Requirements for the final exam for MATH 251

The exam will consist of 8 questions about definitions (5 pts each), 4 questions asking for statements of theorems (5 pts each), and 2 proofs (15 pts each). The questions will cover the following topics.

1. Definition of a vector space, a subspace, examples. Definitions of algebraic sum and external direct sum. Definition of the span of a family of vectors, the proof that it is a subspace. Definition of a spanning family and of a linearly independent family. The proof of the theorem that 3 definitions of a basis are equivalent.

2. The statement and proof of Steinitz substitution lemma. Deduction of the corollaries that the cardinality of a linearly independent set cannot exceed the cardinality of a basis, and consequently that dimension is well-defined. The corollaries that every linearly independent set can be completed to a basis, and hence that the dimension of a subspace is bounded by the dimension of the ambient space. Coordinates with respect to a basis, the matrix of a basis change. Proof of the identities involving that matrix and the corollary that a matrix is invertible iff its columns form a basis.

3. Definition of a linear map, its kernel and image, the proof that they are subspaces, and that trvial kernel is equivalent to injectivity. Examples. The definition of an isomorphism and the proof that every finitely dimensional vector space is isomorphic to F^n. The statement and proof of the theorem on the kernel and image. Applications. The definition of a matrix representing a linear map and the statement and proof of the theorem on its properties. Definition of inner direct sum. Proof of the isomorphism with external direct sum. Definition of a projection. Proof of the theorem on the equivalent definition involving direct sum. Definition of the quotient space.

4. Definition of a permutation, inversion, the sign, transposition and n-cycle. The formula for the sign using decomposition into cycles. The proof of multiplicativity of the sign. The definition of the determinant with 4 axioms. The proof of the theorem on its existence and uniqueness. Corollary on functions satisfying only axioms (1)-(3). Cauchy's theorem with proof. Corollary that a matrix is invertible iff its determinant is nonzero. Laplace theorem on expanding the determinant with proof. The adjoint matrix and Cramer's rule. Definition of an orientation, and an isomorphism preserving orientation. The statement of Binet-Cauchy theorem. The definition of Wronskian.

5. The definition of column and row rank. Proof (you can choose one of the two proofs) that they are equal. The algorithm for computing the inverse matrix.

6. The definition of a linear functional and of the dual space and the calculation if its dimension. The proof of the existence of the dual basis. The definition of the annihilator and the lemma on its (just first three) properties with proof. Proof that the annihilator of U_1+U_2 is the intersection of the annihilators of U_1 and U_2. The definition of the dual map and the calculation of its matrix in the dual basis. The definition of the polar dual to a convex set in R^2 containing 0.

7. The definition of inner product, and associated norm and distance. Pythagorean lemma with proof. Cauchy-Schwarz inequality with proof. The proposition on the properties of the norm with proof. Examples of inner products, in particular the one defined using a Hermitian matrix. Orthogonal and orthonormal bases. The theorem on the existence and formula for the orthogonal projection with proof. Corollaries. The theorem on the existence of an orthonormal basis (Gram-Schmidt process) with proof. The definition of the orthogonal subspace, and the proposition on the decomposition of a finite dimensional inner product space into the direct sum of a subspace and its orthogonal subspace, with proof. Lemma on the orthogonal projection minimizing the distance to the subspace. The statement of the theorem on the volume of an m-parallelopiped in R^n. The definition of Gram matrix. The definition of the approximate solution of the least squares method to the overdetermined system Ax=y where the columns of A are linearly independent.

8. The definition of eigenvalues, eigenvectors, similar matrices. Proof that similar matrices have the same eigenvalues. The definition of the characteristic polynomial. The proof of the proposition on det and tr appearing as coefficients of the characteristic polynomial. The proof of the remark on 3 equivalent definitions of an eigenvalue. Definition of eigenspace, geometric and algebraic multiplicity. Proof of the relation between them. Definition of a diagonalizable matrix and the lemma and theorem on equivalent conditions with proofs. The derivation of the closed formula for the terms of the Fibonacci sequence.

9. The statement of Cayley-Hamilton theorem. The definition of minimal polynomial. The statements of the proposition and theorem on the properties of the minimal polynomial. The definition of a T-invariant subspace. The statement of the primary decomposition theorem (without proof), and the corollary on the condition for diagonalizability with proof. Definition of a nilpotent map, a Jordan block and Jordan form. The statement on the theorem on the Jordan form and the application to compute powers.

10. The definition of Hermitian adjoint, the proof of existence and uniqueness for a finite dimensional inner product space. Definition of a self-adjoint map. The proof that they have real eigenvalues and orthogonal eigenspaces. Definition of Legendre polynomials. Proof of the theorem that a self-adjoint map is diagonal in an orthonormal basis. Definition of a unitary map and matrix, equivalent conditions. Definition of a symmetric bilinear form, principal axis theorem with proof. The theorem that a Hermitian matrix is positive definite iff all its eigenvalues are > 0 with proof. Sylvester's theorem with proof. Classification of quadrics. Definition of a normal map, the proof that it is diagonalizable, the proof of the spectral theorem.