b) does the linear independence of a system $\{a_1, \ldots, a_n\}$ imply the linear independence of the system $\{a_1 + a_2, a_2 + a_3, \ldots, a_{n-1} + a_n, a_n + \lambda a_1\}$?

3403. Prove the linear independence of the systems of functions:

a) $\sin x,\ \cos x$;

b) $1,\ \sin x,\ \cos x$;

c) $\sin x,\ \sin 2x,\ \ldots,\ \sin nx$;

d) $1,\ \cos x,\ \cos 2x,\ \ldots,\ \cos nx$;

e) $1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots,\ \cos nx,\ \sin nx$;

f) $1,\ \sin x,\ \sin^2 x,\ \ldots,\ \sin^n x$;

g) $1,\ \cos x,\ \cos^2 x,\ \ldots,\ \cos^n x$.

3404. Prove the linear independence of the systems of functions:

a) $e^{\alpha_1 x},\ \ldots,\ e^{\alpha_n x}$;

b) $x^{\alpha_1},\ \ldots,\ x^{\alpha_n}$;

c) $(1 - \alpha_1 x)^{-1}, \ldots, (1 - \alpha_n x)^{-1}$,

where $\alpha_1, \ldots, \alpha_n$ are pairwise distinct real numbers.

3405. Prove that in the space of functions of one real variable, vectors $f_1, \ldots, f_n$ are linearly independent if and only if there exist numbers $a_1, \ldots, a_n$ such that $\det\big(f_i(a_j)\big) \neq 0$.

3406.

a) Let there be defined, in a vector space V over the field $\mathbb{C}$, a new multiplication of vectors by complex numbers by the rule $\alpha \circ x = \bar{\alpha} x$. Prove that V with respect to the operations $+$ and $\circ$ is a vector space. Find its dimension.

b) Let $\mathbb{C}^n$ be the abelian group of all rows $(a_1, \ldots, a_n)$ of length n, $a_i \in \mathbb{C}$. If $b \in \mathbb{C}$ we put $b \circ (a_1, \ldots, a_n) = (b\bar{a}_1, \ldots, b\bar{a}_n)$. Is $\mathbb{C}^n$ a vector space with respect to the operations $+$ and $\circ$?

3407. Prove that

a) the group $\mathbb{Z}$ is not isomorphic to the additive group of any vector space;

b) the group $\mathbf{Z}_n$ is isomorphic to the additive group of a vector space over some field if and only if n is a prime number;

c) a commutative group A is a vector space over the field $\mathbf{Z}_p$ if and only if $px = 0$ for any $x \in A$;

d) a commutative group A can be turned into a vector space over $\mathbb{Q}$, if and only if it has no elements of finite order (except zero) and, for any natural number n and any $a \in A$, the equation $nx = a$ has a solution in the group A.

3408. Let F be a field and E be its subfield.

a) Prove that F is a vector space over E.

b) If F is finite then $|F| = |E|^n$, where n is the dimension of F as a vector space over E.

c) If F is finite then $|F| = p^m$, where p is the characteristic of F.

d) Find the basis and dimension of $\mathbb{C}$ over $\mathbb{R}$.

e) Let $m_1, \ldots, m_1$ be distinct square-free natural numbers. Prove that the numbers $1, \sqrt{m_1}, \ldots, \sqrt{m_n}$ are linearly independent in $\mathbb{R}$ over $\mathbb{Q}$.

f) Let $r_1, \ldots, r_n$ be distinct rational numbers in the interval $(0, 1)$. Prove that in the space $\mathbb{R}$ over $\mathbb{Q}$ the numbers $2^{r_1}, \ldots, 2^{r_n}$ are independent.

g) Let α be a complex root of an irreducible polynomial over $\mathbb{Q}$, $p \in \mathbb{Q}[x]$. Find the dimension over $\mathbb{Q}$ of the space $\mathbb{Q}[\alpha]$, consisting of all numbers of the form $f(\alpha)$, $f \in \mathbb{Q}[x]$.

3409. Let M be a set consisting of n elements. On the set of its subsets 2^M let there be defined the operations of addition and multiplication by elements of the field $\mathbf{Z}_2$ as in Exercise 102.

$$1X = X, \quad 0X = \emptyset.$$

a) Prove that with respect to these operations the set 2^M is a vector space over the field $\mathbf{Z}_2$, and find its basis and dimension.

b) Let $X_1, \ldots, X_k$ be subsets of M, neither of which is contained in the union of the others. Prove that $\{X_1, \ldots, X_k\}$ is an independent system.

3410. Let the vectors $e_1, \ldots, e_n$ and x be given, in some basis by coordinates:

a) $e_1 = (1, 1, 1)$, $e_2 = (1, 1, 2)$, $e_3 = (1, 2, 3)$, $x = (6, 9, 14)$;

b) $e_1 = (2, 1, -3)$, $e_2 = (3, 2, -5)$, $e_3 = (1, -1, 1)$, $x = (6, 2, -7)$;

c) $e_1 = (1, 2, -1, -2)$, $e_2 = (2, 3, 0, -1)$, $e_3 = (1, 2, 1, 4)$, $e_4 = (1, 3, -1, 0)$, $x = (7, 14, -1, 2)$.

Prove that $(e_1, \ldots, e_n)$ is also a basis of the space and find the coordinates of x in this basis.