HOMEWORK 2

Due Thursday, October 5

- The same rules as for Assignment 1 apply.
- Working through the assignment on your own will help you to learn the material and identify those areas which you need to study more.
- If you have questions, make sure to clear them up during *office hours* or by asking on the myCourses *discussion boards*.

Readings:

- Read GEB, Chapters 2, 3 and 4 carefully. Skim Chapters 5 and 6 at your leisure.

Problems:

- In general, you should not need to consult other sources for answering these questions. If you do so nevertheless, intellectual honesty requires you to state the sources!

1. *Cardinal arithmetic*. Make sure to understand *all* the notation. Let the set A be countably infinite, i. e., have cardinality $\aleph_0$: $|A| = \aleph_0$.

   (a) Let $B_0, B_1, B_2, \ldots$ be a countably infinite series of sets, all of which have cardinality $\aleph_0$ (i. e., $|B_i| = \aleph_0$, for all $i \in \mathbb{N}$), which have no elements in common (i. e., $B_i \cap B_j = \emptyset$ for all $i, j \in \mathbb{N}$), and also have no element in common with A (i. e., $A \cap B_i = \emptyset$, for all $i \in \mathbb{N}$).
   Claim: The cardinality of the set $C = A \cup B_0 \cup B_1 \cup B_2 \cup \ldots$ (i. e., $C = A \cup B_i$, for all $i \in \mathbb{N}$) is also $\aleph_0$.
   Give a brief justification of why the claim is true.

   (b) If two sets A and D, where D is a subset of A (i. e., $D \subseteq A$) both have cardinality $\aleph_0$, what is the cardinality of the set $A - D$ that contains all those elements that are in A, but not in D?

2. *Formal systems*. Now we will consider three joint formal systems. First, consider the following formal System **V**:

   Alphabet: `P`, `1`.

   Axiom: `P`.

   Inference rule: If x is a **V**-theorem (i. e., a theorem in the System **V**), then so is x`1`.

   (a) State three **V**-theorems.

   Using the theorems of the **V**-system, we define a new System **F**:

Alphabet: P, 1, (,), $\succ$.
Axioms: Any **V**-theorem.
Inference rule: If x and y are **F**-theorems, then so is $(x \succ y)$.

(b) State three **F**-theorems.

Finally, using the theorems of the **F**-system, we define a new System **T**:

Alphabet: P, 1, (,), $\succ$.
Axiom: If x and y are **F**-theorems, then the following is a **T**-theorem:
$(x \succ (y \succ x))$.
Inference rule: If x is a **T**-theorem, then $(x \succ x)$ is also a **T**-theorem.

(c) State three **T**-theorems.
(d) Can you think of a meaningful interpretation of the System **T**?
If so, explain it briefly. If not, explain what causes your difficulties.

3. *Formal systems*.

(a) Invent a formal system **Y**, stating the alphabet, the axiom(s), and the rule(s) of inference, such that it has only *lengthening* rules (see GEB).
(b) State 4 theorems of this system.
(c) Describe a *decision procedure* for the **Y**-theorems (i. e., an algorithm that, when presented with an arbitrary string t, determines whether t is a **Y**-theorem or not).

4. *Interpretations*. Consider the following formal system.

Alphabet: [,], T, F, A, O, E.
Axiom: T E T.
Inference rules:

1.	x T $y \rightarrow x$ [T A T] y	4.	x T $y \rightarrow x$ [T O T] y
2.	x T $y \rightarrow x$ [T O F] y	5.	x F $y \rightarrow x$ [F A T] y
3.	x T $y \rightarrow x$ [F O T] y	6.	x F $y \rightarrow x$ [T A F] y

where x and y are any strings of the alphabet (including empty strings).

(a) State four theorems of this system.
(b) Consider the following *interpretation*:

T $\Longrightarrow$ 5 [$\Longrightarrow$ (
F $\Longrightarrow$ 3] $\Longrightarrow$)
E $\Longrightarrow$ =, equality, e. g., 5 = 5.
O $\Longrightarrow$ max, the maximum-function, e. g., '(5 max 3)' evaluates to 5.
A $\Longrightarrow$ min, the minimum-function, e. g., '(5 min 3)' evaluates to 3.

For example, under this interpretation

the string:	[	T	A	T	]	E	T
is interpreted as:	(	5	min	5	)	=	5
and to be evaluated as:					5	=	5

What are the interpretations of the four theorems you have given in answer (a), when evaluated?

(c) Is this interpretation a *model* (or *meaningful* in the sense of GEB, Ch. 2)? Give an explanation for your answer.

(d) Can you think of a different (if possible, interesting) model?

5. *Recursiveness*. "There exist formal systems whose negative space (set of non-theorems) is not the positive space (set of theorems) of any formal system." (GEB, p. 72)

   (a) Rephrase this claim using the terminology introduced in class: 'set', 'recursively enumerable', and 'recursive'.

   (b) How would you explain this claim to a friend who isn't taking Comp 230?

6. *Formal primes*. Derive the theorem '`P-----`' in the formal system presented in GEB, pp. 73–74.

7. *Fancy nouns*. Give an example of a *fancy noun* consisting of more than four words. (This problem refers to GEB, pp. 131–133).

   Explain why your example is indeed a *fancy noun*.

8. *Propositional logic*. Have a look at Handout 2 'A short introduction to propositional logic.'

   (a) Draw a truth table for the *wff* $\mathsf{P}_1 \supset (\mathsf{P}_0 \supset \mathsf{P}_1)$.

   (b) Do Exercise 4 of the Handout.

9. *History*. Find some information about the life and mathematical achievements of Georg Cantor. Write a short paragraph about some interesting facts about his life and work.

   (Cite all of your sources!)

⋆ *Bonus questions.*

   (a) Read the article "Georg Cantor and transcendental numbers", available here:
   `http://www.maa.org/programs/maa-awards/writing-awards/georg-cantor-and-transcendental-numbers`
   Explain in a brief paragraph what it means that "All transcendantals live on diagonals".

   (b) On p. 168 of GEB, you can see a text in Assyrian cuneiform script. Where in Montreal can you see ancient clay tablets with this script?