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# Euclid

The city of Alexandria, on the mediterranean coast of Egypt, was founded by Alexander the Great in 332 B.C., who brought Greeks, Egyptians and Jews to settle there. One of his generals, Ptolemy I, made Alexandria the capital of his kingdom and founded a dynasty consisting of a long line of rulers, also named 'Ptolemy' and ending with the reign of the famous queen Cleopatra, who picked the wrong side in a Roman civil war.

Ptolemy established a university in Alexandria, called the 'Museum', which was soon to acquire a library holding more than 600,000 papyrus scrolls. For well over 600 years, Alexandria was to be the mathematical and scientific center of the world, with only some schools of philosophy surviving in Athens, although, after the extinction of the Ptolemaic line with Cleopatra, Alexandria was ruled by Rome. It was ultimately conquered by the Arabs in 641 AD.

The first chair of mathematics at the Museum was occupied by Euclid (330 to 275 BC), said to have been a student of a student of Plato. Apart from a couple of anecdotes, we know little about his life, and some ancient authors even thought he was a committee, like the 20th century Nicolas Bourbaki. According to one anecdote, Euclid told the impatient king that 'there is no royal road to learning'. According to another, he gave a small coin to a student who demanded to know the practical value of the lectures he had been attending.

Euclid wrote a number of books, on optics, music, astronomy etc., but his fame rests on the *Elements*, a collection of 13 so-called books (which we would now call chapters), which presented the foundations of all the mathematics known in his day. Nothing like this was to be published again, until the middle of the 20th century, when Nicolas Bourbaki issued a collection of books that purported to cover the elements of all the mathematics we study now.

None of the theorems contained in the 13 books can with certainty be ascribed to Euclid himself. It is believed that the Pythagoreans, including Archytas, were responsible for much of what appears in Books I, II, VI, VII, VIII, IX and XI and that Hippocrates was behind Books III and IV. For Books V and XII we are to thank Eudoxus, and Books X and XIII are said to be based on the work of Theaetetus.

However, the logical organization of the *Elements* is undoubtedly Euclid's contribution. Its success can be measured by the fact that, after more than 2,000 years, it was still used as a textbook in British schools. Moreover, throughout the ages, its structure was often imitated. Thomas Aquinas used a similar axiomatic presentation in his *Summa*, Newton's *Principia* is written in the style of the *Elements* and Spinoza's *Ethics* follows its logical arrangement. Undoubtedly the *Elements* has been the most influential scientific textbook in history.

Euclid's grandiose plan was to deduce all of mathematics from a small number of initial definitions and assumptions. The assumptions are subdivided into *axioms*, dealing with mathematics in general, and *postulates*, dealing with geometry in particular.

His treatment illustrated the ideal described by Aristotle at the beginning of his *Posterior Analytics*: sure knowledge is obtained by the rigorous deduction of the consequences of basic truths. To Euclid, these basic truths were either definitions or basic assumptions, largely assertions of unique existence. Let us take a closer look at his definitions, axioms and postulates.

The *Elements* begins with a list of 23 definitions, of which we will mention the first four:

1. A *point* is that which has no parts.

2. A *line* is length without width.

3. The extremities of a line are points.

4. A *straight line* is a line which lies evenly with the points on itself.

These statements are not definitions in the modern sense, though they make it clear that a point has no extension, that a line is not necessarily straight and that it is of finite length. Today we prefer to regard points and straight lines as undefined primitive concepts and leave the definition of curved lines to more advanced mathematics. The obscurity of Definition 4 may be due to the translation.

Euclid's axioms are intended to apply to all of mathematics, not just to geometry. A typical axiom asserts: 'If equals are added to equals, their sums are equal.' One cannot quarrel with this statement, though today we might derive it from axioms of equality and the view of addition as an operation.

Euclid lists five postulates, which we shall now state and comment upon.

**I.** To draw a straight line from any point to any other point.

Presumably this means that there exists a unique straight line joining two distinct given points. Thus, a 'straight line' cannot be interpreted as referring to a great circle on a sphere, as there are many great circles joining two antipodes, e.g., the meridians passing through the two poles on the globe. The way to get around this objection is to *identify* antipodal points; one then obtains *elliptic geometry*, which also satisfies Postulate I.

**II.** To produce a finite straight line continuously in a straight line.

Here 'continuously' is usually interpreted to imply 'indefinitely', thus ruling out not only spherical, but also elliptic geometry.

**III.** To describe a circle with any center and any distance [as radius].

Like Postulate I, this is a construction, or unique existence statement, the word 'circle' having previously been defined, in Definition 15, as 'a plane figure contained by one line [i.e. curve] such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.' It would appear that by a circle Euclid means not just its circumference but also its interior.

**IV.** That all right angles are equal to one another.

The status of this assertion as a postulate is rather dubious, and it has been argued, already in antiquity, that it should be listed as an axiom instead.

**V.** That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

This is the most famous of Euclid's postulates and it is to his credit that he recognized its significance. It will be discussed at length in Chapter 17.

It is on these definitions and assumptions that Euclid plans to erect his impressive edifice of logical deductions. Here is how he begins:

**Proposition 1.**
On a finite straight line to construct an equilateral triangle.

In his proof he considers a segment $AB$ and constructs circles with centers $A$ and $B$ and radius $AB$. He then considers the point $C$ in which the two circles intersect and goes on to show that $\triangle ABC$ is equilateral.

This proof falls short of modern standards of rigour. In general, two circles may meet in two points, touch at one point, or not meet at all. In the present situation, they do, in fact, meet in two points; but this does not follow from Euclid's explicit assumptions.

Book I concludes with proofs of the Theorem of Pythagoras and its converse. Euclid is careful to show that there is a square on the hypotenuse, before discussing its properties. This is interesting, in view of Legendre's later proof that the existence of such a square implies Euclid's Postulate V.

To prove the Theorem of Pythagoras, Euclid uses a theory of area. Nowadays we are tempted to define the area of a rectangle as 'length times width'. This presupposes a theory which explains what it means to multiply two irrationals. Euclid approached the question of area from a more elementary point of view. He began with the idea that two polygons have the same area if they first can be dissected into triangles which can be reassembled, as in a jigsaw puzzle, to form a polygon exactly like the second polygon. It is only in Book VI, after Euclid has presented Eudoxus's theory of irrationals, that the length times width formula is justified.

However, in Book II, Euclid gives geometric treatments of certain basic algebraic identities, such as $a(b+c) = ab + ac$, using the areas of rectangles to handle products. He also gives a proof of a statement equivalent to what we now call the Law of Cosines.

Book III discusses the basic properties of the circle. Euclid goes to great length to give rigorous proofs. For example, in spite of the fact that it is 'obvious from the diagram', Euclid offers a demonstration of the fact that the points on a chord of a circle lie in the interior of the circle. Euclid is not always successful in his attempt at rigour, but it is clear that he does understand the need for it.

Book IV gives constructions for various regular polygons. It culminates with a treatment of the regular 15-gon. This achievement remained unsurpassed until 1796, when Carl Friedrich Gauss (1777–1855) found a construction for the regular 17-gon.

In Book V, Euclid uses Eudoxus's definitions of proportion to deduce an arithmetic for line segments. The 'commutativity of multiplication' is the subject of Proposition 16.

In Book VI, Euclid uses the material of Book V to derive the basic properties of similar triangles. The book concludes with the theorem that the length of a circular arc is proportional to the angle it subtends at the center of the circle. In talking about 'arclength', Euclid is implicitly presupposing the 'completeness' of the plane.

Books VII to IX present some elementary theorems of number theory. Included are proofs for Euclid's Algorithm (VII 2), the unique factorization of square-free integers (IX 14), the infinitude of primes (IX 20), the formula for the sum of a geometric progression (IX 35), and the formula for even perfect numbers (IX 36).

Book X is occupied with what we might call 'field extensions of degree 4 over rationals'. Euclid is interested in knowing when an expression like $\sqrt{7+2\sqrt{6}}$, which looks like it has 'degree 4' is actually equal to an expression like $1+\sqrt{6}$, which involves only one 'layer' of square roots.

Book XI derives the basic theorems of solid geometry. A 'cone' is defined in terms of the revolution of a right triangle. A 'cube' is 'a solid figure contained by six equal squares'. Proposition XI 21 says that 'any solid angle is contained by plane angles [whose sum is] less than four right angles'. This proposition is used at the end of Book XIII to show that there are at most 5 regular polyhedra.

Book XII is the masterpiece of Eudoxus. Without the help of calculus, he manages to give a rigorous treatment of the volumes of the pyramid, cone and sphere.

Book XIII is the apex of the *Elements*. For each of the five regular polyhedra, Euclid derives the ratio of its side to the radius of the sphere in which it is inscribed. Although Euclid failed to give a complete theory of regular polygons – for example, the construction of the regular 17-gon is missing – he succeeded in giving a complete theory of regular polyhedra.

Euclid's *Elements*, or watered down versions of it, was used for over 2,000 years in universities and schools to teach not only geometry but also rigorous thinking. Not long after World War II, a reaction against this program set in and educators decided that geometry was not the appropriate place for training in logic. Anyway, they argued, Euclid was not rigorous enough and Hilbert's rigorous treatment (Chapter 17) was too cumbersome. So geometry was swept away in favour of 'New Mathematics'. The French mathematician Dieudonné, one of the founding members of the Bourbaki group, suggested that linear algebra should replace what he contemptuously called 'the theory of the triangle'.

## Exercises

1. True or false? If two triangles have the same area, you can cut one of them up into little triangles, which can then be placed side by side to form a triangle congruent to the second triangle. Give a reference or a reason for your answer.

2. How did Euclid construct the regular 15-gon?