Homework #2 : Solutions

(1) Euclid used all five axioms in his proof, though not directly. The circle axiom is used to construct right angles, hence the squares

(2) Suppose $\frac{3}{1+\sqrt{2}}$ were a fraction of whole numbers, say $\frac{3}{1+\sqrt{2}} = \frac{a}{b}$. Then $3b = a(1+\sqrt{2})$

$3b = a + a\sqrt{2}$ $\quad$ $3b - a = a\sqrt{2}$ $\quad$ $\frac{3b-a}{a} = \sqrt{2}$

This is impossible because we know $\sqrt{2}$ cannot be a fraction of whole numbers

(3) Suppose $\sqrt{7} = \frac{a}{b}$, where the fraction is reduced
Then $\sqrt{7}b = a$ and $7b = \sqrt{7}a$.
Note that we know $2 < \sqrt{7} < 3$, so $\sqrt{7} - 2 < 1$

Now $\sqrt{7} = \frac{a}{b} = \frac{(\sqrt{7}-2)}{(\sqrt{7}-2)}\frac{a}{b} = \frac{\sqrt{7}a - 2a}{\sqrt{7}b - 2b}$

$= \frac{7b - 2a}{a - 2b}$. This is impossible because it is

a fraction whose numerator is less than a (because $\sqrt{7}-2 < 1$)

④ A number is not prime if it has non-trivial factors

For example 35 is not prime because $35 = 5 \cdot 7$

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 + 91 = 7\left(2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 + 13\right)$$

$= 7 \cdot 13 \cdot \left(2 \cdot 3 \cdot 5 \cdot 11 \cdot 17 + 1\right)$ so its not prime

⑤ a) just multiply it out. Note that reading the formula backwards it says

$$X^8 - 1 = (X-1)(X^7 + X^6 + X^5 + \ldots + 1)$$, so $X^8 - 1$ factors

This is just a special case of the general result that

$$X^n - 1 = (X-1)(X^{n-1} + X^{n-2} + \ldots + 1)$$ always factors

b) $2^{40} - 1 = (2^5)^8 - 1 = (2^5 - 1)\left((2^5)^7 + (2^5)^6 + \ldots + 1\right)$

so $2^{40} - 1$ factors, so $2^{40} - 1$ is not prime.

You might ask why we didnt just factor

$$2^{40} - 1 = (2-1)(2^{39} + 2^{38} + 2^{37} + \ldots + 1)$$. In this case

the first factor is $2-1$, which is just 1

which is a trivial factorization.

If you read carefully the answer above you'll see we proved two important results

a) If n is not prime, then $2^n - 1$ is not prime

b) If x is a whole number not equal to 2 then $x^n - 1$ is never prime

For example we see that $936^{147} - 1$ is not prime without having to work out what this number is.