HOMEWORK 1

Due Thursday, September 21

- Please, submit your answers in *myCourses* before the beginning of the lecture.
- The assignment will *not* be graded. You receive your portion of the Homework grade by just handing it it.
- All assignments together are worth 5% of your final grade for this course.
- Working through the assignment on your own will help you to learn the material and identify those areas which you need to study more.
- It will also help you considerably for the Quizzes, Midterm, and Final exams!
- If you have questions, make sure to clear them up during *office hours* or by asking on the *myCourses discussion board*.

Readings:

- Read the course *syllabus* (*myCourses*).
- Have a look at the course *schedule* (on course webpage).
- Read *GEB* up to p. 41 (Introduction and Chapter 1).

Problems:

- Please read the questions carefully before you answer them!
- You should *not* need to consult other sources than the textbook for answering these questions. If you do so nevertheless, intellectual honesty requires you to state the sources!
- If you have discussed the assignment with others, please state their full names on the assignment.

1. *A simple mathematical proof.*

   The set of natural numbers $\mathbb{N}$ consists of the numbers $0, 1, 2, 3, \ldots$

   Definition 1: A natural number a is *even*, if and only if there is a natural number b, such that $a = 2b$.

   Definition 2: A natural number a is *odd*, if and only if there is a natural number b, such that $a = 2b + 1$.

   Using these two definitions together with the fact that a natural number is either even or odd (i. e., a number that is not even is odd, and vice versa) prove the following theorem:

   Theorem: For any natural number a, a^2 is even if and only if a is even.

This was used in class as a Lemma in the proof that $\sqrt{2}$ is irrational.

Recall that to prove a *bi-conditional* claim of the form 'A if and only if B', you need to show *two* conditionals: (1) Assuming A you need to prove B, and (2) assuming B you need to prove A.

This proof requires only very simple facts about algebra.

2. *Recursive definitions.* Give recursive definitions of three (infinite) domains of your choice. Try to come up with interesting examples!

3. *Mathematical Induction.* Prove by induction that

$$\sum_{i=1}^{n} 2i \;=\; n(n+1).$$

Recall that $\sum_{i=1}^{n} 2i$ is an abbreviation for the sum of the terms $2i$, starting with $i = 1$ and ending with $i = n$: $2 + 4 + 6 + \ldots + 2(n-1) + 2n$.

4. *Mathematical induction.* Consider the system of well-formed geq-strings determined by the following recursive definition:

Base clause: The string '-geq-' is a well-formed geq-string.

Inductive clause 1: If 'x geq x' is a well-formed geq-string, so is 'x -geq x -', where x is composed of hyphens only.
(Example: If x is the string '--', then x- is the string '---'.)

Inductive clause 2: If 'x geq y' is a well-formed geq-string, so is 'x -geq y', where x and y are composed of hyphens only.

Final clause: No strings other than those obtained from the base clause and inductive clauses are well-formed geq-strings.

(a) State three strings that are well-formed geq-strings and three that are not.

(b) Prove by induction the following claim: If 'x geq y' is a well-formed geq-string, then the number of hyphens in x is greater than or equal to the number of hyphens in y.

(c) If for a natural number a, x is a string of a-many hyphens, is x geq x a well-formed geq-string? Justify your answer.

(d) If for two natural numbers a and b, $a > b$ and if x is a string of a-many hyphens, y a string of b-many hyphens, is x geq y a well-formed geq-string? Justify your answer.

5. *Functions.* Consider the sets $A = \{b, c, d, e\}$ and $B = \{a, b, d, f\}$. Which of the following relations are total functions from A to B?
   And which total functions are injective, surjective, bijective?

   (a) $F_1 = \{\langle b, d\rangle, \langle c, f\rangle, \langle d, b\rangle, \langle e, a\rangle\}$.

   (b) $F_2 = \{\langle b, b\rangle, \langle c, b\rangle, \langle d, b\rangle, \langle e, b\rangle\}$.

   (c) $F_3 = \{\langle a, a\rangle, \langle b, b\rangle, \langle d, d\rangle, \langle e, f\rangle\}$.

   (d) $F_4 = \{\langle b, f\rangle, \langle c, a\rangle, \langle b, d\rangle, \langle e, b\rangle\}$.

   (e) $F_5 = \{\langle b, f\rangle, \langle c, d\rangle, \langle d, b\rangle\}$.

6. *Functions.*

   (a) Prove that on the natural numbers the function $f(x) = x + 7$ is injective.

   (b) Explain why f is not surjective (onto).

7. *Diagonalization.* Prove that the set of subsets of the natural numbers is not denumerable. (See also *GEB*, p. 421–422.)

8. *Cardinal numbers* Make sure to understand *all* the notation. Let the set A be countably infinite, i. e., have cardinality $\aleph_0$: $|A| = \aleph_0$.

   (a) What is $|A \cup \{a\}|$, where $a \notin A$ (i. e., what is the cardinality of the set containing all elements of A together with a new element a)?

   (b) What is the cardinality of $A \cup B$, where $|B| = \aleph_0$ and $A \cap B = \emptyset$ (i. e., A and B have no element in common)?
   Justify your answers with a short argument.

9. *Gödel, Escher, Bach*

   (a) What is *meta-mathematics*?

   (b) Explain the difference between *object language* and *meta-language*.
   Give an example in which there is an object language and a meta-language.

   (c) After reading the Introduction to *GEB*, try to state in your own words what Gödel's Theorem is about and what the Gödel sentence G says.
   (You definitely should *not* consult other sources of information for this question than what's written in the Introduction to our textbook!)

   (d) What is *Principia Mathematica*? Explain in a sentence or two.

   (e) State three English adjectives that are *autological* and three that are *heterological* (see *GEB*, p. 21; do not use the ones mentioned in the book; be creative!).