The Infinitesimal Calculus

During the Renaissance in western Europe, Science and Mathematics began to burst from their seams. The laws of physics as laid out by Aristotle were challenged by Bacon, Copernicus, and Galileo, while Kepler used new astronomical data to develop a new theory of planetary motion. All that was needed to complete this revolution in science was a mathematical description of what was going on. Unfortunately the necessary mathematics had not yet been discovered

Mathematics had gone through enormous changes by the middle of the seventeenth century. A classical Greek would not have recognized the subject. Modern algebraic notation, logarithms, trigonometry, and functions only made their appearance at the end of the sixteenth century. Most importantly Descartes introduced his system of graphs and equations of curves, which gave an algebraic interpretation of geometry and a geometric interpretation of algebra. Still mathematicians were thwarted in their efforts to explicate the new cosmology. They kept banging into two problems pointed out by Zeno two-thousand years earlier.

The problem of infinite sums

Zeno said that a line has no length because the line is made up of points, and each point has no length. You cannot add up nothings and get something. Now Zeno knew full well that lines do have length, but he challenged philosophers and mathematicians to explain this seeming paradox. To understand the problem we draw a section of the number line with the points visualized as little squares (even though we cannot actually see a single point):

Clearly the total length of this segment is five, so the sum of the widths of the points should be five. But one of the basic tenets of geometry is that a point has no width, that is the width of a point is zero, so how can they add up to five? Of course I have only drawn a few dozen points and there are actually infinitely many points between 3 and 8. The problem may be put this way: If you add fifty zeros you get zero, if you add five million zeros you get zero, but if you add infintely many zeros you can get five!

This just shows that infinite arithmetic does not work like kindergarten arithmetic. We cannot base our understanding of infinite sums on our personal experience of the finite world we live in. The first problem of calculus is how to calculate an infinite sum of quantities that we cannot actually see.

In order to avoid writing and rewriting the phrase "width of a single point" we will name this quantity

If x is a point then dx represents its width.

Back in kindergarten when I first learned about numbers I had a set of wooden blocks with numbers written on them, and if someone had asked me to form the "number line" I would have put the blocks in a row like this:

$$\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\ \ldots$$

At that moment I would have said the width of each block was one, that is $dx = 1$ for each number x. Later in life I learned about one-place decimals and my number line looked like this:

$$\boxed{0}\boxed{.1}\boxed{.2}\boxed{.3}\ \ldots\ \boxed{.9}\boxed{1}\boxed{1.1}\boxed{1.2}\ \ldots$$

Then I would have said the width of each point was $.1$, that is $dx = .1$ for each number x. My blocks were smaller.

Now I know there are infinitely many points on the line, but I still think of them as little wooden blocks, and sometimes I can tell you how long a string of them is. The first calculation of an infinite sum is

> The sum of all the values dx, one such value for every x between a and b, is $b-a$
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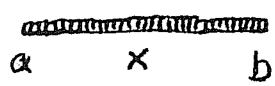
> 
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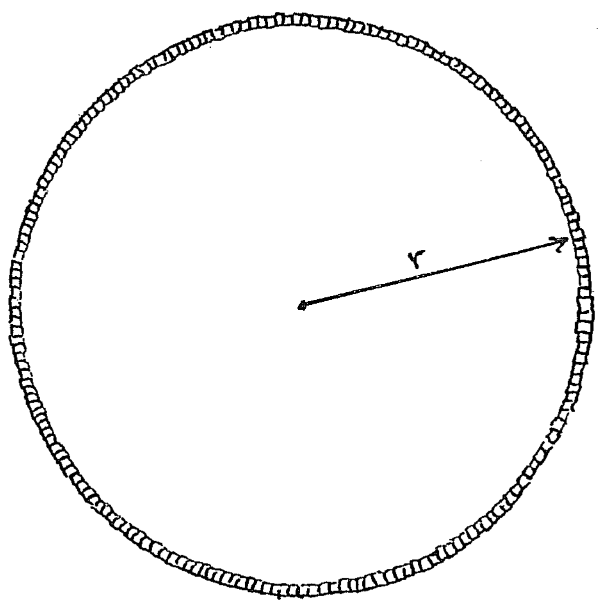
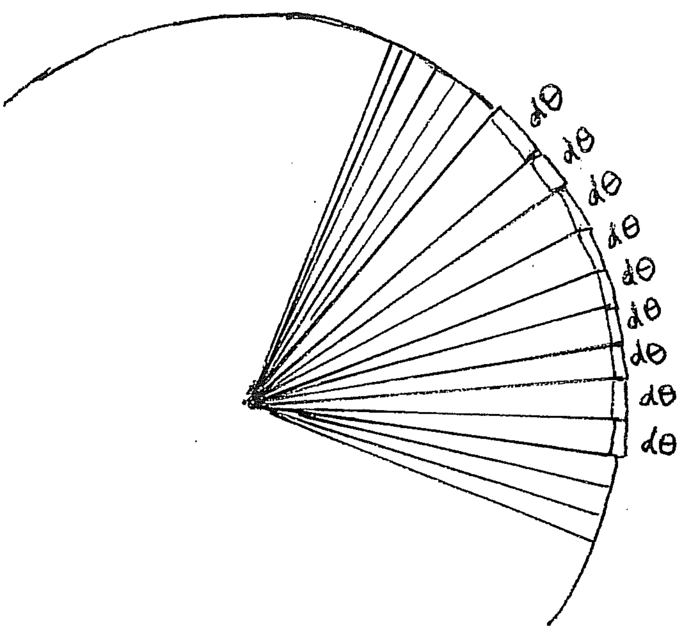
Thinking in terms of infinite sums gives a beautiful way of calculating the area of a circle. Start with a circle of radius r and think of the circumference of the circle as a chain of little points



Here we have only drawn a couple of hundred points just to get the idea. Now suppose each point has width $d\theta$ and draw a little triangle to the center of the circle (of course we have drawn the triangles much bigger than they actually are).



It should be clear that the total area of the circle is the sum of the areas of the little triangles. Now each triangle has base $d\theta$ and height r, so each triangle has area $\frac{1}{2} r\,d\theta$ and

$$\text{Area of circle} = \tfrac{1}{2} r\,d\theta + \tfrac{1}{2} r\,d\theta + \tfrac{1}{2} r\,d\theta + \dots$$

where there is one triangular area $\frac{1}{2} r\,d\theta$ for each point on the circumference. This is an infinite sum, but it is an infinite sum we can calculate:

$$\text{Area} = \tfrac{1}{2} r\,d\theta + \tfrac{1}{2} r\,d\theta + \tfrac{1}{2} r\,d\theta + \tfrac{1}{2} r\,d\theta + \dots$$

$$= \tfrac{1}{2} r\,(d\theta + d\theta + d\theta + d\theta + \dots)$$

We are not at all sure what one $d\theta$ is, but it is clear that if you add together all the lengths $d\theta$ you get the circumference of the circle, so

$$\text{Area of circle} = \tfrac{1}{2} r\,(\text{circumference}) = \tfrac{1}{2} r\,(2\pi r) = \pi r^2$$

which is the formula we all memorized in kindergarten without asking why it is true.

Before continuing with illustrations of infinite sums we will introduce some useful (and famous) notation. We have already seen that

> The sum of all the quantities dx for every x between $x=a$ and $x=b$ is $b-a$

This phrase is long and awkward, so we will use an elongated "s", written $\int$, for the words "sum of all".

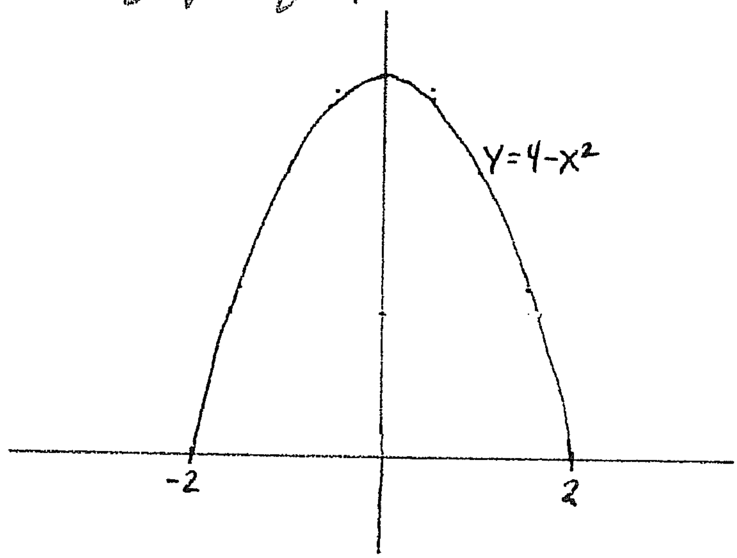
> $\int dx$ means "sum of all the dx"

We want to start the summation with $x=a$ and keep adding until we get to $x=b$. We put the starting point on the bottom of the summation sign and the final point on the top yielding this abbreviation of the phrase above.

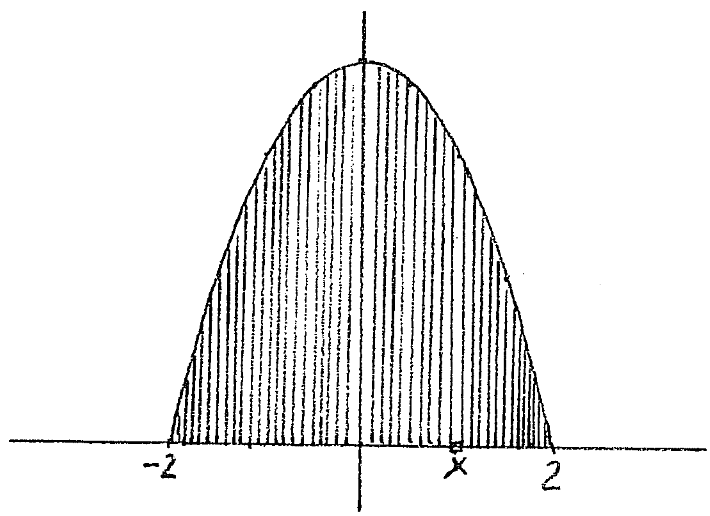
$$\boxed{\int_a^b dx = b-a}$$

Calculation of Area

Just as we used an infinite sum to find the area of a circle, infinite sums can be used to find areas bounded by other curves. For example, consider the region bounded by the graph of $y = 4 - x^2$ and the x-axis:



We want to find the area of this region. Extend each point on the x-axis up to the curve as below:



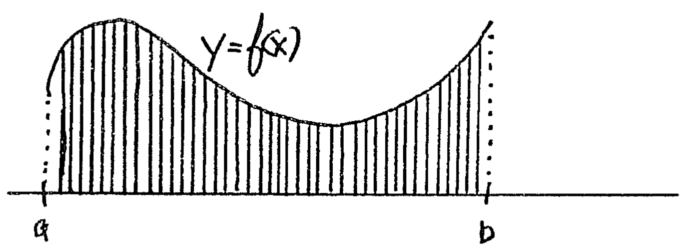
This cuts the total area into many tall thin trapezoids. To find the total area we need only find the area of each trapezoid and then add them up.

Look at one of the trapezoids at a point x between -2 and 2. The width of the trapezoid is the width of the point x, namely dx, and the height of the trapezoid is given by the equation $y = 4 - x^2$. Since the area of a trapezoid is height times width the area of the trapezoid at x is $y\,dx$, or $(4-x^2)\,dx$.

How many trapezoids are there? Infinitely many, one for each point x between -2 and 2, so the total area under the curve is the infinite sum

$$\int_{-2}^{2} (4-x^2)\,dx$$

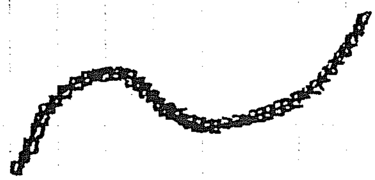
We could have started with any curve $y=f(x)$ and any two endpoints $x=a$ and $x=b$



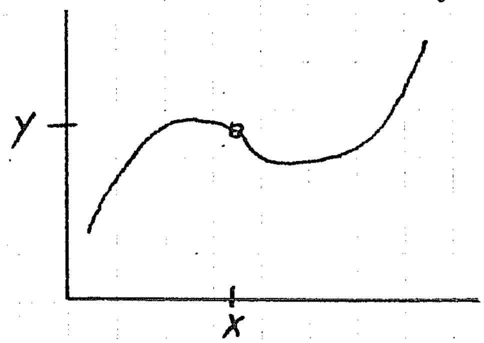
$$\text{Area} = \int_a^b y\,dx$$

Calculation of Length

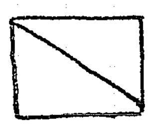
We can also use infinite sums to find the length of a curve. Picture a curve as a string of points



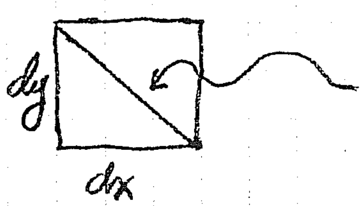
To find the length of the curve we just find the bit of length as the curve goes through each point and then add them all up. Look at a single point (x,y) on the curve



The point (x,y)



The width of the point (x,y) is just dx, while its height is dy, the width of the point y on the y-axis. We can now apply the Pythagorean Theorem to find the length of the curve at the point (x,y)



length $= \sqrt{dx^2 + dy^2}$

Hence to find the total length of the curve we calculate the sum $\int \sqrt{dx^2 + dy^2}$. This is an infinite sum because there are infinitely many points on the curve.

Infinite sums can also be used to find volumes, surface area, center of mass, and many other useful quantities. In each case the basic idea is the same — you break up the problem into little pieces, calculate each little piece and then add them all up. Now we only have to figure out how to compute the value of such a sum.

The key is to step back and think about kindergarten sums, that is pretend that you only use whole numbers and that $dx = 1$. In that case we find, for example,

$$\int_2^7 x^2 dx = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

Here I add all the values x^2, one such value for every x between 2 and 7. Now this is not the true value of the sum $\int_2^7 x^2 dx$ because I'm supposed to use <u>every</u> x between 2 and 7, not just the whole numbers, but thinking of kindergarten sums should make the following clear

<u>Basic summation formulas</u>

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \quad \text{if } c \text{ is constant}$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

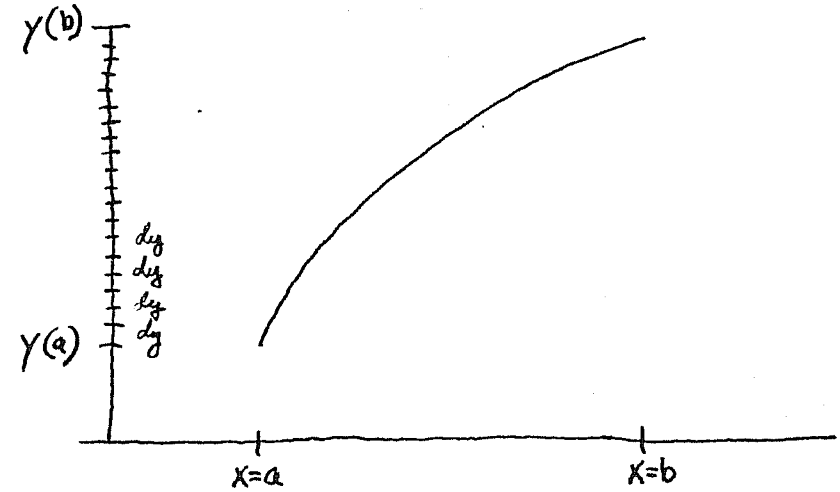
The last formula says (in plain English) "if you add $f(x)dx$ from $x=a$ to $x=b$, then add them from $x=b$ to $x=c$, that's the same as adding all the way from a to c." Even more simply: "if you add from

$x=2$ to $x=5$, then add from $x=5$ to $x=13$, that's the same as adding from $x=2$ to $x=13$." The other formulas above are equally obvious. The next formula is a bit more difficult, but it gives the key to the entire subject

The fundamental theorem of calculus

> If y is a continuous function of x
>
> then $$\int_{x=a}^{x=b} dy = y(b) - y(a)$$

"Continuous" just means there are no holes or breaks in the graph of y. The fundamental theorem comes from looking at the graph of y and asking where the values dy are, and what happens if you add them up:



The quantities dy are just the widths of the points on the y-axis, and those widths add up to the distance from $y(a)$ to $y(b)$, which is $y(b)-y(a)$.

EXAMPLE: Suppose $y=x^2$ is our function. The fundamental theorem says $\displaystyle\int_{x=2}^{5} dy = y(5)-y(2) = 5^2-2^2 = 21$, so here is a sum we can actually evaluate.

The problem with this example is that I don't know what dy is, that is given that $y=x^2$ or any other function of x, how do I find dy? As we saw when we looked at the calculation of length the value of dy will change for different points on the curve, so dy is a function of x, a new function derived from y.

Integrals and differentials

Write the fundamental theorem again: $\displaystyle\int_{x=a}^{x=b} dy = y(b) - y(a)$

Now we make two crucial changes. We will assume $y(a) = 0$, and instead of summing from $x = a$ to a specific $x = b$ we will replace b with an arbitrary x. We get

$$\int_a^x dy = y(x)$$

If you think of dy as y chopped up into little pieces, this makes perfect sense. It says if you chop y into little pieces and then add them up you get y back.

The word "differential" means the difference or separation between two things. If you take a whole thing and chop it up you get a difference, so we say

dy is the differential of y

Conversely, "integral" means "whole" or "unbroken" (it comes from the Latin word "intangere" = "untouched").

Hence to "integrate" means to put things together, so we say

$$\int_a^b dy \text{ is the "integral" of } dy \text{ from } a \text{ to } b$$

If we put an arbitrary x as the top boundary of the summation, then the function $\int^x dy$ is just called the integral of dy, and the fundamental theorem says if y is a continuous function of x then the integral of the differential of y is just y itself.

A bit of notation: The calculation $y(b)-y(a)$ comes up so often (because of the fundamental theorem) that we want to shorten up our notation and write

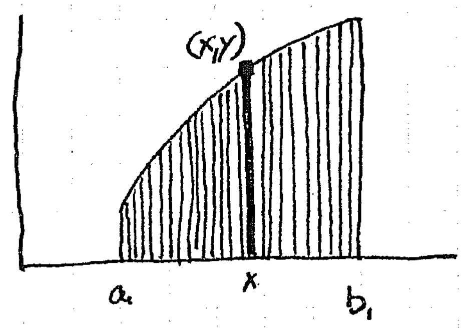
$$y(b)-y(a) = y(x)\Big|_a^b$$

Thus our very first calculation of an infinite sum is written

$$\int_a^b dx = x\Big|_a^b = b-a$$

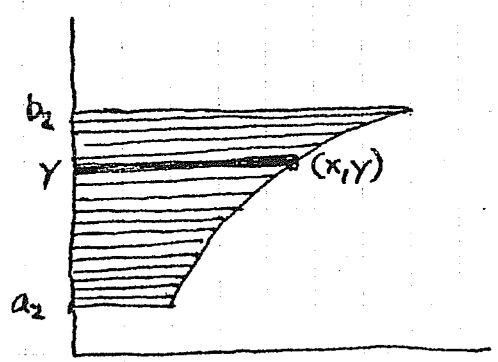
Summation by parts

If y is a function of x, we have already seen that we can use an infinite sum to calculate the area under the graph of y



$$\text{Area} = \int_{a_1}^{b_1} y\,dx$$

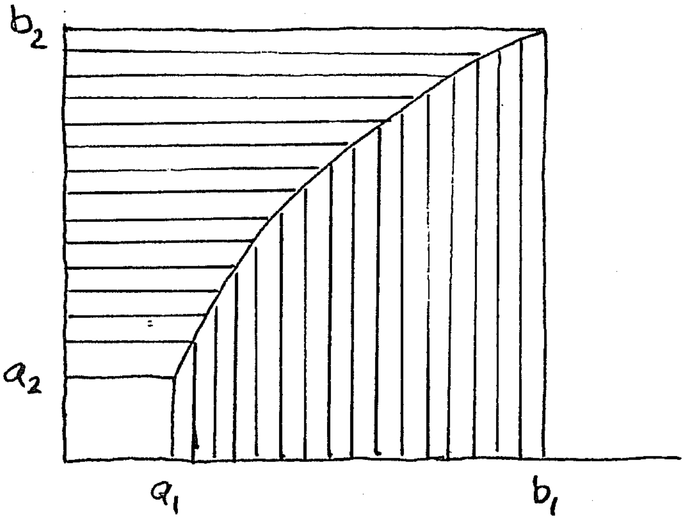
The darkened trapezoid has height y and width dx. Now consider the area to the left of the curve, i.e. between the curve and the y-axis



$$\text{Area} = \int_{a_2}^{b_2} x\,dy$$

The darkened trapezoid has height x (measured sideways) and width dy, and the area is the sum of all such trapezoids.

Now put the two pictures together



The shaded regions added together is clearly the area of the big rectangle less the area of the little rectangle

$$\int_{a_1}^{b_1} x\,dy + \int_{a_2}^{b_2} y\,dx = b_1 b_2 - a_1 a_2 = xy\Big|_a^b$$

Leaving off the boundary values we get the "summation by parts" formula

$$\boxed{\int x\,dy + \int y\,dx = xy}$$

This can also be written

$$\int (x\,dy + y\,dx) = xy$$

Now the fundamental theorem says

$$\int d(xy) = xy$$

so we conclude

$$\boxed{d(xy) = x\,dy + y\,dx}$$

This is known as the "Product Rule for differentials", and it allows us to compute differentials for many important functions. Since differentials and integrals are intimately related this allows us to compute many of our infinite sums.

Example: Consider the very simple function $y = x$. The formula above gives

$$d(x^2) = d(x\,x) = x\,dx + x\,dx = 2x\,dx$$

Therefore $\int 2x\,dx = \int d(x^2) = x^2$. Dividing by 2 gives

$$\int x\,dx = \frac{1}{2}x^2$$

Similarly $d(x^3) = d(x \cdot x^2) = x\,d(x^2) + x^2\,dx$

$$= x \cdot 2x\,dx + x^2\,dx = 3x^2\,dx$$

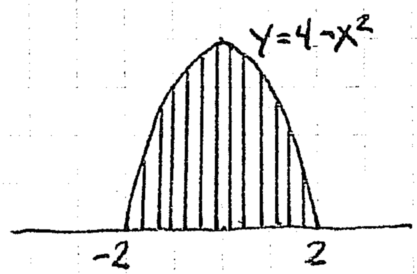
Therefore $\int 3x^2\,dx = \int d(x^3) = x^3$. Dividing by 3 gives

$$\int x^2\,dx = \frac{1}{3}x^3$$

Proceeding in this manner we get the power rule for integrals:

$$\boxed{\int x^n\,dx = \frac{1}{n+1}x^{n+1}}$$

This formula allows us to evaluate many infinite sums. Go back to the example of finding the area under the parabola $y = 4 - x^2$



We saw before this area is $\int_{-2}^{2} y\,dx$ but we did not know how to calculate this infinite sum. Now we can say

$$\int_{-2}^{2} y\,dx = \int_{-2}^{2} (4-x^2)\,dx = \int_{-2}^{2} 4\,dx - x^2\,dx = 4x - \frac{1}{3}x^3 \Big|_{-2}^{2}$$

$$= \left(4\cdot 2 - \frac{2^3}{3}\right) - \left(4\cdot(-2) - \frac{1}{3}\,\frac{(-2)^3}{3}\right) = \frac{32}{3}$$

The problem of instantaneous velocity

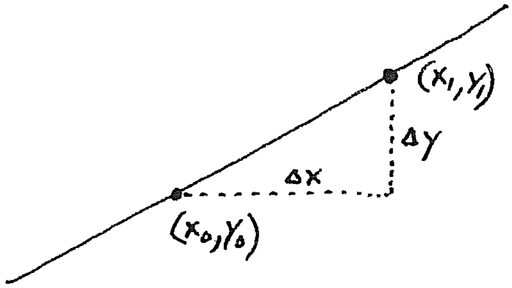
We now turn to the second of Zeno's paradoxes. He said "The flying arrow does not move at any instant because no time passes at a given moment. Therefore the flying arrow never moves." Of course Zeno knew that a flying arrow does move, but he challenged philosophers and mathematicians to make sense out of "velocity at an instant".

The seventeenth century brought renewed interest in this problem because of Galileo's new physics with its emphasis on velocity and acceleration. Descartes saw that the problem of velocity is the same as the problem of the slope of a curve at a given point. To find velocity

$$\text{Velocity} = \frac{\text{CHANGE IN DISTANCE}}{\text{CHANGE IN TIME}} = \frac{\text{METERS}}{\text{SECONDS}}$$

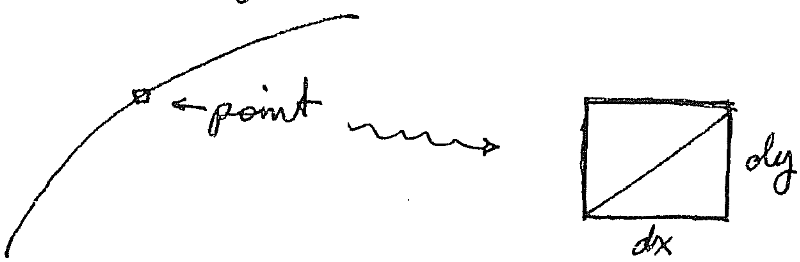
$$\text{Slope} = \frac{\text{RISE}}{\text{RUN}} = \frac{\text{CHANGE IN } Y}{\text{CHANGE IN } X}$$

Zeno's question is "how does this make sense "at an instant", when there is no change in time?" We will look at the corresponding problem for slope: Given the graph of a function, what does slope "at a single point" mean? We know that if the graph is a straight line we can use two points and the slope formula:



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

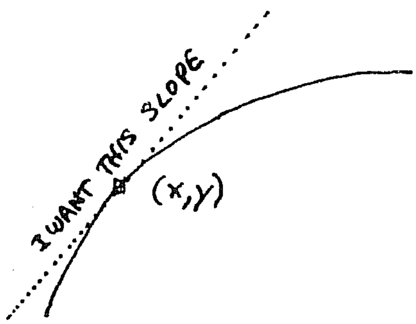
But how can we do this with only one point? This becomes important when we look at a curve instead of a straight line. Pick a point on the curve and imagine the curve at that point:



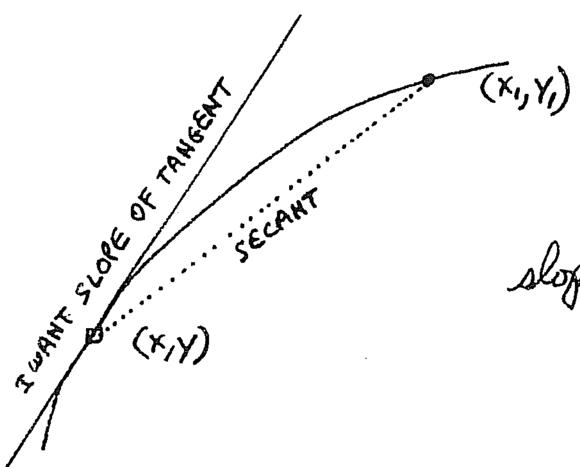
It seems to make sense to say the slope is just $\frac{dy}{dx}$, but when we try to find dy and dx we are back to the same problem we had with infinite sums. The only reasonable value we can give to dx or dy

would be zero, so $\frac{dy}{dx} = \frac{0}{0}$ which is meaningless. Hence we cannot directly calculate the slope of a curve.

Newton and Leibnitz independently discovered a method for computing the slope using Archimedes's technique of successive approximation. Start with a function y of x and draw its graph, and suppose I want to find the slope at some point (x, y) on the curve:

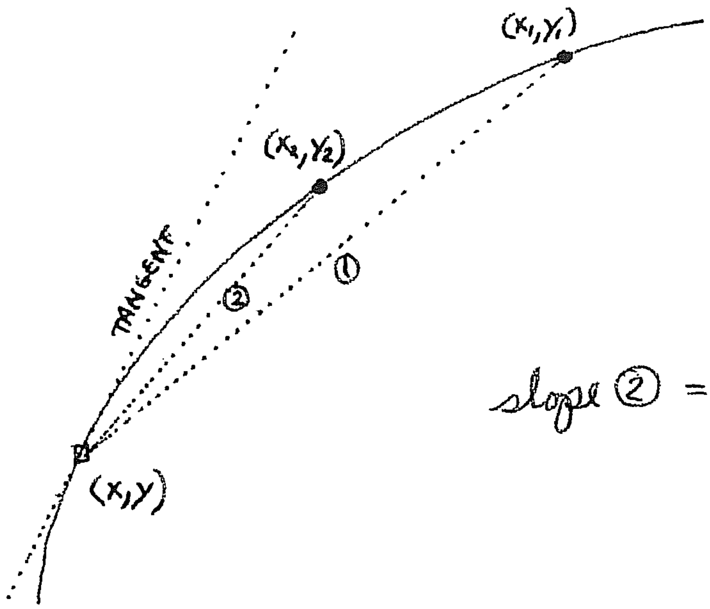


What we want is the slope of the tangent line at (x, y). If we choose a second point (x_1, y_1) on the curve we can use the slope formula to find the slope of the segment from (x, y) to (x_1, y_1) (a "secant" of the curve)



$$\text{slope of secant} = \frac{y_1 - y}{x_1 - x}$$

Obviously the slope of the secant is not what we want, but suppose we choose another point (x_2, y_2) on the curve that is closer to (x,y) and draw the secant from (x,y) to (x_2, y_2)



$$\text{slope ②} = \frac{y_2 - y}{x_2 - x}$$

We want the slope of the tangent. The slope of secant ① is no good. The slope of secant ② is still not right but it is closer to what I want. If I move (x_2, y_2) closer to (x,y) the slope of the secant will be even closer to the slope of the tangent

> As (x_1, y_1) approaches (x,y)
>
> $\frac{y_1 - y}{x_1 - x}$ approaches the slope of the tangent at (x,y)

Of course the secants will never actually reach the tangent, but they will never move past the tangent either. We say that the tangent is the "limit" of the secants, that is the thing they are getting closer and closer to though they may never reach it. The slope of the tangent, denoted y', is the limit of the slopes of the secants as x_1 approaches x. This is written

$$\boxed{y' = \lim_{x_1 \to x} \frac{y_1 - y}{x_1 - x}}$$

Since we are looking at a curve rather than a line, the slope is different at different values of x. Hence y' is a function of x, a new function derived from y which tells you the slope of y at any point on the curve. This new function y' is called the "derivative" of y.

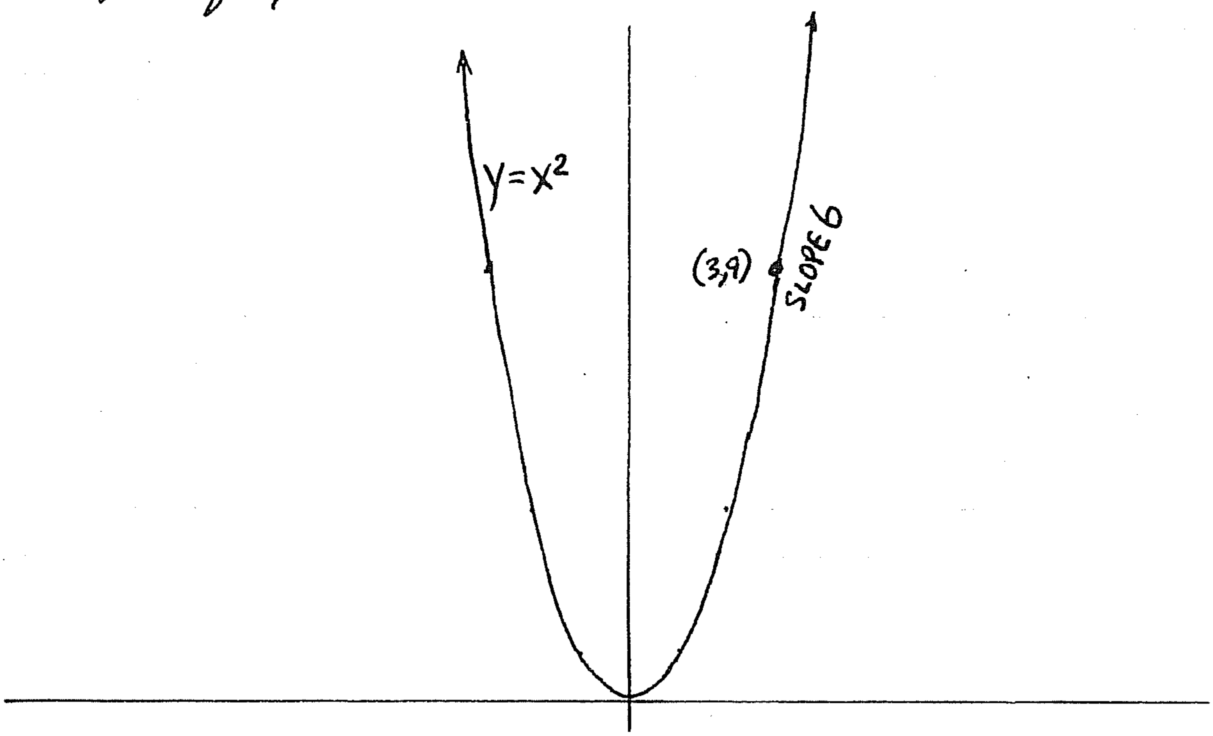
The derivative y' tells me the slope $\frac{dy}{dx}$ at any point on the graph of y.

Example: Suppose our curve is the parabola $y = x^2$.
To find the derivative y' we find the slope of the secant from (x, x^2) to (x_1, x_1^2) and then find the limit as x_1 approaches x

$$\text{slope of secant} = \frac{x_1^2 - x_2^2}{x_1 - x} = \frac{(x_1 - x)(x_1 + x)}{x_1 - x}$$

$= (x_1 + x)$. As x_1 gets closer to x then $x_1 + x$ gets closer to $x + x = 2x$, so $y' = 2x$

This tells me the slope of the parabola at any point I'm interested in. If we want the slope at the point $(3,9)$ we just shoot $x = 3$ into y' : $y'(3) = 6$ so the slope of $y = x^2$ at $(3,9)$ is 6.



Recall that the slope is just the quotient $\frac{dy}{dx}$, so we have found that at the point $(3,9)$ on the graph of $y = x^2$ the quotient $\frac{dy}{dx} = 6$. The derivative y' gives me a tool for finding dy over dx without finding dy or dx alone.

The derivative then gives the solution to the problem of instantaneous velocity. Suppose $y(t)$ gives you distance (say in meters) at time t (say in seconds)

Then $y'(t) = \frac{dy}{dt} = \frac{\text{meters}}{\text{second}} = \text{velocity}$. Similarly

the derivative of the derivative y'' gives acceleration, so the notion of the derivative was exactly what Newton needed when he reinvented physics in the seventeenth century.

The connection

Integrals (infinite sums) and derivatives are connected by the fundamental theorem. Suppose $\frac{dy}{dx} = y'$
Then the differential $dy = y'dx$ and the fundamental theorem says

$$\int y'dx = y$$

Hence if we want to evaluate an infinite sum of the form $\int f(x)dx$ and we recognize $f(x) = y'$ for some y that we know, we can evaluate the infinite sum. This is exactly the reverse of the problem of finding the slope. To find slope $\frac{dy}{dx}$ you start with a function y and you find the derivative y'. To find an integral $\int f(x)\,dx$ you presume $y' = f(x)$ and you try to find y. The terminology is

y' is the derivative of y

y is the antiderivative of y'

Here then is our last restatement of the fundamental theorems of calculus:

> If you can calculate an antiderivative of $f(x)$
> you can evaluate the integral $\int f(x)dx$

Much of the standard calculus course now becomes a long list of rules and tricks for cranking out derivatives and antiderivatives. These, in turn become the basis for much of science, engineering, economics, etcetera. But this hides the true nature of a deep and beautiful subject.

Calculus is really about infinitesimal arithmetic, that is calculating sums and quotients and products of quantities that are infinitesimally small or infinitely numerous. Today we talk about "differential calculus" and "integral calculus", but for hundreds of years the subject was "infinitesimal calculus". This nomenclature fell out of favour with the realization that "infinitesimal" is not well-defined, but it is still the best description of the subject.

In the seventeenth century calculus was invented to explicate the new physics and cosmology. Using calculus Newton was able to explain the reason the Earth rotates around the Sun. But the solution lies in the world of the infinitesimally small and the infinitely many, a world which otherwise would be closed to our poor little finite minds.

Tom Fox -2013