# Goedel's Theorems and the Continuum Hypothesis

In the last section we saw that $\#\mathbb{N} < \#\mathbb{R}$. In order to deal with this and other problems dealing with infinite sets Georg Cantor and his contemporaries developed a general theory of sets – how to define them, how to measure their size. It was believed that any description of a collection of things defined a set, but this led to horrible contradictions that threatened the very foundations of mathematics (Russell's Paradox, for example).

In order to rectify these problems, mathematicians and logicians began to rethink the fundamental notions upon which mathematics is built. The idea was to do for all of mathematics what Euclid had done for geometry – find a minimal number of obviously true statements from which all other mathematical truth will follow by inextricable logic. There were two schools of thought on how this could be done. One could define sets and their properties and build from that, or one could start with the laws of logic. The latter approach was developed by Russell and Whitehead, culminating in their monumental "Principia Matematica", while set theory was set on a firm footing by Hilbert and his school. Today the set theory of Zermelo and Fraenkel is generally accepted as the starting point for modern mathematics.

Two questions remained: Could all mathematical "truth" be derived from the axioms of set theory, and how can we be sure that no further paradoxes can arise from these axioms?

Here is an apparently simple question we can ask about infinite sets: We saw in the last chapter that $\#\mathbb{N} < \#\mathbb{R}$. Can we find a set in between, i.e. can we find a set $S$ so that $\#\mathbb{N} < \#S < \#\mathbb{R}$? Cantor could not find such a set and came to believe such a set didn't exist. That there is no "number" between $\#\mathbb{N}$ and $\#\mathbb{R}$ is his famous "Continuum Hypothesis". In 1960 Paul Cohen shocked the world of mathematics by proving

> The Continuum Hypothesis is independent of the axioms of set theory.

The proof of this result is way beyond our scope, but we can still understand what it says. Think back to our discussion of geometry. From Euclid's first four axioms can we prove the parallel postulate? No, it is independent of the first four axioms. We can develop geometry assuming the parallel postulate is true, or we can develop geometry assuming the parallel postulate is false, and either choice leads to a perfectly valid geometric universe. The parallel postulate is neither true nor false in any universal sense. Cohen demonstrated that the same is true of the Continuum Hypothesis. You can accept it as true and build mathematics from that, or declare it to be false and go from there.

This seems very disturbing because the Continuum Hypothesis says something about the set of real numbers, which we know and love. We feel that we know all about the real numbers, but in fact they are only constructs of the human mind with properties that we can either give or take away.

It would be reasonable to accept the Continuum Hypothesis as a new axiom of set theory and then ask if we can now prove all other mathematical "truths", but this project is made hopeless by Kurt Goedel's incompleteness theorem (1931):

> In any mathematical system powerful enough to include arithmetic we can find statements that are independent, that are neither provable nor disprovable in that system

We will not try to explain how Goedel proved this theorem, but we must understand what it says: From no small number of accepted axioms can we determine the truth or falsity of all mathematical statements. Propositions that follow by deduction from the axioms will be true in any model of the axioms, but other statements will be true in some models and false in others. There will always be creative work to be done, new alternate mathematical universes to be built.

In one sense Goedel's theorem is frightening. We could work for years trying to prove something that is unprovable, but in another sense this only affirms the creative force needed to do mathematics. We are not uncovering God-given truths, we are building edifices of interrelations. Goedel's theorem causes no harm to the edifices we build, it just clarifies what it is we are doing.

Goedel also proved another theorem that has more profound philosophical implications:

> Within no mathematical system (that includes arithmetic) can we prove the consistency of that system.

We have to explain "consistency". Suppose we write down all the axioms for arithmetic, like $x+y=y+x$ and $(x+y)+z=x+(y+z)$ and $x(y+z)=xy+xz$, and suppose we add an axiom that says $(x+y)^2=x^2+y^2$. Very soon we will find that $0=2$, which contradicts the other axioms. Our axioms would be self contradictory, though it takes a bit of work to realize it. We say that this system of algebra is not consistent. It is like writing "$x=3$ and $x\neq 3$"; these statements are not consistent with each other.

If we write down the axioms for set theory, how can we prove there are no built-in self-contradictions, how can we prove our system is consistent? We can't. That's what Goedel's theorem says. We can diligently try to avoid inconsistencies, but if we do have a consistent theory of mathematics we will never be able to prove it.

Once again this leads to a terrifying idea. Perhaps everything we have written is nonsensical. This is very close to the old philosopher's question "how do we know we are not all mad, and our reasoning is not just the raving of lunatics?" We can't, but if we are mad it doesn't matter what we do, so we might as well try to be reasonable. In some sense we must fall back on the Pythagorean faith in the order of the universe and continue to search for meaning and truth in the world of mathematics.

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true"

— Bertrand Russell

"Although the whole of this life were said to be nothing but a dream and the physical world nothing but a phantasm, I should call this dream or phantasm real enough, if, using reason well, we were never deceived by it."

— G. Leibnitz