The real numbers are uncountable

The title of this section should surprise no one. Of course you can't count the real numbers – there are infinitely many of them. But, in fact, there is a way of "counting" certain infinite sets, and that is the subject of this section.

We begin by considering how to "count" a very large set. Suppose you are staging a rock concert in a football stadium and you want to know if there are more men or women there. You could count all the men and count all the women, but this is a huge task. Here is another way: Ask every man to grab one woman, every woman to grab one man. If there are any men left unpaired we know there are more men in the stadium. Of course you must make sure the pairing is one-to-one, but if there are no men left over and no women left over we know there were the same number of men as women.

Using the language of sets, we have compared the set of all men with the set of all women and found they have the same number of elements. This idea of pairing can be used to compare the size of any two sets, even if they have infinitely many elements. Two sets have the same number of members if there is a one-to-one pairing of all the elements of one set with all the elements of the other set. Using the idea of a function we can say

> Two sets, A and B, have the same number of elements if there is a one-to-one function from A onto B. In this case we write $\#A = \#B$.

We start with a simple, but important example: Let $\mathcal{E}$ denote the set of even whole numbers, and let Θ denote the set of odd whole numbers. It is intuitively clear that $\#\mathcal{E} = \#\Theta$, since half the numbers are even and half are odd, but we will verify our intuition is correct by defining a function, or pairing as above. Consider the function f defined by $f(x) = x+1$. If we put an even number in the place of x, then $f(x)$ is an odd number. Every even number is thus paired with an odd number, and no odd number will be left out. Thus $\#\mathcal{E} = \#\Theta$ as claimed.

The pairing defined by f can be illustrated by writing down "all" the even numbers and "all" the odd numbers and then matching corresponding pairs:

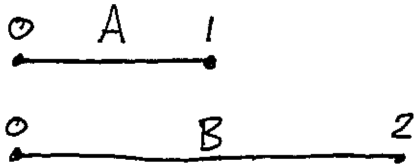
$$
\begin{array}{lcccccccccccc}
\mathcal{E} & \ldots & -4 & -2 & 0 & 2 & 4 & 6 & 8 & \ldots & x & \ldots \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
\mathcal{O} & \ldots & -3 & -1 & 1 & 3 & 5 & 7 & 9 & \ldots & x+1 & \ldots
\end{array}
$$

It is clear that there are no left over, unpaired, odd numbers. However, suppose we used instead the function g defined by $g(x) = x^3+1$. Then the pairing would look like

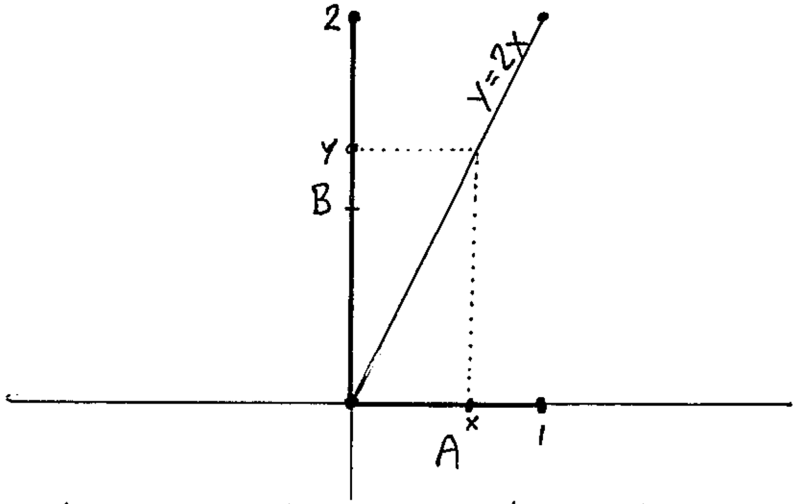
$$
\begin{array}{lcccccccccccc}
\mathcal{E} & & & & -2 & 0 & 2 & 4 & & & & & \\
 & & & & & \downarrow & & & & & & & \\
\mathcal{O} & -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 & 9 & \ldots & 65 & \ldots
\end{array}
$$

and there would be lots of odd numbers left over. This only shows that there are many "bad" ways to pair $\mathcal{E}$ and $\mathcal{O}$. We say $\#\mathcal{E} = \#\mathcal{O}$ because there is a good way to pair the elements, even though there will be other "bad" ways to pair them.

Our next example is more counter-intuitive. Let A denote $\{x : 0 \le x \le 1\}$, the set of all real numbers between zero and one. Let $B = \{y : 0 \le y \le 2\}$, the set of all real numbers between zero and two.



It looks like B is twice as big as A, but in fact they have the same number of points, i.e. $\# A = \# B$. To see this, consider the function f defined by $f(x) = 2x$, or rather the pairing defined by $y = 2x$, which we can draw as a graph:



This graph matches each point x in A with a point y in B, and no points are left over in either set, hence $\# A = \# B$. This example can easily be generalized to show that any two intervals have the same number of points.

Without mentioning it we saw a similar example in the previous section. Let $\mathbb{D}$ denote the open unit disk, and let $\mathbb{R}^2$ denote the entire Euclidean plane (the x-y plane from high-school). The projection defined in section 7 matches each point in $\mathbb{R}^2$ with a unique point in $\mathbb{D}$, and so $^*\mathbb{R}^2 = {}^*\mathbb{D}$. There are just as many points within a unit circle as there are points in the whole infinite plane.

About now you should be thinking that any two infinite sets have the same number of elements, but it is not true. Some infinite sets have more elements than some others. In other words, there is no number "infinity", but many infinite numbers, some bigger than others.

Let $\mathbb{N}$ denote the set of natural numbers: $1, 2, 3, 4, \ldots$. Any set that has the same (infinite) number of elements as $\mathbb{N}$ is called "countable", or countably infinite. Let $[0,1]$ denote the unit interval, the set of all real numbers between zero and one. We will now prove Georg Cantor's famous result:

$$\boxed{[0,1] \text{ is not countable}}$$

We will show that no matter how you pair the elements of $\mathbb{N}$ with the elements of $[0,1]$ there will always be elements of $[0,1]$ left over. Thus $[0,1]$ has a bigger infinite number of elements than does $\mathbb{N}$. It will be handy to write the elements of $[0,1]$ in binary notation, so each number in $[0,1]$ can be written as a sequence of zeros and ones, like $.101101001\ldots$.

Suppose you thought you had a function f from $\mathbb{N}$ to $[0,1]$, and suppose

$$f(1) = .\,a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots$$
$$f(2) = .\,a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots$$
$$f(3) = .\,a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots$$

etcetera. Without knowing what the numbers a_{ij} are we will find a binary expansion that is not on the list above. Hence the pairing defined by f leaves out some elements of $[0,1]$, no matter what f is.

Look at the list (matrix) above, and write down the diagonal number $c = a_{11}\, a_{22}\, a_{33}\, a_{44} \ldots$.

$$
\begin{array}{l}
f(1) = .a_{11}\ a_{12}\ a_{13}\ a_{14}\ \ldots \\
f(2) = .a_{21}\ a_{22}\ a_{23}\ a_{24}\ \ldots \\
f(3) = .a_{31}\ a_{32}\ a_{33}\ a_{34}\ \ldots \\
f(4) = .a_{41}\ a_{42}\ a_{43}\ a_{44}\ \ldots
\end{array}
\qquad = c
$$

Now c may well be on the list above, but $1-c$ is not. To see this, notice that if $c = .10010110\ldots$ then $1-c = .01101001\ldots$, that is $1-c$ has a zero wherever c has a one and vice versa (this is one of the charms of binary arithmetic). Now $1-c$ cannot be $f(n)$ for any n because $f(n)$ and $1-c$ have different n^{TH} binary places. Thus there is no one-to-one function of $\mathbb{N}$ onto $[0,1]$, there are always elements of $[0,1]$ left over.

We have shown then that $\#[0,1] > \#\mathbb{N}$. We will show in the following addendum that this may be generalized to show that given any set we can always find an infinite set with more elements. There is no biggest infinite number.

Addendum

The set of rational numbers is countably infinite

Let $\mathbb{Q}$ denote the set of all rational real numbers. We here show that $\#\mathbb{N} = \#\mathbb{Q}$, that is there are just as many whole numbers as fractions. At first glance this seems very strange, since there are infinitely many fractions between any two whole numbers, but this just shows that our intuition is not adequate when dealing with infinite quantities, of which we have no experience in the physical universe.

We start by writing down all quotients of whole numbers:

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots\ldots \\[4pt]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \frac{7}{2} & \ldots\ldots \\[4pt]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \frac{7}{3} & \ldots\ldots \\[4pt]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \frac{7}{4} & \ldots\ldots \\[4pt]
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \ldots\ldots & & & \\
\vdots & \vdots & \vdots & \vdots & & & &
\end{array}$$

We will now define a pairing of whole numbers n with quotients $\frac{p}{q}$. The matched pair we write as $(n, \frac{p}{q})$

$$\begin{tikzcd}
(1,1) \arrow[r] & (2,2) \arrow[dl] & (6,3) \arrow[r] & (7,4) \arrow[dl] & (15,5) \arrow[r] & (16,6) \arrow[dl] \\
(3,\frac{1}{2}) \arrow[d] & (5,\frac{2}{2}) \arrow[ur] & (8,\frac{3}{2}) \arrow[dl] & (14,\frac{4}{2}) \arrow[ur] & (17,\frac{5}{2}) \arrow[dl, dotted] & \\
(4,\frac{1}{3}) \arrow[ur] & (9,\frac{2}{3}) \arrow[dl] & (13,\frac{3}{3}) \arrow[ur] & & & \\
(10,\frac{1}{4}) \arrow[d] & (12,\frac{2}{4}) \arrow[ur] & & & & \\
(11,\frac{1}{5}) \arrow[ur] & & & & &
\end{tikzcd}$$

Following the arrows we see we have matched 1 with 1, 2 with 2, 3 with $\frac{1}{2}$, 4 with $\frac{1}{3}$, etc. Continuing this process we will eventually get to any quotient on the chart, so no quotients are left over, and clearly no whole numbers are left over. Thus there are just as many quotients as whole numbers. Since the rational numbers are amongst the quotients (we could skip over duplicates if we wanted), there are just as many rational numbers as whole numbers, and we are done.

We will now show that there is no biggest infinite set. Given a set X, let 2^X denote the set of all functions from X to the set with two elements $\{0,1\}$. Thus if f is in 2^X, then $f(x)$ is either zero or one for each x in X.

$$\boxed{\#2^X > \#X}$$

In plain English this says there are more binary functions on X than elements of X. Suppose there were a way of pairing elements x of X with functions in 2^X. Let f_x be the function paired with x. We will find a function that is paired with no x. First look at the "diagonal" function g defined by $g(x) = f_x(x)$. The function g may be paired with some element x, but $1-g$ is not: $1-g$ cannot be f_x for any x because $(1-g)(x) = 0$ if $f_x(x) = 1$ and vice versa. Thus any pairing of X with 2^X will always result in elements of 2^X left over, so $\#2^X > \#X$.