The square root of two is irrational

After proving their eponymous theorem, the Pythagoreans began an investigation of the relationship between number and length. If a square has a side whose length is 1, its diagonal has length $\sqrt{2}$ by the Pythagorean Theorem. But what number is $\sqrt{2}$? Before answering this question you must say what you mean by a number, and by "number" the Pythagoreans meant whole number or a quotient of whole numbers. They had already shown that musical harmonies are describable in terms of ratios of whole numbers, and a central tenet of their philosophy (or cosmology) was that all the secrets of the universe would be revealed through these ratios, so they set about trying to calculate $\sqrt{2}$ as a fraction, a ratio of whole numbers, hereafter called a "rational" number.

It was easy to see that $\frac{7}{5} < \sqrt{2} < \frac{3}{2}$ because $\frac{49}{25} < 2 < \frac{9}{4}$, but the Pythagoreans couldn't find $\sqrt{2}$ exactly.

In fact they found that it couldn't be done

$$\boxed{\text{The square root of two is not rational}}$$

Before giving the proof of the theorem above, let me reiterate its consequences for the theory of numbers. This says $\sqrt{2}$ is not a fraction of whole numbers. It is a new kind of number, a "non-rational" number, an "irrational" number. Are there other such irrational numbers? In what sense are these actual numbers? Are there numbers that cannot be written in terms of fractions and roots? Does every length represent a number. Can every number be written as a decimal? What do we really mean when we say "number"? These and a whole flood of similar questions were unleashed by the theorem above, questions that kept mathematicians busy for the next two thousand five hundred years.

Even so, the real importance of the theorem is in its method of proof. Think about what it says:

You cannot find a fraction $\frac{m}{n}$ of whole numbers so $\left(\frac{m}{n}\right)^2 = 2$

From an epistemological point of view this seems impossible to demonstrate. How can you prove it is impossible to find m and n? It is entirely reasonable to argue "maybe YOU can't find such a fraction, but perhaps in the future someone else will succeed. After all the modern world is full of things that were once thought impossible". This only illustrates the difference between mathematical truth and "truth" in the real world. The mathematical world is immutable and everlasting, while the real world is ephemeral. Zeus may decide to reduce the universe to a black hole and annul all the laws of physics, but He cannot find a fraction of whole numbers whose square is two.

As with all the great theorems of mathematics the key is the interplay amongst algebra, geometry, and logic. The theorem at its core is demonstrated by the following chain of reasoning:

Suppose Zeus (or anybody else) thought he had found a ratio of whole numbers whose square is two.
Then we can find a whole number between zero and 1 (which we prove using some algebra or geometry, or a mix of the two)
This is nonsense because there is no such number, so Zeus was mistaken.

This is the famous method of "reductio ad absurdum" which we all use in everyday life. "What you say can't be true because it would lead to something which is silly." You cannot find a fraction which equals $\sqrt{2}$ because that would lead to a positive whole number less than 1, which is silly.

$\sqrt{2}$ cannot be written as a ratio of whole numbers

proof #1 : Suppose $\sqrt{2} = \frac{a}{b}$. Then $b\sqrt{2}$ is natural, i.e. $b\sqrt{2}$ is a whole number ≥ 1. Now look at the powers of $\sqrt{2}$. Every even number is 2 times something, so may be written as $2n$, and every odd number is one more than an even number, so may be written as $2n+1$. Hence

$$b\sqrt{2}^{\text{even}} = b\sqrt{2}^{2n} = b2^n \text{ is natural}$$

$$b\sqrt{2}^{\text{odd}} = b\sqrt{2}^{2n+1} = b\sqrt{2}\,2^n \text{ is natural}$$

Hence b times any power of $\sqrt{2}$ is natural, so $b(\sqrt{2}-1)^k$ is natural for any power k because $(\sqrt{2}-1)^k$ is made up of powers of $\sqrt{2}$ and powers of 1. Now we already know that $\frac{1}{2} \geq \sqrt{2}-1$ so

$$\frac{b}{2^k} = b\left(\frac{1}{2}\right)^k > b\left(\sqrt{2}-1\right)^k \geq 1$$

This implies $b \geq 2^k$ for every power k, which is silly because there is no such number. Therefore $\sqrt{2} \neq \frac{a}{b}$.

proof #2. We start with an easy preliminary observation

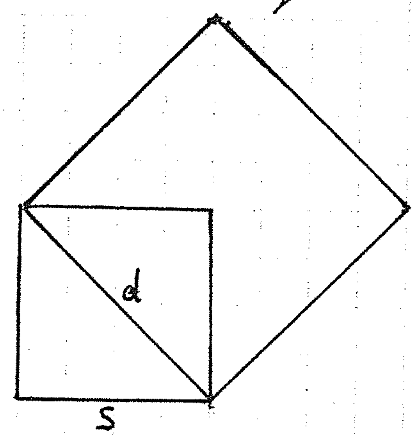
The square of an even number is even,
the square of an odd number is odd

Suppose x is even. Then x is two times some integer. Say $x = 2n$. Then $x^2 = 4n^2 = 2(2n^2)$, so x^2 is two times something and is even. Suppose x is odd. Then x is an even number plus one: $x = 2n+1$. So $x^2 = (2n+1)^2 = 4n^2+4n+1 = 2(2n^2+2n)+1$. Thus x^2 is an even number plus one, so is odd.

Now to the main. Suppose $\sqrt{2} = \frac{a}{b}$ and suppose we have reduced this ratio to lowest terms. Now $\sqrt{2}\,b = a$, so squaring both sides gives $2b^2 = a^2$. Thus a^2 is even (because it is two times something). By the preliminary result a must be even, say $a = 2n$. Now $2b^2 = a^2 = (2n)^2 = 4n^2$, so $b^2 = 2n^2$, so b^2 is even. By the preliminary result b must be even. Thus both a and b are even, but this is silly because we started out with $\frac{a}{b}$ reduced.
Thus $\sqrt{2} \neq \frac{a}{b}$.

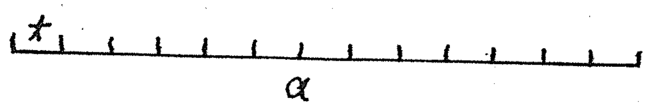
Neither of the proofs given above are true to the spirit of Greek mathematics, which was based on geometry rather than algebra. After all, if you cannot say what number is the square root of two, how can you know it even exists?

For the Greeks $\sqrt{2}$ was a proportion of lengths, not a number. If s is the side of a square and d is the diagonal of the square, the Pythagorean Theorem says the square on d is twice the square on s



In order to deal with this and other similar situations the Greeks developed a whole system of geometric algebra. "Adding" two lengths or areas has a clear meaning. "Multiplying" two lengths meant forming a rectangle having the two lengths as sides, but the meaning of the "division" of two lengths is a bit more problematic.

Definition: A length a is "measured" by a length t if a is a natural multiple of t, that is if there is a whole positive number m such that $a = mt$.



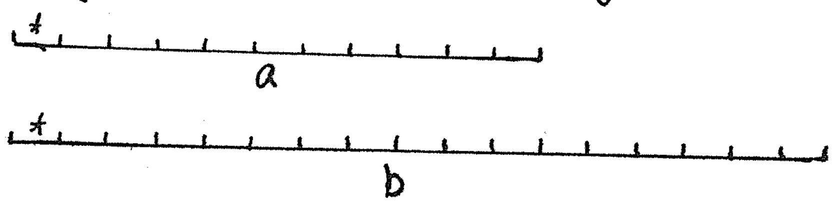
$a = 13t$

In plain English a is measured by t if a is built from a bunch of little ts (if these were numbers rather than lengths we would say t is a divisor of a). For example, you cannot measure a yard in centimeters because one yard is between 91 cm and 92 cm, but you can measure a yard using millimeters because 1 yd = 9144 mm. We say a millimeter measures a yard because you could use a ruler marked in millimeters to exactly measure one yard.

If two lengths, a and b, are both measured by t, we say t is a "common measure" of a and b. Finally

> Two lengths are "commensurable" if they have a common measure.

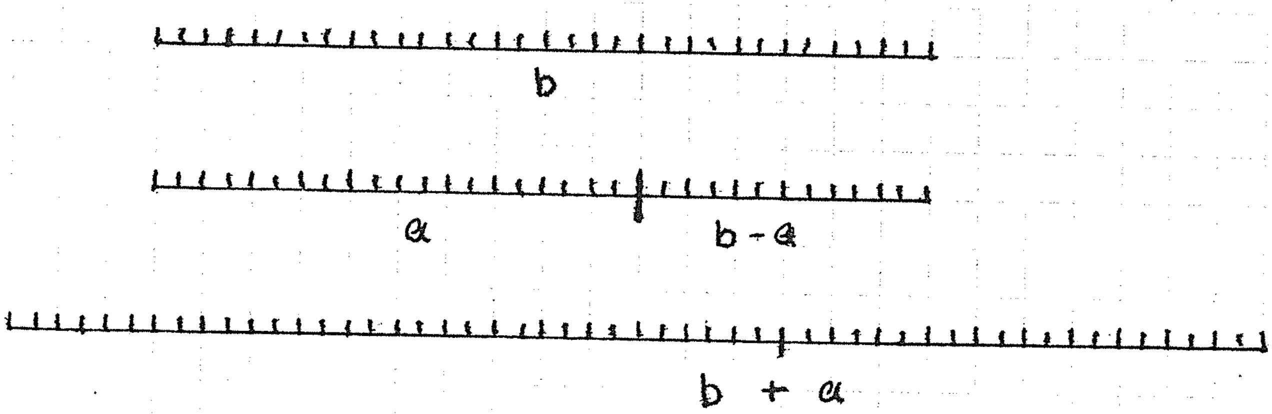
It is very important to keep the geometric view in mind. The lengths a and b are commensurable if they are both built from the same little lengths t



You might say that a and b are commensurable if they can be measured exactly with the same ruler. We must connect this notion of commensurability with the algebra of lengths.

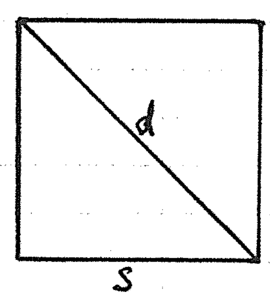
If a and b have common measure t, then t also measures $b-a$ and $b+a$

This is made obvious by the following pictures:



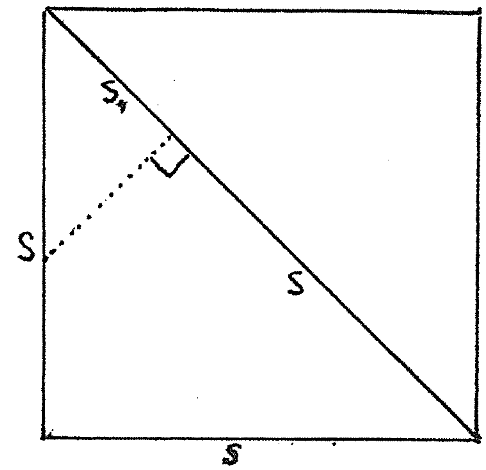
The big question now is whether or not any two lengths are commensurable. For example, a yard may be measured in inches, but a meter may not. However, both may be measured in millimeters. Looking at the picture of commensurable lengths, it appears as if you could always find a length t that measures a and b just by taking t small enough, but it is not true:

$$\boxed{\text{The side of a square and its diagonal are not commensurable}}$$

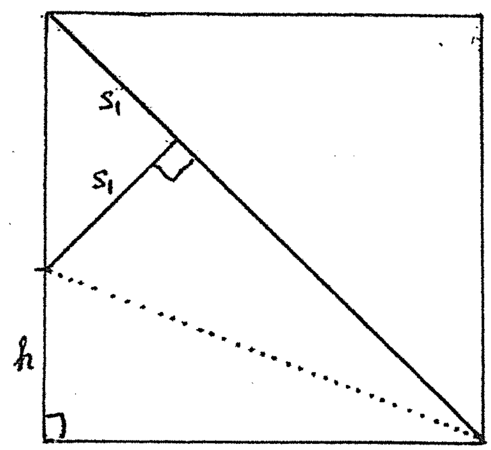


The theorem says you cannot find a ruler that will measure both lengths s and d above. Its proof is a sequence of pictures:

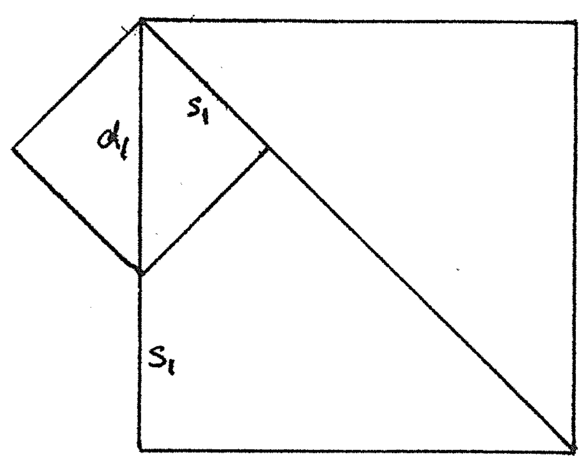
Mark a length s on the diagonal and draw the right angle as marked. The triangle so formed is isosceles, so the dotted line is s_1, which is $d-s$



Draw the dotted line as shown. The two triangles so formed are similar, so the length marked h is in fact s_1 again



We now finish drawing the square in the upper-left corner having sides s_1 and diagonal $d_1 = s - s_1$



We are now prepared to use reductio ad absurdum. Suppose we thought that the side s and diagonal d were commensurable, with common measure t. We have built above a new square with side s_1 and diagonal d_1. Because $s_1 = d - s$, s_1 is measured by t. Because $d_1 = s - s_1$, d_1 is measured by t. Hence all the lengths in the last picture are measured by t. If we repeat the construction using the smaller square we get an even smaller square whose side s_2 and diagonal d_2 are measured by t. We can repeat the construction again and again until the length of the side s_n is smaller than t. But this is silly because t must measure s_n and t cannot measure a length smaller than itself. We are done.

Now you might ask what this idea of commensurable lengths has to do with irrational numbers. As mentioned above, measurability is to length what divisibility is is to number. We can say a number x is measured by a number t if x is a natural multiple of t, which is the same as saying t divides x naturally.

For example, 39 is measured by 3 because $39 = 13 \cdot 3$, and $\sqrt{75}$ is measured by $\sqrt{3}$ because $\sqrt{75} = 5 \cdot \sqrt{3}$. Now $\sqrt{147}$ is also measured by $\sqrt{3}$ because $\sqrt{147} = 7 \cdot \sqrt{3}$, so we can say $\sqrt{75}$ and $\sqrt{147}$ are commensurable with common measure $\sqrt{3}$.

Two numbers, a and b, are commensurable if they are each a natural multiple of the same number. If t is the common measure, then $a = mt$ and $b = nt$ where m and n are whole numbers.

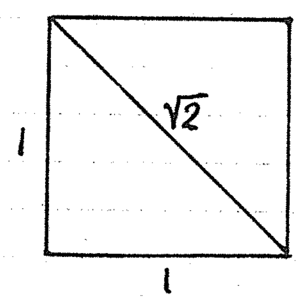
If a and b are commensurable, then $\frac{a}{b}$ can be written as a fraction of whole numbers because

$$\frac{a}{b} = \frac{mt}{nt} = \frac{m}{n}$$

Conversely, if $\frac{a}{b} = \frac{m}{n}$ then $\frac{a}{m} = \frac{b}{n}$ and that is the common measure. Hence we have just shown

> Two numbers, a and b, are commensurable if and only if $\frac{a}{b}$ is rational.

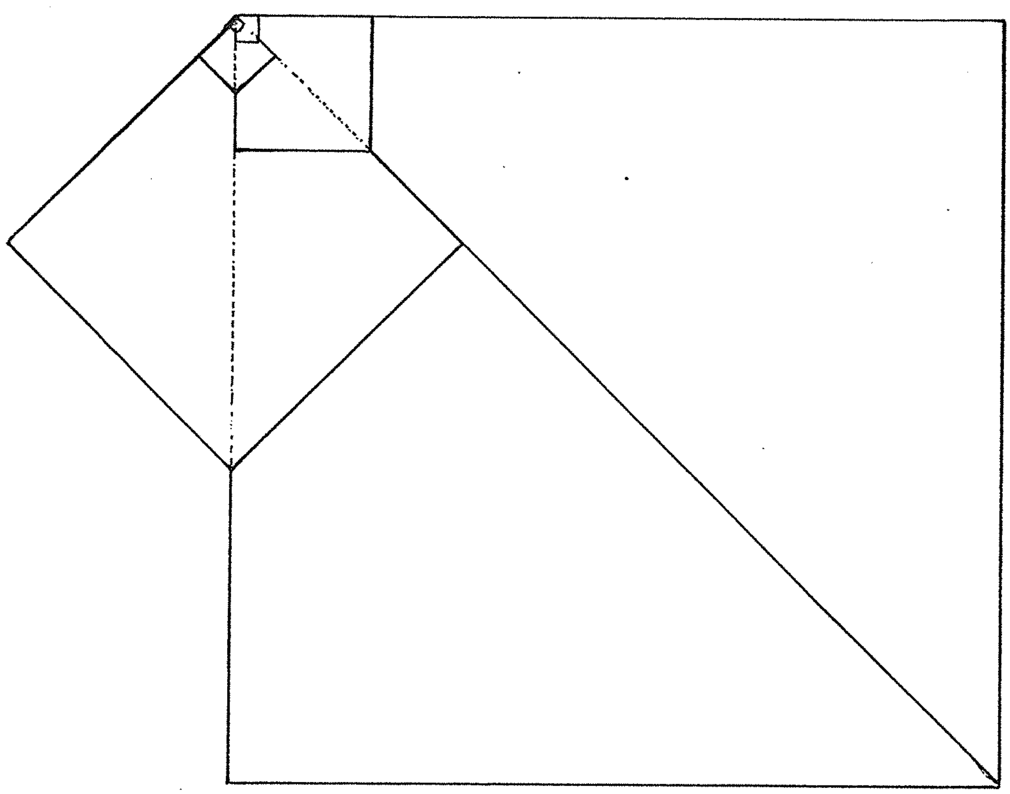
Now back to the square root of two. Look at the square whose side has length 1. By the Pythagorean Theorem its diagonal has length $\sqrt{2}$



But we proved geometrically that the side and diagonal are not commensurable, hence $\frac{\sqrt{2}}{1}$ is not rational, so $\sqrt{2}$ is not rational.

The attentive reader will notice that our geometric proof is, in fact, the "same" as the algebraic proof #1.

It does not really matter which proof we use because the fundamental idea is the same. There is an old saying that "all things are possible in Heaven and Earth", but it is not true in the world of mathematics. It is not possible to find a fraction whose square is two, and it is not possible to find a ruler that will measure both the side and the diagonal of a square.



Supplement

Proof #1 given above (and the geometric proof) may be shortened by assuming the fraction has been already reduced

Proof #1 bis: Suppose $\sqrt{2} = \frac{a}{b}$ and b is the smallest possible denominator. Now $\sqrt{2}\,b = a$, hence $2b = \sqrt{2}\,a$.

Then $\sqrt{2} = \dfrac{a}{b} = \dfrac{a(\sqrt{2}-1)}{b(\sqrt{2}-1)} = \dfrac{a\sqrt{2}-a}{b\sqrt{2}-b} = \dfrac{2b-a}{a-b}$

But this is a fraction of whole numbers whose denominator is less than b because $\sqrt{2}-1 < 1$

I prefer the original Proof #1 or the geometric version because the algebra used above was unknown to the Greeks, and the whole notion of a reduced fraction was not made clear and precise until Gauss proved unique factorization in the nineteenth century.