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Sumerian-Babylonian Mathematics

The Sumerians were a people of unknown linguistic affinity, who lived in the southern part of Mesopotamia (Iraq), and whose civilization was absorbed by the Semitic Babylonians around 2000 BC. Babylonian culture reached its peak in about 575 BC, under Nebuchadnezzar, but most of the mathematical achievements we shall discuss in this chapter and in Chapter 5 are much older, going back as far as 2000 BC — about the time when the biblical patriarch Abraham was said to have been born in the Sumerian city of Ur.

As we shall see, Mesopotamian mathematics is quite impressive. However, we should remember that, like the ancient Egyptians, the Mesopotamians never gave what we would call 'proofs' for their results; the first people to do so were the Greeks.

In representing numbers up to (and including) 59, the Sumerians and Babylonians used a decimal system. For example, they wrote 35 as follows, where we have approximated the original cuneiform figures by ours:

$$<<< \qquad \begin{array}{c} YYY \\ YY \end{array}$$

On the other hand, 60 is again denoted by Y, and so is 60^2, as well as 60^{-1}, 60^{-2}, etc. It is usually clear from the context which is meant. Here are some further examples:

$$<<< \quad = \quad 30, \text{ or } 30/60 = 1/2;$$

$$< YY \quad = \quad 12, \text{ or } 1/5;$$

$$Y << \begin{matrix} Y & Y & Y \\ & Y & \end{matrix} \quad = \quad 84, \text{ or } 7/5.$$

The Babylonian use of scale 60 was taken over into Greek astronomy around 150 BC by Hipparchus of Nicaea and it is still used today in measuring time and angles. To remove ambiguities in the above three examples, we would write

$$30^\circ \text{ or } 30',$$

$$12^\circ \text{ or } 12',$$

$$1^\circ 24' \text{ or } 1'24''.$$

The scale 60, or *sexagesimal system*, was also employed for weights of silver: 60 shekels = 1 mina; 60 minas = 1 talent. The prophet Ezekiel, living in Babylon, wrote in 573 BC:

> The Lord Yahweh says this: ... Twenty shekels, twenty-five shekels and fifteen shekels are to make one mina (*Ezekiel* 45:9–12).

The later Babylonians even introduced a symbol for zero:

$$Y \begin{matrix} < \\ < \end{matrix} \begin{matrix} Y & Y & Y \\ & Y & \end{matrix} = 60^2 + 4 = 3604.$$

Ptolemy (150 AD) replaced this symbol by a small circle, probably from the Greek word 'ouden', meaning 'nothing'.

In order to divide, the Babylonians made use of the fact that $a/b = a \cdot b^{-1}$. To this end, they constructed tables of inverses, like the one given in Table 4.1 (taken from Neugebauer [1969]). Note that the scribe did not list the inverses of any integers having a prime factor other than 2, 3 or 5. It seems he was afraid of repeating sexagesimals!

The Babylonians also had tables of squares, cubes, square roots, cube roots, and even roots of the equations

$$x^2(x \pm 1) = a.$$

Their method for extracting square roots is sometimes called *Heron's method* after Heron of Alexandria (60 AD), who included it in his *Metrica*. Let a_1 be a rational number between $\sqrt{a}$ and $\sqrt{a}+1$, where a is a positive non-square integer; let $a_{n+1} = (a_n + a/a_n)/2$; then $a_n \to \sqrt{a}$ as $n \to \infty$. Indeed, if $e = a_1 - \sqrt{a}$, we have $0 < e < 1$ and

$$0 < a_{n+1} - \sqrt{a} < 2\sqrt{a}(e/2\sqrt{a})^{2^n}$$

(see Exercise 4). As $n \to \infty$, this tends to 0.

b	1/b	b	1/b
2	30′	16	3′45″
3	20′	18	3′20″
4	15′	20	3′
5	12′	24	2′30″
6	10′	25	2′24″
8	7′30′	27	2′13″20″
9	6′40′	30	2′
10	6′	32	1′52″30″
12	5′	36	1′40″
15	4′	40	1′30″

TABLE 4.1. Mesopotamian table of inverses (scale 60)

For example, if $a = 2$, $a_1 = 3/2$, then $a_2 = 17/12$ and $a_3 = 577/408$. In sexagesimal notation, $577/408 = 1°24'51''10'''35'''' \ldots$. The fourth approximation $a_4 = 665857/470832$, which is $1°24'51''10'''7'''' \ldots$ in sexagesimal notation. The difference between a_4 and $\sqrt{2}$ is less than

$$2\sqrt{2}\left(\frac{3/2-\sqrt{2}}{2\sqrt{2}}\right)^{2^4} < 10^{-23}.$$

The Babylonian tablet YBC7289, dating from about 1600 BC, gives $\sqrt{2}$ as $1°24'51''10'''$.

Exercises

1. Write 5000 in the Babylonian manner. (You may use our degrees, minutes and seconds.)

2. Let a/b be a proper, reduced fraction (with a and b positive integers). Let $e_1 = 60a/b$ and $e_{n+1} = 60(e_n - [e_n])$ – where $[e_n]$ is the greatest integer less than or equal to e_n. Prove that the Babylonian sexagesimal expansion for a/b is

$$(.[e_1][e_2][e_3]\ldots)_{60}\,.$$

3. Express 1/7 as a repeating sexagesimal.

4. Prove by mathematical induction that

$$0 < a_{n+1} - \sqrt{a} < 2\sqrt{a}(e/2\sqrt{a})^{2^n}.$$

5. Use the Babylonian method to find $\sqrt{3}$ to within 60^{-10}.

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More about Mesopotamian Mathematics

In *Science Awakening I*, B. L. van der Waerden quotes the beginning of 'AO8862', a Babylonian clay tablet going back to about the same time as the Rhind Papyrus:

> Length, width. I have multiplied the length and the width, thus obtaining the area. Then I added to the area, the excess of the length over the width: 183 was the result. Moreover, I have added the length and the width: 27. Required length, width and area.
>
> 27 and 183, the sums; 15 the length; 180 the area; 12 the width;
>
> One follows this method:
>
> $27 + 183 = 210$, $2 + 27 = 29$.
>
> Take one half of 29 (this gives $14\frac{1}{2}$),
>
> $14\frac{1}{2} \times 14\frac{1}{2} = 210\frac{1}{4}$,
>
> $210\frac{1}{4} - 210 = \frac{1}{4}$.
>
> The square root of 1/4 is 1/2.
>
> $14\frac{1}{2} + \frac{1}{2} = 15$, the length;
>
> $14\frac{1}{2} - \frac{1}{2} = 14$, the width.
>
> Subtract 2, which has been added to 27, from 14, the width. 12 is the actual width. I have multiplied the length 15 by the width 12.
>
> $15 \times 12 = 180$, the area;
>
> $15 - 12 = 3$;
>
> $180 + 3 = 183$.

What is going on here? In modern notation, we would write x and y for length and width, respectively. The problem is to find a solution for the simultaneous equations

$$xy + (x - y) = 183 \quad \text{and} \quad x + y = 27.$$

The answer is given as $x = 15$ and $y = 12$. The scribe's method is this: consider

$$xy + x - y + x + y = x(y + 2) = 210.$$

Putting $y' = y + 2$, we have $xy' = 210$. On the other hand, adding the factors of 210, we get

$$x + y' = x + y + 2 = 29;$$

$$\text{hence } \tfrac{1}{2}(x + y') = \tfrac{1}{2}(29) = 14\tfrac{1}{2};$$

$$\text{hence } \frac{x^2 + 2xy' + y'^2}{4} = (14\tfrac{1}{2})^2 = 210\tfrac{1}{4};$$

$$\text{hence } \frac{x^2 - 2xy' + y'^2}{4} = 210\tfrac{1}{4} - 210 = \tfrac{1}{4} \text{ (the so-called discriminant);}$$

$$\text{hence } \frac{x - y'}{2} = \tfrac{1}{2}.$$

Adding and subtracting $\frac{1}{2}(x + y')$ and $\frac{1}{2}(x - y')$, we get $x = 14\frac{1}{2} + \frac{1}{2} = 15$ and $y' = 14\frac{1}{2} - \frac{1}{2} = 14$. Note that 14 is not really the width; but $y = y' - 2 = 14 - 2 = 12$ is. The scribe then computes the area and checks his work. The scribe did not consider the possibility $x = 14, y + 2 = 15$, which gives the second solution $x = 14$, $y = 13$. He did not know how to take the negative square root of $\frac{1}{4}$.

The Babylonians could solve many kinds of equations, including: $ax = b$, $x^2 \pm ax = b$, $x^3 = a$, $x^2(x + 1) = a$. They could also solve simultaneous equations having the following forms:

$$x \pm y = a, \quad xy = b;$$

$$x \pm y = a, \quad x^2 + y^2 = b.$$

They even managed to solve the following pair of equations:

$$x^3\sqrt{x^2 + y^2} = 3,200,000; \quad xy = 1200. \qquad (*)$$

As we saw just above, the Babylonians knew that

$$a^2 - b^2 = (a + b)(a - b).$$

They also knew that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Like the Egyptians, the Babylonians built pyramids, or *ziggurats*. If each story of a ziggurat consists of a square platform measuring $1 \times m \times m$, then the volume of a ziggurat with a base of length n is

$$(1 \times n \times n) + \cdots + (1 \times 2 \times 2) + (1 \times 1 \times 1) = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

The Babylonians knew that the formula for this sum is

$$n(n+1)(2n+1)/6\,,$$

a result also known to Pythagoras, but perhaps first proved by Archimedes.

According to the biblical story of the Tower of Babel, there was once an attempt to build a ziggurat 'with its top reaching heaven'. Perhaps the people behind this project thought that there was only a finite distance between heaven and earth, or perhaps they thought that they could calculate the sum of $1^2 + 2^2 + 3^2 + \cdots$, not realizing that the series diverges.

A remarkable fact about ancient Babylonian mathematics is that it included not just the so-called theorem of Pythagoras, but a theory of 'Pythagorean triangles'. (A *Pythagorean triangle* is a triple (x, y, z) of positive integers such that $x^2 + y^2 = z^2$, and thus x, y and z are sides of a right angled triangle.) From a clay tablet called 'Plimpton 322' (dating from 1900–1600 BC), we can deduce that the Babylonians used a result of which the following is a modern version:

> Suppose u and v are *relatively prime positive integers*, that is, integers whose greatest common divisor is 1. Assume that not both are odd and that $u > v$. Then, if $a = 2uv, b = u^2 - v^2$ and $c = u^2 + v^2$, we have $\gcd(a, b, c) = 1$ and $a^2 + b^2 = c^2$.

Included in Plimpton 322 is the triangle (13500, 12709, 18541), which is generated by taking $u = 125$ and $v = 54$.

The converse of the above theorem is also true. That is, if a, b and c are relatively prime positive integers, with a even, such that $a^2 + b^2 = c^2$, then there are relatively prime positive integers u and v, not both odd, such that $a = 2uv, b = u^2 - v^2$ and $c = u^2 + v^2$. It is not impossible that the Babylonians knew this, but the earliest record we have of this result is in the solutions of Problems 8 and 9 of Book II of the *Arithmetica* of Diophantus (250 AD). Indeed, since Diophantus explained his ideas in terms of special cases, it is correct to say that the first explicit, rigorous proof of the converse of the Babylonian theorem was given only in 1738, by C. A. Koerbero (Dickson [1971], Vol. II).

According to a tablet found in 1936 in Susa, an ancient city in what is now Iran, the Babylonians sometimes used the value $3\frac{1}{8}$ for π. At other times, they seem to have been satisfied with $\pi \approx 3$. It has been suggested that this Babylonian usage is behind 1 *Kings* 7:23–24:

> He [Hiram of Tyre] made the basin of cast metal, ten cubits from rim to rim, circular in shape and five cubits high; a cord

> thirty cubits long gave the measurement of its girth. Under its rim and completely encircling it were gourds; they went around the basin over a length of thirty cubits.

But perhaps the basin was hexagonal and not circular!

Exercises

1. Consider the simultaneous equations $xy + x - y = a$ and $x + y = b$, where a and b are given integers. What is a necessary and sufficient condition on a and b so that x and y will be integers?

2. Solve the simultaneous pair $(*)$ (from a Susa tablet).

3. Prove by mathematical induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6.$$

4. Rabbi Nehemiah (150 AD) was unhappy with the idea that the Bible had used a very inaccurate value for π and he suggested that 'the diameter of 10 cubits included the walls of the basin, while the circumference excluded them.' Assuming that he was right, and assuming that the Bible used a perfectly accurate value for π, how wide was the wall (or rim) of the basin?

5. Prove that if a triangle has sides of lengths a, b and c, and if $a^2 + b^2 = c^2$, then the triangle is right angled.

6. Prove the Babylonian theorem for Pythagorean triangles.

7. Prove the converse of the Babylonian theorem for Pythagorean triangles.

8. In 1901, L. Kronecker gave the first proof that all positive integer solutions of $a^2+b^2 = c^2$ are given without duplication by $a = 2uvk$, $b = (u^2 - v^2)k$, $c = (u^2+v^2)k$, where u, v and k are positive integers such that $u > v$, u and v are not both odd, and u and v are relatively prime. Prove Kronecker's theorem.

9. The 15 Pythagorean triangles in Plimpton 322 have angles which approximate the 15 whole number angles from 44° to 58° inclusive. Find a Pythagorean triangle, with relatively prime sides, one of whose angles is within 2/5° of 47°. (Hint: see Anglin [1988].)