Surds and Euclid II.14

It is said that philosophers are like recalcitrant children. Disdainful of authority they constantly ask "why", "what does that mean", "how do you know"? Many teachers do not like these little brats and say "be quiet and memorize the formula", but it is by answering the questions of children that we can gain a deeper understanding of what we thought we knew.

So here is a simple question: "Does 49 have a square root?" You answer "yes". I ask "how do you know?". You answer "because $7 \times 7 = 49$, so $\sqrt{49} = 7$" Very good. To answer the question "does 49 have a square root" you write down a number whose square is 49.

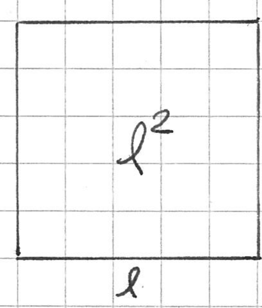
Here is the next question: "Does 2 have a square root?" You answer "yes". I ask "how do you know?"

Your response may well be "because $\sqrt{2} \times \sqrt{2} = 2$, so the square root of 2 is $\sqrt{2}$". But this last statement is completely circular, it says nothing. Logically this is the same as this argument: "Are there any flying elephants?" "Yes, Dumbo". "Who is Dumbo?" "An elephant that can fly." This does not answer the question of the existence of flying elephants, and saying "the square root of two is $\sqrt{2}$" does not speak to its existence.

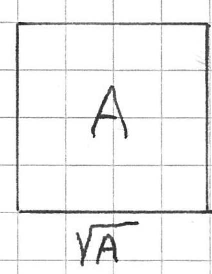
So how do you know that the square root of two exists? You can write down the symbol $\sqrt{2}$, but how do you know this symbol represents a number? After all you can write down the symbol $\sqrt{-4}$ but that doesn't guarantee that this is a number.

To show $\sqrt{2}$ exists you might try to write it down or describe it in a different manner, but you cannot write down $\sqrt{2}$ as a fraction (Pythagoras) and its decimal is infinitely long and doesn't repeat, so you can't write that down either. It is a very real possibility that there is no number whose square is two.

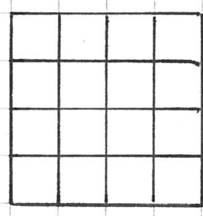
The ancient Babylonians dealt with this question in an entirely different manner. If l is a length then "the square of l" means the square built upon the length l. This is not a numerical statement, this is a geometric picture. l^2 is an actual square.



Conversely, given a square A, its base (root) is $\sqrt{A}$.

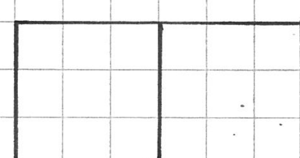


From this point of view, $\sqrt{B}$ only makes sense if B is a square. Hence $\sqrt{16}$ makes sense because I can find a square equal to 16

The square 16  16 unit squares

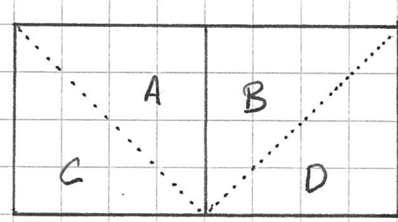
and I can see that the root of this square is the length 4. It is very important to understand that the root of a square is a length, not a number.

Back to the square root of two. This only makes sense if I can find a square 2, that is a square made up of two unit squares. It is easy to find a rectangle made from two unit squares

The rectangle 2 

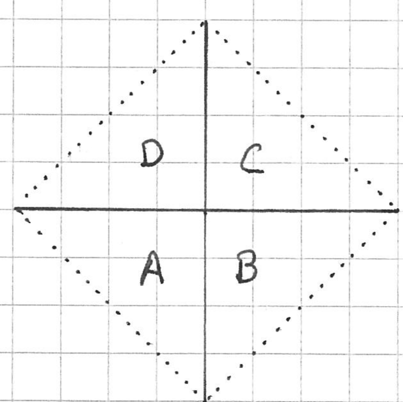
but what we need is a square equal to this rectangle.

The Babylonians solved this problem almost four thousand years ago. Take the rectangle 2 and divide each of the unit squares along its diagonal



rectangle 2

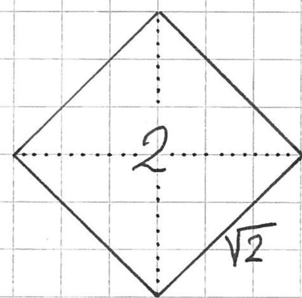
This gives us four triangles, which I have labeled above. Now move the bottom two triangles C and D and put them on the top of the rectangle



square 2

Since all the triangles are right triangles we now have a square. Since it is made from the same triangles as the rectangle 2, I have found a square equal to the rectangle. The figure above is the square 2, and

and its side is $\sqrt{2}$. The picture below was found on a Babylonian clay tablet dating from around 1500 BC



Looking back at our picture of the rectangle 2 we see

The square root of two is the diagonal of a unit square

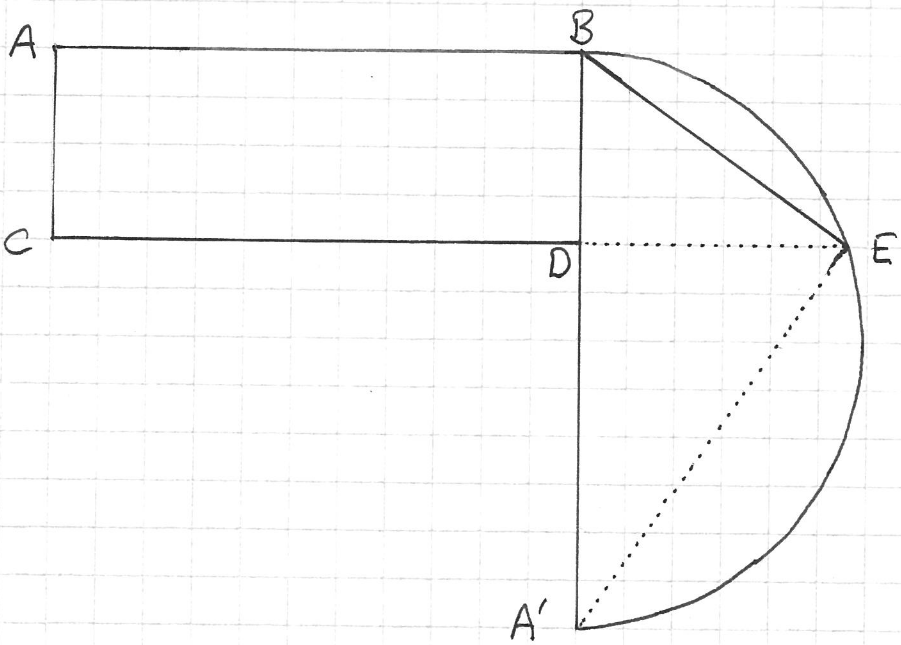
This gives a complete answer to our questions "does 2 have a square root" and "how do you know", but it says $\sqrt{2}$ is a line segment, not a number. We'll get back to this later.

The Greeks inherited mathematics from the Egyptians and Babylonians, but unlike their practical predecessors, the Greeks had a longing for abstraction and generalization. The Babylonians had been satisfied for a thousand years with having found the square 2, but they moved no further. Greek mathematicians pushed on. Is there a square 3, a square made from three unit squares? Obviously we can find a square 4 made from four unit squares, but how about 5?

The key here is to not look at specific numbers. Rather we look for a square of any given size. As Alfred North Whitehead said "generalization is the soul of mathematics", and the Greek philosopher/mathematicians were great at generalization. In his famous book "The Elements" Euclid gave a solution to our problem. After proving that any rectilineal figure is equal to a rectangle, Euclid proved proposition II.14.

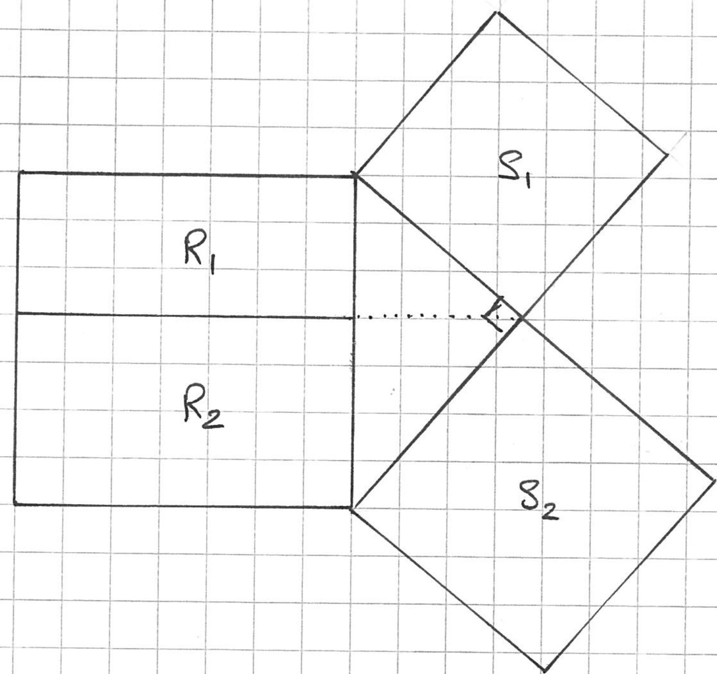
Euclid II.14 : To construct a square equal to a given rectangle.

proof*: Let ABCD below be the given rectangle.



Draw BA′ equal to BA, and draw a semicircle with diameter BA′. Finally let E be the point where the line through CD meets the semicircle. The square on BE equals the rectangle ABCD because if you draw the square you recognize Euclid's proof of the Pythagorean Theorem since A′BE is a right triangle by Thale's Theorem.

Q.E.D.



For completeness I have drawn above the most famous picture in mathematics, Euclid's diagram for the Pythagorean Theorem (proposition I.47). He showed that square S_1 equals rectangle R_1, which we used in our proof of II.14.

<u>Corollary</u>: Any whole number has a square root.

If n is any whole number we can easily build a rectangle from n unit squares, and II.14 ensures it is equal to a square whose side is then $\sqrt{n}$. Similarly it is easy to find a rectangle that is one third of a unit square, so $\frac{1}{3}$ also has a square root.

This can easily be generalized to show that any positive fraction has a square root. Furthermore, since we can find a segment $\sqrt{2}$ it seems we can find a rectangle $\sqrt{2}$ and hence a square root of $\sqrt{2}$, so we are well on our way to showing any number has a square root.

There are two important barriers to completing this program. First, when we found the square root of a number we did not get a number but a line segment. For example, we found a line segment whose square is 5, but we still didn't find a number whose square is five. Is every line segment a number, is every number a line segment? We still cannot write down a number such that when you multiply it times itself you get two.

The second problem is that it is not obvious at all that certain numbers can be made into line segments, most famously the number π. Now we know that the circumference of a circle with diameter one is pi. This is in fact the definition of pi. However this defines pi as a circular arc, not as a line segment. Can we find a straight line segment which is equal to the circular arc pi? Another approach is this: We know a circle with radius one is equal to pi. Can we find a square with the same area? This is the famous problem of "squaring the circle" which plagued mathematicians for twenty centuries.

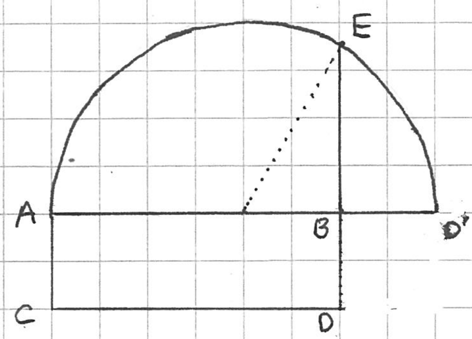
The ancients Greeks decided there was no good reason to think there is a number whose square is two. They decided that if you want to study numbers you must stick to whole numbers and factions, which you can write down, and stay away from mystical numbers, like root seven, which may or may not exist.

It is not that they gave up studying quantities like $\sqrt{3}$. They knew that these were line segments, so their study fell under the rubric of geometry, not number theory. In his "Elements" Euclid spends several chapters explaining the rules for the algebra of numbers, and then shows the same kind of rules work for the algebra of line segments, but for him these were two separate subjects.

Since the "Elements" was the Bible of Mathematics, the Greek point of view held sway for over a thousand years. Not until the middle ages did Arabic mathematicians begin to treat surds as numbers rather than line segments. They simply assumed such numbers exist without trying to justify the assumption.

The unification of geometry and algebra by Descartes in the seventeenth century encouraged Europeans to adopt the Arabic view of surds as numbers, but it was not until the nineteenth century that the simple question "is there a number whose square is 2" was answered to the satisfaction of philosophically inclined mathematicians. It took a thorough reexamination of of the concept of real number by Cantor and Dedekind, but I must say that their explanation that "$\sqrt{2}$ is a certain equivalence class of Cauchy sequences" does not warm my heart. I prefer the Greek explanation that $\sqrt{2}$ is the side of a square, something I can see and touch with my mind's eye.

* Footnote. The proof given here of Euclid II.14 is not that given by Euclid. His picture looks like



and Euclid shows that the square on BE equals the rectangle ABCD. The reasoning necessary is harder tha in our proof, but Euclid's proof has important generalizations to the study of conic sections, where the semi-circle above is replaced by an ellipse.

When comparing our proof and Euclid's proof of II.14 note that our picture could be drawn as below, but the connection to I.47 is not so clear.