1

Egyptian Mathematics

The Greeks believed that mathematics originated in Egypt. As to the reason for this, opinion was divided. Aristotle thought that mathematics was developed by priests, 'because the priestly class was allowed leisure' (*Metaphysics* 981b 23-24). Herodotus believed that the annual flooding of the Nile necessitated surveying to redetermine field boundaries, and thus led to the invention of geometry. In fact, Democritus referred to Egyptian mathematicians as 'rope stretchers'. It may be of interest to note that the Egyptians themselves believed that mathematics had been given to them by the god Thoth. Our only original sources of information on the mathematics of ancient Egypt are the Moscow Mathematical Papyrus and the Rhind Mathematical Papyrus.

The Moscow Papyrus dates from 1850 BC, about the time the Bible dates the life of the patriarch Abraham. In 1893 it was acquired by V. S. Golenishchev and brought to Moscow (Gillings [1972], p. 246). Problem 14 of this papyrus is by far the most interesting. It is the computation of a *truncated pyramid*, a square pyramid with a similar pyramid cut off its top. If a side of the base has length a and a side of the top has length b, then the volume of the truncated pyramid of vertical height h is

$$V = \frac{h}{3}(a^2 + ab + b^2).$$

This is exactly the formula used by the Egyptians. Note that, if $b = 0$, we get the formula for the volume of the complete pyramid.

The Rhind Mathematical Papyrus seems to be based on an earlier work. It was written by one Ahmose in 1650 BC, about the time when, accord-

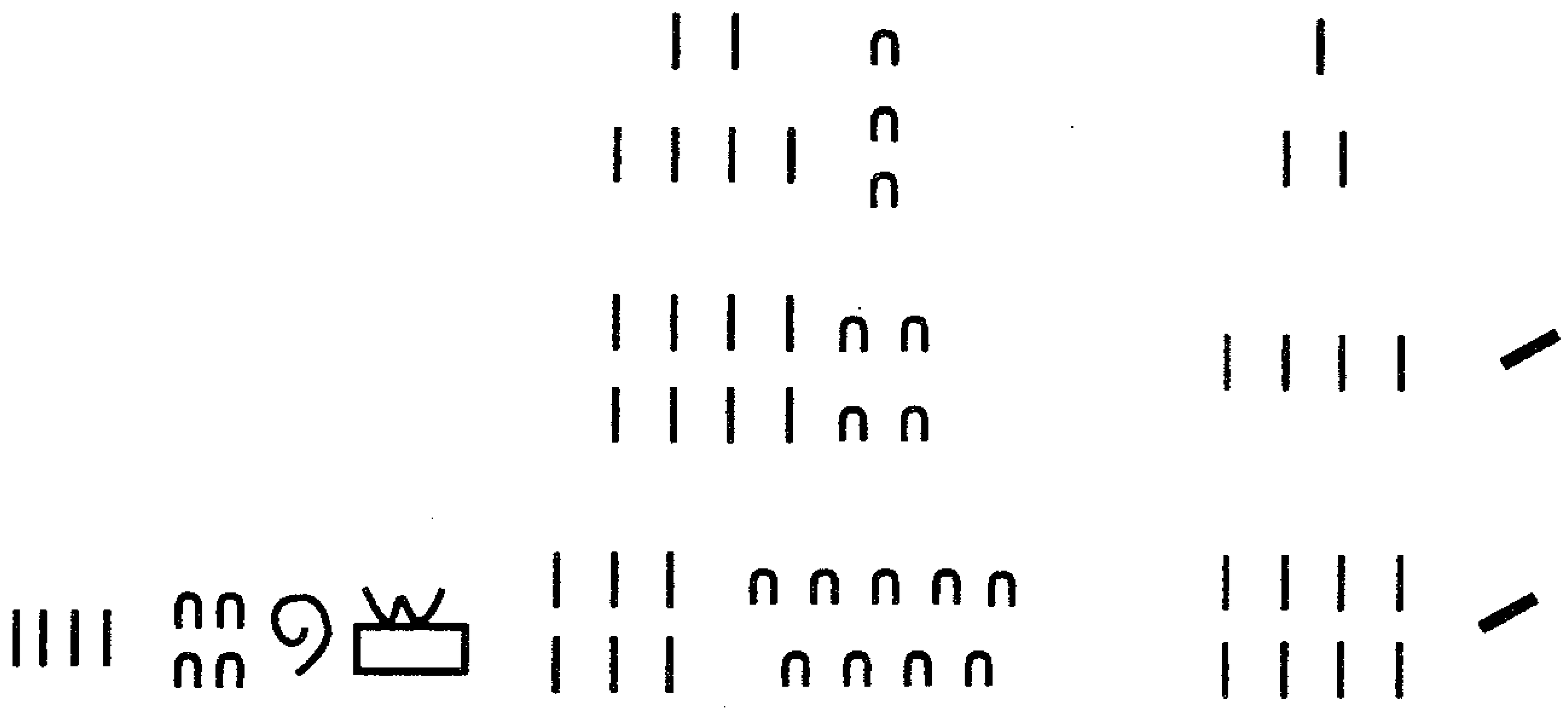

FIGURE 1.1. Rhind Papyrus

ing to the Bible, Joseph was governor of Egypt. Alexander Henry Rhind acquired it in Luxor, Egypt in 1858; the British Museum bought it from his estate in 1865. Complete photographs of the papyrus can be found in *The Rhind Mathematical Papyrus* edited by G. Robins and C. Shute.

The Rhind Papyrus opens by promising the reader 'a thorough study of all things, insight into all that exists, knowledge of all obscure secrets'. It is a bit of a letdown to find that it is, in fact, a sequence of solved problems in elementary mathematics, a sort of Schaum's outline for aspiring scribes. These scribes had to calculate how many bricks were needed to build a ramp of a certain size, how many loaves of bread were required to feed the labourers, and so on.

Problem 32 of the papyrus is an exercise in multiplication written as in Figure 1.1.

Transcribing this into modern notation, we have

12	1	
24	2	
48	4	/
96	8	/

144 = the sum of the checked entries.

Clearly, this is a calculation to show that $12 \times 12 = 144$, using the fact that $12 = 4 + 8$.

By doubling and adding, the Egyptians were able to multiply any two natural numbers – without having to memorize multiplication tables! Sometimes they used a different, yet equivalent, method, as illustrated by the following multiplication of 70 by 13:

70	13	/
140	6	
280	3	/
560	1	/

910 = sum of checked entries.

We let the reader figure out why this works. The method of repeated doubling can also be used for division. In the following example, we divide

184 by 17 (stopping at 136, as the next double exceeds 184):

17	1	
34	2	/
68	4	
136	8	/

The Egyptians would first check off the last row and subtract 136 from 184, obtaining 48. They would then check off the row containing 34, the highest multiple of 17 less than 48. Since $48 - 34 = 14$ is less than 17, they would now add up all the entries in the second column with check marks beside them: $2 + 8 = 10$. This gives the answer: the quotient is 10 and the remainder is 14.

In carrying out these divisions, the Egyptians sometimes interspersed doubling with multiplication by 10 (their language expressed numbers in the base 10, just as ours does). For example, Problem 69 in the Rhind Mathematical Papyrus is to calculate the number of 'ro' of flour in each loaf, if 1120 ro of flour is made into 80 loaves. In other words, we are asked to divide 1120 by 80:

80	1	
800	10	/
160	2	
320	4	/

sum of checked numbers = 14.

The Egyptians also knew how to extract square roots and how to solve linear equations. They used the hieroglyph h much as we use the letter x for the unknown. They used the formula $(\frac{4}{3})^4 r^2$ for the area of a circle (which implies 3.16 as an approximation to π) and they did some interesting work with arithmetic progressions. For example, Problem 64 of the Rhind Papyrus is to find an arithmetic progression with 10 terms, with sum 10, and with common difference 1/8.

In using fractions, the Egyptians were hampered by a curious tradition. They insisted on expressing all fractions (except 2/3) as the sum of distinct *unit* fractions of the form $1/n$, n being a positive integer. Thus 2/9 would be written as $1/6 + 1/18$ and 19/8 as $2 + 1/4 + 1/8$. Even 2/3 is sometimes written as $1/2 + 1/6$.

For us it is easy to divide 5/13 by 12, but for the Egyptians this was a substantial problem. To help with such problems, they had a table listing unit fraction decompositions for fractions of the form $2/n$, with n an odd positive integer. This table (found in the Rhind Papyrus) gives 2/13 as $1/8 + 1/52 + 1/104$. Since $5 = (2 \cdot 2) + 1$, Ahmose would write

$$\begin{aligned} 5/13 &= 2(1/8 + 1/52 + 1/104) + 1/13 \\ &= 1/4 + 1/26 + 1/52 + 1/13. \end{aligned}$$

From this he would obtain

$$(5/13)/12 = 1/48 + 1/312 + 1/624 + 1/156.$$

Actually, any fraction of the form $2/(2m+1)$ can be expressed as a sum of the unit fractions $1/(m+1)$ and $1/(m+1)(2m+1)$. Not that the Egyptians always followed this recipe; for example, Ahmose wrote $2/45 = 1/30+1/90$.

Recently, Paul Erdös proposed the following problem: show that, if n is an odd integer greater than 4, then $4/n$ can be written as a sum of three distinct unit fractions. The problem has not yet been solved. (See Mordell, p. 287.)

Exercises

1. Derive the formula for the volume of a truncated pyramid from that of a pyramid.
2. Explain why the above method for multiplying 70×13 works.
3. Find two ways of writing $1/4$ as the sum of two distinct unit fractions.
4. If m is a positive integer, show that $4/(4m+3)$ can be written as the sum of three distinct unit fractions.