COMP 330 Autumn 2014
Assignment 3
Due Date: 22^{nd} October 2015

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There are **5** questions for credit and one for your spiritual growth plus one relatively easy question that might appeal to algebra nuts. All the regular questions are excellent practice for the mid-term. The alternate questions will not help you prepare for the mid-term. The homework is due in class at the beginning of the class.

Question 1[20 points] Are the following statements true or false? Prove your answer in each case. We have some fixed alphabet Σ with at least two letters.

- If A is regular and $A \subseteq B$ then B must be regular. [3]
- If B is regular and $A \subseteq B$ then A must be regular. [3]
- If A and AB are both regular then B must be regular. [7]
- If $\{A_i | i \in \mathbb{N}\}$ is an infinite family of regular sets then $\bigcup_{i=1}^{\infty} A_i$ is regular. [7]

Question 2[20 points]
Show that the following languages are not regular by using the pumping lemma.

1. $\{a^n b^m a^{n+m} | n, m \geq 0\}$,
2. $\{x | x = x^R, x \in \Sigma^*\}$, where x^R means x reversed; these strings are called *palindromes*. An example is *abba* a non-example is *baba*.

Alternate Question 2[20 points]
Show that $\{a^n b^m | \mathsf{gcd}(n, m) = 1; n, m > 0\}$ is not regular.

Question 3[20 points] Show that the language

$$F = \{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$$

is not regular. Show, however, that it satisfies the statement of the pumping lemma as I proved it in class, i.e. there is a p such that all three conditions for the pumping lemma are met. Explain why this does not contradict the pumping lemma.

Question 4[20 points] Let D be the language of words w such that w has an even number of a's and an odd number of b's and does not contain the substring ab.

1. Give a DFA with *only five* states, including any dead states, that recognizes D.
2. Give a regular expression for this language.

Alternate Question 4[20 points] In assignment 1 we had an alternate question that asked you to prove that if a language is regular then the lefthalf of the language is also regular. Similarly, if I define the *middle thirds* of a regular language by

$$\mathsf{mid}(L) = \{y \in \Sigma^* | \exists x, z \in \Sigma^* \text{ s.t. } xyz \in L \text{ and } |x| = |y| = |z|\}$$

then $\mathsf{mid}(L)$ is also regular. *I am not asking you to prove this; it is too easy after you have done left-half.* What if I delete the "middle" and keep the outer portions. More precisely define,

$$\mathsf{outer}(L) = \{xz | \exists y \in \Sigma^*, xyz \in L, \text{ and } |x| = |y| = |z|\}$$

then is it true that $\mathsf{outer}(L)$ is regular if L is regular? Give a proof if your answer is "yes" and a counter-example, with a proof that it is not regular, if your answer is "no."

Question 5[20 points] Consider the language $L = \{a^n b^m | n \neq m\}$; as we have seen this is not regular. Recall the definition of the equivalence $\equiv_L$ which we used in the proof of the Myhill-Nerode theorem. Since this language is not regular $\equiv_L$ cannot have finitely many equivalence classes. Exhibit explicitly, infinitely many distinct equivalence classes of $\equiv_L$.

Alternate Question 5[20 points] Suppose that we have a *one-letter* alphabet, say $\{a\}$. Consider *any* subset B of a^*; B could be *very* non-regular. Show that B^*, defined in the obvious way, is *always* regular.
(Hint: Euclid's algorithm, suitably extended.)

Please turn over for the (distracting) extra questions.

Question 6[0 points] Consider a probabilistic variant of a finite automaton. Come up with a formalization of what this might mean. Suppose that you have a reasonable definition and now you define acceptance to mean that your word causes the machine to reach an accept state with probability at least $\frac{2}{3}$. Show that such automata can recognize non-regular languages.

Question 7[0 points] A Kleene algebra $\mathcal{K}$is a structure $(K, +, \cdot, *, 0, 1)$ where K is a set, $(K, +, 0)$ forms a commutative monoid, $(K, \cdot, 1)$ forms a monoid, 0 is an annihilator for $\cdot$, $\cdot$ distributes over $+$, $+$ is idempotent and the $*$ operation satisfies the following conditions:

$$1 + aa^* = a^*, 1 + a^*a = a^*.$$

In addition we have a natural partial order defined by

$$a \leq b \Leftrightarrow a + b = b.$$

Using this order we can state the final two axioms:

$$b + ac \leq c \Rightarrow a^*b \leq c; b + ca \leq c \Rightarrow ba^* \leq c.$$

Prove that in any Kleene algebra a^*b is the least solution to the equation $x = ax + b$ where x is the unknown for which we are solving the equation.