Mathematical induction

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1 Recursive (inductive) definition [*Setting up the dominos.*]

A recursive or inductive ('hereditary' in *GEB*) definition consists of:

1. A *base* clause:
   Defines the basic elements of our set.
2. One or more *inductive* clauses:
   Tell us how to generate complex elements from parts.
3. And a *final* clause:
   States that all elements are either basic elements or generated by the inductive clause.

Example. The natural numbers $\mathbb{N}$ can be defined as follows:

1. (Base clause) 0 is a natural number.
2. (Inductive clause) If n is a natural number, then the successor of n (formally: $s(n)$ or $n+1$) is a natural number, too.
3. (Final clause) Nothing else is a natural number.

For an inductive definition of well-formed strings of propositional logic, see *GEB*, p. 182; for an inductive definition of numerals, see *GEB*, p. 213.

Watch out for recursive definitions in *GEB* and in your other courses!
Examples: Typographical systems, formulas, derivations, proofs, lists.

2 Mathematical induction [*Making all dominos fall.*]

The method of mathematical induction (*GEB*: 'an argument based on heredity') may only be applied to an inductively defined domain. To prove that a property holds for all elements in the domain we proceed as follows:

1. *Base case:*
   Show that the property holds for the basic element(s).
2. *Induction step:*
   Assume that the property holds for some element n (Induction Hypothesis).

   Show that is also holds for any element generated from n by the inductive clauses.
3. *Conclusion:* The property holds for *all* elements.

The above is also called *weak* mathematical induction.

Sometimes one would like to assume the induction hypothesis not only for the previous element, but also for all smaller elements. This leads to a logically equivalent method of proof, called *strong*, or *complete*, mathematical induction:

1. *Induction step:*
   Assume that the property holds for all elements less than n (Induction Hypothesis).

   Show that it also holds for any element n, generated by the inductive clauses.

   Induction step (Alternative version):
   Assume that the property holds for all elements less or equal than n (Induction Hypothesis).

   Show that it also holds for any element generated from n by the inductive clauses.
2. *Conclusion:* The property holds for *all* elements.

Note that strong induction does not require to prove the base case separately, because one has to take care of this in the induction step (see Example 5.5).

Also, some books refer to induction on the natural numbers as 'mathematical induction' and call induction on other inductively defined objects *structural induction*.

3 Proofs by mathematical induction

The proofs of all the following examples will have the *same structure*. Try to remember it!

For any proof by mathematical induction:

- Always state the *claim* that you are proving.
- Always say whether you are showing the *base case* or the *induction step*.
- Always state the *induction hypothesis*.
- Always make clear where you *apply* the induction hypothesis.
- Always state the *conclusion*.

If any of these are missing in your homework or exams, points will automatically be deducted.

4 Formalizations of mathematical induction

More on this later in the course!

For the natural numbers, the principle of *(weak) mathematical induction* can be formalized as:

$$\Big(\underbrace{P(0)}_{\textit{Base case}} \wedge \underbrace{\forall n\,\big[\overbrace{P(n)}^{\textit{IH}} \to P(n+1)\big]}_{\textit{Inductive step}}\Big) \to \underbrace{\forall x\ P(x)}_{\textit{Conclusion}}.$$

The principle of *strong induction*, which is equivalent to the above, is:

$$\underbrace{\forall n\big[\overbrace{\big(\forall m < n.\, P(m)\big)}^{\textit{IH}} \to P(n)\big]}_{\textit{Inductive step}} \to \underbrace{\forall x\ P(x)}_{\textit{Conclusion}}.$$

Or, alternatively:

$$\underbrace{\forall n\big[\overbrace{\big(\forall m \leq n.\, P(m)\big)}^{\textit{IH}} \to P(n+1)\big]}_{\textit{Inductive step}} \to \underbrace{\forall x\ P(x)}_{\textit{Conclusion}}.$$

(For a proof that weak and strong induction are indeed equivalent, see Machover, *Set Theory, Logic, and their Limitations*, Cambridge University Press (1996), Theorems 3.2, 4.1, and 4.2.)

5 Examples

Remark: In general, the method of induction does not tell us how to arrive at the induction hypothesis, but only shows us that it holds for all elements of the domain.

5.1 $\sum_{i=1}^{n} i = n(n+1)/2$ (sum of the first n natural numbers)

Theorem 1 *For any natural number $n \geq 1$, 1+2+3+...+n $=$ $n(n+1)/2$.*

Proof: (By mathematical induction)

- **Base case:** The claim holds for $n = 1$: $1 = 1(1+1)/2$.
- **Induction step:**
  Induction hypothesis: Assume that the claim holds for some natural number n.
  Need to show that $\sum_{i=1}^{(n+1)} i = (n+1)((n+1)+1)/2$ also holds.
  Assume that $\sum_{i=1}^{n} i = n(n+1)/2$.

$$\begin{aligned} \sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{by induction hypothesis} \\ &= \frac{(n^2+n)+(2n+2)}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

- **Conclusion:** The claim holds for all natural numbers ≥ 1.

□

5.2 $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Theorem 2 *For any natural number n, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.*

Proof: (By mathematical induction)

- **Base case:** The claim holds for $n = 0$: $2^0 = 1 = 2^{(0+1)} - 1 = 2^1 - 1$.

- **Induction step:**
  Induction Hypothesis: Assume that the claim holds for some natural number n.
  Need to show that $\sum_{i=0}^{n+1} 2^i = 2^{(n+1)+1} - 1$ also holds.
  Assume that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{(n+1)} \\
&= \left(2^{n+1} - 1\right) + 2^{(n+1)} \quad \text{by induction hypothesis} \\
&= 2 \times \left(2^{n+1}\right) - 1 \\
&= 2^{n+2} - 1 \\
&= 2^{(n+1)+1} - 1
\end{aligned}$$

- **Conclusion:** The claim holds for all natural numbers.

□

5.3 $\sum_{i=0}^{n} i^3 = (n(n+1)/2)^2$ (sum of the first n cubes)

Theorem 3 *The sum of the first n cubes is equal to $(n(n+1)/2)^2$.*

Proof: (By mathematical induction)

- **Base case:** The claim holds for $n = 0$:
  $0^3 = 0 = 0^2$.

- **Induction step:**
  Assume that the claim holds for some natural number n (induction hypothesis) and prove that it also holds for $n+1$.
  Assume: $1^3 + 2^3 + 3^3 + \cdots + n^3 = (n(n+1)/2)^2$ (Induction Hypothesis).
  Need to show: $1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 = (n(n+1)/2)^2 + (n+1)^3$.

$$\begin{aligned}
&= (n+1)^2 \times ((n/2)^2 + (n+1)) \\
&= (n+1)^2 \times (n^2/4 + n + 1) \\
&= (n+1)^2 \times (n^2 + 4n + 4)/4 \\
&= (n+1)^2 \times (n+2)^2/4 \\
&= (n+1)^2 \times ((n+2)/2)^2 \\
&= \left((n+1) \times (n+2)/2\right)^2
\end{aligned}$$

- **Conclusion:** The claim holds for all natural numbers.

□

5.4 Divisibility by 7

Theorem 4 *For any natural number $n \geq 1$, 7 divides $8^n - 1$.*

Proof: (By mathematical induction)

- **Base case:** The claim holds for $n = 1$:

$$8^1 - 1 = 7.$$

- **Induction step:**
  Induction Hypothesis: Assume that the claim holds for some natural number n.
  Need to show that $8^{(n+1)} - 1$ is also divisible by 7.
  Assume, 7 divides $8^n - 1$, i.e., $\exists k \in \mathbb{N}$, such that

$$\begin{array}{rcll}
8^n - 1 &=& 7 \times k & \text{Induction Hypothesis.} \\
8 \times (8^n - 1) &=& 8 \times (7 \times k) & \text{Multiply both sides with 8.} \\
8 \times (8^n - 1) + 7 &=& 8 \times (7 \times k) + 7 & \text{Add 7 to both sides.} \\
(8^{(n+1)} - 8) + 7 &=& 8 \times (7 \times k) + 7 & \text{Multiply and law of exponents.} \\
8^{(n+1)} - 1 &=& (7 \times 8 \times k) + 7 & \text{Addition/Subtraction, comm. of } \times. \\
8^{(n+1)} - 1 &=& 7 \times (8 \times k + 1) & \text{Factoring out 7.} \\
8^{(n+1)} - 1 &=& 7 \times j & \text{Replace } 8 \times k + 1 \text{ by } j.
\end{array}$$

Therefore, 7 divides $8^{(n+1)} - 1$.

Conclusion: The claim holds for all natural numbers ≥ 1. □

5.5 Fibonacci numbers

The Fibonacci numbers $f_0, f_1, f_2, \ldots$ are defined by

$$\begin{array}{rcll} f_0 &=& f_1 = 1, & \text{and} \\ f_n &=& f_{n-1} + f_{n-2} & \text{for all } n \geq 2. \end{array}$$

Theorem 5 *For all $n \geq 0$, $f_n \leq (7/4)^n$.*

Proof: (By strong mathematical induction)

- **Induction step:**
  Induction Hypothesis: Let k be ≥ 0, and the claim holds for all numbers i, such that $0 \leq i < k$.

  Show: $f_k \leq (7/4)^k$.

  Three cases need to be distinguished:

  Case 1: $k = 0$.
  $f_0 = 1$ and $(7/4)^0 = 1$, so $f_0 \leq (7/4)^0$.

  Case 2: $k = 1$.
  $f_1 = 1$ and $(7/4)^1 = 7/4$, so $f_1 \leq (7/4)^1$.

  Case 3: $k > 1$.

$$\begin{array}{rcll} f_k &=& f_{k-1} + f_{k-2} & \\ &\leq& (7/4)^{k-1} + (7/4)^{k-2} & \text{(by Induction Hypothesis)} \\ &=& ((7/4)^{k-2} \times (7/4)) + (7/4)^{k-2} & \\ &=& (7/4)^{k-2} \times ((7/4) + 1) = (7/4)^{k-2} \times (11/4) & \\ &=& (7/4)^{k-2} \times (44/16) & \\ &\leq& (7/4)^{k-2} \times (49/16) = (7/4)^{k-2} \times (7/4)^2 & \\ &=& (7/4)^k & \end{array}$$

- **Conclusion:** The claim holds for all natural numbers.

□

5.6 Addition is associative

Suppose we defined the natural numbers with zero ('0') and successor function ('s') as follows: 0, $s(0)$, $ss(0)$, $sss(0)$, ...

Let addition ('+') on the numbers be defined as:

1. $0 + A = A$, and
2. $s(A) + B = s(A + B)$, for all natural numbers A and B.

Theorem 6 *Addition is associative, i. e. for all natural numbers $(A+B)+C = A+(B+C)$.*

Proof: (By mathematical induction on A)

- **Base case:** The claim holds of $A = 0$:
  $(0 + B) + C = B + C = 0 + (B + C)$.
- **Induction step:**
  Induction Hypothesis: Assume the claim holds for some natural number n and prove that it also holds for $n + 1$.

$$\begin{array}{rcll} \text{Assume: } (A + B) + C &=& A + (B + C) & \text{Induction Hypothesis.} \\ s((A + B) + C) &=& s(A + (B + C)) & \text{Apply s().} \\ s(A + B) + C &=& s(A) + (B + C) & \text{Rule 2.} \\ (s(A) + B) + C &=& s(A) + (B + C) & \end{array}$$

- **Conclusion:** The claim holds for all natural numbers.

□

(Challenging problem: In the same setup as above, prove that addition is commutative, i. e. $A + B = B + A$. Hint: Prove as a lemma that $A + 0 = A$.)