MATH323 - Tutorial 1 (January 11th)

Univariate Differentiation Rules

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

more generally, $\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \dots\right)$

$$= 0 + 1 + 2x + 3x^2 + \dots$$
$$= \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} nx^{n-1}$$

2. $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

3. Product Rule. Let f, g be two differentiable functions then

$$\frac{d}{dx} h(x) = \frac{d}{dx} f(x)g(x) = \left(\frac{d}{dx} f(x)\right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx} g(x)\right)$$
$$= f'(x)g(x) + f(x)g'(x)$$

4. Quotient Rule. Let f, g be differentiable functions with $g(x) \neq 0$ then

$$\frac{d}{dx} h(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{d}{dx} f(x)\right) g(x) - \left(\frac{d}{dx} g(x)\right) f(x)}{(g(x))^2}$$
$$= \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

5. Chain Rule. Let f and g be differentiable functions then

$$\frac{d}{dx} h(x) = \frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx} f(g(x)) \cdot \frac{d}{dx} g(x)$$
$$= f'(g(x)) \cdot g'(x)$$

Examples:

1. Let $\alpha, \beta \in \mathbb{R}$ be fixed constants and let $m(t) = (1-\beta t)^{-\alpha}$. Find $m'(0)$ and $m''(0)$.

Solution: $m(t) = (1-\beta t)^{-\alpha}$

$$m'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta)$$
$$= \alpha\beta(1-\beta t)^{-\alpha-1}$$
$$m'(0) = \alpha\beta(1-0)^{-\alpha-1} = \alpha\beta$$

$$m''(t) = \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)$$
$$= \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2}$$
$$m''(0) = \alpha(\alpha+1)\beta^2(1-0)^{-\alpha-2} = \alpha(\alpha+1)\beta^2$$

2. Let $\theta_1 < \theta_2 \in \mathbb{R}$ be fixed constants. Let $m(t) = \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2-\theta_1)}$.

Calculate $m'(1)$.

Solution: $m(t) = \frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2-\theta_1)}$

$$m(t) = \frac{1}{\theta_2-\theta_1}\left(\frac{e^{t\theta_2}-e^{t\theta_1}}{t}\right)$$

$$m'(t) = \frac{1}{\theta_2-\theta_1}\left(\frac{[\theta_2 e^{t\theta_2} - \theta_1 e^{t\theta_1}]t - [e^{t\theta_2} - e^{t\theta_1}]}{t^2}\right)$$

$$m'(t) = \frac{(\theta_2-1)e^{t\theta_2} + (1-\theta_1)e^{t\theta_1}}{(\theta_2-\theta_1)t^2}$$

$$m'(1) = \frac{(\theta_2-1)e^{\theta_2} + (1-\theta_1)e^{\theta_1}}{\theta_2-\theta_1}$$

3. Let $p \in [0,1]$ be a fixed constant. Let $m(t) = [pe^t + (1-p)]^n$.
Find $m'(0)$.

Solution: $m(t) = [pe^t + (1-p)]^n$

$$m'(t) = n[pe^t + (1-p)]^{n-1}[pe^t]$$
$$m'(0) = n[pe^0 + (1-p)]^{n-1}[pe^0]$$
$$= np$$

Univariate Integration Rules

1. $\int_a^b x^n \, dx = \frac{x^{n+1}}{n+1}\Big|_a^b = \left(\frac{b^{n+1}}{n+1}\right) - \left(\frac{a^{n+1}}{n+1}\right)$

2. $\int_a^b e^x \, dx = e^x\Big|_a^b = e^b - e^a$

3. Integration by Parts

$$\int_a^b (du)\, v \, dx = uv\Big|_a^b - \int_a^b u\,(dv)\, dx$$

Examples

1. $\int_0^1 \frac{1}{2} \, dx = \frac{1}{2} x\Big|_0^1 = \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}$

2. Let $\lambda \in \mathbb{R}$ be a fixed constant. Calculate $\int_0^\infty \lambda e^{-\lambda x} \, dx$.

Solution: $\int_0^\infty \lambda e^{-\lambda x} \, dx = -e^{-\lambda x}\Big|_0^\infty$

$$= \left(\lim_{x\to\infty} -\lambda e^{-\lambda x}\right) - \left(-e^{-\lambda(0)}\right)$$

$$= 0 - (-1) = 1$$

3. Let $\lambda \in \mathbb{R}$ be a fixed constant. Calculate $\int_0^\infty \lambda x e^{-\lambda x} \, dx$.

Solution: $\int_0^\infty \lambda x e^{-\lambda x} \, dx$ (integration by parts) Let $u = x$, $dv = \lambda e^{-\lambda x}$

$\Rightarrow du = 1$, $v = -e^{-\lambda x}$

$$= -xe^{-\lambda x}\Big|_0^\infty - \int_0^\infty -e^{-\lambda x} \, dx$$

$$= \left(\lim_{x\to\infty} -xe^{-\lambda x}\right) - \left(-0e^{-\lambda 0}\right) + \left[-\frac{1}{\lambda} e^{-\lambda x}\right]\Big|_0^\infty = \left(\lim_{x\to\infty} -\frac{1}{\lambda} e^{-\lambda x}\right) + \left(\frac{1}{\lambda} e^0\right) = \frac{1}{\lambda}$$

4. Let $\mu, \sigma \in \mathbb{R}$ be fixed constants. Let $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$. Using the fact that $\int_{-\infty}^{\infty} f(y)\,dy = 1$, $\forall \mu, \sigma \in \mathbb{R}$.

calculate $\int_{-\infty}^{\infty} y f(y)\,dy$.

Solution: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} y e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} y e^{-(y-\mu)^2/2\sigma^2} dy$

Use change of variables. Let $z = y - \mu$

$\Rightarrow y = z + \mu$

$\Rightarrow dy = dz$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (z+\mu) e^{-\frac{z^2}{2\sigma^2}} dz$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \left\{ \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2\sigma^2}} dz + \mu \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\sigma^2}} \right\}$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(-\sigma^2 e^{-\frac{z^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} \right) + \mu \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\sigma^2}} dz}_{= \int_{-\infty}^{\infty} f(z)\,dz \text{ when } \mu = 0}$

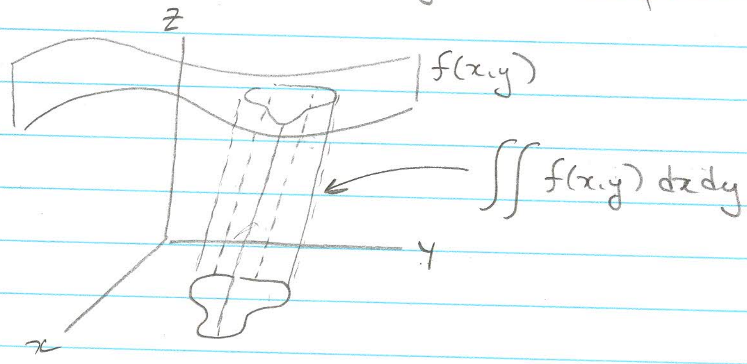
$= 1$

$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\lim_{z\to\infty} -\sigma^2 e^{-\frac{z^2}{2\sigma^2}} - \lim_{z\to-\infty} -\sigma^2 e^{-\frac{z^2}{2\sigma^2}} \right) + \mu$

$= 0 + \mu = \mu$

Multivariate Integration

Calculating the volume from a region in the plane to a surface.



In univariate integration, we integrate along a 1-d line and so use 1 integral.

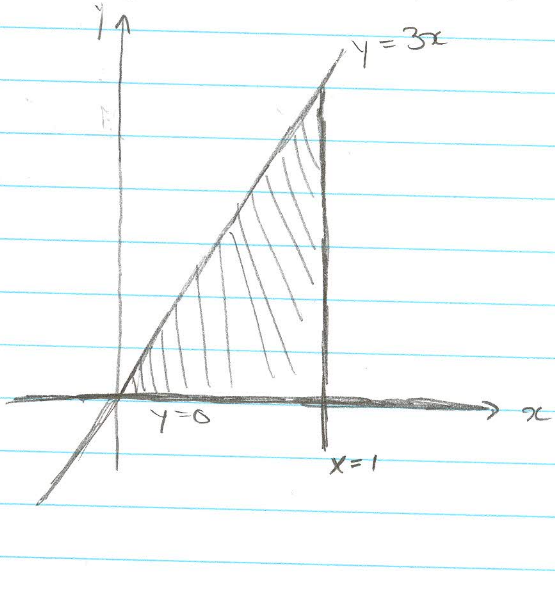
In bivariate integration, we integrate over a 2-d region and so use 2 integrals. Integration rules do not change. Variables other than the one over which we are integrating are treated as fixed constants.

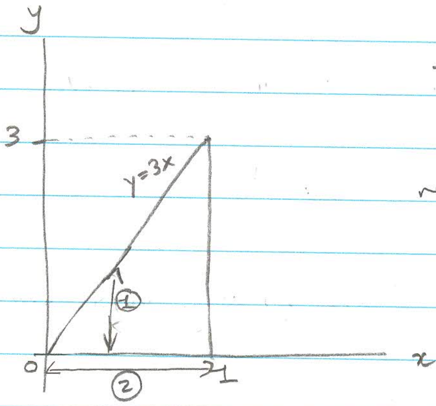
Examples

1. Find the volume projected by the function $f(x,y) = 2x^2y$ onto the region bounded inside the lines $y=0$, $x=1$, $y=3x$.

Solution:

Picture





If we integrate with respect to y first (innermost integral) then with respect to x, we obtain:

$$\int\int 2x^2 y \, dy \, dx$$

This order determines the bounds of the integrals:

$$\int_0^1 \int_0^{3x} 2x^2 y \, dy \, dx$$

Now, let's evaluate:

$$\int_0^1 \left\{ \int_0^{3x} 2x^2 y \, dy \right\} dx$$

in this integral, we treat x^2 as a constant because we integrate with respect to y

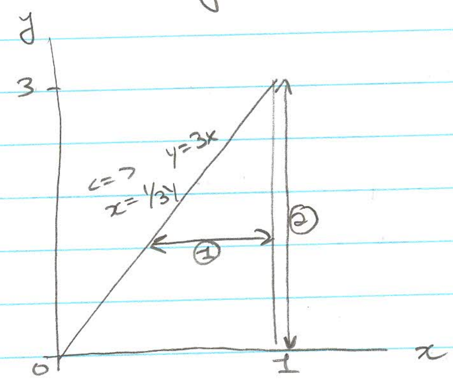
$$= \int_0^1 \left\{ 2x^2 \left(\frac{1}{2} y^2\right) \Big|_0^{3x} \right\} dx$$

$$= \int_0^1 2x^2 \left(\frac{1}{2}(3x)^2 - \frac{1}{2}(0) \right) dx = \int_0^1 \frac{(2x^2)(3x)^2}{2} \, dx$$

$$= \int_0^1 9x^4 \, dx$$

$$= \frac{9}{5} x^5 \Big|_0^1 = \frac{9}{5}$$

Let's change the order with which we integrate. x first then y



$$\iint 2x^2y \, dx \, dy$$

The ① line goes from $\frac{1}{3}y$ to 1.

The ② line goes from 0 to 3, so

$$\int_0^3 \int_{1/3 y}^{1} 2x^2y \, dx \, dy$$

Now, let's evaluate:

$$\int_0^3 \left\{ \int_{1/3 y}^{1} 2x^2y \, dx \right\} dy$$

in this integral, we treat y as a constant because we integrate with respect to x

$$= \int_0^3 \left\{ y \left(\frac{2}{3} x^3 \right) \Big|_{1/3 y}^{1} \right\} dy$$

$$= \int_0^3 y \left(\frac{2}{3}(1)^3 - \frac{2}{3}\left(\frac{1}{3}y\right)^3 \right) dy$$

$$= \int_0^3 \frac{2}{3}y - \frac{2}{81}y^4 \, dy = \frac{2}{3}\left(\frac{1}{2}y^2\right) - \frac{2}{81}\left(\frac{1}{5}y^5\right) \Big|_0^3$$

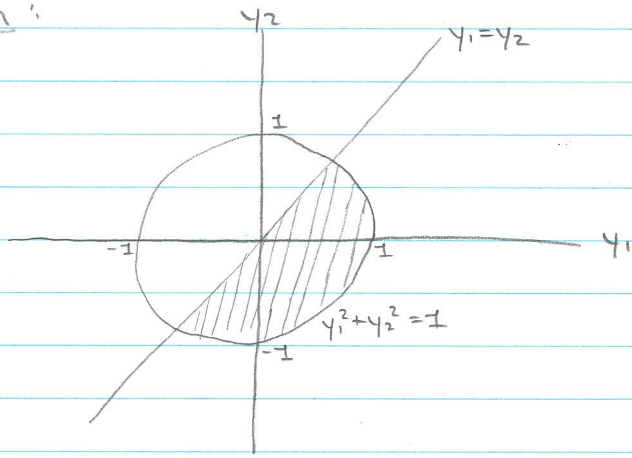
$$= \left(\frac{\cancel{2}}{3}\right)\left(\frac{9}{\cancel{2}}\right) - \left(\frac{2}{\cancel{81}}\right)\left(\frac{\cancel{243}}{5}\right)^3$$

$$= 3 - \frac{6}{5} = \frac{15-6}{5} = \frac{9}{5}$$

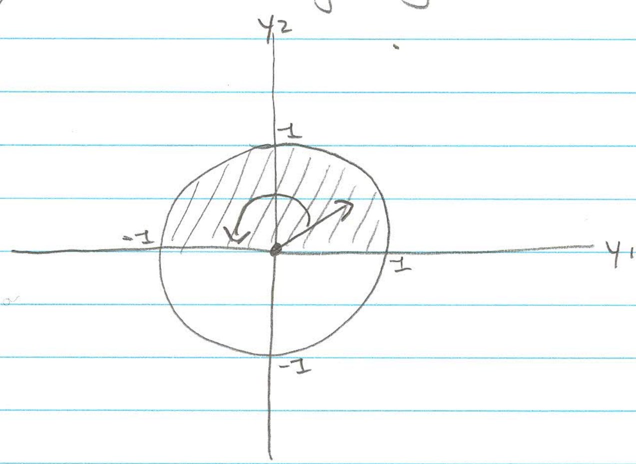
2. Let $f(y_1, y_2) = \begin{cases} \frac{1}{\pi}, & y_1^2 + y_2^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$

Find the volume projected by $f(y_1, y_2)$ for the region bounded by $y_1^2 + y_2^2 \le 1$ and $y_1 \le y_2$.

Solution:



Note that this region is difficult to integrate over. Observe that $f(y_1, y_2)$ is constant over the whole circle and the line $y_1 = y_2$ simply cuts the circle in half, So, this integration is equivalent to integrating over the following region:



Now, we could integrate with respect to y_1, y_2 however the bounds are difficult. We will use polar coordinates and integrate with respect to the radius and an angle. Intuitively, we are integrating using a "clockhand".

Original Integral : $\iint \frac{1}{\pi}\, dy_1\, dy_2$

New Integral after change of variables : $\iint \frac{1}{\pi} J \, dr\, d\theta$

where

$$J = \det \begin{pmatrix} \frac{dy_1}{dr} & \frac{dy_1}{d\theta} \\ \frac{dy_2}{dr} & \frac{dy_2}{d\theta} \end{pmatrix}$$

Letting $y_1 = r\cos\theta$
$y_2 = r\sin\theta$

$$J = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$
$$= r$$

Integral : $\int_0^{\pi}\int_0^{1} \frac{r}{\pi}\, dr\, d\theta$

$$= \int_0^{\pi} \frac{r^2}{2\pi}\Big|_0^1 \, d\theta = \int_0^{\pi} \frac{1}{2\pi}\, d\theta = \frac{\theta}{2\pi}\Big|_0^{\pi}$$
$$= \frac{1}{2}$$