The sum of the reciprocals of the squares is $\frac{\pi^2}{6}$

An "infinite series" is a never-ending sum. We have already seen two:

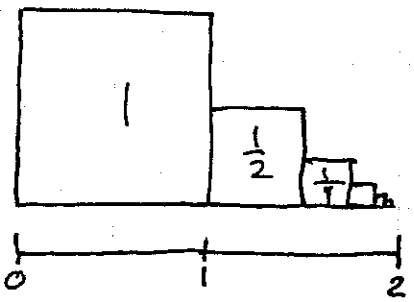
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots = \infty$$

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots = \frac{1}{3}$$

The first "diverges to infinity", that is as you add more and more numbers the total sum gets bigger and bigger. The second series, on the other hand is just $.3333\ldots = \frac{1}{3}$. Another famous series is

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2$$

It is quite easy to see that the sum is two. Visualize adding up wooden blocks like in kindergarten. The first block has width 1, the second block has width $\frac{1}{2}$, etc:



Each time you add another block you only add half the distance to 2, so you will never pass 2. On the other hand, you won't ever stop adding blocks before you get to 2. The sum is not less than 2 and it's not bigger than 2, so it must be 2.

Infinite series have been a central theme in mathematics ever since Zeno brought up Achilles and the arrow two-thousand five-hundred years ago. Philosophers and mathematicians debated their meaning century after century. With the advent of the calculus, new tools enabled mathematicians to solve a host of problems involving infinite arithmetic in general, including infinite series. In 1735 the great Leonhard Euler proved

$$\boxed{1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\dots=\frac{\pi^2}{6}}$$

In order to show this we need two preliminary elementary facts about factoring. In high-school you used factoring to find the zeros of a polynomial. For example, let

$$p(x) = x^2-7x+12$$

$$p(x) = (x-3)(x-4)$$

so you know $x=3$ and $x=4$ will make this zero. Factoring tells you the zeros of the polynomial, but the reverse is also true. If you didn't know how to factor $x^2-7x+12$ and I told you the zeros were 3 and 4 you'd know the factors were $(x-3)$ and $(x-4)$.

Here is another example: Can you factor this polynomial?

$$1-4x-36x^2+166x^3-397x^4-210x^5$$

Probably not, but if I tell you the zeros are $1, \frac{1}{3}, -\frac{1}{5}, -\frac{1}{2}, \frac{1}{7}$ then you can easily find the factors

$$1-4x-36x^2+166x^3-397x^4-210x^5=(1-x)(1-3x)(1+5x)(1+2x)(1-7x)$$

Notice that each factor must start with a one so that they can multiply out to the one on the left.

If you know the zeros of a polynomial you can find the factors

To understand the second elementary result look at these products:

$$(1+ax)(1+bx) = 1+(a+b)x+abx^2$$

$$(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+ac+bc)x^2+abcx^3$$

$$(1+ax)(1+bx)(1+cx)(1+dx) = 1+(a+b+c+d)x+\text{lots of } x^2+\text{lots of } x^3+abcdx^4$$

Look at the coefficient of x in the product and you see

If you multiply out $(1+ax)(1+bx)(1+cx)(1+dx)(1+ex)\dots$ the coefficient of x is $a+b+c+d+e\dots$

Euler's master-stroke was to apply these two elementary results to infinitely long polynomials. You learned in calculus that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$. Substituting $x = \pi\theta$ gives

$$\sin \pi\theta = \pi\theta - \frac{\pi^3}{3!}\theta^3 + \frac{\pi^5}{5!}\theta^5 - \frac{\pi^7}{7!}\theta^7 + \dots$$

Just to get warmed up, consider the function f defined by

$$f(\theta) = 1 - 2\sin\pi\theta = 1 - 2\pi\theta + \frac{2\pi^3}{3!}\theta^3 - \dots$$

We use our first elementary result: What values of θ will make this zero? Because $\sin\left(\frac{1}{6}\pi\right) = \frac{1}{2}$ we see that

$$f\left(\tfrac{1}{6}\right) = 1 - 2\sin\left(\tfrac{\pi}{6}\right) = 0$$

But also $\frac{1}{2} = \sin\left(\frac{5}{6}\pi\right) = \sin\left(-\frac{7}{6}\pi\right) = \sin\left(-\frac{11}{6}\pi\right) = \sin\left(\frac{13}{6}\pi\right) \dots$,

so the zeros of $f(\theta)$ are $\frac{1}{6}, \frac{5}{6}, -\frac{7}{6}, -\frac{11}{6}, \frac{13}{6}$, etc.

Since we know the zeros of $f(\theta)$ we know the factors:

ZEROS: $\frac{1}{6} \quad \frac{5}{6} \quad -\frac{7}{6} \quad -\frac{11}{6} \quad \frac{13}{6} \quad \frac{17}{6} \quad -\frac{19}{6} \quad -\frac{23}{6} \dots$

FACTORS: $(1-6\theta)\left(1-\frac{6}{5}\theta\right)\left(1+\frac{6}{7}\theta\right)\left(1+\frac{6}{11}\theta\right)\left(1-\frac{6}{13}\theta\right)\left(1-\frac{6}{17}\theta\right)\dots$

$$f(\theta) = 1 - 2\pi\theta + \frac{2\pi^3}{3!}\theta^3 \ldots = (1-6\theta)(1-\tfrac{6}{5}\theta)(1+\tfrac{6}{7}\theta)(1+\tfrac{6}{11}\theta)(1-\tfrac{6}{13}\theta)(1-\tfrac{6}{17}\theta)\cdots$$

Now use the second elementary result. When we multiply out the product on the right the coefficient of θ will be $-6 - \frac{6}{5} + \frac{6}{7} + \frac{6}{11} - \frac{6}{13} - \frac{6}{17} + \ldots$, while the coefficient of θ on the left is -2π, so we have

$$-2\pi = -6 - \frac{6}{5} + \frac{6}{7} + \frac{6}{11} - \frac{6}{13} - \frac{6}{17} + \frac{6}{19} + \ldots$$

$$\pi = 3\left(1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} \cdots\right)$$

This is a remarkable formula for π, but reading it backwards we see that we have been able to evaluate an infinite series:

$$1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} + \ldots = \frac{\pi}{3}$$

By replacing the formula for $f(\theta)$ with expressions like $1 - \sqrt{2}\sin\pi\theta$ or $1 - \frac{2}{\sqrt{3}}\sin\pi\theta$ or just $1 - \sin\pi\theta$, Euler was able to evaluate a whole class of infinite series, each sum of which results in a multiple of pi.

Back to our main purpose. Again we start with

$$\sin(\pi\theta) = \pi\theta - \frac{\pi^3}{3!}\theta^3 + \frac{\pi^5}{5!}\theta^5 - \frac{\pi^7}{7!}\theta^7 + \dots$$

Now divide by $\pi\theta$ and we get

$$\frac{\sin(\pi\theta)}{\pi\theta} = 1 - \frac{\pi^2}{3!}\theta^2 + \frac{\pi^4}{5!}\theta^4 - \frac{\pi^6}{7!}\theta^6 + \dots$$

What are the zeros of this function? Because $\sin(\pi) = \sin(-\pi) = \sin(2\pi) = \sin(-2\pi) = \sin(3\pi) = \sin(-3\pi) = \dots = 0$ we see that the zeros are all the non-zero integers $1, -1, 2, -2, 3, -3, 4, -4, \dots$

Hence we can factor the polynomial:

$$1 - \frac{\pi^2}{3!}\theta^2 + \frac{\pi^4}{5!}\theta^4 - \frac{\pi^6}{7!}\theta^6 + \dots = (1+\theta)(1-\theta)(1+\tfrac{1}{2}\theta)(1-\tfrac{1}{2}\theta)(1+\tfrac{1}{3}\theta)(1-\tfrac{1}{3}\theta)\dots$$

Now use the fact that $(1+\theta)(1-\theta) = 1-\theta^2$, $(1+\frac{1}{2}\theta)(1-\frac{1}{2}\theta) = 1-\frac{1}{4}\theta^2$, etc and we get

$$1 - \frac{\pi^2}{3!}\theta^2 + \frac{\pi^4}{5!}\theta^4 - \frac{\pi^6}{7!}\theta^6 + \dots = (1-\theta^2)(1-\tfrac{1}{4}\theta^2)(1-\tfrac{1}{9}\theta^2)(1-\tfrac{1}{16}\theta^2)\dots$$

By our second elementary result the coefficient of θ^2 on the right will be $-1-\frac{1}{4}-\frac{1}{9}-\frac{1}{16}-\frac{1}{25}-\dots$, but the coefficient of θ^2 on the left is $-\frac{\pi^2}{3!}$, so

$$-\frac{\pi^2}{3!} = -1-\frac{1}{4}-\frac{1}{9}-\frac{1}{16}-\frac{1}{25}\dots$$

$$\frac{\pi^2}{6} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\dots$$

which is what we wanted to show.

Euler used his factorization to find values for $\sum \frac{1}{n^p}$ for every even integer p. We will look at how he took care of $\sum \frac{1}{n^4}$, but we need two more elementary results:

If you multiply out $(1+ax)(1+bx)(1+cx)(1+dx)\ldots$ the coefficient of x^2 is the sum of the letters taken two at a time, that is $ab+ac+ad+bc+bd+cd+\ldots$

The square of a sum is the sum of the squares plus twice the sum of the terms taken two at a time.

As an example of the second statement notice that

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd).$$

Now apply these two observations to the factorization

$$1-\frac{\pi^2}{3!}\theta^2+\frac{\pi^4}{5!}\theta^4+\ldots = (1-\theta^2)(1-\tfrac{1}{4}\theta^2)(1-\tfrac{1}{9}\theta^2)(1-\tfrac{1}{16}\theta^2)\ldots$$

The first statement above says that $\frac{\pi^4}{5!}$ is the sum of the reciprocals of the squares taken two at a time. Denote that sum by the letter A. The second statement says

$$\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\right)^2 = 1^2+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\ldots+2A$$

Since we already know $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots = \frac{\pi^2}{6}$ and $A=\frac{\pi^4}{5!}$
we have

$$\left(\frac{\pi^2}{6}\right)^2 = 1^2+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\ldots+2\frac{\pi^4}{5!} \quad \text{and so}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots = \frac{\pi^4}{90}$$

Proceeding in this manner Euler showed that $\sum \frac{1}{n^6} = \frac{\pi^6}{945}$ $\sum \frac{1}{n^8} = \frac{\pi^8}{9450}$ and so on. In the subsequent two hundred and eighty-five years, no one has been able to improve on these results. In particular it is still not known how to deal with $\sum \frac{1}{n^3}$.

Modern mathematicians have objected to Euler's rather free use of infinite algebra, but there are now numerous precise proofs verifying Euler's results. Euler seems to have had a direct line to the gods of mathematics who told him what would work and what wouldn't.

Conclusion

These notes only touch upon a few of the important themes of mathematics. I could have easily included sections on logic, coordinate geometry, calculus, probability, group theory, topology, four-dimensional space, etcetera. My aim has been to convince the reader that mathematics is a living creative art and is not the repetitious, boring techniques for solving problems (as taught in our schools). Like all art forms, mathematics can be enjoyed and appreciated by everyone. I have given below references for further exploration.

Tom Fox

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"The pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings" — A.N. Whitehead

References

I have listed below a few of the many books written about mathematics for the non-expert. The anthology by Newman remains the best source of material, while the autobiography of G.H. Hardy gives a glimpse into a pure mathematical soul. The last five references are more advanced

R. Courant and H. Robbins, "What is Mathematics" (Oxford Press)

W. Dunham, "Journey through Genius" (Penguin)

G. H. Hardy and C.P. Snow, "A Mathematician's Apology" (Cambridge)

J. Newman (editor), "The World of Mathematics", 4 vols (Dover)

H. Rademacher and O. Toeplitz, "The Enjoyment of Mathematics" (Princeton)

A.N. Whitehead, "An Introduction to Mathematics" (Michigan)

M. Aigner and G. Ziegler, "Proofs from the Book" (Springer)

Euclid, "The Elements", T.L. Heath editor, 3 vols. (Dover)

L. Euler, "Introduction to Analysis of the Infinite", 2 vols. (Springer)

S. Hawking (editor), "God Created the Integers" (Running Press)

E. Nagel and J. Newman, "Goedel's Proof" (NYU)