MATH 251, homework 3, due date Monday Jan 26.

Problem 1. Are the following maps linear? Justify your answer.

(i) $T\colon \mathbf{R}^3 \to \mathbf{R}^2,\ T((x_1, x_2, x_3)) = (1 + x_1, x_2)$;

(ii) $T\colon \mathbf{R}^3 \to \mathbf{R}^2,\ T((x_1, x_2, x_3)) = (x_3, x_1 + x_2)$;

(iii) $T\colon \mathbf{R}^3 \to \mathbf{R}^2,\ T((x_1, x_2, x_3)) = (x_3, x_1^2 + x_2^2)$;

(iv) $T\colon \mathbf{R}[t]_2 \to \mathbf{R}[t]_3,\ T(f(t)) = t^2 + f(t)$;

(v) $T\colon \mathbf{R}[t]_2 \to \mathbf{R}[t]_3,\ T(f(t)) = tf(t) + t^2 f'(t)$.

Problem 2. Let V be the subspace of $\mathcal{C}[0, 1]$ spanned by the vectors of the linearly independent sequence $B = (e^x, xe^x, x^2e^x)$. Let D be the differentiation operator on V. Compute ${}_B[D]_B$.

Problem 3. Let $T\colon U \to V, R\colon V \to W$ be linear maps between finite dimensional vector spaces U, V, W, with $\dim V = n$. Denoting $\mathrm{rk}(T) = \dim \mathrm{Im}(T)$ etc, prove

$$\mathrm{rk}(R) + \mathrm{rk}(T) - n \leq \mathrm{rk}(RT).$$

Hint: first prove

$$\dim \ker(RT) \leq \dim \ker(R) + \dim \ker(T).$$