Directed Graphical Models

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Directed Graphical Models

- Probabilistic graphical models (PGMs) are a rich framework for encoding probability distributions over complex domains [Koller and Friedman, 2009].
- In this class we will focus on directed graphical models (DGMs), which are one type of PGM.
- Directed graphical models (DGMs) are a family of probability distributions that admit a compact parametrization that can be naturally described using a directed graph.
- DGMs are also known as Bayesian networks.
- Statistical inference for DGMs can be performed using frequentist or Bayesian methods, so it is misleading to call them Bayesian networks [Wasserman, 2013].

Directed Acyclic Graphs (DAGs)

- A directed graph consists of a set of nodes with arrows between some nodes.
- Graphs are useful for representing independence relations between variables.
- More formally, a directed graph G consists of a set of vertices V and an edge set E of ordered pairs of vertices.
- For our purposes, each vertex corresponds to a random variable.
- If $(Y, X) \in E$ then there is an arrow pointing from Y to X.

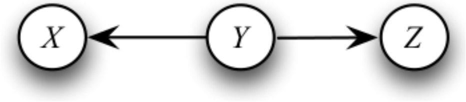


Figure: A directed graph with vertices $V = \{X, Y, Z\}$ and edges $E = \{(Y, X), (Y, Z)\}$.

Directed Acyclic Graphs (DAGs)

- If an arrow connects two variables X and Y (in either direction) we say that X and Y are adjacent.
- If there is an arrow from X to Y then X is a parent of Y and Y is a child of X.
- The set of all parents of X is denoted by π_X or $\pi(X)$.
- A directed path between two variables is a set of arrows all pointing in the same direction linking one variable to the other such as the chain shown below:

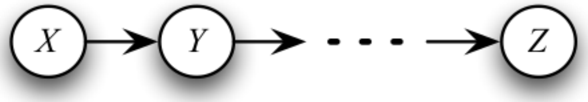
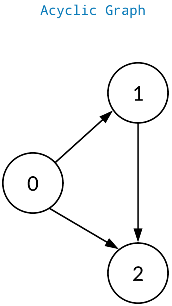


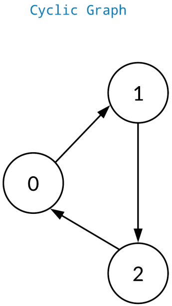
Figure: A chain graph with a directed path.

- X is an ancestor of Y if there is a directed path from X to Y (or X = Y).
- We also say that Y is a descendant of X.

Directed Acyclic Graphs (DAGs)

- A directed path that starts and ends at the same variable is called a cycle.
- A directed graph is acyclic if it has no cycles.
- In this case we say that the graph is a directed acyclic graph or DAG.





- From now on, we only deal with directed acyclic graphs since it is very difficult to provide a coherent probability semantics over graphs with directed cycles.
- For the remainder of this class we will use the terms Bayesian Network, directed graphical model (DGM) and directed acyclical graph (DAG) interchangeably.

Probability and DAGs

- An important concept we need to introduce to understand DAGs is the chain rule of probability.
- For a set of random variables $X_1, \ldots, X_n$ we can write the joint probability function $f(x_1, x_2, \ldots, x_n)$ as

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2|x_1)\ldots f(x_n|x_{n-1}, \ldots, x_2, x_1).$$

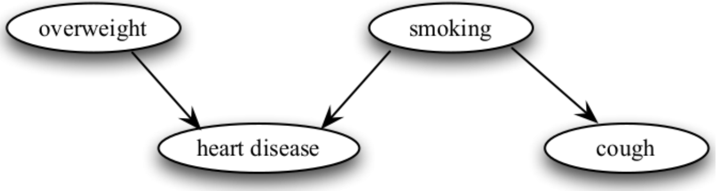
- A DAG is a distribution in which each factor on the right hand side depends only on a small number of ancestor variables $\pi(x)$. [Ermon and Kuleshov,]
- Let G be a DAG with vertices $V = (X_1, \ldots, X_d)$.
- If P is a distribution for V with probability function $f(x)$ (density or masss), we say that G represents P , if

$$f(x) = \prod_{j=1}^{d} f(x_j|\pi_{x_j})$$

where π_{x_j} is the set of parent nodes of X_j

Probability and DAGs

- The next figure shows a DAG with four variables.



- The probability function takes the following decomposition:
- $f(\text{overweight, smoking, heart disease, cough}) = f(\text{overweight}) \times f(\text{smoking}) \times f(\text{heart, disease}|\text{ overweight, smoking}) \times f(\text{cough}|\text{smoking})$.

Conditional Independence

- Let X, Y and Z be random variables.
- X and Y are conditionally independent given Z, written $X \perp Y|Z$, if:

  $$f(x, y|z) = f(x|z)f(y|z)$$

  for all x, y and z.
- Notice that f can be either a density function for continous random variables or a probability mass function for discrete random variables.
- Intuitively, this means that, once you know Z, Y provides no extra information about X.

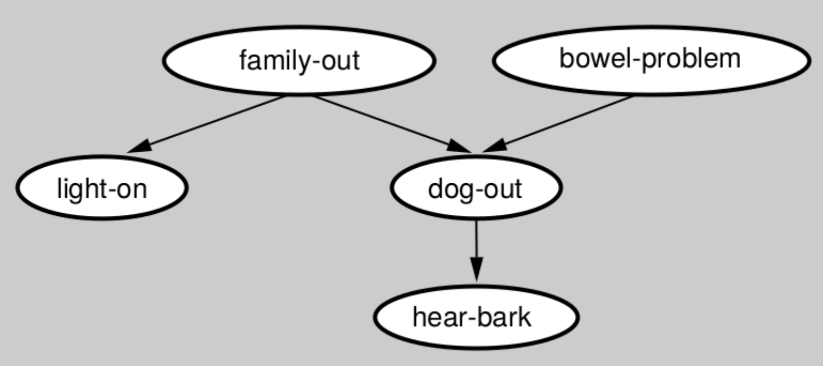
An Example

- The best way to understand DAGs is to imagine trying to model a situation in which causality plays a role.
- And also our understanding of what is actually going on is incomplete
- So we need to describe things probabilistically.
- The following example is based on [Charniak, 1991].
- Eugene Charniak is a famous AI researcher who's got the following situation.
- When he goes home at night, he wants to know if his family is home before trying the doors.
- Often when his wife leaves the house, she turns on an outdoor light.

An Example

- Eugene’s wife can also turn on the outdoor light if she is expecting a guest.
- Also, they have a female dog.
- When nobody is home, the dog is put in the back yard.
- The same is true if the dog has bowel troubles.
- Finally, if the dog is in the backyard, Eugene’s will probably hear her barking
- The next slide shows a DAG encoding all the above causal relationships.

An Example



- The DAG can help to predict what will happen in a particular scenario (if his family goes out, the dog goes out)
- Or to infer causes from observed effects (if the light is on and the dog is out, then his family is probably out).

D-separation

- sdsad

Plate Notation

- sdsad

Estimation for DAGs

- Two estimation questions arise in the context of DAGs.
- First, given a DAG $\mathcal{G}$ and data $d_1, \ldots, d_n$ from a distribution f consistent with $\mathcal{G}$, how do we estimate f?
- Second, given data $d_1, \ldots, d_n$ how do we estimate $\mathcal{G}$?
- The first question is pure estimation while the second involves model selection.
- These are very involved topics and are beyond the scope of this course.
- We will just briefly mention the main ideas.

Estimation for DAGs

- If we are doing frequentist inference, we typically use some parametric model $f(x|\pi_x;\theta_x)$ for each conditional density.
- The likelihood function is then

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f(d_i;\theta)) = \prod_{i=1}^{n}\prod_{j=1}^{m} f(X_{ij}|\pi_j;\theta_j)$$

- where X_{ij} is the value of X_j for the ith data point and j are the parameters for the-jth conditional density.
- We can then estimate the parameters by maximum likelihood.
- On the other hand, if we want to perform Bayesian inference we must set priors for all our variables $X_1,\ldots,X_m$ and estimate the posterior accordingly.

Estimation for DAGs

- To estimate the structure of the DAG itself, we could fit every possible DAG using maximum likelihood and use AIC (or some other method) to choose a DAG.
- However, there are many possible DAGs so we would need much data for such a method to be reliable.
- Also, searching through all possible DAGs is a serious computational challenge.
- Producing a valid, accurate confidence set for the DAG structure would require astronomical sample sizes.
- If prior information is available about part of the DAG structure, the computational and statistical problems are at least partly ameliorated [Wasserman, 2013].

Conclusions

- Blabla

References I

Charniak, E. (1991).
Bayesian networks without tears.
AI magazine, 12(4):50–50.

Ermon, S. and Kuleshov, V.
Cs228 notes.

Koller, D. and Friedman, N. (2009).
Probabilistic graphical models: principles and techniques.
MIT press.

Wasserman, L. (2013).
All of statistics: a concise course in statistical inference.
Springer Science & Business Media.