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MATH 417/487 - Mathematical Programming

Homework Set No. 2

2.1 (Convexity preserving operations)

a) (Intersection) Let I be an arbitrary index set (possibly uncountable) and let $C_i \subset \mathbb{R}^n$ $(i \in I)$ be convex sets. Show that $\bigcap_{i\in I} C_i$ is convex.

b) (Linear images and preimages) Let $A \in \mathbb{R}^{m\times n}$ and let $C \subset \mathbb{R}^n$, $D \subset \mathbb{R}^m$ be convex. Show that

$$A(C) := \{Ax \mid x \in C\} \quad \text{and} \quad A^{-1}(D) := \{x \mid Ax \in D\}$$

are convex.

(2+2 P.)

2.2 (Characterization of extreme points) Let $C \subset \mathbb{R}^n$ be convex. Show that for $x \in C$ the following are equivalent:

i) For all $x_1, x_2 \in C$ we have that $\frac{1}{2}x_1 + \frac{1}{2}x_2 = x$ imnplies $x_1 = x_2$.

ii) x is an extreme point of C.

iii) $x = \sum_{i=1}^r \lambda_i x_i$ for some $r \in \mathbb{N}$, $x_i \in C$ $(i = 1, \dots, r)$, $\lambda_i \geq 0$ $(i = 1, \dots, r)$ and $\sum_{i=1}^r \lambda_i = 1$ implies $x_i = x$ for all $i = 1, \dots, r$.

iv) $C \setminus \{x\}$ is convex.

(4 P.)

2.3 (Projection on subspaces) Let $U \subset \mathbb{R}^n$ be a subspace. Then it is known that for every $x \in \mathbb{R}^n$ there exist unique vectors $u \in U$ and $u' \in U^\perp$ such that $x = u+u'$. Show the following:

a) $P_U(x) = u$.

b) $P_U : \mathbb{R}^n \to U$ is linear.

(2+2 P.)

***2.4 (Level-boundedness and existence of minimizers)** Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous such that for all $\alpha \in \mathbb{R}$ the *sublevel set*

$$\mathrm{lev}_\alpha f := \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$$

is bounded (possibly empty). Show that f takes a minimum on $\mathbb{R}^n$.

(4 P.)

2.5 (Convex Optimization) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad (x, y \in \mathbb{R}^n,\ \lambda \in [0,1]).$$

For a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a convex set $X \subset \mathbb{R}^n$ consider the optimization problem

$$\min f(x) \quad \text{s.t.} \quad x \in X.$$

Prove the following:

a) $\mathrm{argmin}_X f$ is convex (possibly empty).

*b) A point $\bar{x} \in X$ is a *local minimizer of f over X*[1] if and only if $\bar{x} \in \mathrm{argmin}_X f$.

(2+3 P.)

Remarks

- *This homework has to be submitted in class on **September 19, 2017**.*
- *The problems marked with '*' are only compulsory for the honors students (MATH 487). It goes without saying that everybody else is also encouraged to tackle them.*
- *You may (and are encouraged to) work with other students in the class on the assignments, however, the work you submit must be your own. You must also cite any materials you have used to complete your work. Since both the midterm and the final exam will, in part, be heavily based on the homework assignments, it is strongly recommended to take the homework seriously.*

[1] There exists $\varepsilon > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in B_\varepsilon(\bar{x}) \cap X$.