MATH323 Review Session (April 12th /2017)

These notes are meant to be used as a study guide through the course material of MATH323. Note that these notes <u>do not</u> contain <u>all</u> concepts from the course but rather highlight some of the main points that have covered either in the textbook / lectures.

In studying for the exam, it is important to compartmentalize the course material into the following modules:

Module ① : Fundamental Rules of Probability & Counting
Module ② : A single random variable (Discrete & Continuous)
Module ③ : Two random variables (Discrete & Continuous)
Module ④ : Functions of random variables and the CLT

These notes will address each module separately and will make connections between modules where appropriate.

Module ① : Fundamental Rules of Probability & Counting

- S : sample space, $E \subseteq S$: event
- Distributive Properties of Sets : $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- Axioms of Probability, $A \subseteq S$, 1) $\mathbb{P}(A) \geq 0$
  2) $\mathbb{P}(S) = 1$
  3) $A_1, A_2, \ldots$ with $A_i \cap A_j = \emptyset$, $i \neq j$, $\mathbb{P}(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$
- Rules of Probability :
  (1) $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$, (2) $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$
  (3) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, (4) $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$
  (5) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B})$, (6) $\mathbb{P}(A \triangle B) = \mathbb{P}(A \cap \bar{B}) + \mathbb{P}(B \cap \bar{A})$
  (7) $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B) / \mathbb{P}(B)$, if $\mathbb{P}(B) > 0$, (8) $\mathbb{P}(\bar{A} \mid B) = 1 - \mathbb{P}(A \mid B)$
  (9) If A is independent of B then $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ or $\mathbb{P}(B \mid A) = \mathbb{P}(B)$
  or $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

- Counting: Assume $S = \{\omega_1, \omega_2, \dots, \omega_n\}$ (i.e S is finite) where $\mathbb{P}(\{\omega_i\}) = \frac{1}{n}$, $i = 1, 2, \dots, n$ (i.e. each sample point ω_i is equally likely).

If $A \subseteq S$ (A is an event) then since each ω_i is equally likely, we just need to count how many distinct ω_i are in A and then divide by the size of S, i.e. $\mathbb{P}(A) = |A| / |S|$

4 Rules: 1) Multiplication Rule

→ with m elements and n elements from two groups, there are a possible $m \times n$ pairs

2) Permutation Rule

→ # of ordered sets of size r chosen from n distinct objects

$$P_r^n = \frac{n!}{(n-r)!}$$

3) Combination Rule

→ # of unordered sets of size r chosen from n distinct objects

$$C_r^n = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

4) Partition Rule

→ # of ways to divide n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$, respectively, where each object appears exactly in one group with $n_1 + n_2 + \dots + n_k = n$, is given by

$$\binom{n}{n_1\ n_2\ \dots\ n_k} = \frac{n!}{n_1!\,n_2! \cdots n_k!} = \binom{n}{n_1}\binom{n-n_1}{n_2} \cdot \ldots \cdot \binom{n-n_1-n_2-\dots-n_{k-1}}{n_k}$$

<u>Example</u>: Given a fleet of taxis numbered 1 through 9 and 3 airports A, B, C where 4 taxis must go airport A, 2 to airport B and 3 to airport C, find the probability that taxis 4 and 6 go to airport A and taxi 1 goes to airport B.

Solution:

With any counting problem, we approach the problem in steps:

① Find $|S|$. If 4 taxis go to airport A, 2 to B and 3 to C, then

$$|S| = \underset{\substack{\uparrow \\ 4 \text{ to } A}}{\binom{9}{4}} \cdot \underset{\substack{\uparrow \\ 2 \text{ to } B}}{\binom{5}{2}} \cdot \underset{\substack{\uparrow \\ 3 \text{ to } C}}{\binom{3}{3}}$$

② "Put yourself in the situation of the problem". What this statement means is to consider the event of interest as already happened and count the ways everything else can be arranged.

Taxis 4 and 6 $\longrightarrow$ A
Taxi 1 $\longrightarrow$ B

Total Number of Taxis remaining: 6
to A = 2, # to B = 1, # to C = 3

Thus, let E be the event of interest:

$$|E| = \binom{6}{2} \cdot \binom{4}{1} \cdot \binom{3}{3}$$

③ Answer the question,

$$\boxed{\mathbb{P}(E) = \frac{\binom{6}{2}\binom{4}{1}\binom{3}{3}}{\binom{9}{4}\binom{5}{2}\binom{3}{3}}}$$

Module ②: A single random variable (Discrete and Continuous)

- A discrete random variable X either takes values $\{x_1, x_2, \ldots, x_n\}$ or $\{x_1, x_2, \ldots\}$ with an associated probability mass function (pmf) given by $\mathbb{P}(X = x) = p(x)$ which satisfies
  (1) $0 \le p(x) \le 1$ for all x, (2) $\sum_x p(x) = 1$

The cumulative distribution function (cdf) is given by

$$\mathbb{P}(X \le y) = \sum_{x : x \le y} p(x)$$

- A continuous random variable X is defined on an interval I and can take uncountably infinite many values. If the cdf, F, of X is continuous, then X is a continuous random variable with probability density function, $f(y) = \frac{d}{dy} F(y)$.

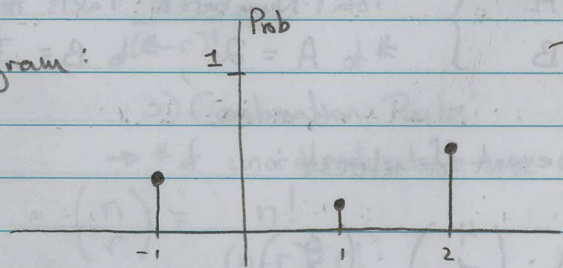
Note: $f(y)$ does **NOT** represent a probability by itself.

Example: Let $X \in \{-1, 1, 2\}$ with respective probabilities $1/3$, $1/6$, $1/2$. Define the pmf and cdf of X.

Solution: $p(-1) = 1/3$, $p(1) = 1/6$, $p(2) = 1/2$

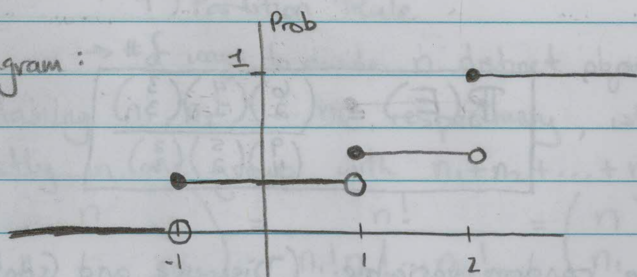
We have thus defined the pmf.

Diagram:



This is a visual representation of the pmf.

Diagram:



This is a visual representation of the cdf.

$$F(y) = \begin{cases} 0 & , \ y < -1 \\ 1/3 & , \ -1 \le y < 1 \\ 1/2 & , \ 1 \le y < 2 \\ 1 & , \ y \ge 2 \end{cases}$$

Object	Discrete	Continuous
Probability	$\mathbb{P}(a \le X \le b) = \sum'_{x: a \le x \le b} p(x)$	$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx$
Expectation	$\mathbb{E}(X) = \sum'_x x p(x)$	$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$
Expectation of Function	$\mathbb{E}(g(X)) = \sum'_x g(x)p(x)$	$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$
Variance	$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum'_x x^2 p(x) - \left(\sum'_x x p(x)\right)^2$	$V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2$
Moment Generating Function	$m(t) = \mathbb{E}(e^{tX}) = \sum'_x e^{tx} p(x)$	$m(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

Property: $m'(0) = \mathbb{E}(X)$, $m''(0) = \mathbb{E}(X^2)$, ..., $m^{(k)}(0) = \mathbb{E}(X^k)$

CDF	$F(y) = \mathbb{P}(X \le y) = \sum'_{x: x \le y} p(x)$	$F(y) = \int_{-\infty}^{y} f(x)\,dx$
Total Probability	$\sum'_x p(x) = 1$	$1 = \int_{-\infty}^{\infty} f(x)\,dx$

Markov Inequality $\quad \mathbb{P}(|X| \ge a) \le \dfrac{\mathbb{E}(|X|^\alpha)}{a^\alpha}, \ \forall a > 0, \ \alpha > 0$

Chebyshev's Inequality $\quad \mathbb{P}(|X - \mathbb{E}(X)| \ge a) \le \dfrac{V(X)}{a^2}$, if $\mathbb{E}(X^2) < \infty$

Chernoff Bound $\quad \mathbb{P}(X \ge a) \le e^{-ta} m_X(t), \ \forall t \ge 0.$

Useful Results

- Gamma Function : $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx$
  - (i) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \ \forall \alpha > 0$ (ii) $\Gamma(n+1) = n! \ \forall n \in \mathbb{N}$
  - (iii) $\Gamma(1/2) = \sqrt{\pi}$

- Expectation/Variance : if X is an r.v. and α is a constant then
$$\mathbb{E}(\alpha X) = \alpha \mathbb{E}(X) \quad \text{and} \quad V(\alpha X) = \alpha^2 V(X)$$

- Quantile : if $0 < p < 1$, the p^{th} quantile of the r.v. X is defined as $Q_p := \min\{y : F(y) \geq p\}$ (i.e. it is the smallest number y for which $\mathbb{P}(X \leq y)$ is at least p)

- Standard Normal and Sums of Normals : if $Z \sim N(0,1)$, then it has pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$ and we denote its CDF by
$$\Phi(y) = \int_{-\infty}^{y} f(x)\,dx.$$

If $Z_1 \sim N(0,1)$ and $a_1, b_1 \in \mathbb{R}$ then $a_1 Z_1 + b_1 \sim N(b_1, a_1^2)$
$Z_2 \sim N(0,1)$ and $a_2, b_2 \in \mathbb{R}$ then $a_2 Z_2 + b_2 \sim N(b_2, a_2^2)$
and if Z_1 and Z_2 are independent then
$$(a_1 Z_1 + b_1) + (a_2 Z_2 + b_2) \sim N(b_1 + b_2,\ a_1^2 + a_2^2)$$

Example : Let $X_1 \sim N(5,1)$ and $X_2 \sim N(4,2)$. Define the independent events $A = \{X_1 > 2\}$ and $B = \{X_2 < -1\}$. Find the probability that both A and B occur by writing your final answer in terms of Φ.

Solution :
By the rules of probability,
$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A \cap B)$$
which by independence
$$= \mathbb{P}(A)\mathbb{P}(B)$$
$$= \mathbb{P}(X_1 > 2)\,\mathbb{P}(X_2 < -1)$$
We will standardize each random variable to use the symbol Φ.
$$= \mathbb{P}\Bigg(\underbrace{\frac{X_1 - 5}{\sqrt{1}}}_{\sim N(0,1)} > \frac{2-5}{\sqrt{1}}\Bigg)\,\mathbb{P}\Bigg(\underbrace{\frac{X_2 - 4}{\sqrt{2}}}_{\sim N(0,1)} < \frac{-1-4}{\sqrt{2}}\Bigg)$$
$$= \mathbb{P}\left(\frac{X_1 - 5}{\sqrt{1}} > -3\right) = \mathbb{P}\left(\frac{X_2 - 4}{\sqrt{2}} < \frac{-5}{\sqrt{2}}\right)$$

$$= \left(1 - \mathbb{P}\left(\frac{X_1 - 5}{\sqrt{1}} \le -3\right)\right) \mathbb{P}\left(\frac{X_2 - 4}{\sqrt{2}} \le \frac{-5}{\sqrt{2}}\right)$$

$$= \left(1 - \Phi(-3)\right) \Phi\left(-\frac{5}{\sqrt{2}}\right)$$

- Binomial / Normal Approximation: if $X \sim \text{Bin}(n, p)$, then we can always approximate X by $Y \sim \text{Normal}(np, np(1-p))$

Module ③ Two random variables (Discrete & Continuous)

Object	Discrete	Continuous
Joint pmf	$p_{X_1,X_2}(x_1,x_2)$	—
Joint pdf	—	$f_{X_1,X_2}(x_1,x_2)$
Marginal pmf/pdf	$p_2(x_2) = \sum_{x_1} p_{X_1,X_2}(x_1,x_2)$	$f_2(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1$
Conditional pmf/pdf	$p_{X_1 \mid X_2 = x_2}(x_1) = \frac{p_{X_1,X_2}(x_1,x_2)}{p_2(x_2)}$	$f_{X_1 \mid X_2 = x_2}(x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_2(x_2)}$
Joint cdf	$F(x_1,x_2) = \sum_{x: x \le x_1} \sum_{y: y \le x_2} p_{X_1,X_2}(x_1,x_2)$ $(= \mathbb{P}(X_1 \le x_1, X_2 \le x_2))$	$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(u,v)\,dv\,du$
Expectation	$\mathbb{E}(g(X,Y)) = \sum_x \sum_y g(x,y) p_{X_1,X_2}(x,y)$	$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X_1,X_2}(x,y)\,dx\,dy$
Independence	$p_{X_1,X_2}(x,y) = p_1(x) p_2(y)$	$f_{X_1,X_2}(x,y) = f_1(x) f_2(y)$
Conditional Expectation	$E(g(X,Y) \mid Y = y) = \sum_x g(x,y) p_{X \mid Y=y}(x)$	$= \int_{-\infty}^{\infty} g(x,y) f_{X \mid Y=y}(x)\,dx$

- Conditional Rules: $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y))$, $V(X) = \mathbb{E}(V(X \mid Y)) + V(\mathbb{E}(X \mid Y))$
- Covariance: $\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$
- Correlation Coefficient: $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$

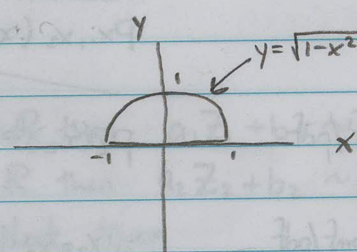
- Multinomial Distribution
  - → n independent trials with k classes
  - → probability to fall into class i is p_i with $p_1+p_2+\dots+p_k=1$
  - → random variables are $Y_1, Y_2, \dots, Y_k$ where Y_i are the number of trials for which the outcome falls into cell i, $Y_1+Y_2+\dots+Y_k=n$

$p(y_1,y_2,\dots,y_k) = \frac{n!}{y_1!y_2!\cdots y_k!} p_1^{y_1} p_2^{y_2}\cdots p_k^{y_k}$ where $y_i = 0,1,\dots,n$ and $\sum_{i=1}^{k} y_i = n$

Example: Let (X,Y) be uniformly distributed over the region $A = \{(x,y): x^2+y^2 \leq 1,\ y \geq 0\}$. Find $\mathbb{P}(X^2+Y^2 \leq (0.5)^2)$ and $\mathbb{E}(X)$.

Solution:

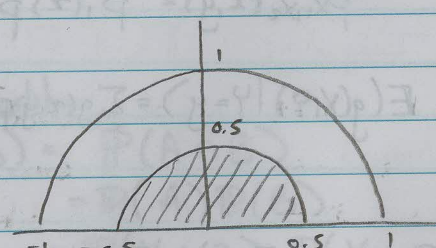
① Draw a picture



② Define $f(x,y)$. Since it is uniform over A, then we take $\frac{1}{\text{Area of } A}$ as the joint pdf. Area of $A = \frac{\pi r^2}{2} = \frac{\pi}{2}$

$\Rightarrow f(x,y) = \frac{2}{\pi}$

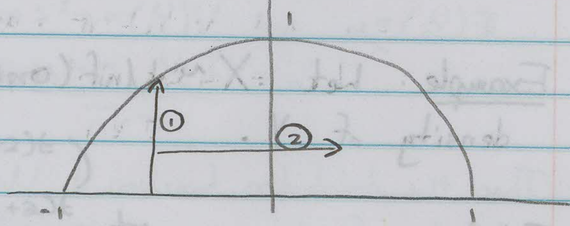
Solving for $\mathbb{P}(X^2+Y^2 \leq (0.5)^2)$, we draw another picture:



Since it is uniform, we can take the area of the shaded region and multiply by the height to get the volume/probability.

$$\mathbb{P}(X^2+Y^2 \leq (0.5)^2) = \left(\frac{\pi(0.5)^2}{2}\right)\left(\frac{2}{\pi}\right) = (0.5)^2 = 0.25$$

In solving for the expectation, we'll draw another picture to get the correct bounds for the integration process



Line ① goes from $y=0$ to $y=\sqrt{1-x^2}$

Line ② goes from $x=-1$ to $x=1$

$$\mathbb{E}(X) = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x\frac{2}{\pi}\,dy\,dx = \int_{-1}^{1}\frac{2x}{\pi}\left(\int_{0}^{\sqrt{1-x^2}} dy\right)dx$$

$$= \int_{-1}^{1}\frac{2x}{\pi}\sqrt{1-x^2}\,dx$$

$$= \frac{-\sqrt{1-x^2}}{\pi}\Bigg|_{-1}^{1}$$

$$= 0$$

! Make Sure you study a joint pmf example!

Module ④: Functions of random variables and the CLT

This module is concerned with finding the density function of a transformed random variable. Let X be a continuous r.v. with density function $f(x)$ and let $Y=g(X)$ for some function g. There are 3 methods

1) Find CDF of Y then differentiate to obtain density of Y
2) Find mgf of Y, then match to another mgf of a distribution you know of another random variable W. Then Y and W have the same density
3) If $g(x)$ is increasing or decreasing for all x then
$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dg^{-1}}{dx}\right|$$

where $\frac{dg^{-1}}{dx} = \frac{d}{dx}\left(g^{-1}(x)\right)$

We will work through an example that presents all 3 methods...

<u>Example</u>: Let $X \sim Unif(0,2)$ and let $Y = 3X+2$. Find the density of Y.

<u>Solution</u>:

Before even using any of the methods, always state the support of the new transformed variable. In our case,

$$f_X(x) = \begin{cases} \frac{1}{2} & , x \in (0,2) \\ 0 & , \text{elsewhere} \end{cases}$$

Since X lives in $(0,2)$ then $3X+2$ lives in $(2,8)$ and so our density will have the form

$$f_{X^2}(y) = \begin{cases} \text{something} & , y \in (2,8) \\ 0 & , \text{elsewhere} \end{cases}$$

Let's now find "something" using our 3 methods:

① Cumulative Distribution function:

$$\mathbb{P}(3X+2 \le y) \quad , \quad y \in (2,8)$$

$$= \mathbb{P}\left(X \le \frac{y-2}{3}\right) \quad , \quad \frac{y-2}{3} \in (0,2)$$

$$= \int_0^{\frac{y-2}{3}} \frac{1}{2}\, dx$$

$$\frac{1}{2}\left(\frac{y-2}{3}\right) = \frac{y-2}{6} = F_{3X+2}(y) \quad , \quad y \in (2,8)$$

$$\Rightarrow f_{3X+2}(y) = \frac{d}{dy} F_{3X+2} = \frac{1}{6}$$

Therefore, $f_{3X+2}(y) = \begin{cases} 1/6, & y \in (2,8) \\ 0, & \text{elsewhere} \end{cases}$

② Moment Generating function:

$$m_{3X+2}(t) = \mathbb{E}_X\left(e^{(3X+2)t}\right)$$
$$= \int_0^2 \frac{1}{2} e^{(3x+2)t}\, dx$$
$$= \frac{1}{2}\left(\frac{1}{3t} e^{3xt+2t}\right)\Big|_0^2$$
$$= \frac{1}{6t}\left(e^{6t+2t} - e^{2t}\right) = \frac{1}{t(8-2)}\left(e^{8t} - e^{2t}\right)$$

Observe the mgf of an arbitrary Unif(a,b) for $t \neq 0$ is $\frac{1}{t(b-a)}(e^{tb} - e^{ta})$ and so we can conclude that $3X+2$ is uniform over $(2,8)$. Thus yielding,

$$f_{3X+2}(y) = \begin{cases} 1/6, & y \in (2,8) \\ 0, & \text{elsewhere} \end{cases}$$

③ Direct calculation:

$$g(x) = 3x+2 \implies \frac{g(x)-2}{3} = x$$
$$\implies \frac{x-2}{3} = g^{-1}(x)$$

Is $g(x)$ strictly increasing or decreasing for $x \in (0,2)$. Yes!

$$\frac{d}{dx} g^{-1}(x) = \frac{d}{dx}\left(\frac{x-2}{3}\right) = \frac{1}{3}$$

$$\implies f_{3X+2}(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dx} g^{-1}(y)\right| \\ 0 \end{cases} = \begin{cases} \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}, & y \in (2,8) \\ 0, & \text{elsewhere} \end{cases}$$

· Central Limit Theorem (CLT): Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables with $\mathbb{E}(Y_i) = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \quad \text{where } \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n\to\infty} \mathbb{P}(U_n \le u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \quad \text{for all } u.$$