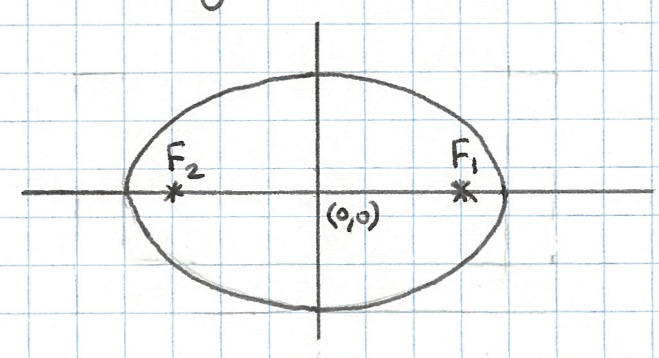
Equations of conic sections

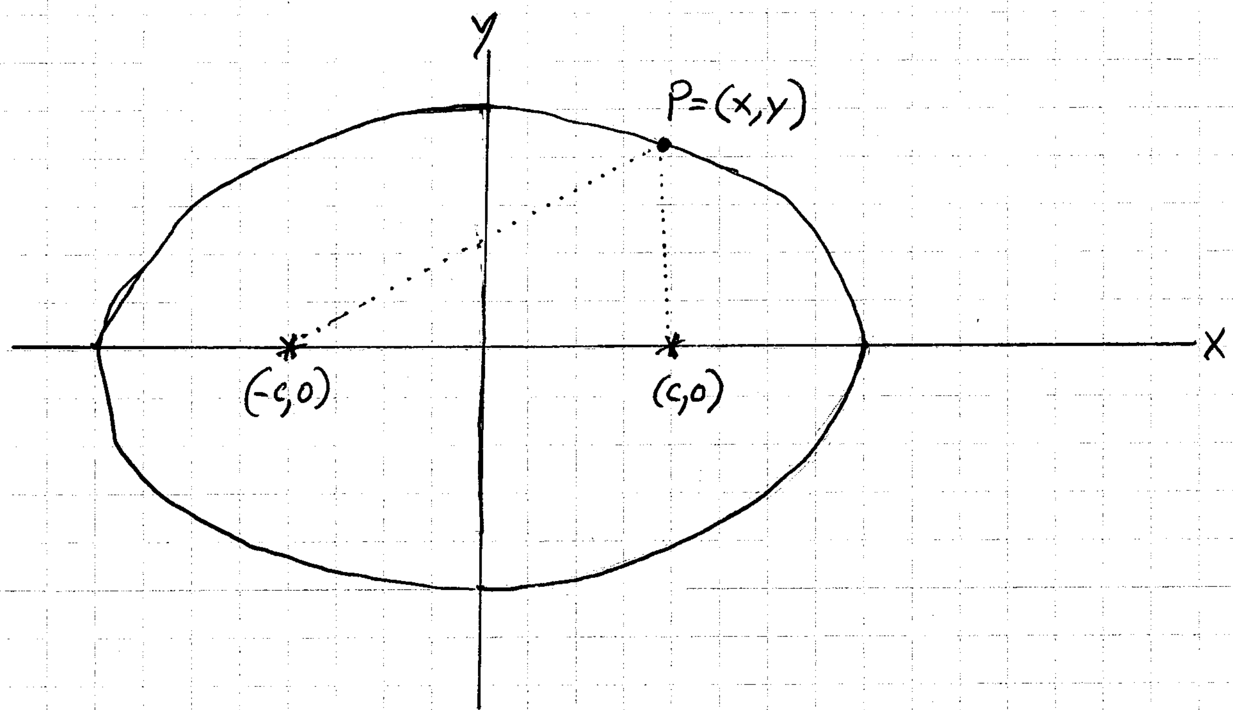
We now have a quasi-algebraic description of conic sections. An ellipse is given by two foci F_1 and F_2 and a constant such that any point P on the ellipse satisfies

$$\boxed{PF_1 + PF_2 = \text{the constant}}$$

Using a coordinate system we can translate this into an equation, a purely algebraic description of the ellipse. The first step is to choose an appropriate set of axes. We know there is a point half way between F_1 and F_2, the center of the ellipse, and we choose that to be the origin of our coordinate system with the x-axis being the line through F_1 and F_2



Suppose c is the distance from the center to the focus. Then F_1 has coordinates $(c,0)$ and F_2 has coordinates $(-c,0)$.



We let $P=(x,y)$ be an arbitrary point on the ellipse, and we let $2a$ denote the constant PF_1+PF_2. The defining equation $PF_1+PF_2=2a$ says

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

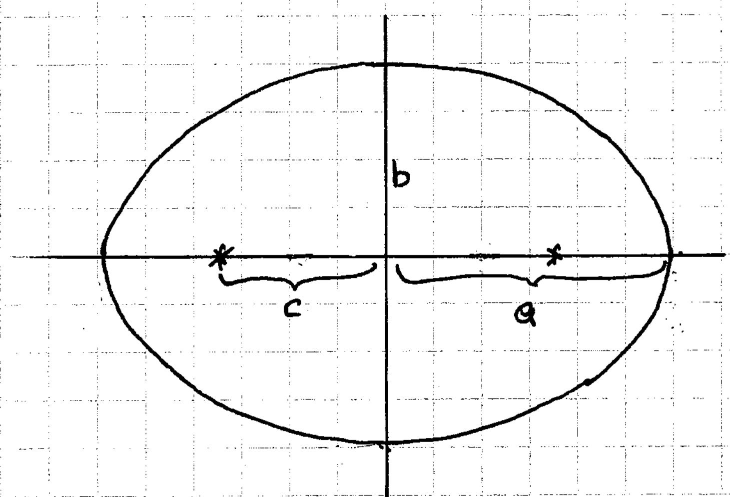
Rationalizing this equation yields

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

Finally we set $a^2-c^2=b^2$ and get the general equation of our ellipse

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

What is the geometric meaning of the constants a and b? Putting $y=0$ into the equation yields $x=\pm a$, so the points $(\pm a, 0)$ are on the ellipse. Similarly putting $x=0$ yields points $(0, \pm b)$ on the y-axis, so now we have a complete picture



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$c^2=a^2-b^2$$

We can now use algebra or calculus applied to the equation to investigate the geometry of the ellipse. This illustrates the power of analytic geometry. It gives us the ability to switch back and forth between a geometric perspective and an algebraic perspective. The equations come from the geometry, but then we use the equations to gain new geometric insight.

The procedure used above to find the equation of an ellipse may also be used on a hyperbola. As we already know a hyperbola has two Foci F_1 and F_2, and if P is any point on the hyperbola

$$\boxed{|PF_1 - PF_2| = \text{a constant}}$$

We must put absolute value signs because we don't know which focus is closer to P. Once again we choose a coordinate system so that the center of the hyperbola is at (0,0) and the foci are at (c,0) and (-c,0). The defining equation above can then be written

$$\left|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}\right| = 2a$$

Rationalizing this yields $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1$

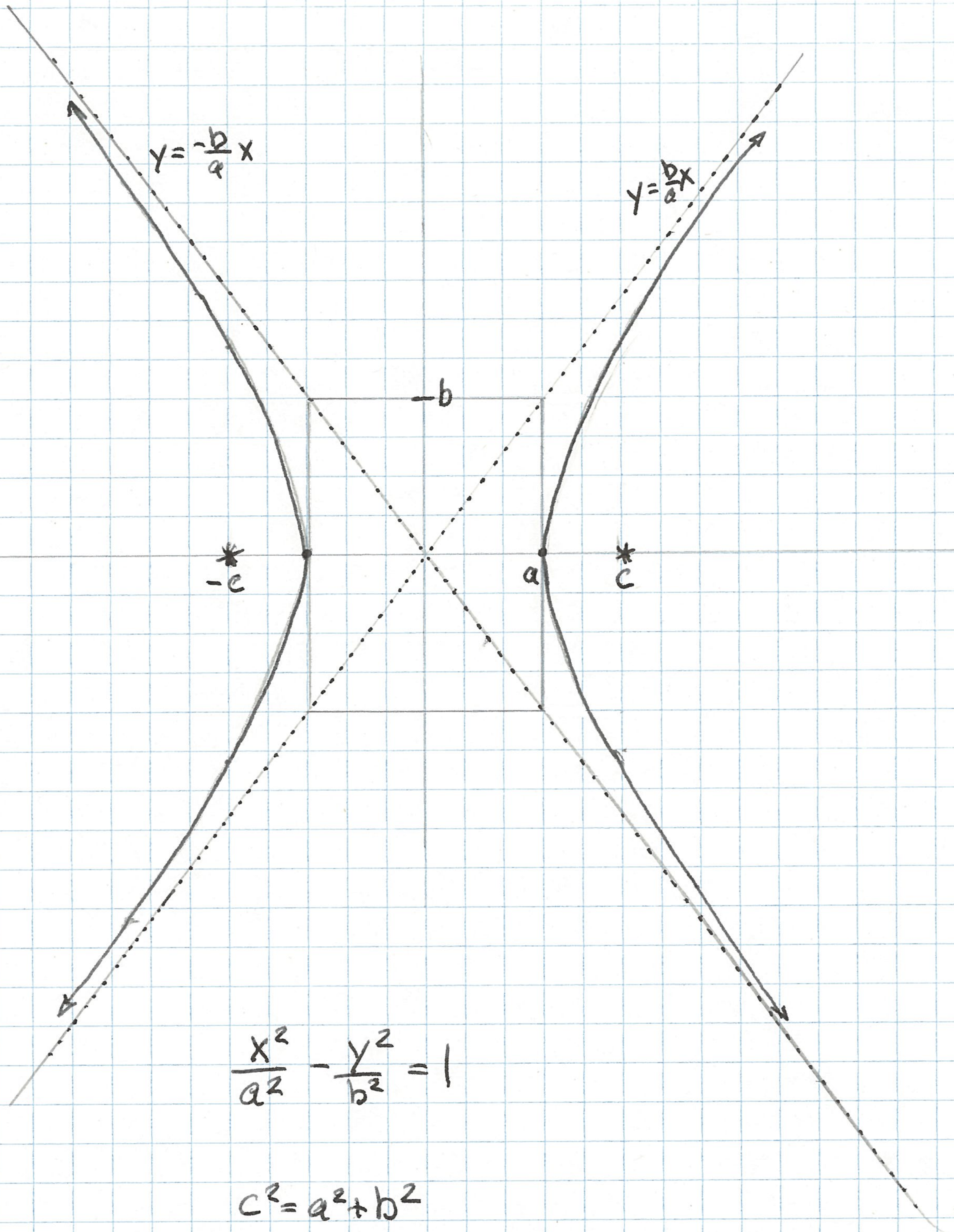
and setting $c^2-a^2=b^2$ we get the general equation of a hyperbola

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Setting $y=0$ gives $x=\pm a$, so just like the ellipse our hyperbola has vertices $(-a,0)$ and $(a,0)$ on the x-axis, but setting $x=0$ gives $y^2=-b^2$, so there are no points on the y-axis.

To understand the constant b rewrite the hyperbola equation as $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$. When x is very big $\sqrt{x^2-a^2}$ and $\sqrt{x^2}$ are very close to each other. This says that when x is very big $y=\pm\frac{b}{a}\sqrt{x^2-a^2}\approx\pm\frac{b}{a}\sqrt{x^2}=\pm\frac{b}{a}x$

Hence when x is very big the hyperbola is very close to the lines $y=\pm\frac{b}{a}x$. The lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$ are asymptotes of the hyperbola. Since we already know the center of the hyperbola is $(0,0)$, the vertices are $(\pm a,0)$, the foci are $(\pm c,0)$, and the asymptotes are $y=\pm\frac{b}{a}x$, we can draw a complete picture:



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

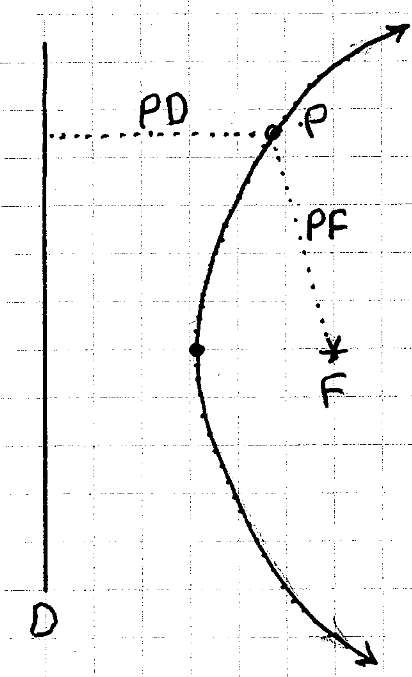
$$c^2 = a^2 + b^2$$

Note that we cannot apply this same reasoning to a parabola because a parabola has only one focus, so it makes no sense to talk about choosing (0,0) to be the center.

The eccentric equation

Conic sections may also be defined in terms of a point (the focus F), a line (the directrix D), and a constant (the eccentricity ε). Any point P on the conic then satisfies the fundamental equation

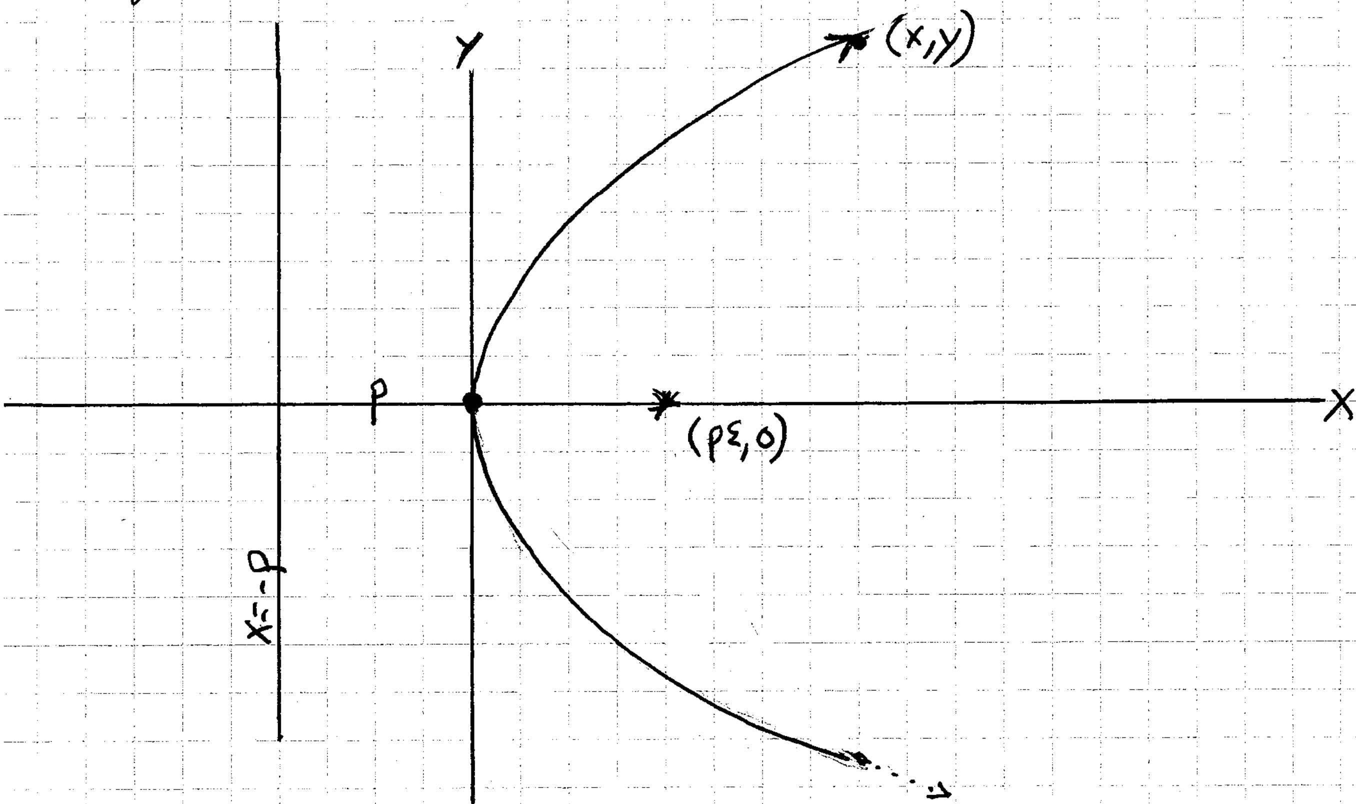
$$\boxed{\frac{PF}{PD} = \varepsilon}$$



Once again we want to choose a coordinate system so that we can translate the equation above into an equation in x and y. We choose as $(0,0)$ the point on the conic between the directrix and focus, and we choose the directrix to be the line $x=-p$ (so the y-axis is parallel to the directrix). Letting $P=(0,0)$ in the fundamental equation we get

$$\frac{PF}{PD} = \frac{PF}{p} = \varepsilon$$

, so $PF = p\varepsilon$ and the coordinates of the focus F are $(p\varepsilon, 0)$.



If $P=(x,y)$ is any point on the conic section the fundamental equation may be written

$$\frac{\sqrt{(x-p\varepsilon)^2+y^2}}{|x+p|} = \varepsilon$$

$$(x-p\varepsilon)^2+y^2 = \varepsilon^2(x+p)^2$$

$$\boxed{y^2 = (\varepsilon^2-1)x^2 + 2p\varepsilon(1+\varepsilon)x} \qquad ☆$$

and this is the equation of our conic section. If $\varepsilon=1$ the eccentric equation becomes $y^2=4px$, and this is the standard equation of a parabola with focus at $(p,0)$, vertex $(0,0)$, and directrix $x=-p$. Note that the equation $y^2=4px$ implies the only point on the horizontal axis $y=0$ is the point $(0,0)$

If $\varepsilon \neq 1$ we can set $y=0$ in equation ☆ and solve for x to find points of the conic on the x-axis:

$$(\varepsilon^2-1)x^2 + 2p\varepsilon(1+\varepsilon)x = 0$$

$$\Longrightarrow \begin{cases} x=0 \\ x=\dfrac{2p\varepsilon}{1-\varepsilon} \end{cases}$$

These two points are the two vertices of the conic, and the point halfway in between, where $x = \frac{p\varepsilon}{1-\varepsilon}$, is the center. Notice that if $\varepsilon < 1$ the center is to the right of $(0,0)$, but if $\varepsilon > 1$ the center is to the left.
In either case the distance from the center to either vertex is $\left|\frac{p\varepsilon}{1-\varepsilon}\right|$, and this is the length a from the previous section.

Since the center is at $x = \frac{p\varepsilon}{1-\varepsilon}$ and the focus is at $x = p\varepsilon$, the distance from the center to the focus is $\left|\frac{p\varepsilon}{1-\varepsilon} - p\varepsilon\right| = \left|\frac{p\varepsilon^2}{1-\varepsilon}\right|$, and this is the length c from the previous section. Finally note that

$$\frac{c}{a} = \left|\frac{p\varepsilon^2}{1-\varepsilon}\right| \div \left|\frac{p\varepsilon}{1-\varepsilon}\right| = \varepsilon$$

Thus from the focal definition of a central conic we can find the eccentricity, and from the eccentric definition we can find the vertices and center.