Formal Theories of Arithmetic

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The names (Π_0, Π_1, Π_2, Π and 'baby', 'junior') are due to Machover 1996. The terminology 'Robinson's Q' and 'DPA' are fairly standard. References to theorems are to (Machover 1996).

Π_0 (Baby arithmetic)

Numerals: S_n is the numeral that stands for n; S_0 is $\mathsf{0}$, S_1 is $\mathsf{S0}$, and S_{n+1} is SS_n.

Π_0 is the theory that is based on the following four postulate schemata:

1. $\mathsf{S}_m + \mathsf{S}_0 = \mathsf{S}_m$. $\quad \forall m, n \in \mathbb{N}$.
2. $\mathsf{S}_m + \mathsf{S}_{n+1} = \mathsf{S}(\mathsf{S}_m + \mathsf{S}_n)$.
3. $\mathsf{S}_m \times \mathsf{S}_0 = \mathsf{S}_0$.
4. $\mathsf{S}_m \times \mathsf{S}_{n+1} = \mathsf{S}_m \times \mathsf{S}_n + \mathsf{S}_m$.

- **Example of a proof in** Π_0**.** Show: $\mathsf{S}_1 + \mathsf{S}_1 = \mathsf{S}_2 \in \Pi_0$.

$$\mathsf{S}_1 + \mathsf{S}_1 = \mathsf{S}_1 + \mathsf{S}_{0+1} \overset{\text{Post. 2}}{=} \mathsf{S}(\mathsf{S}_1 + \mathsf{S}_0) \in \Pi_0$$

$(*)$ So, $\quad \mathsf{S}_1 + \mathsf{S}_1 = \mathsf{S}(\mathsf{S}_1 + \mathsf{S}_0) \in \Pi_0$

By Postulate 1 ($m = 1$) $\quad \mathsf{S}_1 + \mathsf{S}_0 = \mathsf{S}_1 \in \Pi_0$

Axiom 6 of first-order logic: $\quad \mathsf{s} = \mathsf{t} \supset \mathsf{fs} = \mathsf{ft}$, applied to the previous line:

$$\mathsf{S}(\mathsf{S}_1 + \mathsf{S}_0) = \mathsf{SS}_1 \in \Pi_0$$

By Def. of numerals: $\quad \mathsf{SS}_1 = \mathsf{S}_2$

$(**)$ So, $\quad \mathsf{S}(\mathsf{S}_1 + \mathsf{S}_0) = \mathsf{S}_2 \in \Pi_0$

Combining lines $(*)$ and $(**)$ we get, by first-order logic,

$$\mathsf{S}_1 + \mathsf{S}_1 = \mathsf{S}_2 \in \Pi_0. \qquad \square$$

- **Power.** Can represent all r. e. relations weakly (Machover, Thm. 10.9.12).
- **Limitation.** Cannot prove $\mathsf{S}_0 \neq \mathsf{S}_1$.

Π_1 (Junior arithmetic)

To get to Π_1: Define $\leq$ and add postulates.

Definition of $\mathsf{r} \leq \mathsf{t}$: For any terms r and t, $\mathsf{r} \leq \mathsf{t} \Leftrightarrow_{df} \exists \mathsf{z}(\mathsf{r} + \mathsf{z} = \mathsf{t})$, where z is the first variable in alphabetic order that occurs neither in r nor in t.

Postulates for Π_1: Postulates for Π_0 together with

5. $\mathsf{S}_m \neq \mathsf{S}_n$.
6. $\forall \mathsf{v}_1\ (\mathsf{v}_1 \leq \mathsf{S}_n \longleftrightarrow ((\mathsf{v}_1 = \mathsf{S}_0) \vee (\mathsf{v}_1 = \mathsf{S}_1) \vee \cdots \vee (\mathsf{v}_1 = \mathsf{S}_n)))$.
7. $\forall \mathsf{v}_1\ ((\mathsf{S}_n \leq \mathsf{v}_1) \vee (\mathsf{v}_1 \leq \mathsf{S}_n))$.

for all natural numbers n, m, with $n \neq m$.

- **Power.** Can represent all recursive relations strongly (Machover, Thm. 10.10.14).
- **Limitation.** $\Pi_1 \nvdash (\forall \mathsf{v}_1\ \mathsf{Sv}_1 \neq \mathsf{S}_0)$.

Π_2 (A finitely axiomatized theory: Robinson's Q)

Π_2 postulates (nine postulates, modified from Robinson, 1950):

I.	$\forall \mathsf{v}_1\ (\mathsf{Sv}_1 \neq \mathsf{S}_0)$.	Zero has no predecessor.
II.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ (\mathsf{Sv}_1 = \mathsf{Sv}_2 \supset \mathsf{v}_1 = \mathsf{v}_2)$.	Successor function is injective.
III.	$\forall \mathsf{v}_1\ (\mathsf{v}_1 + \mathsf{S}_0 = \mathsf{v}_1)$.	Recursive definition of addition.
IV.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ [\mathsf{v}_1 + \mathsf{Sv}_2 = \mathsf{S}(\mathsf{v}_1 + \mathsf{v}_2)]$.	
V.	$\forall \mathsf{v}_1\ (\mathsf{v}_1 \times \mathsf{S}_0 = \mathsf{S}_0)$.	Recursive definition of multiplication.
VI.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ [\mathsf{v}_1 \times \mathsf{Sv}_2 = \mathsf{v}_1 \times \mathsf{v}_2 + \mathsf{v}_1]$.	
VII.	$\forall \mathsf{v}_1\ (\mathsf{v}_1 \leq \mathsf{S}_0 \supset \mathsf{v}_1 = \mathsf{S}_0)$.	Nothing is less than zero.
VIII.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ (\mathsf{v}_1 \leq \mathsf{Sv}_2 \supset \mathsf{v}_1 \leq \mathsf{v}_2 \vee \mathsf{v}_1 = \mathsf{Sv}_2)$.	Less or equal.
IX.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ (\mathsf{v}_1 \leq \mathsf{v}_2 \vee \mathsf{v}_2 \leq \mathsf{v}_1)$.	Relation is total.

- **Limitation.** $\forall \mathsf{v}_1\ (\mathsf{Sv}_1 \neq \mathsf{v}_1) \notin \Pi_2$.

Π (First-order Dedekind-Peano arithmetic: DPA)

Peano 1889, but Dedekind 1888!

Hofstadter: 'TNT'.

Π consists of the first six of the postulates of Π_2, together with an *induction schema*:

I.	$\forall \mathsf{v}_1\ (\mathsf{S}\mathsf{v}_1 \neq \mathsf{S}_0)$	(0 has no predecessor.)	
II.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ (\mathsf{S}\mathsf{v}_1 = \mathsf{S}\mathsf{v}_2 \rightarrow \mathsf{v}_1 = \mathsf{v}_2)$	(The successor function is injective.)	
	(Addition:)		(Multiplication:)
III.	$\forall \mathsf{v}_1\ (\mathsf{v}_1 + \mathsf{S}_0 = \mathsf{v}_1)$	V.	$\forall \mathsf{v}_1\ (\mathsf{v}_1 \times \mathsf{S}_0 = \mathsf{S}_0)$
IV.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ (\mathsf{v}_1 + \mathsf{v}_2 = \mathsf{S}(\mathsf{v}_1 + \mathsf{v}_2))$	VI.	$\forall \mathsf{v}_1 \forall \mathsf{v}_2\ (\mathsf{v}_1 \times \mathsf{S}\mathsf{v}_2 = \mathsf{v}_1 \times \mathsf{v}_2 + \mathsf{v}_1)$

Postulate schema of induction. For every formula α with one free variable:

$$\alpha(\mathsf{S}_0) \rightarrow \big[\forall \mathsf{v}_1\ (\alpha(\mathsf{v}_1) \rightarrow \alpha(\mathsf{S}\mathsf{v}_1)) \rightarrow \forall \mathsf{v}_1\ \alpha(\mathsf{v}_1)\big].$$

- **Power.** Most formulas you will ever care about are in DPA.
- **Limitation.** Some formulas are not…

See (Gödel 1931) and (Paris and Harrington 1977).

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