COMP 330 Autumn 2015
Assignment 1
Due Date: 24^{th} Sept 2015

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10^{th} September 2015

Please attempt all questions. There are **5** questions for credit and one for your spiritual growth. The homework is due in class **at the beginning of the class.** There are alternate versions of questions 2, 3 and 4 if you want more challenging questions or questions requiring more mathematical background. Do not do them unless you are *very* confident. If you attempt the alternate questions we will ignore any answers to the regular versions of the questions, *even if they are correct and your answers to the alternate versions are wrong.* Question 6 should not be handed in, but discussed privately with me. You will get no extra credit or other benefit related to your grade for doing it; it is for your spiritual growth.

Question 1[20 points] We fix a finite alphabet Σ for this question. As usual, Σ^* refers to the set of all finite strings (words) over Σ.

(a) Given $x, y \in \Sigma^*$ we say that x is a **prefix** of y if $\exists z \in \Sigma^*\ y = xz$. If x is a prefix of y and y is a prefix of x what can you *deduce* about the relationship between x and y? [5 points]

(b) For this part we assume that $\Sigma = \{a, b\}$. We write $\#_a(x)$ for the number of occurrences of the letter a in the word x and similarly for $\#_b$. We claim that

$$\forall x \in \Sigma^*, \exists y, z \in \Sigma^* \text{ such that } x = yz \wedge [\#_a(y) = \#_b(z)].$$

Is this true? If so prove it, if not disprove it. [15 points]

Question 2[20 points] Fix a finite alphabet Σ and let $\emptyset \neq L \subseteq \Sigma^*$. We define the following relation R on words from Σ^*:

$$\forall x, y \in \Sigma^*, xRy \text{ if } \forall z \in \Sigma^*, xz \in L \text{ iff } yz \in L.$$

Prove that this is an equivalence relation.

Alternate Question 2[20 points] The collection of strings Σ^* with the operation of concatenation forms an algebraic structure called a *monoid*. A monoid is a set with a binary associative operation and with an identity element (necessarily unique) for the operation. Every group is a monoid but there are many monoids that are not groups because they do not have inverses; a natural example is the non-negative integers. A monoid *homomorphism* is a map between monoids that preserves the identity and the binary operation. Let Σ be any finite set and let M be *any* monoid. Show that *any* function $f : \Sigma \to M$ can be extended in a unique way to a monoid homomorphism from $\Sigma^* \to M$. This is an example of what is called a *universal property.*

Question 3[20 points] Consider, pairs of natural numbers $\langle m, n\rangle$ where $m, n \in \mathbf{N}$. We order them by the relation $\langle m, n\rangle \sqsubseteq \langle m', n'\rangle$ if $m < m'$ or $(m = m') \wedge n \leq n'$, where $\leq$ is the usual numerical order.

1. Prove that the relation $\sqsubseteq$ is a partial order. [10 points]
2. Prove that $\sqsubseteq$ is a well-founded order. [10 points]

Alternate Question 3[20 points] Recall that a *well-ordered* set is a set equipped with an order that is well-founded as well as linear (total). For any poset $(S, \leq)$ and monotone function $f : S \to S$, we say f is *strictly monotone* if $x < y$ implies that $f(x) < f(y)$; recall that $x < y$ means $x \leq y$ and $x \neq y$. A function $f : S \to S$ is said to be *inflationary* if for every $x \in S$ we have $x \leq f(x)$. Suppose that W is a well-ordered set and that $h : W \to W$ is strictly monotone. *Prove* that h must be inflationary.

Question 4[20 points] Give deterministic finite automata accepting the following languages over the alphabet $\{0, 1\}$.

1. The set of all words ending in 00. [6 points]
2. The set of all words ending in 00 *or* 11. [6 points]
3. The set of all words such that the *second* last element is a 1. By "second last" I mean the second element counting backwards from the end. Thus, 0001101 is not accepted and 11101010 is accepted. [8 points]

Alterenate Question 4[20 points] Suppose that L is a language accepted by a DFA (i.e. a regular language) show that the following language is also regular:

$$\text{lefthalf}(L) := \{w_1 | \exists w_2 \in \ \Sigma^* \text{ such that } w_1w_2 \in L \text{ and } |w_1| = |w_2|\}.$$

[Hint: use nondeterminism.]

Question 5[20 points]

1. Give a deterministic finite automaton accepting the following language over the alphabet $\{0, 1\}$: The set of all words containing 100 or 110. [5 points]
2. Show that *any* dfa for recognizing this language must have at least 5 states. [15 points]

Question 6[0 points] Suppose that L is a language accepted by a DFA (i.e. a regular language) show that the following language is also regular:

$$\text{LOG}(L) := \{x | \exists y \in \ \Sigma^* \text{ such that } xy \in L \text{ and } |y| = 2^{|x|}\}.$$