USING MACHINE LEARNING FOR NON LINEAR EDDY-VISCOSITY MODELS

Lars Davidson

NON-LINEAR $k - \varepsilon$ MODEL

- Non-linear $k-\omega$ model by Craft *et al.* [1]. $a_{ij} \equiv \frac{\overline{v_i' v_j'}}{k} - \frac{2}{3}\delta_{ij}$

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\begin{aligned}
a_{ij} = & \boxed{-2c_\mu \tau \bar{S}_{ij}} + c_1\tau^2\left(\bar{S}_{ik}\bar{S}_{kj} - \frac{1}{3}\bar{S}_{\ell k}\bar{S}_{\ell k}\delta_{ij}\right) + c_2\tau^2\left(\bar{\Omega}_{ik}\bar{S}_{kj} - \bar{S}_{ik}\bar{\Omega}_{kj}\right) \qquad (1)\\
& + c_3\tau^2\left(\bar{\Omega}_{ik}\bar{\Omega}_{jk} - \frac{1}{3}\bar{\Omega}_{\ell k}\bar{\Omega}_{\ell k}\delta_{ij}\right) + c_4\tau^3\left(\bar{S}_{ik}\bar{S}_{k\ell}\bar{\Omega}_{\ell j} - \bar{\Omega}_{i\ell}\bar{S}_{\ell k}\bar{S}_{kj}\right) + c_5\tau^3\left(\bar{\Omega}_{i\ell}\bar{\Omega}_{\ell m}\bar{S}_{mj} + \bar{S}_{i\ell}\bar{\Omega}_{\ell m}\right.\\
& - c_5\frac{2}{3}\bar{\Omega}_{mn}\bar{\Omega}_{n\ell}\bar{S}_{\ell m}\delta_{ij} + c_6\tau^3\bar{S}_{k\ell}\bar{S}_{k\ell}\bar{S}_{ij} + c_7\tau^3\bar{\Omega}_{k\ell}\bar{\Omega}_{k\ell}\bar{S}_{ij}, \quad \bar{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} - \frac{\partial \bar{v}_j}{\partial x_i}\right), \quad \tau = k/\varepsilon
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- Fully developed channel flow includes only $c_1 - c_3$

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\begin{aligned}
a_{11} &= \frac{1}{12}\tau^2\left(\frac{\partial \bar{v}_2}{\partial x_1}\right)^2(c_1 - 6c_2 + c_3), \quad a_{22} = \frac{1}{12}\tau^2\left(\frac{\partial \bar{v}_2}{\partial x_1}\right)^2(c_1 + 6c_2 + c_3) \\
a_{33} &= -\frac{1}{6}\tau^2\left(\frac{\partial \bar{v}_2}{\partial x_1}\right)^2(c_1 + c_3)
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
\overline{v_1'^2} &= \frac{k\nu_t}{12\tilde{\varepsilon}}\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2 (c_0+6c_2)+\frac{2}{3}k, \quad \overline{v_2'^2} = \frac{k\nu_t}{12\tilde{\varepsilon}}\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2 (c_0-6c_2)+\frac{2}{3}k \\
\overline{v_3'^2} &= -\frac{k\nu_t}{6\tilde{\varepsilon}}\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2 c_0+\frac{2}{3}k, \quad \overline{v_1' v_2'} = -\nu_t\frac{\partial \bar{v}_1}{\partial x_2}, \quad \tilde{\varepsilon}=\varepsilon-\nu\left(\frac{\partial k^{1/2}}{\partial y}\right)^2, \quad \nu_t = c_\mu\frac{k^2}{\tilde{\varepsilon}} \\
c_\mu &= \frac{0.667 r_\eta\left[1-\exp(-0.415\exp(1.3\eta^{5/6})\right]}{1+1.8\eta}, \quad \eta = r_\eta\frac{k}{\tilde{\varepsilon}}\frac{\partial \bar{v}_1}{\partial x_2}, \quad R_t=\frac{k^2}{\nu\tilde{\varepsilon}} \\
A_2 &= a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-2a_{12}a_{21} \\
f_\mu &= 1.1(\tilde{\varepsilon}/\varepsilon)^{1/2}\frac{1-0.8\exp(-R_t/30))}{1+0.6A_2+0.2A_2^{3.5}} \\
r_\eta &= 1+(1-\exp(-(2A_2)^3))\left[1+4\left(\exp(-R_t/20)\right)^{1/2}\right] \\
f_q &= \frac{r_\eta}{1+0.0086\eta^2)^{1/2}}, \quad c_1=-0.05\frac{f_q}{f_\mu}, \quad c_2 = 0.11\frac{f_q}{f_\mu}, \quad c_3=0.21\frac{f_q}{f_\mu}
\end{aligned}
$$

THE LOW-RE NUMBER MODEL BY CRAFT *et al.* [1]: PREDICTIONS

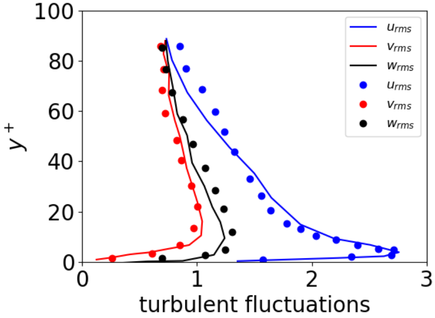

FIGURE: Turbulent fluctuations in fully developed-channel flow at $Re_b = 13\,750$. Lines: predictions taken from [1]; symbols: DNS data taken from [1]

With $c_0 \equiv c_1 + c_3$, Eq. 2 at Slide 2 reads

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\overline{v_3'^2} &= -\frac{k}{6}\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2 c_0+\frac{2}{3}k, \quad \overline{v_1'v_2'} = -c_\mu\tau\frac{\partial \bar{v}_1}{\partial x_2}, \quad c_\mu = 0.09, \quad \tau = k/\varepsilon
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c_0 = -\frac{6a_{33}}{\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2}, \quad c_2 = \frac{2a_{11}+a_{33}}{\tau^2\left(\frac{\partial \bar{v}_1}{\partial x_2}\right)^2}
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Now let's train a Neural Network model (`pytorch` in Python) using DNS data to compute c_0 and c_2.

Output: c_0 and c_2 (`pytorch` accepts multiple output variables).

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Input: I've tried four options

Input (influence) variable and output variables

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Input: I've tried four options

1. $(\partial U^+/\partial y^+)^2$

Output: c_0 and c_2 (`pytorch` accepts multiple output variables).

Input: I've tried four options

1. $(\partial U^+/\partial y^+)^2$
2. $(\partial U^+/\partial y^+)^2$ and $(\partial U^+/\partial y^+)^{-1}$

Output: c_0 and c_2 (`pytorch` accepts multiple output variables).

Input: I've tried four options

1. $(\partial U^+/\partial y^+)^2$
2. $(\partial U^+/\partial y^+)^2$ and $(\partial U^+/\partial y^+)^{-1}$
3. $T^2\,(\partial U/\partial y)^2$ and $T\,(\partial U^+/\partial y^+)^{-1}$, $T = k/\varepsilon$

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4. $(\partial U^+/\partial y^+)^2$ and k^+/ε^+

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Other possible input options:

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Other possible input options:

1. $\nu_t^+ \equiv \nu_t/\nu$

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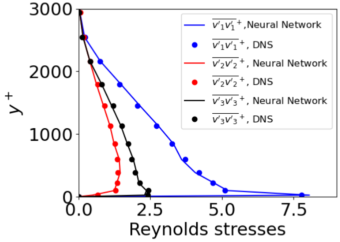
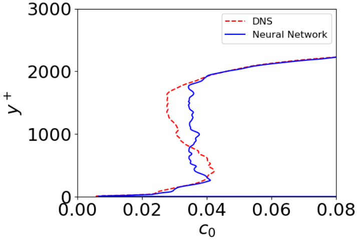
Other possible input options:

1. $\nu_t^+ \equiv \nu_t/\nu$
2. y^+

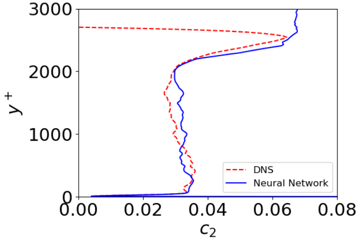
Neural Network: DNS data, boundary-layer $Re_\theta = 8\,180$ [4]



(A) Reynolds stresses.

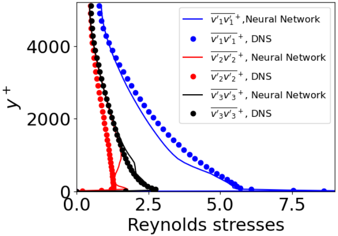


(B) Predicted c_0 coefficient.



(C) Predicted c_2 coefficient.

- Predicted Reynolds stresses of channel flow with trained NN (no CFD)



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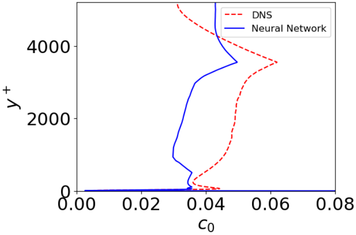

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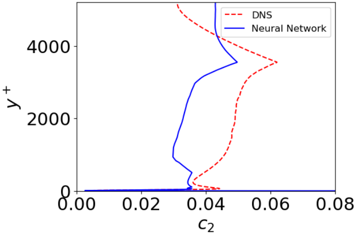
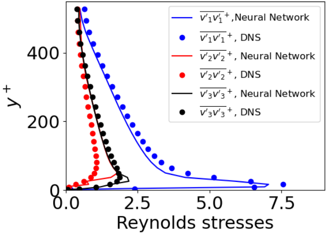


(C) Predicted c_2 coefficient.

- Below I show predicted Reynolds stresses with trained NN (no CFD)



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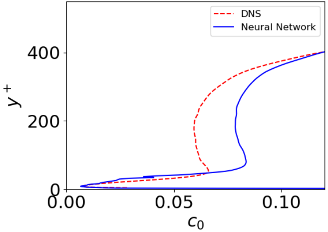

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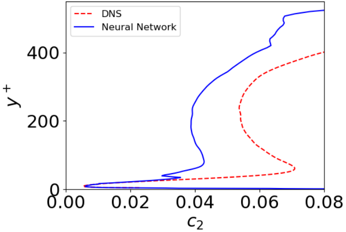

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[1] T. J. Craft, B. E. Launder, and K. Suga. Prediction of turbulent transitional phenomena with a nonlinear eddy-viscosity model. *International Journal of Heat and Fluid Flow*, 18:15–28, 1997.

[2] L. Davidson. pyCALC-LES: a Python code for DNS, LES and Hybrid LES-RANS. Division of Fluid Dynamics, Dept. of Mechanics and Maritime Sciences, Chalmers University of Technology, Gothenburg, 2021.

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[4] G. Eitel-Amor, R. Orlu, and P. Schlatter. Simulation and validation of a spatially evolving turbulent boundary layers up to $Re_\theta = 8300$. *International Journal of Heat and Fluid Flow*, 47:57–69, 2014.

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