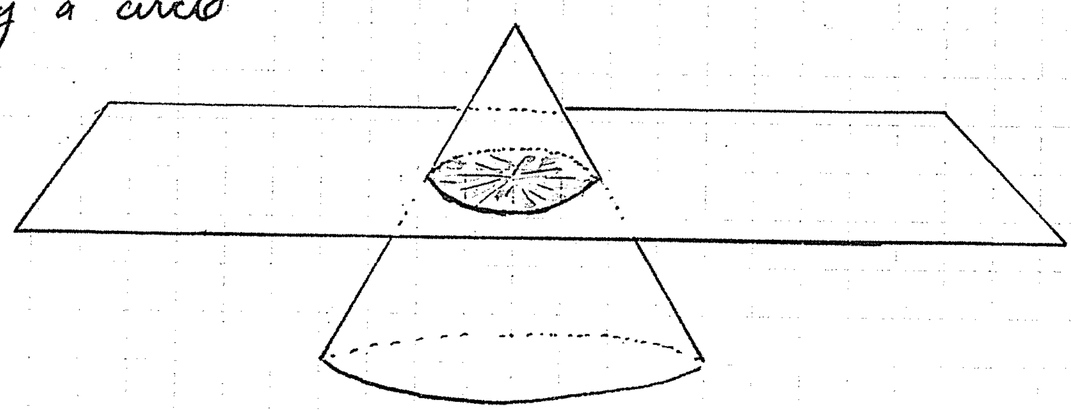
Conic Sections

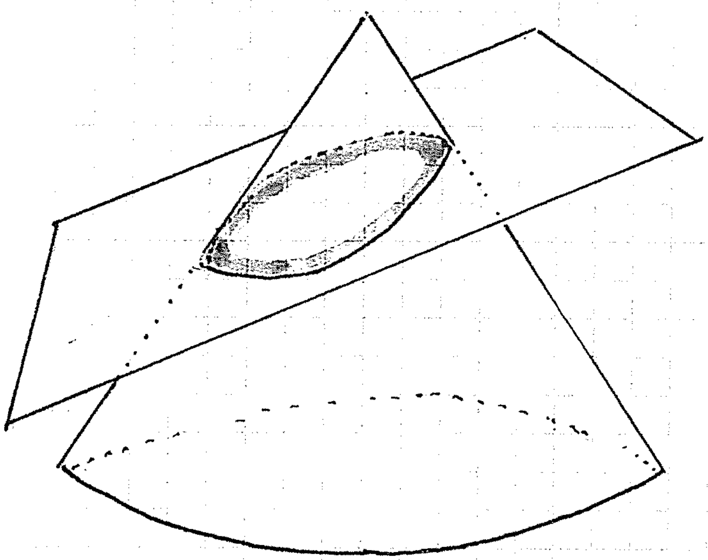
A study of conic sections has formed part of the standard mathematics curriculum for many centuries. Today circles ellipses, hyperbolas, and parabolas appear in the context of "analytic geometry" along with the equations and graphs of other types of curves, but this gives a distorted view of the subject because conic sections were thoroughly understood two thousand years before graphs and equations were invented.

Conic sections were defined in the fourth century BC to be the intersection of a cone and a plane. If the plane is parallel to the base of the cone the intersection is clearly a circle

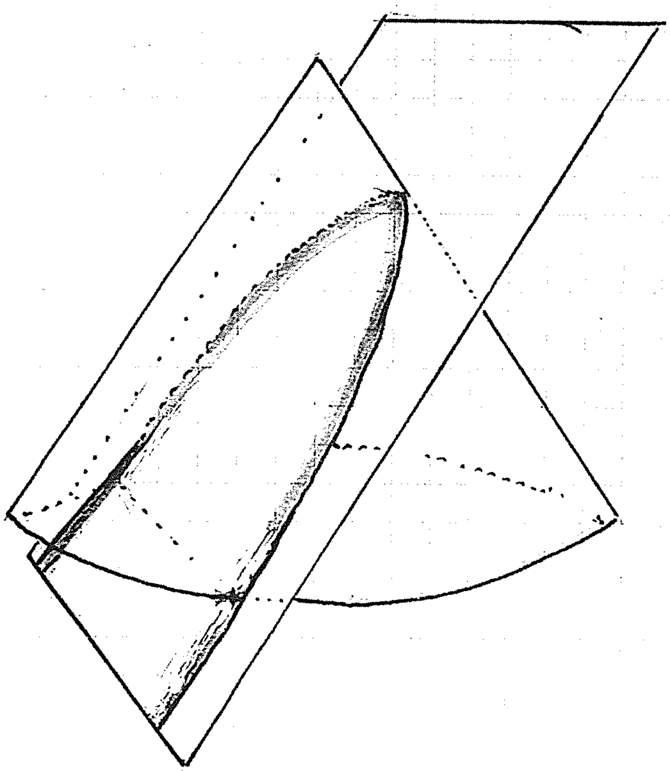


There are three other ways in which the plane could cut the cone, and these lead to the three other conic sections.

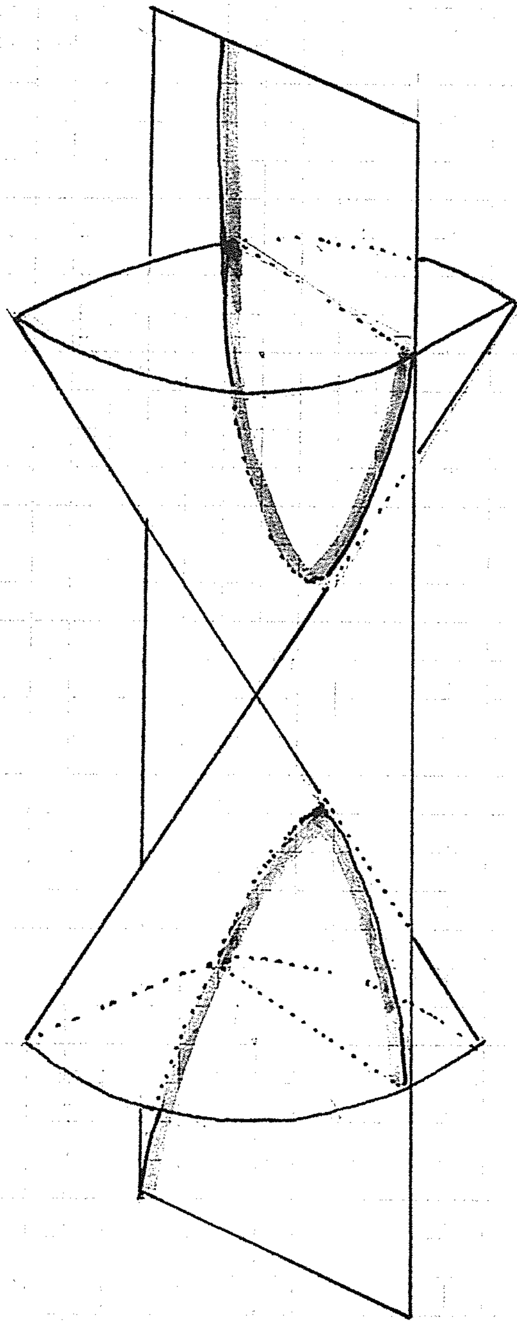
Ellipse



Parabola

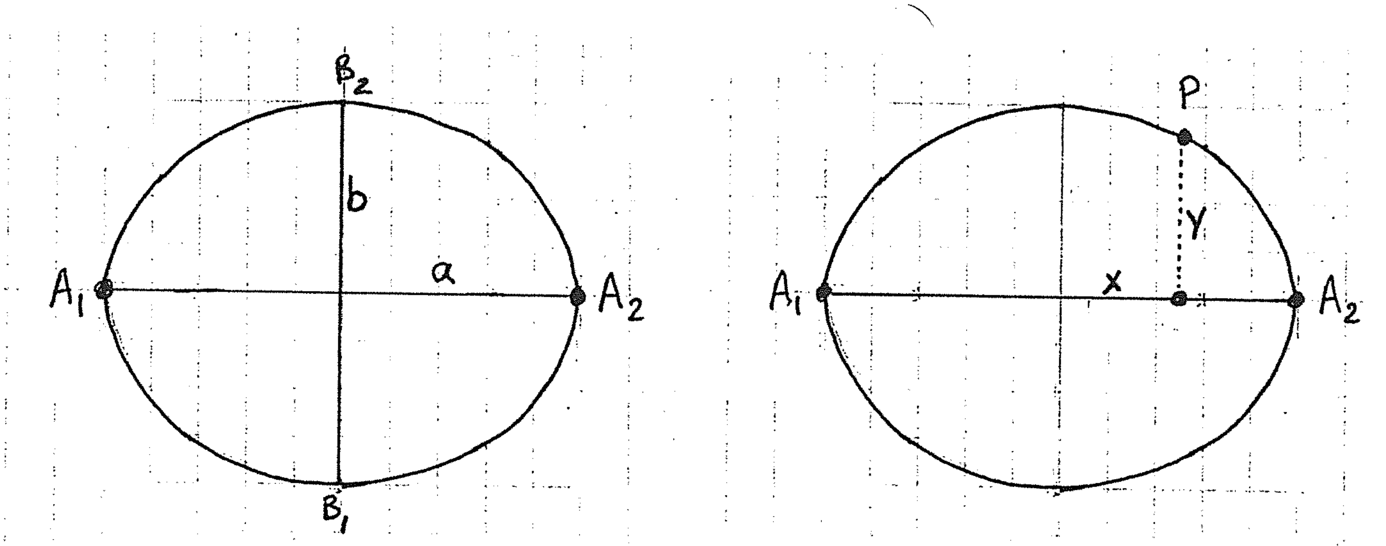


Hyperbola



"The Great Geometer", Apollonius, explored the properties of these curves in his book "Conics", written around 200 BC. To illustrate his methods we will look at the ellipse, though the same methods apply to hyperbolas and parabolas.

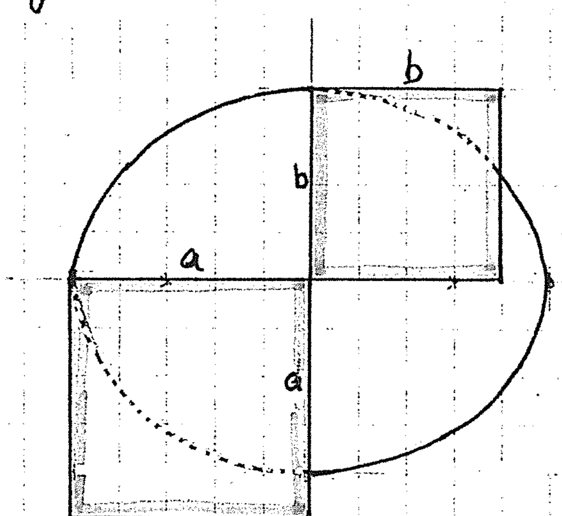
First we identify the major and minor axes of the ellipse, A_1A_2 and B_1B_2 below, then the two segments a and b:

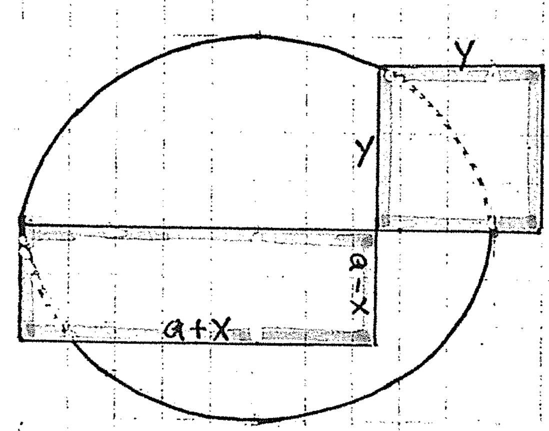


On the right we have chosen a point P on the ellipse and dropped a segment y from P perpendicular to the major axis. This gives us the segment x, and Apollonius proved that

$$\boxed{\frac{y^2}{(a+x)(a-x)} = \frac{b^2}{a^2}} \quad \star$$

and this relation characterizes the ellipse. This should be very satisfying to the modern reader, for we can easily derive the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which everyone memorized in high school. However, this completely misrepresents Apollonius' formula because graphs and equations of curves were not invented until the seventeenth century AD. Where we wrote "y^2" Apollonius wrote "the square on y", that is, he drew a real geometric square. Where we wrote "$(a+x)(a-x)$" Apollonius wrote "the rectangle with sides $a+x$ and $a-x$", and he drew a rectangle. "b^2" means "the square on b" and "a^2" means "the square on a", so our pictures look like

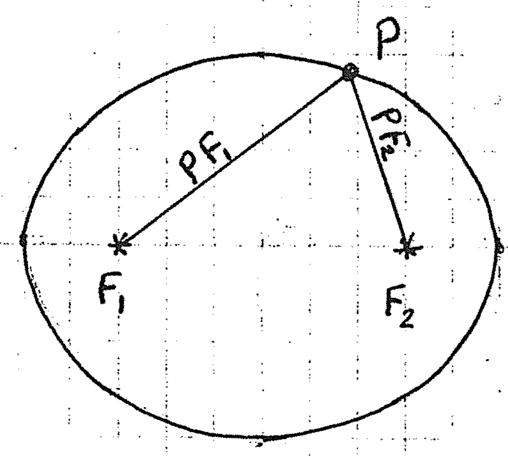




Apollonius' formula says the ratio of the square to the rectangle on the right is the same as the ratio of the squares on the left.

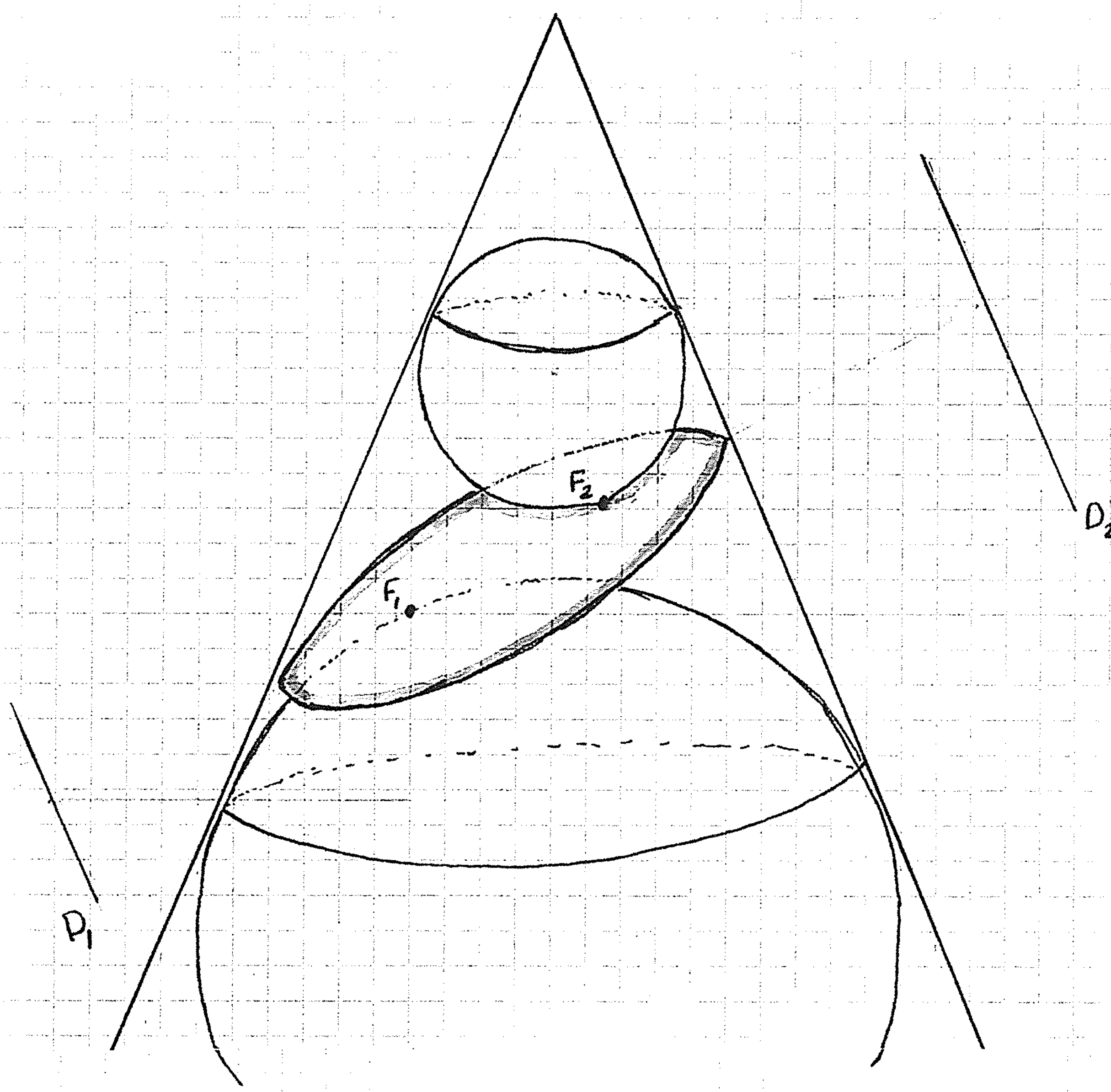
Using this geometric approach Apollonius was able to expose most of the properties of the conics, including tangent lines, chords, diameters, areas of segments, et cetera. The fact that he could do this without modern notions of algebra, graphs, or equations is one of the marvels of ancient mathematics.

Apollonius went on to explore the "focal" properties of ellipses and hyperbolas. Just as a circle has a focal point not on the circle, its center, ellipses (and hyperbolas) have two focal points, the foci of the ellipse. If F_1 and F_2 are the foci and P is any point on the ellipse, then the sum of the distances from P to the foci does not depend on the choice of the point P

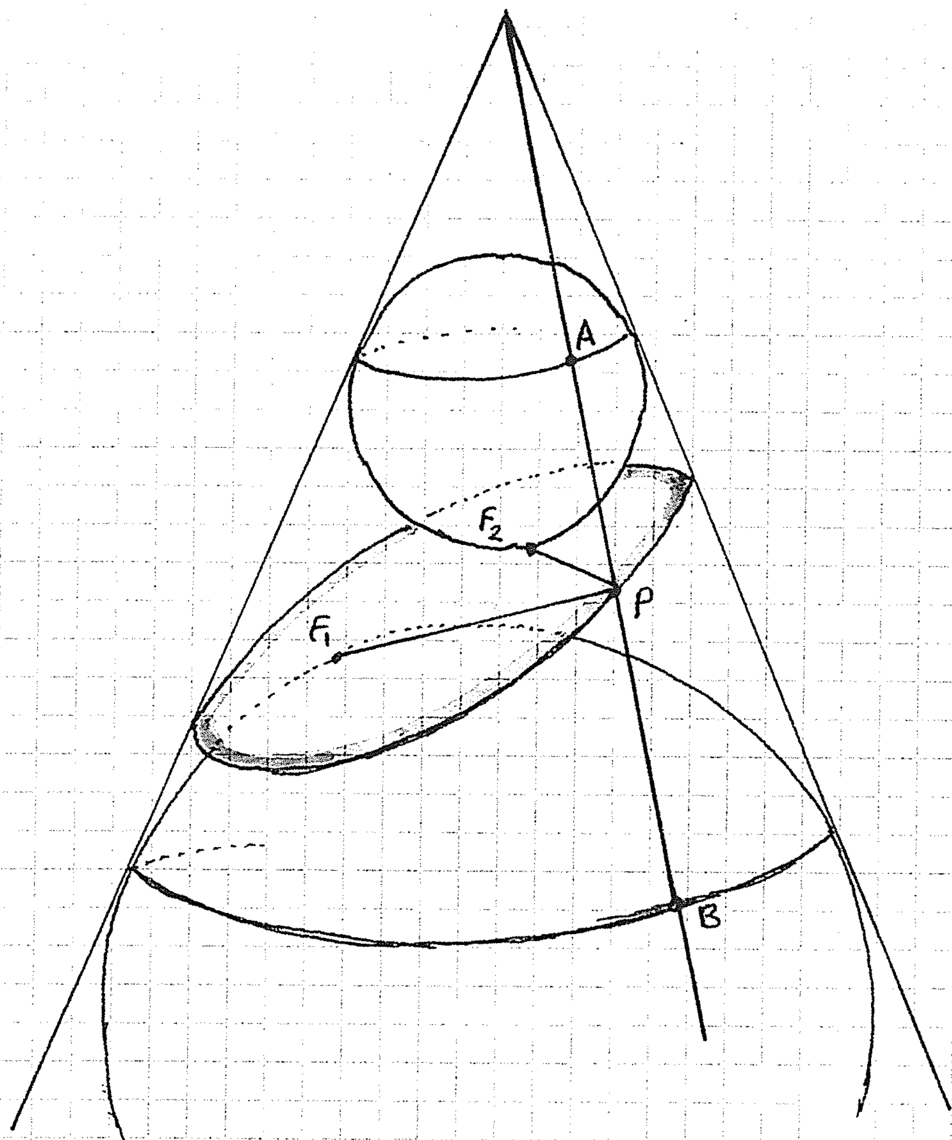


$$\boxed{PF_1 + PF_2 = \text{constant}}$$

Here is a particularly beautiful (and recent) proof of Apollonius' formula. We start with two spheres inscribed in the cone and tangent to the ellipse.



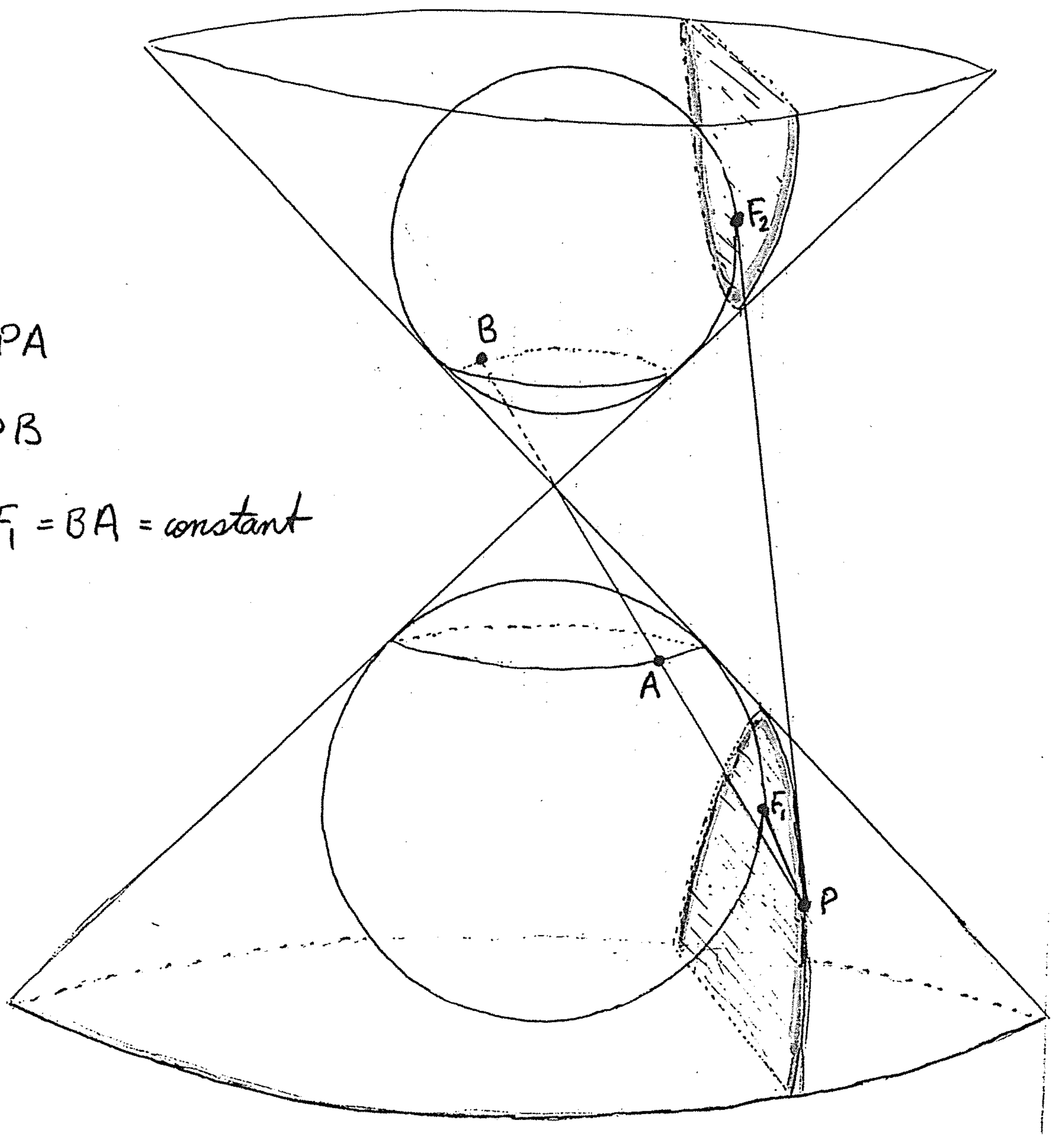
Where the spheres touch the ellipse, F_1 and F_2, are the foci of the ellipse (The lines D_1 and D_2 are the two directrixes of the ellipse, which we will discuss later).



Let P be any point on the ellipse and draw the line through P and the vertex of the cone. It hits the upper sphere at A and the lower sphere at B. Now PF_1 and PB are both tangent to the lower sphere, so $PF_1 = PB$. Likewise both PF_2 and PA are tangent to the upper sphere, so $PF_2 = PA$. Hence $PF_1 + PF_2 = PB + PA$. But PB+PA is the distance between two parallel circles so does not depend on the point P. Hence $PF_1 + PF_2$ is a constant.

A similar argument shows that a hyperbola has two foci, F_1 and F_2, and any point P on the hyperbola satisfies

$$PF_1 - PF_2 = \text{constant}$$



$$PF_1 = PA$$

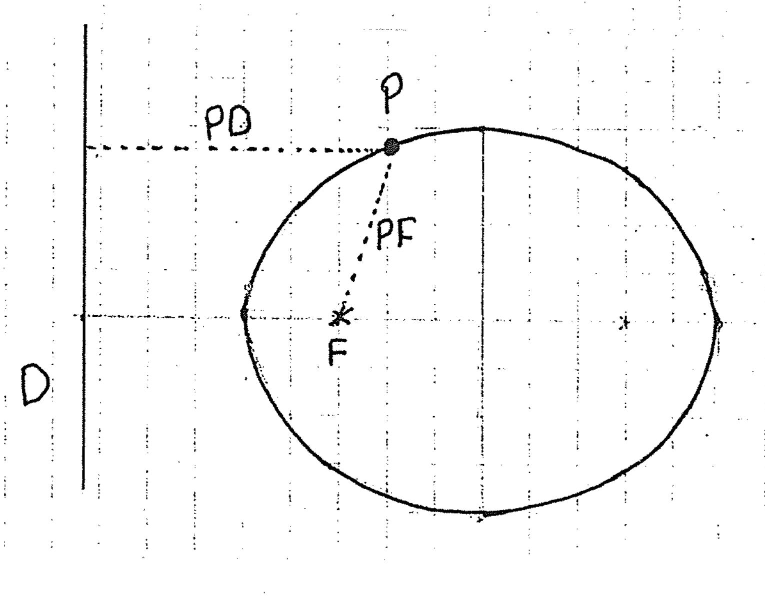
$$PF_2 = PB$$

$$PF_2 - PF_1 = BA = \text{constant}$$

Apollonius was not quite satisfied with the focal definition of conics because parabolas do not fit the pattern ; a parabola has only one focus. On the other hand Euclid had shown that a parabola can be described as the set of points equidistant from a given point and a given line. Pappus (around 350 AD) took the last step towards a unified definition of conics when he found ellipses and hyperbolas could also be described using a given point and a given line.

Going back to our ellipse, there is a special line outside the ellipse and perpendicular to the major axis, the "directrix" D of the ellipse. Pappus showed that the ratio $\frac{PF}{PD}$ does not depend on the point P on the ellipse, i.e. the ratio is a constant. The ratio $\frac{PF}{PD}$ is the "eccentricity" and is denoted ε.

* $$\boxed{\frac{PF}{PD} = \varepsilon}$$



Pappus showed that for any ellipse the eccentricity ε is between zero and one. The eccentricity measures how far the ellipse is from being a circle. If ε is close to zero the ellipse is almost circular, and for this reason we say a circle has eccentricity zero even though a circle has no directrix (sometimes we say the directrix of a circle is the line "at infinity", an idea that is made explicit in projective geometry).

If we use the same equation * and set $\varepsilon = 1$ we get a parabola instead of an ellipse, and if $\varepsilon > 1$ the equation defines a hyperbola. Hence Pappus' formula gives a unified description of all conic sections. As A.N. Whitehead wrote "generality is the soul of mathematics", and Pappus' exposition is a perfect example. When, a thousand years later, Kepler needed to understand conic sections to explicate his new discription of planetary motion, he found everything spelled out in the work of Apollonius and Pappus.