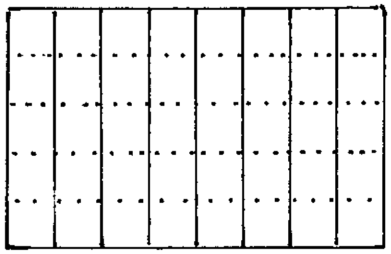
The area of a circle is pi times the square of the radius

Everybody has learned the formula for the area of a circle : $A = \pi r^2$. Our teachers told us it was true, but who told the Greeks it was true? They had no teachers, they had to discover this formula on their own and then show it is correct. In order to do so they invented a new way of thinking, a new way of getting to an answer.

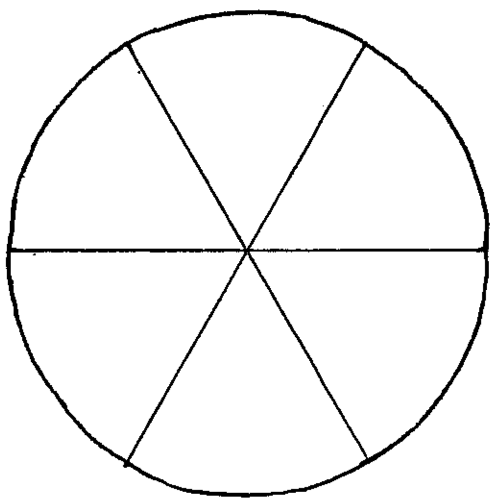
Before looking at a circle, how do we know the area of a rectangle is width times height? It is easy to picture a rectangle as cut into little squares :



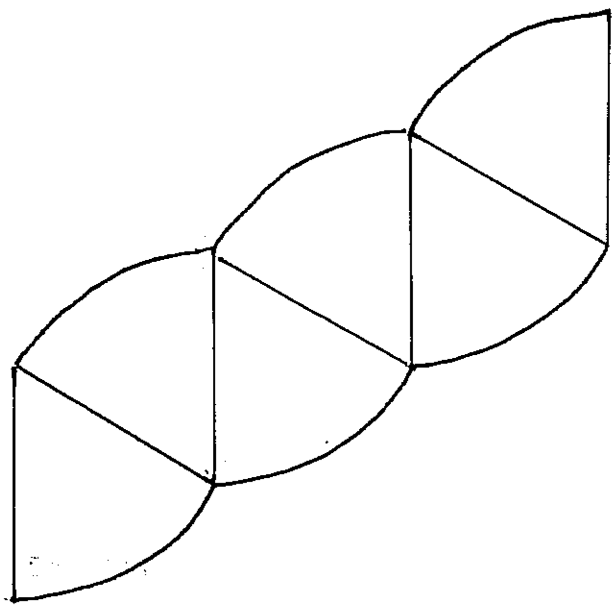
Now you count the width, count the height and multiply to get the number of little squares, which is the area.

You cannot do the same trick with a circle because it doesn't neatly divide into little squares you can count.

The natural way to cut up a circle is into wedges, like slices of pizza :



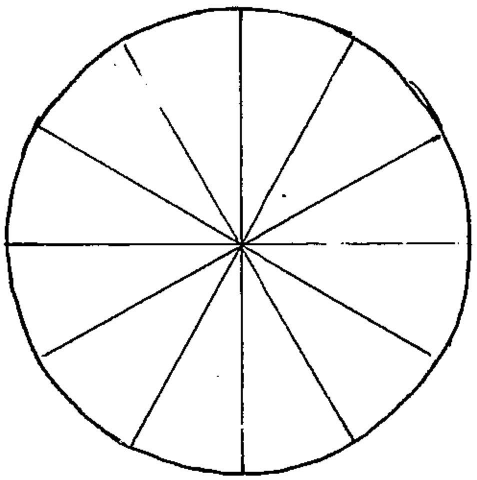
Now if we knew the area of each slice we could add them up to get the area of the circle, but without knowing the area of the circle we can't find the area of a slice, so there must be another way. The first trick is to stack the slices as below :

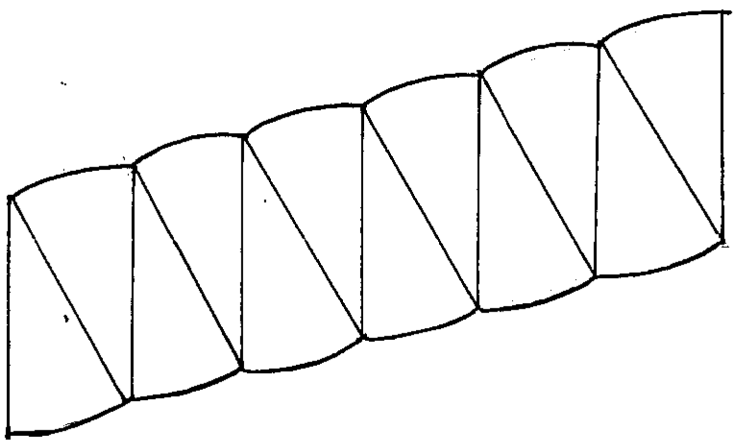


This looks like a sort of parallelogram 

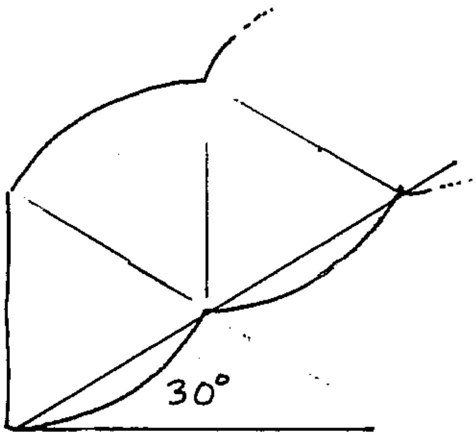
but with scalloped edges on the top on the bottom.
Note that the height of this scalloped "parallelogram"
is just the radius "r" of the circle, while the
lower edge has total length πr since it is half
the circumference.

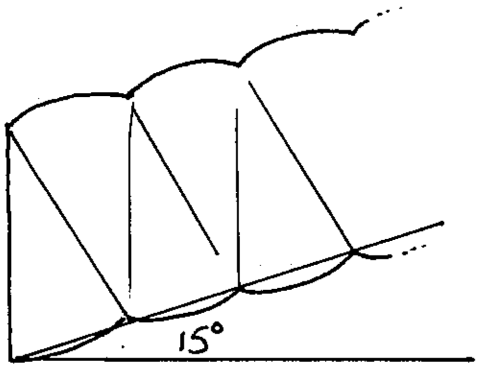
The circle above we cut into six slices. Suppose
we instead cut it into twelve slices and stack the slices
as before:





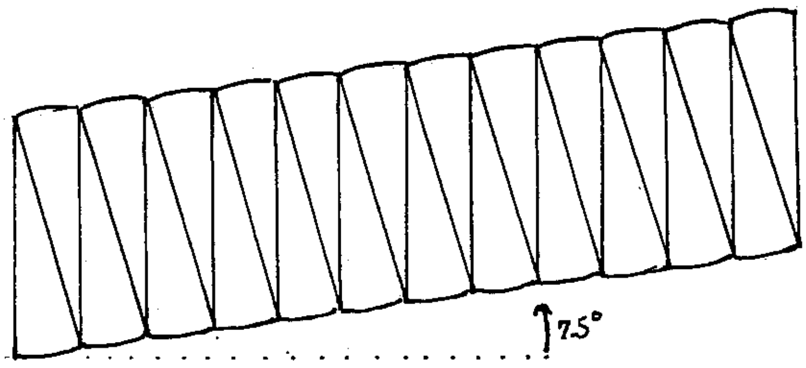
Again we get a scalloped "parallelogram", and again its height is r and the total length of the lower edge is πr, but there are two important changes. Our first "parallelogram" angled up from the horizontal at 30° while the second was only 15° off the horizontal.





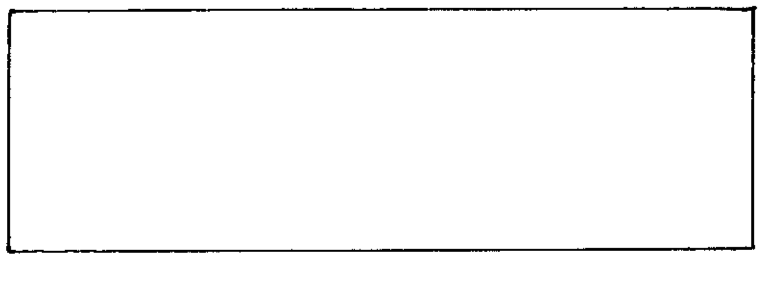
Also, the scalloping is much less pronounced in the second picture, that is the lower edge is closer to being a straight line. Remember that these two "parallelograms" have the same area, namely the area of the original circle.

We repeat the process once more. Cut the circle into twenty four wedges and arrange the wedges in a row as before.



This is the last scolloped "parallelogram" we will draw. Note that it is tilted up from the horizontal by only 7.5°, and remember that the height is the radius r, the lower edge has total length πr, and the area is the area of the circle (which we are trying to find).

Here now is the key new idea. These scolloped "parallelograms" are looking more and more like an actual rectangle.

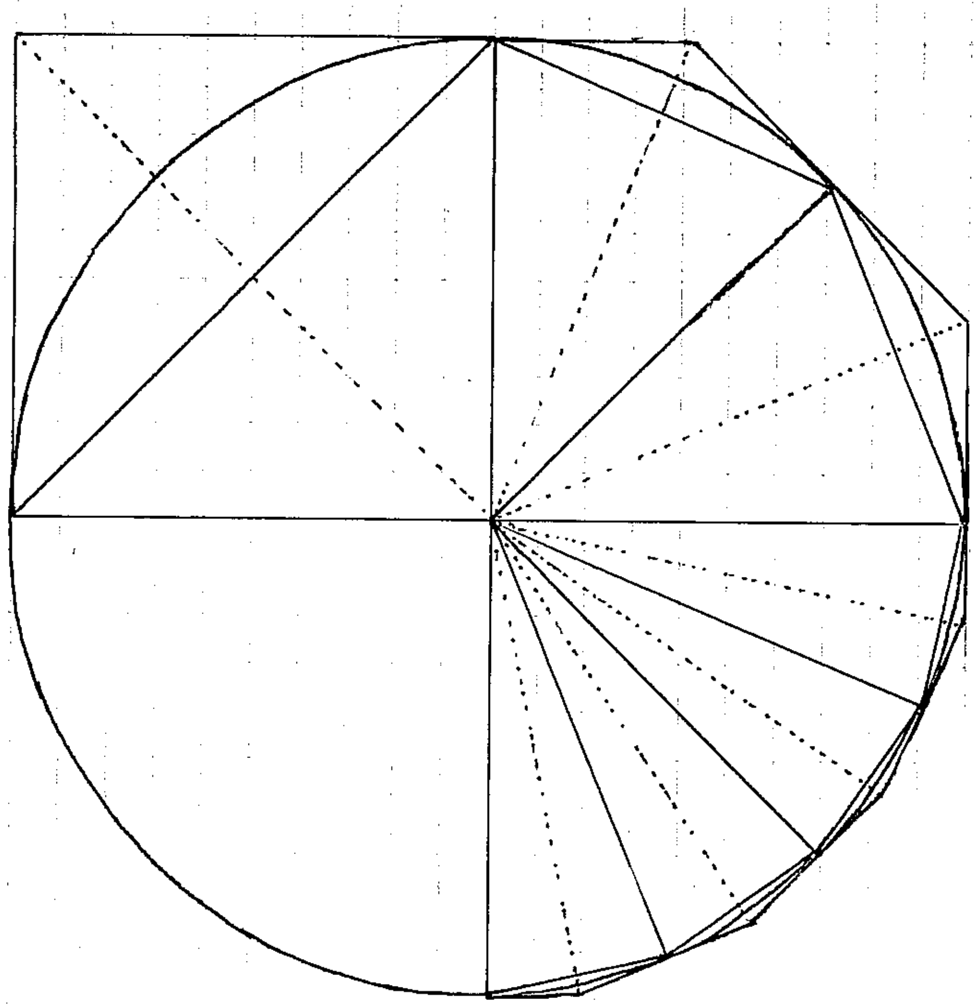


Certainly the scolloped "parallelograms" will never actually reach the rectangle, no matter how many wedges into which we cut the circle, but the scolloped "parallelograms" will get closer and closer to the rectangle, and can be made as close as you want. In particular, it is easy to see that the angle of inclination, which we've already see go from $30°$ to $15°$ to $7.5°$, is just $\frac{180}{n}$, where n is the number of wedges. If we use a billion wedges the angle will be $\frac{180}{1000000000} =$ $.00000018$ degrees. Note also that the scolloping will never pass below the horizontal since the horizontal is tangent to the arc in the bottom left. Now the lower edge of each scolloped "parallelogram" has length πr as we have seen. Those lower edges are getting closer and closer to the horizontal edge of the rectangle, so the width of the rectangle must also be πr. Its height is still r, the height of the scolloped "parallelograms", so the area of the rectangle is width times height which is $\pi r \cdot r = \pi r^2$. Finally, the areas of the scolloped "parallelograms" are all the same, namely the area of the original circle, and they are getting closer and closer to the area of the rectangle, so the area of the circle must also be πr^2, and we are done!

This is the method of "successive approximation". We did not calculate the area of the circle directly; we made finer and finer approximations of the area. Since these approximations are getting closer and closer to πr^2, that must be the actual area.

Archimedes used this method to solve a plethora of similar problems, finding the area of an ellipse, the area under a parabola, the volume of a cone, etc. In 1667 Newton used the same method, combined with Descartes' coordinate geometry, to invent the calculus and discover all the laws of classical physics, the fruition of the seeds planted by the Greeks two thousand years before.

Supplement: Archimedes' Proof



Archimedes did not cut the circle into wedges as I did. He used circumscribed and inscribed polygons as drawn above. The area of the circle is between the areas of the circumscribed and inscribed polygons, but as we make polygons with more and more sides the inner and outer areas both approach πr^2. Since the area of the circle is between πr^2 and πr^2, it must be πr^2.