

Maximum Bipartite Matching

Abrar Jahin Sarker¹
Abdullah Muhammed Amimul Ehsan²
Mostafa Rifat Tazwar³

¹2005012

²2005017

³2005020

^{1,2,3}Department of Computer Science and Technology, BUET

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Introduction

Definition

A bipartite graph is one whose vertices can be split into two independent groups U, V such that every edge connects between U and V .

Note

There can not be any edge in between U and in between V .
The graph is two colourable and doesn't have cycles of odd length.

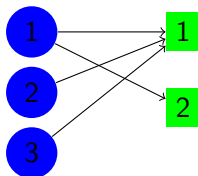


Fig: Visualization of bipartite graph

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The Problem

Maximum Bipartite Matching

Given a bipartite graph $G = (A \cup B, E)$, find an
 $\{ S \subseteq A \times B : S \text{ is a matching and is as large as possible. } \}$
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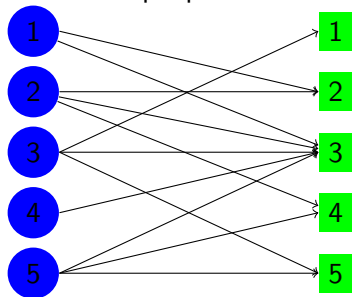
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Example

In a picnic, there are 5 people and 5 food items. Some people express interest in some of the items. How can we satisfy maximum number of people while wasting minimum number of items?



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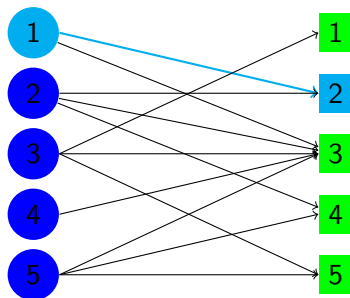
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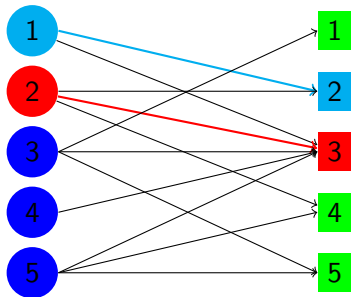
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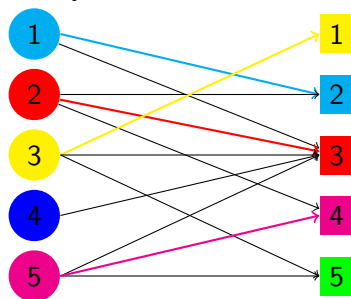


Greedy Approach



Greedy Approach

Here, person 4 can not have anything and person 5 got a match luckily.



We could only satisfy four guests. One guest is unhappy.

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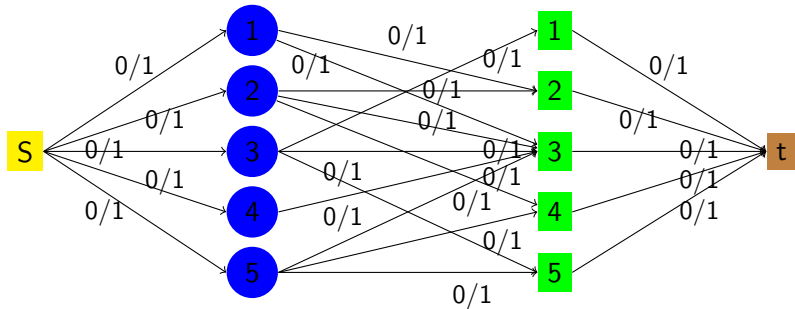
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Reducing to Network Flow Problem

Approach:

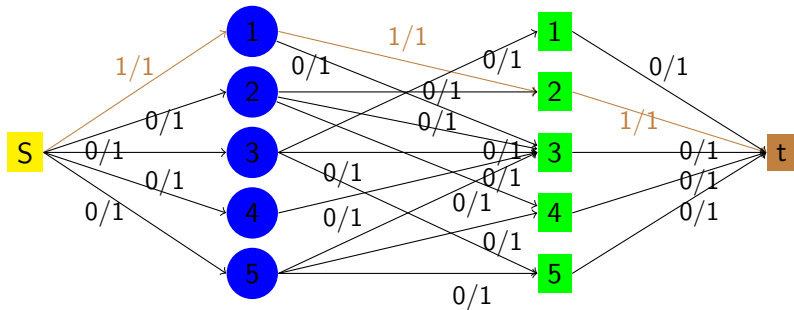
- Make all the edges directed if not and add 0 flow and 1 capacity for all edges.Expressed as 0/1 in Flow/Capacity format
- add two new nodes:
 - 1 Source(S)
 - 2 Destination(t)
- Add nodes from source to People with flow/capacity=0/1.
- Add nodes from food items to destination with 0 flow and 1 capacity
- After applying network flow algorithm, flow_i0 between the pairs indicates a matching.



Edmonds-Karp Algorithm to Solve Network Flow Problem

- Initialize flow $f(u, v) = 0$ for all edges (u, v) in the graph.
- Repeat the following steps until no augmenting paths can be found:
 - Use BFS to find the shortest augmenting path from source to sink.
 - If no augmenting path is found, terminate.
 - Let P be the augmenting path found by BFS.
 - Let $cf(P)$ be the minimum residual capacity along path P .
 - For each edge (u, v) in P :
 - Update flow $f(u, v) = f(u, v) + cf(P)$.
 - Update flow $f(v, u) = f(v, u) - cf(P)$ for the reverse edge.
- The value of the maximum flow is the sum of flow values leaving the source.

Solution to Network Flow Problem



The diagram illustrates a flow network with a source node S (yellow square), five intermediate nodes (blue circles labeled 1 to 5), five intermediate nodes (green squares labeled 1 to 5), and a sink node t (brown square). Edges are labeled with flow/capacity pairs. The network structure is as follows:

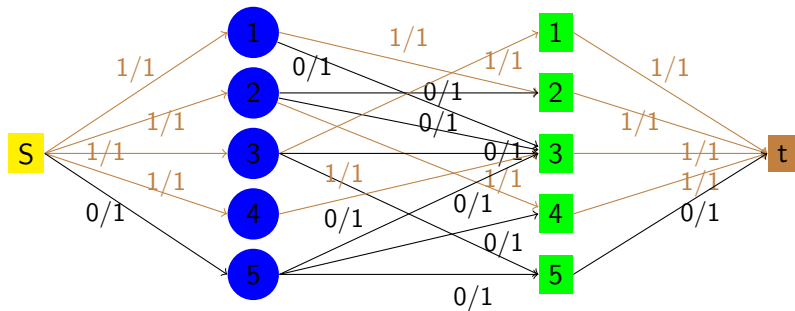
- Source S connects to intermediate nodes 1, 2, 3, 4, and 5 (blue circles) with edges labeled $1/1$, $1/1$, $0/1$, $0/1$, and $0/1$ respectively.
- Intermediate nodes 1, 2, 3, 4, and 5 (blue circles) connect to intermediate nodes 1, 2, 3, 4, and 5 (green squares) with edges labeled $0/1$, $0/1$, $0/1$, $0/1$, and $0/1$ respectively.
- Intermediate nodes 1, 2, 3, 4, and 5 (green squares) connect to sink t with edges labeled $0/1$, $1/1$, $0/1$, $1/1$, and $0/1$ respectively.

The flow path highlighted in orange is: $S \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow t$.

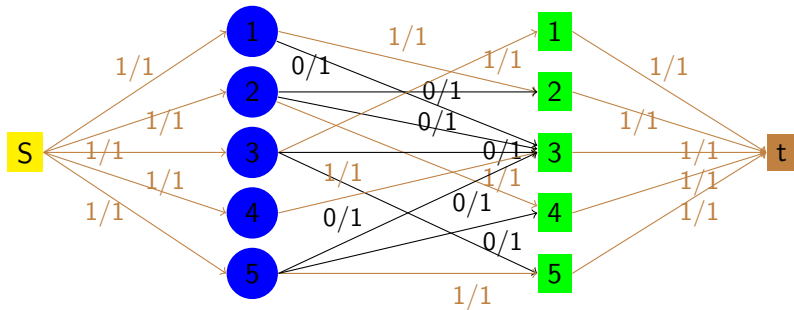
The diagram illustrates a flow network with a source node S (yellow square) and a sink node t (brown square). There are two columns of intermediate nodes, each containing nodes 1 through 5. The left column nodes are blue circles, and the right column nodes are green squares. Edges connect S to the left column, between the two columns, and from the right column to t . Each edge is labeled with a flow value in the form a/b , where a is the flow and b is the capacity. Orange edges represent a flow of 1, and black edges represent a flow of 0. The network structure and flow values are as follows:

- From S to left column: $S \rightarrow 1$ (1/1), $S \rightarrow 2$ (1/1), $S \rightarrow 3$ (1/1), $S \rightarrow 4$ (0/1), $S \rightarrow 5$ (0/1).
- Between columns:
 - Left 1 to Right 1: 1/1
 - Left 1 to Right 2: 0/1
 - Left 1 to Right 3: 0/1
 - Left 2 to Right 1: 0/1
 - Left 2 to Right 2: 0/1
 - Left 2 to Right 3: 0/1
 - Left 3 to Right 1: 0/1
 - Left 3 to Right 2: 0/1
 - Left 3 to Right 3: 0/1
 - Left 4 to Right 1: 0/1
 - Left 4 to Right 2: 0/1
 - Left 4 to Right 3: 0/1
 - Left 5 to Right 1: 0/1
 - Left 5 to Right 2: 0/1
 - Left 5 to Right 3: 0/1
 - Left 5 to Right 4: 0/1
 - Left 5 to Right 5: 0/1
- From right column to t : Right 1 $\rightarrow t$ (1/1), Right 2 $\rightarrow t$ (1/1), Right 3 $\rightarrow t$ (0/1), Right 4 $\rightarrow t$ (1/1), Right 5 $\rightarrow t$ (0/1).

Solution to Network Flow Problem



Solution to Network Flow Problem



Analyzing the solution

Here, we can not augment the paths anymore. We have already found the max flow of the network. The max flow = 5. Hence, Number of matching = 5.

Pairs are :

- 1 Person 1, Item 2
- 2 Person 2, Item 4
- 3 Person 3, Item 1
- 4 Person 4, Item 3
- 5 Person 5, Item 5

Using greedy approach, we found 4 pairs which was not maximum.

Extension of Maximum Bipartite matching

Allowing multiple items for a person

What if a person is allowed to take multiple items?

Solution

The capacity of edge connected the source and a person would be equal to the number of items that person is allowed to take. Then Network flow algorithm should be applied as usual.

Extension of Maximum Bipartite matching

Multiple copies of item available

What if an item has multiple instances?

Solution

The capacity of edge connected the item and Destination would be equal to the number of copies of the item . Then Network flow algorithm should be applied as usual.

Extension of Maximum Bipartite matching

Taking same item multiple times?

What if an item can be taken multiple times by a person?

Solution

The capacity of edge connected between the person and the item would be equal to the number of times the item can be taken by that person. Then Network flow algorithm should be applied as usual.

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References



Jon Kleinberg and Eva Tardos, *Algorithm design*, Pearson Education India, 2006.