Maximum Bipartite Matching

Abrar Jahin Sarker[1]
Abdullah Muhammed Amimul Ehsan[2]
Mostafa Rifat Tazwar[3]

[1]2005012

[2]2005017

[3]2005020

[1,2,3]Department of Computer Science and Technology, BUET

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Outline

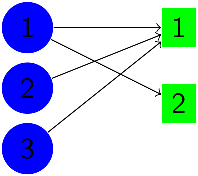
Introduction

Bipartite Graph

A bipartite graph is one whose vertices can be split into two independent groups U,V such that every edge connects vertices of different groups.

Note

There can not be any edge between two vertices of U or two vertices of V .
The graph is two colourable and doesn't have cycles of odd length.



Outline

The Problem

Maximum Bipartite Matching

Given a bipartite graph $G = (A \cup B, E)$, find an $\{\ S \subseteq A \times B\ :\ S$ is a matching and is as large as possible. $\}$ [KT06]

Outline

Example

In a picnic,there are 5 people and 5 food items.Some people express interest in some of the items.How can we satisfy maximum number of people while wasting minimum number of items?

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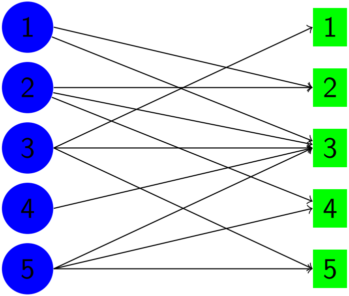
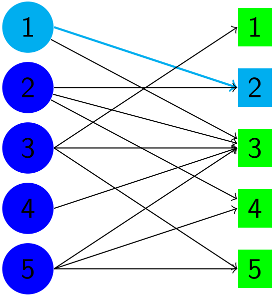


Outline

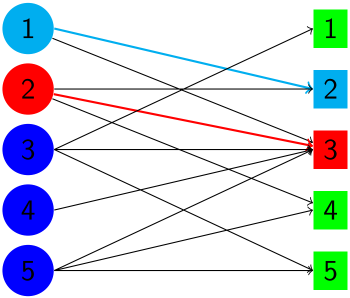
Greedy Approach

Person 1 starts by taking the first available item



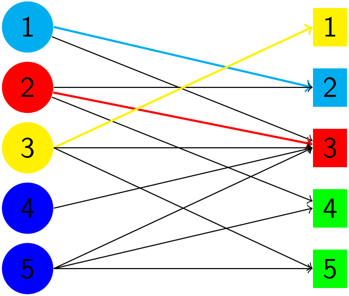
Greedy Approach

Making item 3, the only valid choice for person 2



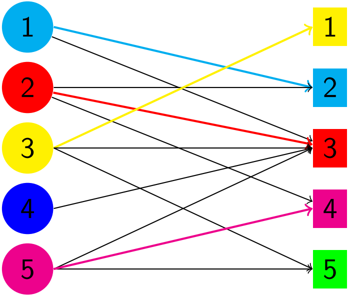
Greedy Approach

Here,item 1 was luckily unoccupied



Greedy Approach

Here,person 4 can not have anything and person 5 got a match luckily.



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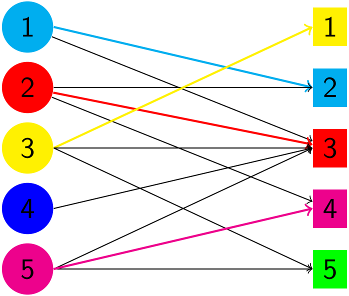

We could only satisfy four guests.One guest is unhappy.

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- Reduce it to Max Flow Problem.

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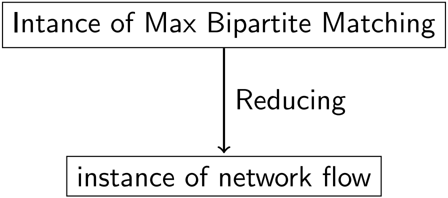
- Where the solution to the network flow problem can easily be used to find the solution to the bipartite matching.



Reducing to Network Flow Problem

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- Make all the edges directed if not and add 0 flow and 1 capacity for all edges.Expressed as 0/1 in Flow/Capacity format

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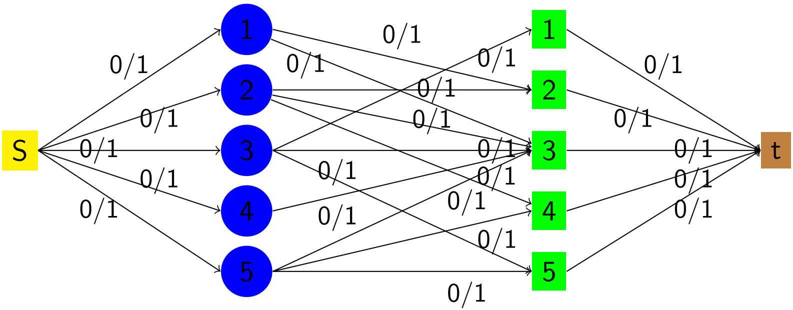
- Add nodes from food items to destination with 0 flow and 1 capacity
- After applying network flow algorithm, flow > 0 between the pairs indicates a matching.

Reduced to Network Flow

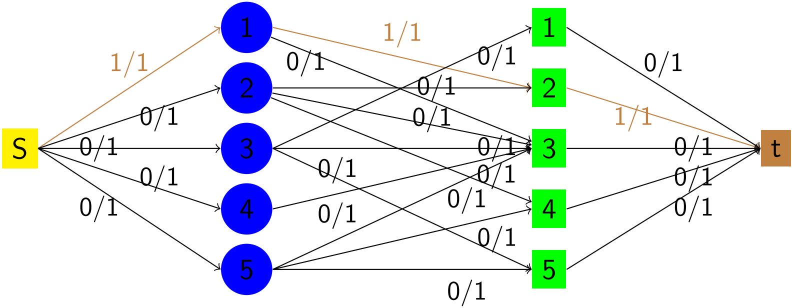


Edmonds-Karp Algorithm to Solve Network Flow Problem

- Initialize flow $f(u, v) = 0$ for all edges (u, v) in the graph.
- Repeat the following steps until no augmenting paths can be found:
  - Use BFS to find the shortest augmenting path from source to sink.
  - If no augmenting path is found, terminate.
  - Let P be the augmenting path found by BFS.
  - Let $cf(P)$ be the minimum residual capacity along path P.
  - For each edge (u, v) in P:
    - Update flow $f(u, v) = f(u, v) + cf(P)$.
    - Update flow $f(v, u) = f(v, u) - cf(P)$ for the reverse edge.
- The value of the maximum flow is the sum of flow values leaving the source.

Outline

Solution to Network Flow Problem



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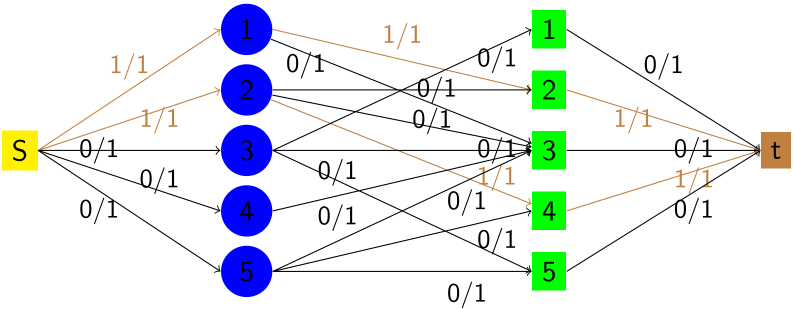

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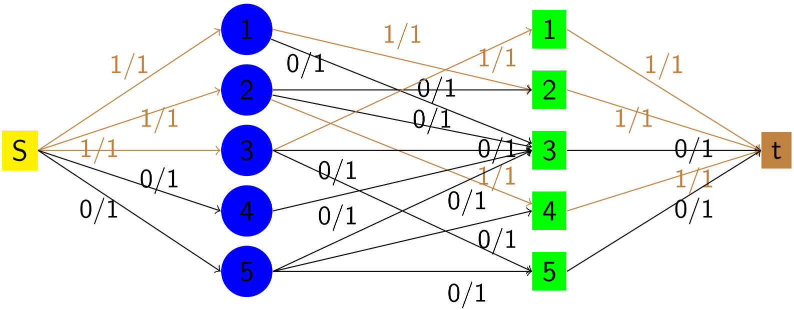

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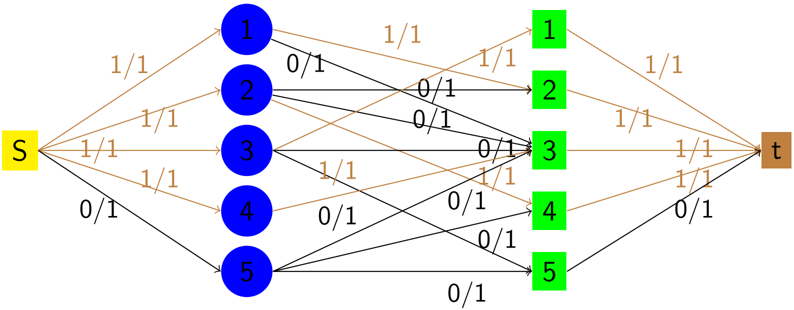

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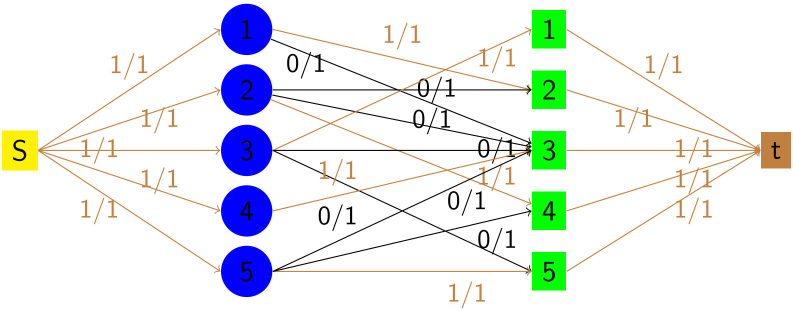

Analyzing the solution

Here,we can not augment the paths anymore.We have already found the max flow of the network.The max flow=5.Hence,Number of matching=5.

Pairs are :

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2. Person 2,Item 4
3. Person 3,Item 1

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2. Person 2,Item 4
3. Person 3,Item 1
4. Person 4,Item 3

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2. Person 2,Item 4
3. Person 3,Item 1
4. Person 4,Item 3
5. Person 5,Item 5

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2. Person 2,Item 4
3. Person 3,Item 1
4. Person 4,Item 3
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Using greedy approach,we found 4 pairs which is not maximum.

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3. C $=$ number of persons $=$ n
4. $m' = 2n + m$ where m $=$ number of edges in the original graph and n $=$ number of people.
5. Hence, the running time of the algorithm is $O((2n + m)n) = O(mn + n^2) = O(nm)$.

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Extension of Maximum Bipartite Matching

Allowing multiple items for a person

What if a person is allowed to take multiple items?

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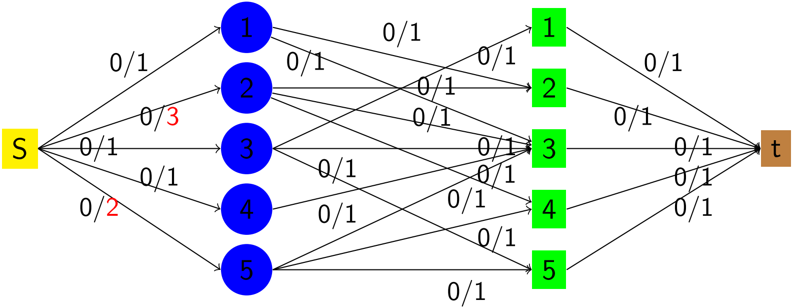
What if a person is allowed to take multiple items?

Solution

The capacity of edge connected the source and a person would be equal to the number of items that person is allowed to take.Then Network Flow algorithm should be applied as usual.

Extension of Maximum Bipartite Matching

For example if Person 2 is allowed to take 3 items and person 5 is allowed to take 2 items:



Extension of Maximum Bipartite Matching

Multiple copies of item available

What if an item has multiple instances?

Extension of Maximum Bipartite Matching

Multiple copies of item available

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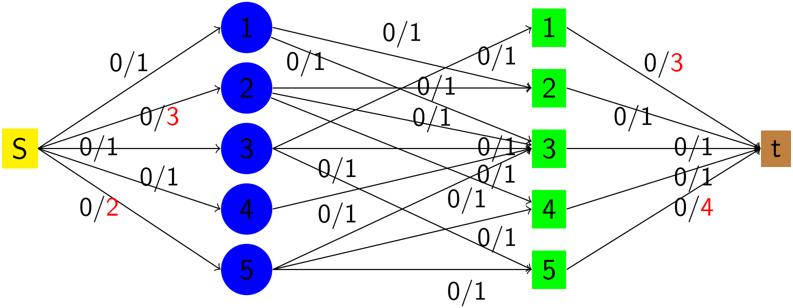
Solution

The capacity of edge connected the item and Destination would be equal to the number of copies of the item. Then Network Flow algorithm should be applied as usual.

Extension of Maximum Bipartite Matching

For example if item 1 has 3 copies and item 5 has 4 copies:



Extension of Maximum Bipartite Matching

Taking same item multiple times?

What if an item can be taken multiple times by a person?

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Taking same item multiple times?

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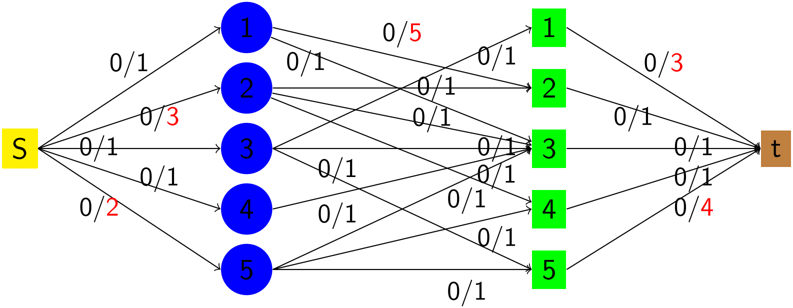
Solution

The capacity of edge connected between the person and the item would be equal to the number of times the item can be taken by that person. Then Network Flow algorithm should be applied as usual.

Extension of Maximum Bipartite Matching

For example if Person1 can take Item 2 upto 5 times:



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References

Jon Kleinberg and Eva Tardos, *Algorithm design*, Pearson Education India, 2006.