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MATH 417/487 - Mathematical Programming

Homework Set No. 1

1.1 (Some Linear Algebra)

a) For $b_1 = (1, 0, 1)^T$ and $b_2 = (0, 1, 0)^T$ find $b_3 \in \mathbb{R}^3$ such that b_1, b_2, b_3 form a basis of $\mathbb{R}^3$. Is b_3 uniquely determined?

b) Let $z_1, \dots, z_r \in \mathbb{R}^n \setminus \{0\}$ such that $z_i^T z_j = 0$ for all i, j with $i \neq j$. Show that $z_1, \dots, z_r$ are linearly independent.

c) Determine $\operatorname{rank}(zz^T)$ for $z \in \mathbb{R}^n$.

d) For real matrices A, B such that AB exists, show that

$$\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

(2+2+2+2 P.)

1.2 (Positive (semi)definiteness) Let $A \in \mathbb{R}^{n \times n}$ be symmetric with the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (see spectral theorem). Show the following:

a) $x^T Ax \geq 0 \ (x \in \mathbb{R}^n) \iff \lambda_i \geq 0 \ (i = 1, \dots, n)$;

b) $x^T Ax > 0 \ (x \in \mathbb{R}^n \setminus \{0\}) \iff \lambda_i > 0 \ (i = 1, \dots, n)$.

(2+1 P.)

1.3 (Symmetric matrices, quadratic functions and infima) Let $A \in \mathbb{R}^{n \times n}$ be symmetric (i.e. $A^T = A$) and $b \in \mathbb{R}^n$.

a) Show that $\ker A \cap \operatorname{rge} A = \{0\}$.

b) Prove that $\ker A + \operatorname{rge} A = \mathbb{R}^n$.

(2+4+5 P.)

Please turn over!

*c) For $q : \mathbb{R}^n \to \mathbb{R},\ q(x) = \frac{1}{2}x^T Ax + b^T x$ show that the following are equivalent:

i) $\inf_{\mathbb{R}^n} q > -\infty$;

ii) A is positive semidefinite (i.e. $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$) and $b \in \operatorname{rge} A$;

iii) $\operatorname{argmin}_{\mathbb{R}^n} q \neq \emptyset$.

Hint: You may use (if needed) without proof that a positive semidefinite matrix has only nonnegative eigenvalues.

1.4 (Minimizing a linear function over the unit ball) Let $g \in \mathbb{R}^n \setminus \{0\}$. Compute the solution of the optimization problem

$$\min\ \langle g,\, d\rangle \quad \text{s.}t. \quad \|d\| \leq 1.$$

(4 P.)

Remarks

- *This homework has to be submitted in class on* ***September 12, 2017****.*
- *The problems marked with '*' are only compulsory for the honors students (MATH 487). It goes without saying that everybody else is also encouraged to tackle them.*
- *You may (and are encouraged to) work with other students in the class on the assignments, however, the work you submit must be your own. You must also cite any materials you have used to complete your work. Since both the midterm and the final exam will, in part, be heavily based on the homework assignments, it is strongly recommended to take the homework seriously.*