Theorem

A prominent example of a "system" to which the incompleteness theorem is sometimes thought to be applicable is "human thought," also known as "the human mind," "the human brain," or "the human intellect." Here the term "human thought" will be used. The same general comments apply as in the case of supposed applications of the theorem to the Bible, the Constitution, the philosophy of Ayn Rand—there is no such thing as the formally defined language, the axioms, and the rules of inference of "human thought," and so it makes no sense to speak of applying the incompleteness theorem to "human thought." There are however special factors that influence this particular invocation of the incompleteness theorem and may seem to lend the view that the incompleteness theorem applies to "human thought" a certain appeal. One formulation of such a view is the following:

> Insofar as humans attempt to be logical, their thoughts form a formal system and are necessarily bound by Gödel's theorem.

Here by "attempt to be logical" one could perhaps mean "try to argue in such a way that all conclusions reached are formal logical consequences of a specified set of basic assumptions stated in a formal language." Within certain narrowly delimited areas of thought, this description is indeed applicable. For example, in attempting to factor large numbers, the conclusions reached, of the form "the natural number $n$ has the factor $m$," are intended to be logical consequences of some basic arithmetical principles. This does not mean, however, that one is restricted to applying logical rules starting from those basic principles in arriving at the conclusion. On the contrary, any mathematical methods may be used, and in the case of factorization, there would be nothing wrong with using divination in tea leaves if this turned out to be useful, since it can be checked whether a claim of the form "the natural number $n$ has the factor $m$" is true or not (and thus whether it is a logical consequence of basic principles). On a stricter interpretation, "attempting to be logical" might consist in the actual application by hand of certain specific formal rules of reasoning, as when we carry out long division on paper, and then indeed it may be reasonable to say that our thinking is closely associated with a formal system. But these are not typical intellectual activities. If by "attempting to be logical" one means only, as in the everyday meaning of "logical," attempting to make sense, to be consistent, not jump to conclusions, and so on, we can't point to any formal systems that have any particular relevance to our thinking. Formal systems are studied and applied in mathematical contexts and in programming computers, not in political debates, in legal arguments, in formal or informal discussions about sports, science, the news, or the weather, in problem solving in everyday life, or in the laboratory. And even when we strive to be as mindlessly computational as we can, to say that "our thoughts form a formal system" is metaphorical at best.

A line of thought that is often put forward in this connection is the following. Let us accept that the incompleteness theorem can only be sensibly applied when we are talking about proving mathematical statements, and not in the context of Bible reading, philosophical or political discussion, and so on. But human beings *do* prove mathematical, and in particular arithmetical, statements. If we say that "human thought," when it comes to proving arithmetical statements, is *not* bound by the incompleteness theorem, are we then not obliged to hold that there is something essentially noncomputable about human mathematical thinking which allows it to transcend the limitations of computers and formal systems, and perhaps even some irreducibly spiritual, nonmaterial component of the human mind? Such a conclusion is as welcome to many as it is unwelcome to others, and this tends to influence how people view attempts to apply the incompleteness theorem to human thought.

The view here to be argued is the following. It doesn't matter, when we talk about the incompleteness theorem and its applicability to human thought, whether people are similar to Lieutenant Commander Data of *Star Trek* fame, with "positronic brains" whirring away in their heads, influenced from time to time by their "emotion chips," or are on the contrary irreducibly spiritual creatures transcending all mechanisms.

The basic assumption made in disputes over whether or not human thought is constrained by the incompleteness theorem is that we can sensibly speak about "what the human mind can prove" in arithmetic. Assuming this, we may ask whether the set $M$ of all "humanly provable" arithmetical sentences is computably enumerable. If it is, "the human mind" is subject to Gödel's incompleteness theorem, and there are arithmetical statements that are not "humanly decidable" (given that "human thought" is consistent). If $M$ is not computably enumerable, human thought surpasses the powers of any formal system and is in this sense not constrained by the incompleteness theorem. To ask whether or not $M$ is computably enumerable is, given this assumption, to pose a challenging and highly significant problem, a solution to which would be bound to be enormously interesting and illuminating.

But is there such a set $M$ of the "humanly provable" arithmetical statements? This assumption is implausible on the face of it and has very little to support it. Actual human minds are of course constrained by many factors, such as the limited amount of time and energy at their disposal, so in considering what is "theoretically possible" or "possible in principle" for "the human mind" to prove, some theory or principle of the workings of "the human mind" is presupposed, one that allows us to speak in a theoretical way of the "humanly provable" arithmetical statements. What theory or set of principles this might be is unclear, and the fact of the variability and malleability of "the human mind" makes it highly unlikely that any such theory is to be had.

Let us consider the question what "the human mind" *has* proved. Some human minds reject infinitistic set theory as meaningless, whereas others find it highly convincing and intuitive, with a corresponding sharp disagreement over whether or not certain consistency statements have been proved or made plausible. The question what "the human mind can prove" presupposes an agreement on what is or is not a proof. Lacking any theoretical characterization of all possible proofs, and even any general agreement on what existing arguments *are* proofs, we can only ask what "the human mind can prove" using certain formally specified methods of reasoning, in which case the human mind becomes irrelevant, and the question is one about what is provable in certain formal systems.

As for malleability, Errett Bishop, who supported and worked in what is known as constructive mathematics, spoke of "the inevitable day when constructive mathematics will be the accepted norm." From what we know about the human mind, it is perfectly conceivable that such an event could take place. Indeed, we know nothing to rule out the possibility that the accepted mathematical norm will in the future be such that even Bishop's constructive mathematics is regarded as partly unjustifiable. It is equally conceivable that people can convince themselves, in one way or another, of the acceptability of extremely nonconstructive principles that are today not considered evident by anybody. In this sense, then, there may be no limit at all on the "capacity of the human mind" for proving theorems, but of course there is nothing to exclude the possibility that false statements will be regarded as proved because principles that are not in fact arithmetically sound will come to be regarded as evident.

The question of the actual or potential reach of the human mind when it comes to proving theorems in arithmetic is not like the question how high it is possible for humans to jump, or how many hot dogs a human can eat in five minutes, or how many decimals of $\pi$ it is possible for a human to memorize, or how far into space humanity can travel. It is more like the question how many hot dogs a human can eat in five minutes without making a totally disgusting spectacle of himself, a question that will be answered differently at different times, in different societies, by different people. We simply don't have the necessary tools to be able to sensibly pose large theoretical questions about what can be proved by "the human mind." This point will be considered again in Chapter 6, in connection with Gödelian arguments in the philosophy of mind.

There is yet another approach to the application of the incompleteness theorem to human thought, which does not seek undecidable statements either in ordinary informal reasoning or in mathematics, but suggests that Gödel's *proof* of the first incompleteness theorem can be carried through in nonmathematical contexts. This suggestion will be considered next.

## 4.3 Generalized Gödel Sentences

### Mathematical

As mentioned in Chapter 2, the technique of introducing provable fixpoints that Gödel invented has been used since in very many arguments in logic, not just in the proof of the first incompleteness theorem. An example follows.

The basic property of a formal system, as emphasized in Section 3.4, is that its set of theorems is computably enumerable. If we no longer require this property, but retain the formal language, the set of axioms, and the rules of inference, we get the concept of a formal theory in a wider sense, also studied in logic. A theory in this generalized sense need not be relevant to mathematical knowledge, since we may not have any method for deciding whether something is a proof in the theory or not, but the concept is very useful in logical studies.

Using this more liberal notion of "theory," we can define $T$ as the theory obtained by adding *every* true statement of the form "the Diophantine equation $D(x_1, \ldots, x_n) = 0$ has no solution" to PA as an axiom. The set of Gödel numbers of true statements of this form can be defined in the language of arithmetic, so using the fixpoint construction we can formulate an arithmetical statement $A$ for which it is provable in PA that $A$ is true if and only if it is not provable in $T$. (This $A$ will not be a Goldbach-like statement.) If $A$ is false, then it is provable in $T$, which is impossible since all axioms of $T$ are true. So $A$ is in fact true but unprovable in $T$.

In this way, the fixpoint construction used in Gödel's proof can be extended to show that many theories that do not constitute formal systems are still incomplete. But suppose we add *every* true arithmetical sentence as an axiom to PA. The resulting theory $T$ (known as "true arithmetic") is obviously complete, so where does the argument break down in this case? It breaks down because we can no longer form a "Gödel sentence" for $T$, since (as this very argument shows) the property of being the Gödel number of a true arithmetical sentence cannot be defined in the language of arithmetic.

## Nonmathematical

So why not, instead of seeking to apply the incompleteness theorem to nonmathematical systems, just mimic Gödel's proof of the theorem by formulating a "Gödel sentence" for such systems? Thus, we might come up with the following:

- The truth of this sentence cannot be established on the basis of the Bible.

- The truth of this sentence cannot be inferred from anything in the Constitution.

- This sentence cannot be shown to be true on the basis of Ayn Rand's philosophy.

Before considering these dubious statements, we need to dispose of a particular objection that is often raised against them, regarding the use of the phrase "this sentence" to denote a sentence in which the phrase occurs. Such self-reference has been thought to be suspect for various reasons. In particular, the charge of leading to an infinite regress when one attempts to understand the statement has been leveled against it. Such a charge makes good sense if one takes "this" to refer to what is traditionally called a "proposition" in philosophy, the *content* of a meaningful sentence. But in the above statements, as in the corresponding construction in the original Gödel sentence, "this sentence" refers to a syntactic object, a sequence of symbols, and there is no infinite regress involved in establishing the reference of the phrase. As in the case of the Gödel sentence, we can make this fact apparent by eliminating the phrase "this sentence" in favor of the use of syntactical operations, such as substitution or "quining" (see Section 2.7), but we can also simply replace the first sentence with

> The truth of the sentence $P$ cannot be established on the basis of the Bible

together with the explicit definition

> $P =$ "The truth of the sentence $P$ cannot be established on the basis of the Bible."

This explicit definition is as unproblematic as any in logic or mathematics, since we are simply introducing a symbol to denote a particular string of symbols. (In particular, the definition of $P$ does not presuppose that the string of symbols on the right means anything.) In the following, the shorter version of self-referential sentences, containing "this sentence," will be used.

Now, in considering the three proposed "Gödel sentences," one's first impulse may be to regard them as unproblematically true, but not as presenting any inadequacy of either the Bible, the Constitution, or Ayn Rand's philosophy. After all, it is by no means obvious that the statements even make sense in the context of Bible studies, etc. The Gödel sentence for PA is in a different case, since that sentence is itself an arithmetical sentence, and the inability of PA to settle it indicates a gap in the mathematical power of PA. That the Constitution (let us suppose) does not suffice to establish the truth of "The truth of this sentence cannot be inferred from anything in the Constitution" indicates no gap whatsoever in the Constitution, which after all was not designed as an instrument for the discussion of logical puzzles.

However, the three statements are not as unproblematic as they may seem, since we might as well go on to formulate a more far-reaching "Gödel sentence" of this kind:

> This sentence cannot be shown to be true using any kind of sound reasoning.

If this sentence is false, it can be shown to be true using sound reasoning, but sound reasoning cannot establish the truth of a false sentence. So sound reasoning leads to the conclusion that the sentence is true—but then it is false. Perhaps we should describe it, rather, as meaningless? But then, surely, it cannot be shown to be true using any kind of sound reasoning, so....

A popular variant of this argument seeks to establish the existence of a "Gödel sentence" for any particular person, one that can unproblematically be held to be true by everybody except that person: for example,

> John will never be able to convince himself of the truth of this sentence.

And of course, in the version using "cannot be shown to be true," we are approaching the classical paradox of the Liar, which leads to the conclusion that the Liar sentence

> This sentence is false

is true if and only if it is false.

The many arguments and ideas surrounding the Liar sentence and the various "Gödel sentences" formulated above will not be discussed in this book. The following observations are however relevant. The incompleteness theorem is a mathematical theorem precisely because the relevant notions of truth and provability are mathematically definable. Nonmathematical "Gödel sentences" and Liar sentences give rise to prolonged (or endless) discussions of just what is meant by a proof, by a true statement, by sound reasoning, by showing something to be true, by convincing oneself of something, by believing something, by a meaningful statement, and so on. In spite of the similarities in form between these other sentences and the fixpoints of arithmetically (or more generally, mathematically) definable properties of sentences in a formal language, we are again not dealing with any application of the incompleteness theorem or its proof, but with considerations or conundrums inspired by the incompleteness theorem. It is an open question whether the pleasant confusion that statements like "John will never accept this statement as true" tend to create in people's minds has any theoretical or philosophical significance.

## 4.4 Incompleteness and the TOE

The TOE is the hypothetical Theory of Everything, which is sometimes thought to be an ideal or Holy Grail of theoretical physics. The incompleteness theorem has been invoked in support of the view that there is no such theory of everything to be had, for example, by eminent physicists Freeman Dyson and Stephen Hawking.

In a book review in the New York Review of Books, Dyson writes:

> Another reason why I believe science to be inexhaustible is Gödel's theorem. The mathematician Kurt Gödel discovered and proved the theorem in 1931. The theorem says that given any finite set of rules for doing mathematics, there are undecidable statements, mathematical statements that cannot either be proved or disproved by using these rules. Gödel gave examples of undecidable statements that cannot be proved true or false using the normal rules of logic and arithmetic. His theorem implies that pure mathematics is inexhaustible. No matter how many problems we solve, there will always be other problems that cannot be solved within the existing rules. Now I claim that because of Gödel's theorem, physics is inexhaustible too. The laws of physics are a finite set of rules, and include the rules for doing mathematics, so that Gödel's theorem applies to them. The theorem implies that even within the domain of the basic equations of physics, our knowledge will always be incomplete.

It seems reasonable to assume that a formalization of theoretical physics, if such a theory can be produced, would be subject to the incompleteness theorem by incorporating an arithmetical component. However, as emphasized in Section 2.3, Gödel's theorem only tells us that there is an incompleteness in the arithmetical component of the theory. The basic equations of physics, whatever they may be, cannot indeed decide every arithmetical statement, but whether or not they are complete considered as a description of the physical world, and what completeness might mean in such a case, is not something that the incompleteness theorem tells us anything about.[1]

Another invocation of incompleteness goes further:

> Not to mention there are an infinite number of other attributes of the world which are simply not quantifiable or computable, such as beauty and ugliness, happiness and misery, intuition and inspiration, compassion and love etc. These are completely outside the grasp of any mathematical Theory of Everything. Since scientific theories are built upon mathematical systems, incompleteness must be inherited in all our scientific knowledge as well. The incompleteness theorem reveals that no matter what progress is made in our science, science can never in principle completely disclose Nature.

Here the connection with the actual content of the incompleteness theorem is tenuous in the extreme: "Since scientific theories are built upon mathematical system, incompleteness must be inherited in all our scientific knowledge as well." This doesn't follow, since nothing in the incompleteness theorem excludes the possibility of our producing a complete theory of stars, ghosts, and cats all rolled into one, as long as what we say about stars, ghosts, and cats cannot be interpreted as statements about the natural numbers. That science cannot be expected to disclose to us everything about beauty and ugliness, intuition and inspiration, and so on, is a reasonable view which neither needs nor is supported by Gödel's theorem.

Stephen Hawking, in a talk entitled "Gödel and the End of Physics," also mentions Gödel's theorem:

> What is the relation between Gödel's theorem, and whether we can formulate the theory of the universe, in terms of a finite number of principles? One connection is obvious. According to the positivist philosophy of science, a physical theory is a mathematical model. So if there are mathematical results that cannot be proved, there are physical problems that cannot be pre-

[1] Dyson conceded this point in a gracious response to similar remarks made by Solomon Feferman in a letter to the New York Review of Books (July 15, 2004).

> dicted. One example might be the Goldbach conjecture. Given an even number of wood blocks, can you always divide them into two piles, each of which cannot be arranged in a rectangle? That is, it contains a prime number of blocks. Although this is incompleteness of sorts, it is not the kind of unpredictability I mean. Given a specific number of blocks, one can determine with a finite number of trials, whether they can be divided into two primes. But I think that quantum theory and gravity together introduce a new element into the discussion, one that wasn't present with classical Newtonian theory. In the standard positivist approach to the philosophy of science, physical theories live rent-free in a Platonic heaven of ideal mathematical models. That is, a model can be arbitrarily detailed and can contain an arbitrary amount of information, without affecting the universes they describe. But we are not angels who view the universe from the outside. Instead, we and our models are both part of the universe we are describing. Thus, a physical theory is self-referencing, like in Gödel's theorem. One might therefore expect it to be either inconsistent, or incomplete. The theories we have so far, are both inconsistent, and incomplete.

Here the upshot is that physical theory is "self-referencing," apparently in the sense that physical theories are "part of the universe" and that one might therefore expect them to be inconsistent or incomplete, considering that Gödel proved his first incompleteness theorem using a self-referential statement. Again, the relevance of the incompleteness theorem is here at most a matter of inspiration or metaphor. But Hawking also touches on another subject, the relevance of arithmetic to predictions about the outcome of physical experiments. Given $104,729$ wooden blocks, will we succeed in an attempt to arrange them into a rectangle? A computation shows $104,729$ to be a prime, so we conclude that no such attempt will succeed. Or, to take a somewhat more realistic example, consider the 15-puzzle, the still-popular sliding square puzzle that Sam Lloyd introduced in 1873, which has long been a favorite among AI researchers when testing heuristic search algorithms. Lloyd offered a $1,000 reward for the solution of the "15-14 problem," the problem of rearranging the squares so that only the last two squares were out of place. He well knew that his money was not at risk, since a combinatorial argument shows that the problem has no solution. Thus, he could set people to work on the problem and confidently predict, on the basis of arithmetical reasoning, the eventual outcome (their giving up).

Do such examples show that arithmetical incompleteness can entail an incompleteness in our description of the physical world? Not really. Suppose the Diophantine equation $D(x_1, \ldots, x_n) = 0$ has no solution, but this fact is not provable in our mathematics. We then have no basis for a prediction of the outcome of any physical experiment describable as "searching for a solution of the equation $D(x_1, \ldots, x_n) = 0$." (Such an experiment might consist in people rearranging wooden blocks or doing pen-and-paper calculations, or it might consist in having a computer execute a program.) This does not, however, indicate any incompleteness in our description of the physical systems involved. Our predictions of the outcome of physical experiments using arithmetic are based on the premise that arithmetic provides a good model for the behavior of certain actual physical systems with regard to certain observable properties (which in particular implies that physical objects like blocks of wood have a certain stability over time, that there are no macroscopic tunneling effects that render arithmetic inapplicable, that eggs do not spontaneously come into existence in baskets, and so on). The relevant description of the physical world amounts to the assumption that this premise is correct. The role of the arithmetical statement is as a premise in the application of this description to arrive at conclusions about physical systems.

## 4.5 Theological Applications

Gödel sometimes described himself as a theist and believed in the possibility of a "rational theology," although he did not belong to any church. In [Wang 87] he is quoted as remarking that "I believe that there is much more reason in religion, though not in the churches, that one commonly believes." It should not be supposed that Gödel's theism agreed with that expressed in established theistic religions. Theistic religions usually involve a God or several gods assumed to stand in a relationship to human beings that makes it meaningful to pray to the God(s), to thank the God(s), to obey the God(s), and more generally to communicate with the God(s). Gödel's "rational theology" was not concerned with such matters. Among his unpublished papers was a version of St. Anselm's ontological proof of the existence of God. More precisely, the conclusion of the argument is that there is a God-like individual, where $x$ is defined to be God-like if every essential property of $x$ is positive and $x$ has every positive property as an essential property. As this explanation of "God-like" should make clear, Gödel's idea of a rational theology was not of an evangelical character, and Oskar Morgenstern relates ([Dawson 97, p. 237]) that he hesitated to publish the proof "for fear that a belief in God might be ascribed to him, whereas, he said, it was undertaken as a purely logical investigation, to demonstrate that such a proof could be carried out on the basis of accepted principles of formal logic."

Although Gödel was thus not at all averse to theological reasoning, he did not attempt to draw any theological conclusions from the incompleteness theorem. However, others have invoked the incompleteness theorem in theological discussions. *Bibliography of Christianity and Mathematics*, first edition 1983, lists 13 theological articles invoking Gödel's theorem. Here are some quotations from the abstracts of these articles:

> Nonstandard models and Gödel's incompleteness theorem point the way to God's freedom to change both the structure of knowing and the objects known.
>
> Uses Gödel's theorem to indicate that physicists will never be able to formulate a theory of physical reality that is final.
>
> Stresses the importance of Gödel's theorems of incompleteness toward developing a proper perspective of the human mind as more than just a logic machine.
>
> ...theologians can be comforted in their failure to systematize revealed truth because mathematicians cannot grasp all mathematical truths in their systems, either.
>
> If mathematics were an arbitrary creation of men's minds, we can still hold to eternal mathematical truth by appealing to Gödel's incompleteness result to guarantee truths that can be discovered only by the use of reason and not by the mechanical manipulation of fixed rules—truths which imply the existence of God.
>
> It is argued by analogy from Gödel's theorem that the methodologies, tactics, and presuppositions of science cannot be based entirely upon science; in order to decide on their validity, resources from outside science must be used.

As can be seen from these quotations, appeals to the incompleteness theorem in theological contexts are sometimes invocations of Gödelian arguments in the philosophy of mind, which will be considered in Chapter 6, and sometimes follow the same line of thought as the arguments considered in Section 4.4 in connection with "theories of everything." But there are also some more specifically theological appeals to the theorem. These are sometimes baffling:

> For thousands of years people equated consistency with determinism, holding that a logically consistent sequence of propositions could have only one outcome. This feeling lies behind the notion that God knows and controls everything. Kurt Gödel, working on a question asked by David Hilbert, showed that consistency does not always mean determinism.

It is difficult to know what to make of the idea that "a logically consistent sequence of propositions can have only one outcome" or of its relation to negation completeness.

Other theological invocations of Gödel are more easily made sense of. The following reflections by Daniel Graves are taken from an essay on the "Revolution against evolution" website:

> Gödel showed that "it is impossible to establish the internal logical consistency of a very large class of deductive systems—elementary arithmetic, for example—unless one adopts principles of reasoning so complex that their internal consistency is as open to doubt as that of the systems themselves." (Here the author is quoting [Nagel and Newman 59]). In short, we can have no certitude that our most cherished systems of math are free from internal contradiction.
>
> Take note! He did not prove a contradictory statement, that $A$ = non-$A$, (the kind of thinking that occurs in many Eastern religions). Instead, he showed that no system can decide between a certain $A$ and non-$A$, even where $A$ is known to be true. Any finite system with sufficient power to support a full number theory cannot be self-contained.
>
> Judeo-Christianity has long held that truth is above mere reason. Spiritual truth, we are taught, can be apprehended only by the spirit. This, too, is as it should be. The Gödelian picture fits what Christians believe about the universe. Had he

> been able to show that self-proof was possible, we would be in deep trouble. As noted above, the universe could then be self-explanatory.
>
> As it stands, the very real infinities and paradoxes of nature demand something higher, different in kind, more powerful, to explain them just as every logic set needs a higher logic set to prove and explain elements within it.
>
> This lesson from Gödel's proof is one reason I believe that no finite system, even one as vast as the universe, can ultimately satisfy the questions it raises.

A main component of these reflections is the observation that any consistent system incorporating arithmetic cannot prove itself consistent and cannot answer every question it raises, and is in these respects "not self-contained." It is only in a wider, stronger system that every question raised by the first system can be answered and more besides, and the first system can also be proved consistent. As a description of the incompleteness theorem, this is unobjectionable, and in fact in his 1931 paper Gödel had a footnote 48a that should be quite congenial to the author of passage quoted:

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (see *Hilbert 1926*, page 184), while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type $\omega$ to the system P). An analogous situation prevails for the axiom system of set theory.

While this part of Graves' comments is thus unobjectionable, they are highly dubious in other aspects. The idea that the consistency of arithmetic is in doubt will be commented on in Chapter 6. It is not clear that this idea is theologically relevant. The formulation "no system can decide between a certain $A$ and non-$A$" is incorrect, since there is no $A$ such that no system can decide between $A$ and non-$A$, but rather for any given system there is an $A$ depending on the system such that the system cannot decide between $A$ and non-A. But the main thrust of the passage is (unsurprisingly) to point to the incompleteness theorem as providing an analogy to a Christian perception of the relation between the universe and God: the universe needs something higher to explain it. Finding such analogies is perfectly legitimate, but of course it is tendentious in the extreme to speak of analogies as "lessons." It is also obscure on what grounds the author claims that "Had he been able to show that self-proof was possible, we would be in deep trouble." If "self-proof" means a consistency proof for $S$ carried out within $S$ itself, it is difficult to believe that any theologian would have concluded from a consistency proof for PA carried out within PA that since arithmetic needs nothing higher than itself to support or explain it, neither does the universe, and so God is unnecessary.

The author makes a further comment:

> As a third implication of Gödel's theorem, faith is shown to be (ultimately) the only possible response to reality. Michael Guillen has spelled out this implication: "the only possible way of avowing an unprovable truth, mathematical or otherwise, is to accept it as an article of faith." In other words, scientists are as subject to belief as non-scientists. [The reference is to *Bridges to Infinity*, Los Angeles: Tarcher, 1983, p.117.]

Here, the reference to Gödel's theorem is pointless. That science involves faith is a standard argument in discussions of theology and religion, but one to which Gödel's theorem is irrelevant. As much or as little faith is needed to accept the axioms of a theory whether or not that theory is complete and the necessity of accepting some basic principles without proof is not something that was revealed by Gödel's theorem.

Another example of a theological invocation of Gödel's theorem is given by the following comments by Najamuddin Mohammed:

> It is pointed out that, no matter how you describe the world (with logical rules) there will always be "some things" that you cannot determine as true or false. And whether you select the answer to these "some things" as true or false doesn't affect the validity of your logical rules. Strange but true!
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> For example let's say: you and I have agreed upon a set of logical rules, then there will always be some thing, lets call it $A$, that we cannot determine as true or false, using our logical rules. You can take $A$ to be true and I can take $A$ to be false, but in either case we are both logically consistent with our new set of logical rules respectively.

> But now we have two sets of self-consistent rules and again there will always be something called $B$ that we cannot agree upon....and so on. This is the basis of Gödel's Incompleteness Theorem.
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> If we rely on logic or reason alone we can end up in utter confusion, with many contradictory but logically self-consistent systems of reasoning/logic. Which is correct? Does everything depend on our current psychological disposition as to what is right and wrong? Correctness has no meaning in these cases, all this can lead to agnostic and atheistic stances.

There is a similarity between these reflections and the ideas about a "postmodern condition" created by the incompleteness theorem commented on in Section 2.8. Incompleteness, it is argued, leads to a profusion of different consistent theories, and nobody knows where truth—or "truth"—is to be had. Thus, reason alone cannot put us on the right path, and religious faith is the way to go. Again, the weakness of this line of thought is that there is not in fact any such branching off into various directions in mathematical thinking, no floundering in a sea of undecidability. The "utter confusion" in mathematical thinking is a theological dream only.

What remains to be considered in connection with theological invocations of the incompleteness theorem are two lines of thought that are not specifically theological but are often thought congenial from a theological point of view: the skeptical conclusions thought to follow from the second incompleteness theorem, and the conclusions about the nature of the human mind claimed to follow from the first incompleteness theorem. These will be considered in later chapters.