MATH 251, homework 6, due date Monday Feb 23.

Problem 1. Let S be the standard basis of $\mathbf{R}^2$ and let $S^* = (x_1, x_2)$ be its dual basis. Let $B = ((5,2),(7,3))$ be another basis of $\mathbf{R}^2$. Compute B^* in terms of x_1 and x_2.

Problem 2. Let $V = \mathcal{C}[0,1]$ and let U be the subspace of functions of the form $y(x) = ax + b$ for some a, b depending on the function. Give an explicit family of functionals $\mathcal{F} \subset U^\perp$ such that for any $y \in V$ satisfying $f(y) = 0$ for all $f \in \mathcal{F}$, we have $y \in U$. In other words, in V^{**} we have

$$\mathrm{Span}\mathcal{F}^\perp \cap \phi(V) = \phi(U).$$

Problem 3. Show that if a linear map T is injective, then T^* is surjective, and if T is surjective, then T^* is injective.

Hint: show that every linear map defined on a subspace of a vector space V extends to a map defined on V.