From Lambek

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The Eighteenth Century

We shall only mention some of the most important mathematicians of the eighteenth century:

- Brook Taylor (1685–1731),
- Colin Maclaurin (1689–1746),
- Abraham de Moivre (1667–1754),
- Leonhard Euler (1707–1783),
- Joseph Louis Lagrange (1736–1813),
- Pierre Simon Laplace (1749–1827),
- Adrien Marie Legendre (1752–1833).

Brook Taylor, an ardent admirer of Newton, discovered the *Taylor series*

$$f(a+x) = f(a) + xf'(a) + x^2 f''(a)/2! + \ldots,$$

publishing it in 1715.

Colin Maclaurin, a Scotsman, is best known for the special case $a = 0$ of Taylor's series. This appeared in his *Treatise of Fluxions* (1742). In his book, Maclaurin tried to be sufficiently rigorous to answer Berkeley's objections to the Calculus, but he did not even get to the point of demonstrating conditions under which his *Maclaurin series* converges.

Abraham de Moivre was born in France, but lived in England. He is famous for his formula

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx\ ,$$

which is easily proved, for natural numbers n, by mathematical induction. De Moivre published an important book on the theory of probability, called the *Doctrine of Chances*.

Society failed de Moivre: in spite of letters of recommendation from both Newton and Leibniz, he was never given a proper job in mathematics. He had to earn a meagre living by private tutoring and answering gamblers' questions on probability. It is said that, as he approached the end of his life, de Moivre slept fifteen minutes longer each day. When he reached a full twenty four hours, he died.

Although Leonhard Euler was Swiss, he spent part of his professional life in Berlin and most of it in St. Petersburg. Towards the end of his life he became blind, but this did not slow down his mathematical output. He found many interesting and exciting results in mathematics. Indeed, it has been said that Euler picked all the raisins out of the mathematical cake. Some of his results are the following:

1. If a convex polyhedron has V vertices, F faces and E edges, then $V + F - E = 2$. For example, a cube has 8 vertices, 6 faces and 12 edges; we have $8 + 6 - 12 = 2$. (Descartes came close to this formula, but he did not actually state it.)

2. $e^{i\pi} = -1$, where e is the 'Euler number':
$$e = \lim_{n\to\infty} (1 + 1/n)^n .$$

3. $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots = \pi/6.$

   Euler's proof of this was not rigorous but, before Euler, no one even guessed that the sum of the series was $\pi/6$.

4. Every even perfect number has the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime.

5. If n is a positive integer, let $\phi(n)$ be the number of natural numbers less than or equal to n and relatively prime to it. Then, if a is a positive integer relatively prime to n, it follows that n is a factor of $a^{\phi(n)} - 1$. Fermat's Little Theorem is a corollary of this.

6. The circumcenter, orthocenter and centroid of a triangle are collinear. The line that passes through them is called the *Euler line*.

(The *circumcenter*, *orthocenter* and *centroid* of a triangle are the meeting points of the right bisectors, altitudes and medians, respectively.)

7. Fermat was wrong when he conjectured that all natural numbers of the form $2^{2^n} + 1$ are primes.

Euler recognized the importance of convergence in dealing with infinite series, but he did not always pay attention to it. For example, he would write

$$1/(x-1) = 1/x + 1/x^2 + 1/x^3 + \cdots$$

(which is correct when $|x| > 1$) and put $x = 1/2$ to obtain $-2 = 2 + 4 + 8 + \cdots$. He also showed a lack of rigour in employing his principle of 'conservation of form', according to which a theorem true for natural number exponents also holds for any real exponent. In this way, he obtained facile 'proofs' of the generalized Binomial Theorem, and the generalized de Moivre's Theorem.

In addition to his numerous discoveries in pure mathematics, by no means all of which have been discussed here, Euler also made important contributions to mechanics. He elaborated the Principle of Least Action. Finally, he worked out a theory of lunar motion. His collected works run to about 75 volumes.

The following proof of Euler's formula $V + F - E = 2$ was suggested by H. S. M. Coxeter.

Let O be a point in the interior of the convex polyhedron. About O as center describe a sphere which contains the polyhedron. Now imagine a source of light placed at O. Rays emanating from O will project the polyhedron onto the surface of the sphere, mapping each flat polygon onto a spherical polygon whose sides are arcs of great circles. (This idea is said to be due to the Arabic mathematician Abu'l Wafa.) Choose a point in the interior of each spherical polygon and join it to the vertices by arcs of great circles, thus dividing each spherical polygon into as many spherical triangles as it has sides. Then

$$\begin{aligned}
720^\circ &= \text{area of sphere} \\
&= \text{sum of areas of spherical triangles} \\
&= \text{sum of angles of spherical triangles} - 180^\circ \times \text{number of triangles} \\
&= \text{sum of angles at interior points} + \\
&\quad\ \text{sum of angles at vertices} - 180^\circ \times 2E \\
&= 360^\circ \times F + 360^\circ \times V - 360^\circ \times E.
\end{aligned}$$

Dividing by 360°, we obtain Euler's formula.

Joseph Lagrange was born in Italy of mixed French and Italian parentage. His father lost the family fortune through speculation, but Lagrange later commented that, if it had not been for this, he might never have turned to mathematics. He was converted to mathematics through an essay by Halley.

At age 23, Lagrange was able to explain, on the basis of Newton's theory of gravitation, why the moon always shows the same face to the earth.

Having acquired an early fame, Lagrange spent 25 years in Prussia at the invitation of Frederick II. After Frederick's death, Lagrange moved to Paris, where he became a favourite of Marie Antoinette. He had mixed feelings about the Revolution, especially when his friend, the chemist Lavoisier, was guillotined, but he stuck it out. He was involved in the introduction of the decimal system for weights and measures. When people pleaded the advantages of the base 12, he would ironically defend the base 11. He became professor of mathematics at the Ecole Polytechnique.

Lagrange was a universal mathematical genius, his interests ranging from number theory to physics. Among his achievements are the following:

1. The first proof of Wilson's Theorem that, if p is a prime number, then it is a factor of $(p-1)!+1$;

2. The first complete solution of the Diophantine equation $x^2 - Ry^2 = 1$, where R is a given nonsquare positive integer; Lagrange generalized this to give a complete treatment of Diophantine equations of the form
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$
where A, B, C, D, E, and F are given integers;

3. The first proof that every natural number is a sum of four squares of natural numbers (e.g., $7 = 2^2+1^2+1^2+1^2$ and $9 = 3^2+0^2+0^2+0^2$);

4. A systematic theory of differential equations;

5. The *Mécanique*, which he conceived at the age of 19 but only published at 52, in which he expressed the dynamics of a rigid system by the equations
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0\,,$$
where T is the total kinetic energy, V is the potential energy, t is the time, θ is any coordinate, and $\dot{\theta} = d\theta/dt$; Lagrange observed that his equations expressed the fact that the total *action* $\int_a^b (T-V)dt$ was minimal; to justify this observation, he had to invent the calculus of variations.

When Lagrange wrote to Euler about his results in the calculus of variations, Euler was so impressed that he withheld his own results from publication so that Lagrange could publish first. Sad to say, such unselfish acts are rare.

After his wife died in 1783, Lagrange wore himself out publishing the *Mécanique*. The excesses of the Revolution upset him and he became subject to fits of depression. From these the lonely genius was rescued by the love of a teenaged girl, Renée Le Monnier, who insisted on marrying him in 1792. For the remaining twenty years of his life, Lagrange was both happy and mathematically productive.

Laplace was the son of humble parents but ended up as a marquis under the restored Bourbons. Politically, he was an opportunist, but occasionally he stood up for his principles. Napoleon once told him, 'you have written a big book on the universe without mentioning its creator', to which Laplace replied: 'I don't need that hypothesis'.

Laplace was more of a mathematical physicist than a pure mathematician. He introduced the potential V and showed that it satisfied

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

His greatest contribution to mathematics was the useful phrase 'it is easy to see', which peppers his *Mécanique Céleste*. In *The History of Mathematics*, David Burton reports:

> The American astronomer Nathaniel Bowditch (1773-1838), who translated four of the five volumes into English, observed, "I never came across one of Laplace's 'Thus it plainly appears' without feeling sure that I had hours of hard work before me to fill up the chasm and find out and show how it plainly appears."

Legendre was a great promoter of Euclid. He showed that the Parallel Postulate follows from the assumption that the plane contains real squares (i.e., quadrilaterals with four equal sides, each of whose angles is a right angle.) He also did work on the method of least squares.

Legendre is best known for his work in number theory. He was the first to prove that the Diophantine equation

$$x^5 + y^5 = z^5$$

has no nonzero integer solutions. He introduced the Legendre symbol $\left(\frac{n}{p}\right)$, where p is a prime and n an integer not divisible by p. He wrote $\left(\frac{n}{p}\right) = 1$ when n has the form $kp + r^2$ (with k and r integers) and $\left(\frac{n}{p}\right) = -1$ when n does not have this form.

Euler had conjectured a theorem, called the Law of Quadratic Reciprocity. Using the the Legendre symbol, it can be expressed as follows: if p and q are distinct odd primes, $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$. Euler could not prove this, but Legendre found proofs for some special cases. It was Gauss who published the first complete proof, in 1801. Gauss later gave five other proofs of the same result.

The Legendre symbol $\left(\frac{n}{p}\right)$ can be calculated quite easily, in view of the following observation due to Euler:

$$n^{\frac{p-1}{2}} = \left(\frac{n}{p}\right) + \text{a multiple of } p.$$

Indeed, it follows from Fermat's Little Theorem that

$$(n^{\frac{p-1}{2}} - 1)(n^{\frac{p-1}{2}} + 1) = n^{p-1} - 1$$

is a multiple of p, so p divides either $n^{\frac{p-1}{2}} - 1$ or $n^{\frac{p-1}{2}} + 1$ (but not both, since it does not divide their difference). We claim that p divides the former if and only if $\left(\frac{n}{p}\right) = 1$, hence p divides the latter if and only if $\left(\frac{n}{p}\right) = -1$, from which facts the observation follows.

To see this, at least in one direction, suppose $\left(\frac{n}{p}\right) = 1$, that is, $n = r^2 + kp$ for some integers r and k. Then

$$n^{\frac{p-1}{2}} = (r^2 + kp)^{\frac{p-1}{2}} = r^{p-1} + k'p = 1 + k''p$$

for some integers k' and k'', hence p divides $n^{\frac{p-1}{2}} - 1$. The converse implication, though not difficult, is a little tricky, and we shall omit its proof.

Exercises

1. Give conditions sufficient for the convergence of the Maclaurin series.
2. Prove de Moivre's formula for positive integers n.
3. Show that e, as defined above, is bound below by 2 and above by 3.
4. Prove that the circumcenter, orthocenter and centroid of any triangle are collinear.
5. Give an example of a formula with exponents which is true when the exponents are natural numbers but not always true when the exponents are rationals.
6. Prove Wilson's Theorem.
7. Check the Law of Quadratic Reciprocity for $p = 5$ and $q = 13$.