A short introduction to propositional logic

Dirk Schlimm

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1 Propositional (or sentential) logic as a language

1.1 The alphabet and well-formed formulas

You can think of propositional logic as a formal language to express *logical* relations between *propositions* or statements (not: names, exclamations, questions, etc.) in a concise way. Propositions can be true or false; e.g., 'Grass is green' and '$3+6=10$'.

The *alphabet* of propositional logic consists of

- *Propositional variables:* P_0, P_1, $\mathsf{P}_2, \dots$. [GEB *has* $\mathsf{P}, \mathsf{Q}, \mathsf{R}$ *and primes, like* P', P''.]
- *Logical connectives:* $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\sim$ (negation),[1]
- *Parentheses:* (,). [GEB *uses* $<$ *and* $>$.]

Since we do not want to consider all possible combinations of these symbols, e.g., '$\supset\supset \mathsf{P}_0$', we restrict ourselves to the *well-formed formulas (wff)* that are defined by the following recursive definition: [*See* GEB, *p. 182*.]

1. *Base clause:* P_i is a *wff*, for $i \in \mathbb{N}$. These *wff*s are also called *atomic wff*s.
2. *Inductive clauses:* If A and B are *wff*s, so are $\sim\mathsf{A}$, $(\mathsf{A} \wedge \mathsf{B})$, $(\mathsf{A} \vee \mathsf{B})$, and $(\mathsf{A} \supset \mathsf{B})$.
3. *Final clause:* Only strings of symbols that are obtained by the above two rules are *wff*s.

Note, that 'A' and 'B' [GEB *uses* 'x' *and* 'y'] are *not wff*s themselves, but only *metalogical variables* that stand for *wff*s (just as in arithmetic 'x' is not a number, but a variable that stands for a number)!

Exercise 1: Which of the following are *wff* of propositional logic: P_0, P_i, (P_{27}), $(\mathsf{A} \supset \mathsf{B})$, $(\sim(\mathsf{P}_5 \vee (\mathsf{P}_0 \supset \mathsf{P}_5)))$, $(\sim\mathsf{P}_5 \supset (\mathsf{P}_0 \vee \mathsf{P}_5))$?

Exercise 2: Write three propositional formulas as *syntax trees.*

1.2 Parentheses

To avoid clutter, we will leave out parentheses when it is clear how the formulas are intended to be read, i.e., we'll write '$\mathsf{P}_0 \supset (\mathsf{P}_1 \supset \mathsf{P}_0)$' instead of '$(\mathsf{P}_0 \supset (\mathsf{P}_1 \supset \mathsf{P}_0))$'.[2]

[1] Alternative symbols used in the literature: & (conjunction), $\rightarrow$ (implication), and $\neg$ or $\overline{}$ (negation).

[2] Some books have rules that allow to leave out *all* parentheses; e.g., $\mathsf{P}_0 \supset \mathsf{P}_1 \supset \mathsf{P}_2$ for $(\mathsf{P}_0 \supset (\mathsf{P}_1 \supset \mathsf{P}_2))$.

1.3 Truth values and truth tables

As interpretations of propositional variables, we will consider only two truth values, namely *true* and *false*, which we abbreviate as T and F. (Also used: $\top$ and $\bot$, or 1 and 0.)

In *classical logic* every proposition is either true or false (principle of bivalence).

Moreover, the connectives of classical logic are *truth-functional*, i.e., given a interpretation of the components of a *wff* into the set of truth values (i.e., a truth assignment), we can determine the truth value of a more complex *wff* as follows:

A	B	A ∧ B
F	F	F
F	T	F
T	F	F
T	T	T

As before, A and B are here *meta*-variables that stand for propositional formulas. So, for example, if P_0 is true, and P_1 is false, then $\mathsf{P}_0 \wedge \mathsf{P}_1$ is also false. In fact, $\mathsf{P}_0 \wedge \mathsf{P}_1$ is only true if both P_0 and P_1 are true. For this reason, we consider the logical connective $\wedge$ to be the formal analogue to the English sentential connective "and."

For similar reasons $\vee$ and $\sim$ are formalizations of "or" and "not," given by the following truth tables:

A	B	A ∨ B	∼A
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	F

The truth table for $\mathsf{A} \supset \mathsf{B}$, which is read as "if A then B," is slightly more difficult, because in English we often think of A and B to be causally connected, while they need not be in logic. To distinguish the meaning of $\supset$ from that of implication in English, it is called "material implication."

Having said this, the meaning of $\mathsf{A} \supset \mathsf{B}$ is fairly intuitive. Consider the statement "If it is your birthday, then I give you a present," which we can formalize as $\mathsf{P}_0 \supset \mathsf{P}_1$. In which cases is this statement *false*? Well, it is certainly false if it is your birthday, but I don't give you a present, right? Now, what about if it isn't your birthday, but I still give you a present? Certainly, this situation does not make the statement false. Accordingly, the truth table for material implication is as follows:

A	B	A ⊃ B
F	F	T
F	T	T
T	F	F
T	T	T

Finally, to determine the truth value of a complex formula, we assign truth values to the propositional variables and then determine the value of subformulas and of the formula using the truth tables.

Exercise 3: Draw truth tables for the *wff*s: $\sim \mathsf{P}_0 \vee \mathsf{P}_1$, $\sim(\mathsf{P}_0 \wedge \mathsf{P}_1)$, and $\mathsf{P}_0 \supset (\mathsf{P}_1 \supset \mathsf{P}_0)$.

Exercise 4: If "The moon is made of green cheese" is represented by P_0 , "$2+2=4$" is represented by P_1 , and "I like red" by P_2 , how are the following statements formalized in the language of propositional logic and what are their truth values (given the truth values of $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2$ in the real world; what are they)?

(a) "The moon is not made of green cheese or I don't like red,"
(b) "If the moon is made of green cheese, then I like red,"
(c) "If it not the case that $2+2=4$, then the moon is made of green cheese."

1.4 Tautologies

If a *wff* is always true, no matter what the truth values of its components are, it is called a *tautology* (sometimes, but misleadingly, such formulas are simply called *valid*).

Exercise 5: Can you always decide whether a *wff* of propositional logic is a tautology or not (i. e., is there a *decision procedure* for tautologies)? How?

Exercise 6: State five different formulas that are tautologies.

1.5 Connectives and logical equivalence

Using truth tables, we can see that, in classical logic, we can form the truth table for $\wedge$ just in terms of $\supset$ and $\sim$ (make sure to understand this concise way of filling out truth tables):

A	B	A ∧ B	∼(A ⊃ ∼B)
F	F	**F**	**F** F T T F
F	T	**F**	**F** F T F T
T	F	**F**	**F** T T T F
T	T	**T**	**T** T F F T
		↑	↑

In other words, we do not need to introduce conjunction as a separate connective, but we can introduce $\mathsf{A} \wedge \mathsf{B}$ as being just an abbreviation for $\sim(\mathsf{A} \supset \sim \mathsf{B})$.

Two formulas that yield exactly the column in a truth table are called *logically equivalent*.

Similarly, also $\vee$ can be given an explicit definition in terms of implication and negation:

$$(\mathsf{A} \vee \mathsf{B}) \quad \Leftrightarrow_{df} \quad \sim\mathsf{A} \supset \mathsf{B}.$$

In fact, one can show that

$$\{\sim, \supset\}, \quad \{\sim, \wedge\}, \quad \text{and} \quad \{\sim, \vee\}$$

are all *minimal sets of connectives*, i. e., that any of these pairs is sufficient to define the other connectives.[3] Surprisingly, one can also define a single operation, such that it suffices to generate all possible combinations of truth values, the *Sheffer stroke* (introduced in 1913 by H. M. Sheffer with the name *rejection*; in digital electronics this operation is also known as *NAND* operation). It is defined by the following truth table:

A	B	A | B
F	F	T
F	T	T
T	F	T
T	T	F

Exercise 7: State a formula that uses only the connective $\mid$ and is equivalent to $\mathsf{A} \vee \mathsf{B}$.

2 Propositional logic as a calculus

We have seen above how we can use the language of propositional logic to *express* complex statements, now we will see how we can make use of it to *conclude* or *infer* certain statements from others.

Say, you know that the statement "If it is Tuesday, then I go to my favorite class" is true, and it is also the case that it is Tuesday. What follows from these premises? That you go to your favorite class!

In general, whenever it is impossible for the premises to be true and the conclusion to be false we say that the inference is *valid*.

2.1 Semantic consequence (entailment)

The notion of validity (which is a *semantic* notion, since it refers to the meaning of *wff*s — in terms of its truth values) can be used to define when a *wff* A (the conclusion) follows from as set of *wff*s (the premises): A *wff* A is a *(semantic) consequence* of a set of *wff*s $\mathsf{A}_0, \ldots, \mathsf{A}_n$ (or, $\mathsf{A}_0, \ldots, \mathsf{A}_n$ *entails* A) if it is impossible that $\mathsf{A}_0, \ldots, \mathsf{A}_n$ are all true and A is false at the same time. In this case, we write

$$\mathsf{A}_0, \ldots, \mathsf{A}_n \models \mathsf{A}.$$

Thus, for example, B is a semantic consequence of A and $\mathsf{A} \supset \mathsf{B}$, i. e., $\mathsf{A},\ \mathsf{A} \supset \mathsf{B} \models \mathsf{B}$. Test it with a truth table! Similarly, $\mathsf{A} \wedge \mathsf{B} \models \mathsf{A}$.

Exercise 8: Is $\mathsf{P}_1, \mathsf{P}_2 \models \sim(\mathsf{P}_1 \supset \mathsf{P}_2) \supset (\mathsf{P}_2 \supset \mathsf{P}_1)$?
(Draw a truth table and check whether the conclusion is T, whenever the premises are T.)

If the conclusion A is a tautology, i. e., it is always true no matter what the truth values of the premises are, then it is the semantic consequence of the empty set, written just as $\models \mathsf{A}$.

Exercise 9: Can you always determine whether a *wff*s of propositional logic is a semantic consequence of a set of formulas or not? How?

[3] Note: This is *not* possible for Intuitionistic Logic (here a different, more complicated, semantics is necessary—truth tables won't do).

2.2 The Natural Deduction calculus

That B follows from $\mathsf{A} \supset \mathsf{B}$ and A was realized a long time ago and it was expressed as the syntactic *rule of inference* called *modus ponens*. In general, *inference rules* can be written with *premises* above a horizontal line and the conclusion below. For example:

$$\begin{array}{r} \text{Premises:} \\ \text{Conclusion:} \end{array} \qquad \frac{\mathsf{A} \qquad \mathsf{A} \supset \mathsf{B}}{\mathsf{B}} \ (\textit{Modus ponens})$$

If we read this rule from top to bottom it licences us to formally *deduce* the conclusion from the premises. In the example, B is deduced from the two premises A and $\mathsf{A} \supset \mathsf{B}$.

Modus ponens is the best known rule of inference, but we can introduce more. The following two rules allow us to *eliminate* a conjunction symbol and are thus called '$\wedge$-Elim':

$$\boxed{\frac{\mathsf{A} \wedge \mathsf{B}}{\mathsf{A}} \ (\wedge\textit{-Elim})} \qquad \qquad \boxed{\frac{\mathsf{A} \wedge \mathsf{B}}{\mathsf{B}} \ (\wedge\textit{-Elim})}$$

To *introduce* a conjunction symbol, we need to have both A and B as premises:

$$\boxed{\frac{\mathsf{A} \qquad \mathsf{B}}{\mathsf{A} \wedge \mathsf{B}} \ (\wedge\textit{-Intro})}$$

By applying one or more inference rules we obtain a *derivation* or *proof* in the form of a tree. For example, **Natural Deduction proof 1** ($\mathsf{A} \wedge \mathsf{B} \vdash \mathsf{B} \wedge \mathsf{A}$):

$$\frac{\dfrac{\mathsf{A} \wedge \mathsf{B}}{\mathsf{B}} \ (\wedge\textit{-Elim}) \qquad \dfrac{\mathsf{A} \wedge \mathsf{B}}{\mathsf{A}} \ (\wedge\textit{-Elim})}{\mathsf{B} \wedge \mathsf{A}} \ (\wedge\textit{-Intro})$$

The formulas on the top of a derivation tree are the *premises* (which can be used multiple times in a derivation), while the *conclusion* is at the bottom of the tree. If A is obtained by a derivation from the premises $\mathsf{A}_0, \ldots, \mathsf{A}_n$, we write

$$\mathsf{A}_0, \ldots, \mathsf{A}_n \vdash \mathsf{A}.$$

Modus ponens eliminates an implication formula, so it is also called '$\supset$-Elim'. In what follows we shall always give introduction and elimination rules for new sentential connectives.

$$\boxed{\frac{\mathsf{A} \qquad \mathsf{A} \supset \mathsf{B}}{\mathsf{B}} \ (\supset\textit{-Elim})} \qquad \qquad \boxed{\frac{\begin{array}{c} [\mathsf{A}] \\ \vdots \\ \mathsf{B} \end{array}}{\mathsf{A} \supset \mathsf{B}} \ (\supset\textit{-Intro})}$$

The implication introduction rule is a little bit more difficult. It allows us to infer $\mathsf{A} \supset \mathsf{B}$, from a proof of B from the assumption A, which is represented by the '$\vdots$'. However, the original assumption A becomes *internalized* in the conclusion, and so it is *discharged* from the proof. This is expressed by the square brackets around A in the rule. A discharged assumption does not count as an assumption in the proof and cannot be used further down the proof tree. Assumptions that are not discharged are called *open* assumptions.

By applying the $\supset$-*Intro* rule to the derivation in the Natural Deduction proof 1 above, *both* occurrences of the assumption $\mathsf{A} \wedge \mathsf{B}$ are discharged, because they both appear above the formula in the tree. In this way we obtain the theorem $(\mathsf{A} \wedge \mathsf{B}) \supset (\mathsf{B} \wedge \mathsf{A})$.

Natural Deduction proof 2: Using $\supset$-*Intro* we can also derive $\mathsf{A} \supset (\mathsf{B} \supset \mathsf{A})$ depending on *no* open assumptions (the subscripts are used to show which assumption is discharged by which rule):

$$\frac{\dfrac{[\mathsf{A}]_2 \qquad [\mathsf{B}]_1}{\mathsf{B} \supset \mathsf{A}} \ (\supset\textit{-Intro}_1)}{\mathsf{A} \supset (\mathsf{B} \supset \mathsf{A})} \ (\supset\textit{-Intro}_2)$$

Formulas like $(\mathsf{A} \wedge \mathsf{B}) \supset (\mathsf{B} \wedge \mathsf{A})$ and $\mathsf{A} \supset (\mathsf{B} \supset \mathsf{A})$, which can be proved in derivations without open assumptions, are called *logical theorems*. We write $\vdash \mathsf{A}$ for a logical theorem A.

Here are the introduction rules for disjunction:

$$\boxed{\frac{\mathsf{A}}{\mathsf{A} \vee \mathsf{B}} \ (\vee\textit{-Intro})} \qquad \qquad \boxed{\frac{\mathsf{B}}{\mathsf{A} \vee \mathsf{B}} \ (\vee\textit{-Intro})}$$

And the elimination rule (note that this is the structure of a 'proof by cases'):

$$\boxed{\frac{\mathsf{A} \vee \mathsf{B} \qquad \begin{array}{c} [\mathsf{A}] \\ \vdots \\ \mathsf{C} \end{array} \qquad \begin{array}{c} [\mathsf{B}] \\ \vdots \\ \mathsf{C} \end{array}}{\mathsf{C}} \ (\vee\textit{-Elim})}$$

To formulate the introduction and elimination rules for negation in a concise way, we add a new symbol '$\perp$' to our alphabet, which stands for a proposition that is always false (e. g., a contradiction).

$$\boxed{\frac{\begin{array}{c} [\mathsf{A}] \\ \vdots \\ \perp \end{array}}{\sim\mathsf{A}} \ (\sim\textit{-Intro})} \qquad \qquad \boxed{\frac{\mathsf{A} \qquad \sim\mathsf{A}}{\perp} \ (\sim\textit{-Elim})}$$

Finally, we have two more rules for the new symbol and one rule to introduce formulas without any assumptions:

$$\boxed{\frac{\perp}{\mathsf{A}} \ (\textit{Ex falso quotlibet})} \qquad \qquad \boxed{\frac{\begin{array}{c} [\sim\mathsf{A}] \\ \vdots \\ \perp \end{array}}{\mathsf{A}} \ (\textit{Reductio ad absurdum})}$$

The above rules were introduced by the logician Gerhard Gentzen in 1935 and are called the *Natural Deduction* calculus. These rules can be justified semantically or taken as conventions.

Natural Deduction proof 3: Derive $\mathsf{B} \vee \mathsf{D}$ from the assumptions $\mathsf{A} \vee \mathsf{C}$, $\mathsf{A} \supset \mathsf{B}$, and $\mathsf{C} \supset \mathsf{D}$ (argument form of 'dilemma').

$$\frac{\mathsf{A} \vee \mathsf{C} \qquad \dfrac{\dfrac{[\mathsf{A}] \qquad \mathsf{A} \supset \mathsf{B}}{\mathsf{B}} \ (\supset\textit{-Elim})}{\mathsf{B} \vee \mathsf{D}} \ (\vee\textit{-Intro}) \qquad \dfrac{\dfrac{[\mathsf{C}] \qquad \mathsf{C} \supset \mathsf{D}}{\mathsf{D}} \ (\supset\textit{-Elim})}{\mathsf{B} \vee \mathsf{D}} \ (\vee\textit{-Intro})}{\mathsf{B} \vee \mathsf{D}} \ (\vee\textit{-Elim})$$

Natural Deduction proof 4: $A \lor \sim A$ (*tertium non datur*; law of the excluded middle) is a theorem.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]}{A \lor \sim A} \quad [\sim(A \lor \sim A)]}{\bot}}{\sim A}}{A \lor \sim A} \qquad [\sim(A \lor \sim A)]}{\bot}}{A \lor \sim A}$$

Exercise 10: Write the name to each inference rule used in the above proof (use subscripts to indicate which assumptions are discharged by which rule).

Exercise 11: Use $A \lor \sim A$ to show that the *Reductio ad absurdum* rule can be dispensed with. In other words, assume $A \lor \sim A$ and a proof of $\bot$ from $\sim A$, and deduce A.

Exercise 12: Prove $A \supset \sim\sim A$ and $(A \land B) \supset (B \land A)$. [*See* GEB*, p. 184*.]

2.3 An axiomatic calculus

Notice that in the Natural Deduction calculus we have many rules of inference. However, it is also possible to give an equivalent *proof system* that has only a single inference rule, namely *modus ponens* and a rule for *substitutions*.

- *Inference rule:* From A and $A \supset B$ infer B (*modus ponens*).
- *Substitution rule:* Any *wff* can be substituted for the metalogical variables A, B, and C (of course, the same metalogical variable must be substituted by the same *wff*).

This calculus is called *axiomatic*, because it consists of the following 10 axiom schemata that can serve as starting points for derivations (taken from *Epstein and Carnielli*, p. 162):

1. $\sim A \supset (A \supset B)$
2. $B \supset (A \supset B)$
3. $(A \supset B) \supset ((\sim A \supset B) \supset B)$
4. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
5. $A \supset (B \supset (A \land B))$
6. $(A \land B) \supset A$
7. $(A \land B) \supset B$
8. $A \supset (A \lor B)$
9. $B \supset (A \lor B)$
10. $((A \lor B) \land \sim A) \supset B$

This system allows for a concise recursive definition of *proofs*: A *proof* of a *wff* A from a set of assumptions $A_0, \ldots, A_n$ is a sequence of *wff*s, such that each of them is either

a) an assumption, or
b) an instantiation (the result of applying a substitution) of one of the axiom schemata, or
c) obtained from previous *wff*s in the sequence by *modus ponens*.

As before, if we prove A without any assumptions, we write $\vdash A$. Although there is only one inference rule to apply, proofs are much harder to find in this system, because the axiom schemata can be instantiated in myriads of ways! (Before the Natural Deduction calculus was invented in 1935, this was the *main* calculus for doing formal proofs.)

Axiomatic proof 1: $\vdash A \supset A$.

1.	$A \supset ((A \supset A) \supset A)$	Axiom 2 $[(A \supset A)/A,\ A/B]$[4]
2.	$A \supset (A \supset A)$	Axiom 2 $[A/A,\ A/B]$
3.	$(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$	Axiom 4 $[A/A,\ (A \supset A)/B,\ A/C]$
4.	$(A \supset (A \supset A)) \supset (A \supset A)$	MP 1, 3
5.	$A \supset A$	MP 2, 4

Axiomatic proof 2: $A \supset A \vdash B \supset (A \supset A)$.

1.	$A \supset A$	Hypothesis.
2.	$(A \supset A) \supset (B \supset (A \supset A))$	Axiom 2. $[A \supset A/B, \quad B/A]$
3.	$B \supset (A \supset A)$	MP 1,2

Axiomatic proof 3: $A \supset B,\ B \supset C \vdash A \supset C$.

1.	$B \supset C$	Hypothesis.
2.	$\underbrace{(B \supset C)}_{B} \supset (\underbrace{A}_{A} \supset \underbrace{(B \supset C)}_{B})$	Axiom 2. $[B \supset C/B, \quad A/A]$
3.	$A \supset (B \supset C)$	MP 1,2
4.	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	Axiom 4. $[A/A, \quad B/B, \quad C/C]$
5.	$(A \supset B) \supset (A \supset C)$	MP 3, 4
6.	$A \supset B$	Hypothesis
7.	$A \supset C$	MP 5, 6

Axiomatic proof 4: $\vdash (A \land \sim A) \supset B$. See Appendix, p. 12, below.

Using the Natural Deduction calculus we can prove this theorem by applying the rules $\sim$*-Elim* and *Ex falso quotlibet*. This theorem essentially says that 'everything follows from a contradiction.' That's why contradictions are bad. (Compare with GEB, p. 196.)

Exercise 13: Indicate all substitutions (see footnote 4) for the applications of the axiom schemata in the Axiomatic Proof 4 (p. 12, below).

[4] Substitutions are indicated within square brackets; A/B means that the metalogical variable B in the axiom schema is replaced (substituted) here by A. Substitutions are made simultaneously (not successively).

2.4 The GEB system

The language is the same as above. Note that Hofstadter uses $<$ and $>$ for parentheses, and x and y for the metalogical variables A and B. The inference rules are (GEB, pp. 188–189):

Joining Rule: If A and B are theorems, then $(\mathsf{A} \wedge \mathsf{B})$ is a theorem.

Separation Rule: If $(\mathsf{A} \wedge \mathsf{B})$ is a theorem, then both A and B are theorems.

Double-tilde Rule: The string '$\sim\sim$' can be deleted from any theorem. It can also be inserted into any theorem, provided that the resulting string is itself well-formed.

Fantasy Rule: If B can be derived when A is assumed to be a theorem, then $(\mathsf{A} \supset \mathsf{B})$ is a theorem.

Carry-over Rule: Inside a fantasy, any theorem from the 'reality' one level higher can be brought in and used.

Rule of Detachment: If A and $(\mathsf{A} \supset \mathsf{B})$ are both theorems, then B is a theorem.

Contrapositive Rule: $(\mathsf{A} \supset \mathsf{B})$ and $(\sim\mathsf{B} \supset \sim\mathsf{A})$ are interchangeable.

De Morgan's Rule: $(\sim\mathsf{A} \wedge \sim\mathsf{B})$ and $\sim(\mathsf{A} \vee \mathsf{B})$ are interchangeable.

Switcheroo Rule: $(\mathsf{A} \vee \mathsf{B})$ and $(\sim\mathsf{A} \supset \mathsf{B})$ are interchangeable.

By 'interchangeable' in the foregoing rules, the following is meant: If an expression of one form occurs as either a theorem or part of a theorem, the other form may be substituted, and the resulting string will also be a theorem. It must be kept in mind that the symbols 'A' and 'B' always stand for well-formed strings of the system.

This system might be a bit easier to use, but it is not very elegant, since it allows to replace parts of theorems, and it also doesn't have clear introduction and elimination rules.

GEB proof 1: $(\mathsf{A} \wedge \mathsf{B}) \supset (\mathsf{B} \wedge \mathsf{A})$ (Compare with the Natural Deduction proof 1.)

[	push[5]
$(\mathsf{A} \wedge \mathsf{B})$	premise of outer fantasy (assumption)
A	Separation Rule
B	Separation Rule
$(\mathsf{B} \wedge \mathsf{A})$	Joining Rule
]	pop out of outer fantasy, reach real world!
$((\mathsf{A} \wedge \mathsf{B}) \supset (\mathsf{B} \wedge \mathsf{A}))$	fantasy rule

Exercise 14: Deduce the theorems 1–4 proved for the Natural Deduction system (Section 2.2) in the GEB-system.

Exercise 15: Translate the GEB-proof of GEB, p. 185 into the Natural Deduction calculus.

[5] See the *Fantasy Rule* (GEB, p. 183–184) and also GEB, Chapter 5 on recursion.

3 Equivalence of syntactic and semantic notions of consequence

We have seen the following notions of consequence (let A be a *wff* and Γ be a set of *wff*s):

1. Semantic ($\Gamma \models \mathsf{A}$; Γ *entails* A; A *is a logical consequence* of Γ)
   (a) Truth tables
2. Syntactic ($\Gamma \vdash \mathsf{A}$; Γ *proves* A; A *is derivable* from Γ)
   (b) Natural Deduction calculus
   (c) Axiomatic calculus
   (d) GEB system

Big question: What is the relation between (a), (b), (c) and (d)?

3.1 Equivalence of proof systems

It turns out that the system of Natural Deduction, the axiomatic proof system, and the GEB system prove exactly the same theorems!

How can this be proved?

E. g., to show the equivalence of ND with the axiomatic calculus, you need to show:

a) Use the ND system to derive all the axioms of the axiomatic system.

b) Use the axiomatic system to justify all the rules of the ND system.

3.2 Soundness and completeness

What is the relation between the semantic method of proof and the syntactic one?

The following two theorems establish that our semantic and syntactic notions of proof coincide:

Soundness Theorem: If Γ is any set of formulas and there is a proof of a formula A from Γ, then Γ entails A.

$$\Gamma \vdash \mathsf{A} \quad \Longrightarrow \quad \Gamma \models \mathsf{A}$$

Completeness Theorem: If a formula A is entailed by a set of formulas Γ, then there is a proof of A from formulas in Γ.

$$\Gamma \models \mathsf{A} \quad \Longrightarrow \quad \Gamma \vdash \mathsf{A}$$

(This is much more difficult to prove. See Phil 310, if you're interested.)

4 The Deduction Theorem

This is an example of reasoning by induction on the length of axiomatic proofs!

We will prove this using the axiomatic calculus from Section 2.3. This makes the induction proof fairly straightforward, because we have only one rule of inference to consider in the induction step.

Deduction Theorem: $\Gamma \cup \{\mathsf{A}\} \vdash \mathsf{B} \iff \Gamma \vdash \mathsf{A} \supset \mathsf{B}$.

Proof: Since this is an "iff"-claim, we have to prove the equivalence in two directions.

($\Leftarrow$ direction, easy) Assume: $\Gamma \vdash \mathsf{A} \supset \mathsf{B}$. Then, surely, also $\Gamma \cup \{\mathsf{A}\} \vdash \mathsf{A} \supset \mathsf{B}$.

Obviously, $\Gamma \cup \{\mathsf{A}\} \vdash \mathsf{A}$.

From these two, we obtain $\Gamma \cup \{\mathsf{A}\} \vdash \mathsf{B}$ by an application of *modus ponens*.

($\Rightarrow$ direction) Assume: $\Gamma \cup \{\mathsf{A}\} \vdash \mathsf{B}$.

This means that there is an axiomatic derivation of B , say

$$\begin{array}{l} \mathsf{B}_1 \\ \mathsf{B}_2 \\ \vdots \\ \mathsf{B}_n\ (= \mathsf{B}). \end{array}$$

We show *by induction* on i , that for each i , $\Gamma \vdash \mathsf{A} \supset \mathsf{B}_i$.

Base case: B_1 can either be (1) an axiom, (2) an element of Γ , or (3) A itself.

- For all cases: Instantiate Axiom Schema 2 as follows: $\mathsf{B}_1 \supset (\mathsf{A} \supset \mathsf{B}_1)$
  Since we have B_1 , *modus ponens* gives us the desired result: $\mathsf{A} \supset \mathsf{B}_1$.
- Case (3) could also be solved as follows: We have proved $\vdash (\mathsf{A} \supset \mathsf{A})$ (axiomatic proof 1), so we're done.

Induction step: *Induction Hypothesis:* Suppose that for all $k < i$, $\Gamma \vdash \mathsf{A} \supset \mathsf{B}_k$.

Need to show, that $\Gamma \vdash \mathsf{A} \supset \mathsf{B}_i$.

If B_i is an axiom, an element of Γ , or A itself, then this is just like the base case, and we're done.

The only case that is left is if B_i is a consequence by *modus ponens* from two earlier lines in the proof, say B_p and B_q , where $\mathsf{B}_q = \mathsf{B}_p \supset \mathsf{B}_i$, and $p, q < i$.

By the Induction Hypothesis (applied to B_p and B_q), we have:

$$\Gamma \vdash \mathsf{A} \supset \mathsf{B}_p, \tag{1}$$

and

$$\Gamma \vdash \mathsf{A} \supset \mathsf{B}_q, \quad \text{i.e.,} \quad \Gamma \vdash \mathsf{A} \supset (\mathsf{B}_p \supset \mathsf{B}_i). \tag{2}$$

Next we instantiate Axiom Schema 4

$$\vdash (\mathsf{A} \supset (\mathsf{B} \supset \mathsf{C})) \supset ((\mathsf{A} \supset \mathsf{B}) \supset (\mathsf{A} \supset \mathsf{C}))$$

by

$$\vdash (\mathsf{A} \supset (\mathsf{B}_p \supset \mathsf{B}_i)) \supset ((\mathsf{A} \supset \mathsf{B}_p) \supset (\mathsf{A} \supset \mathsf{B}_i)) \tag{3}$$

substituting A/A , B_p/B , and B_i/C .

Now everything is in place and we apply *modus ponens* on (2) and (3), which yields

$$\Gamma \vdash (\mathsf{A} \supset \mathsf{B}_p) \supset (\mathsf{A} \supset \mathsf{B}_i) \tag{4}$$

Finally, *modus ponens* on (1) and (4) yields

$$\Gamma \vdash \mathsf{A} \supset \mathsf{B}_i,$$

which is what we wanted to prove.

Conclusion: The claim holds for all i , so it also holds for $i = n$, i. e., for $\mathsf{B}_i = \mathsf{B}_n = \mathsf{B}$.

Both directions of the if-and-only-if claim have been proved, so the theorem holds. ☐

Appendix

Axiomatic proof 4: $\vdash (\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{B}$.

1.	$(\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{A}$	Axiom 6
2.	$(\mathsf{A} \wedge \sim\mathsf{A}) \supset \sim\mathsf{A}$	Axiom 7
3.	$(\sim\mathsf{A} \supset (\mathsf{A} \supset \mathsf{B})) \supset ((\mathsf{A} \wedge \sim\mathsf{A}) \supset (\sim\mathsf{A} \supset (\mathsf{A} \supset \mathsf{B})))$	Axiom 2
4.	$\sim\mathsf{A} \supset (\mathsf{A} \supset \mathsf{B})$	Axiom 1
5.	$(\mathsf{A} \wedge \sim\mathsf{A}) \supset (\sim\mathsf{A} \supset (\mathsf{A} \supset \mathsf{B}))$	MP 3, 4
6.	$((\mathsf{A} \wedge \sim\mathsf{A}) \supset (\sim\mathsf{A} \supset (\mathsf{A} \supset \mathsf{B}))) \supset$ $(((\mathsf{A} \wedge \sim\mathsf{A}) \supset \sim\mathsf{A}) \supset ((\mathsf{A} \wedge \sim\mathsf{A}) \supset (\mathsf{A} \supset \mathsf{B})))$	Axiom 4
7.	$((\mathsf{A} \wedge \sim\mathsf{A}) \supset \sim\mathsf{A}) \supset ((\mathsf{A} \wedge \sim\mathsf{A}) \supset (\mathsf{A} \supset \mathsf{B}))$	MP 5, 6
8.	$(\mathsf{A} \wedge \sim\mathsf{A}) \supset (\mathsf{A} \supset \mathsf{B})$	MP 2, 7
9.	$((\mathsf{A} \wedge \sim\mathsf{A}) \supset (\mathsf{A} \supset \mathsf{B})) \supset (((\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{A}) \supset ((\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{B}))$	Axiom 4
10.	$((\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{A}) \supset ((\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{B})$	MP 8, 9
11.	$(\mathsf{A} \wedge \sim\mathsf{A}) \supset \mathsf{B}$	MP 1, 10