

Descriptive Statistics

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Exploratory Data Analysis

- Descriptive Statistics or Exploratory Data Analysis or (EDA) encompasses a set of techniques to quickly understand the nature of a data collection or **dataset**.
- The main goal of descriptive statistics is to explore the data to find some patterns that can be exploited to generate hypotheses.
- It was proposed by the statistician John Tukey.
- It is based mainly on two types of techniques: **summary statistics** and **data visualization**.
- In this class you will see both types of techniques, in addition to their application in R for some toy datasets.
- This class is partially based on chapter 3 of [Tan et al., 2016].

The Iris dataset

- We will work with a well-known dataset called **Iris**.
- The dataset consists of 150 observations of iris plant flowers.
- There are three types of iris flower classes: **virginica**, **setosa** and **versicolor**.
- There are 50 observations of each.
- The variables or attributes measured for each flower are:
 - 1 The type of flower as a categorical variable.
 - 2 The length and width of the petal in cm as numerical variables.
 - 3 The length and width of the sepal in cm as numeric variables.



The Iris dataset



Figure: Virginica - Setosa - Versicolor

- The dataset is available in R:

```
> data(iris) # to load the dataset into the workspace  
> names(iris)  
[1] "Sepal.Length" "Sepal.Width"  "Petal.Length"  
     "Petal.Width"  "Species"
```

- In order to access the variables directly in our workspace we use the command `attach(iris)`.

Summary Statistics

- Summary statistics are values that explain properties of the data.
- Some of these properties include: frequencies, measures of central tendency and dispersion.
- Example:
 - Central tendency: mean, median, mode.
 - Variation: measure the variability of the data, such as standard deviation, range, etc..
- Most summary statistics can be calculated by making a single pass through the data.

Frequency and Mode

- The absolute frequency of an attribute value is the number of times it is observed.
- The relative frequency is the absolute frequency divided by the total number of examples.
- In R we can count the frequencies of occurrence of each distinct value of a vector using the command `table`:

```
> table(iris$Species)
      setosa versicolor  virginica
      50         50         50
> vec<-c(1,1,1,0,0,3,3,3,3,2)
> table(vec)
vec
0 1 2 3
2 3 1 4
```

- Exercise: Calculate the relative frequencies of the above vector.

```
> table(vec)/length(vec) # Relative frequencies
vec
0 1 2 3
0.2 0.3 0.1 0.4
```

Frequency and Mode (2)

- The mode of an attribute is the most frequent value observed.
- The mode function is not implemented natively in R, but it is easy to calculate using `table` and `max`:

```
my_mode<-function(var) {  
  freq.var<-table(var)  
  value<-which(freq.var==max(freq.var))  
  as.numeric(names(value))  
}  
> my_mode(vec)  
[1] 3  
> my_mode(iris$Sepal.Length)  
[1] 5
```

- We generally use frequencies and mode to study categorical variables.

Central Tendency Measures

- These measures attempt to summarize the observed values into a single value associated with the centrally located value.
- The mean is the most common measure of central tendency for a numeric variable.
- If we have n observations it is calculated as the arithmetic mean or average.

$$\text{mean}(x) = \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

- The major problem with the mean is that it is very sensitive to **outliers** or outliers.
- We take a random vector of mean 20 and then add a random element that comes from a distribution of much larger mean. We see that the mean is strongly affected by noise:

```
> vec<-rnorm(10,20,10)
> mean(vec)
[1] 16.80036
> vec.noise<-c(vec,rnorm(1,300,100))
> mean(vec.noise)
[1] 35.36422
```


Central Tendency Measures (2)

- We can robust the mean by removing a fraction of the extreme values using the **trimmed mean**.
- In R we can give a second parameter to the function `mean` called `trim` that defines the fraction of extreme elements to discard.
- Example: We discard 10% of the extreme values in the previous example:

```
> mean(vec,trim=0.1)
[1] 17.78799
> mean(vec.noise,trim=0.1)
[1] 19.51609 # much more robust
```

Central Tendency Measures (3)

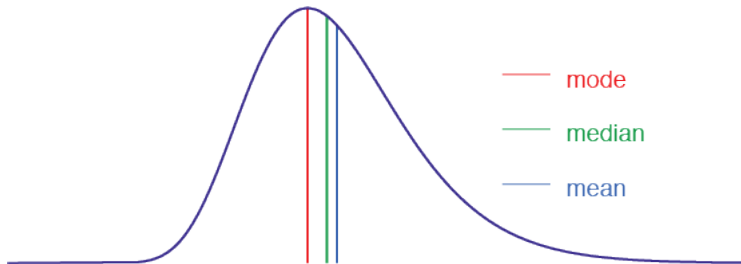
- The median represents the central ranking position of the variable that separates the lower half and the upper half of the observations.
- Intuitively, it consists of the value where for one half of the observations all values are greater than it, and for the other half all are less.

$$\text{median}(x) = \begin{cases} x_{r+1} & \text{If } |x| \text{ (vector length) is odd, } |x| = 2r + 1 \\ \frac{1}{2}(x_r + x_{r+1}) & \text{If } |x| \text{ is even, } |x| = 2r \end{cases}$$

- For the above example, we see that the median is more robust to noise than the mean:

```
> median(vec)
[1] 17.64805
> median(vec.noise)
[1] 17.64839
```

Comparison between mode, median and mean



Percentiles or Quantiles

- The k -th percentile of a numerical variable is a value such that $k\%$ of the observations are below the percentile and $(100 - k)\%$ are above this value.
- Quantiles are equivalent to percentiles but expressed in fractions instead of percentages.
- In R they are calculated with the command `quantile`:

```
# All percentiles  
quantile(Sepal.Length, seq(0, 1, 0.01))
```

- In addition it is very common to talk about the **quantiles** which are three specific percentiles:
 - The first quartile Q_1 (lower quartile) is the percentile with $k = 25$.
 - The second quartile Q_2 is with $k = 50$ which is equivalent to the median.
 - The third quartile Q_3 (upper quartile) is with $k = 75$.

```
# The minimum, the three quartiles and the maximum.  
> quantile(Sepal.Length, seq(0, 1, 0.25))  
 0%   25%   50%   75%  100%  
4.3   5.1   5.8   6.4   7.9
```

Summarizing a Data Frame

- In R we can summarize various summary statistics of a variable or of a data.frame using the command `summary`.
- For numerical variables it gives us the minimum, the quartiles, the mean and the maximum.
- For categorical variables it gives us the frequency table.

```
> summary(iris)
Sepal.Length      Sepal.Width      Petal.Length      Petal.Width
Min.      :4.300      Min.      :2.000      Min.      :1.000      Min.      :0.100
1st Qu.:5.100      1st Qu.:2.800      1st Qu.:1.600      1st Qu.:0.300
Median :5.800      Median :3.000      Median :4.350      Median :1.300
Mean    :5.843      Mean    :3.057      Mean    :3.758      Mean    :1.199
3rd Qu.:6.400      3rd Qu.:3.300      3rd Qu.:5.100      3rd Qu.:1.800
Max.    :7.900      Max.    :4.400      Max.    :6.900      Max.    :2.500

Species
setosa      :50
versicolor:50
virginica   :50
```

Exercise

- Using the command `tapply` analyze the mean, median and quartiles for the three species of **Iris** for the four variables.
- Do you notice any differences in the different species?

```
tapply(iris$Petal.Length, iris$Species, summary)
tapply(iris$Petal.Width, iris$Species, summary)
tapply(iris$Sepal.Length, iris$Species, summary)
tapply(iris$Sepal.Width, iris$Species, summary)
```

Variability Measures

- Variability measures or dispersion measures tell us how different or similar the observations tend to be with respect to a particular value. Usually this value refers to some measure of central tendency.
- The range is the difference between the maximum and minimum value:

```
> max(Sepal.Length) - min(Sepal.Length)
[1] 3.6
```

- The standard deviation is the square root of the variance that measures the mean squared differences of the observations from the mean.

$$\text{var}(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

$$\text{sd}(x) = \sqrt{\text{var}(x)}$$

```
> var(Sepal.Length)
[1] 0.6856935
> sd(Sepal.Length)
[1] 0.8280661
```

Variability Measures (2)

- Like the mean, the standard deviation is sensitive to outliers.
- The most robust measures are usually based on the median.
- Let $m(x)$ be a measure of central tendency of x (usually the median), we define the **average absolute deviation** (AAD) as:

$$\text{AAD}(x) = \frac{1}{n} \sum_{i=1}^n |x_i - m(x)|$$

- Exercise: Program the function `aad` in R, as a function that receives a vector `x` and a central mean function `fun`. The absolute value is calculated with the command `abs`:

```
aad<-function(x, fun=median) {  
  mean(abs(x-fun(x)))  
}  
> aad(Sepal.Length)  
[1] 0.6846667  
> aad(Sepal.Length, mean)  
[1] 0.6875556
```


Variability Measures (3)

- Let b be a constant we define the **mean absolute deviation** as:

$$\text{MAD}(x) = b \times \text{median}(|x_i - m(x)|)$$

- In R is calculated with the command `mad` with the parameters `center` as a function measuring the central tendency of the variable and `constant` as the constant b . By default the median and the value 1.482 is used.

```
> mad(Sepal.Length)
[1] 0.7
```

- Finally, the inter-quartile range (IQR) is defined as the difference between the third and the first quartile ($Q_3 - Q_1$).

```
IQR(Sepal.Length)
[1] 1.3
```

Multivariate Summary Statistics

- To compare how one variable varies with respect to another, we use multivariate measures.
- The covariance $cov(x, y)$ measures the degree of joint linear variation of a pair of variables x, y :

$$cov(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x - \bar{x})(y - \bar{y})$$

- Where $cov(x, x) = var(x)$
- In R it is computed with the command `cov`:

```
> cov(Sepal.Length, Sepal.Width)
[1] -0.042434
```

- If we give it a matrix or a data.frame of numeric variables, it computes a covariance matrix:

```
> cov(iris[,1:4])
```

	Sepal.Length	Sepal.Width	Petal.Length	Petal.Width
Sepal.Length	0.6856935	-0.0424340	1.2743154	0.5162707
Sepal.Width	-0.0424340	0.1899794	-0.3296564	-0.1216394
Petal.Length	1.2743154	-0.3296564	3.1162779	1.2956094
Petal.Width	0.5162707	-0.1216394	1.2956094	0.5810063

Multivariate Summary Statistics (2)

- If two variables are independent of each other, their covariance is zero.
- A measure of relationship that does not depend on the scale of each variable is the **linear correlation**.
- The linear correlation or **Pearson's** correlation coefficient $r(x, y)$ is defined as:

$$r(x, y) = \frac{\text{cov}(x, y)}{\text{sd}(x)\text{sd}(y)}$$

- The linear correlation varies between -1 to 1 .

Multivariate Summary Statistics (3)

- A value close to 1 indicates that as one variable grows the other also grows in a linear proportion.
- A value close to -1 indicates an inverse relationship (one is growing and the other is decreasing).
- If the correlation is close to zero we have linear independence.
- Note that a correlation of zero does not imply that there cannot be a non-linear relationship between the variables.
- In R, the correlation is calculated with the command `cor`.

```
> cor(iris[,1:4])
```

	Sepal.Length	Sepal.Width	Petal.Length	Petal.Width
Sepal.Length	1.0000000	-0.1175698	0.8717538	0.8179411
Sepal.Width	-0.1175698	1.0000000	-0.4284401	-0.3661259
Petal.Length	0.8717538	-0.4284401	1.0000000	0.9628654
Petal.Width	0.8179411	-0.3661259	0.9628654	1.0000000

Contingency Tables

- To analyze the relationship between categorical variables we use **contingency tables**.
- The table is filled with the marginal frequencies of all pairs of values between two categorical variables.
- In R they are created using the command `table` that we used before for computing frequencies, but now giving two vectors as input:

```
gender<-c("Male", "Female", "Male", "Female", "Female", "Male")
studies<-c("college", "postgraduate", "high school",
           "postgraduate", "high school", "college")
```

```
>table(gender, studies)
      studies
gender college high school postgraduate
Female      0          1           2
Male       2          1           0
```

Skweness and Kurtosis

There are other two summary statistics that focus on more complex properties of the data distribution called skeweness and kurtosis [Pipis, 2020].

Skeweness

- Skeweness is a measure of the **asymmetry** of the probability distribution.

$$\text{skewness}(x) = \frac{\sum_{i=1}^n (x_i - \bar{x})^3}{(n-1)sd(x)^3}$$

- It indicates how much our underlying distribution deviates from the normal distribution since the normal distribution has skewness 0.
- Generally, we have three types of skewness:
 - 1 Symmetrical:** the skewness is close to 0 and the mean is almost the same as the median.
 - 2 Negative skew:** the majority of the observations are concentrated on the right tail (the median is greater than the mean).
 - 3 Positive skew:** the majority of the observations are concentrated on the left tail (the median is less than the mean).

Skweness

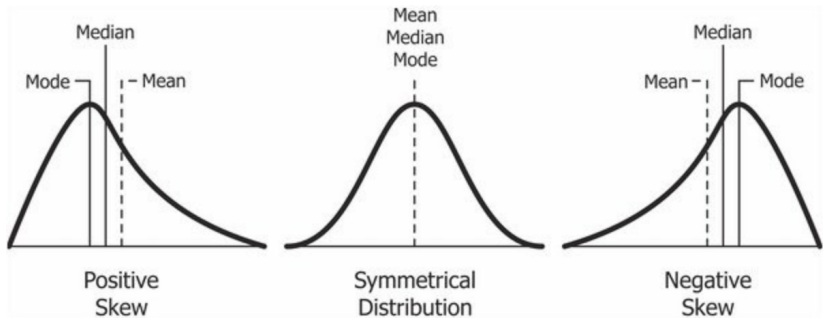


Figure: Source: Wikipedia

Skweness

- In R, we can calculate skweness with the formula **skewness** from the library **moments**:

```
> library(moments)
#positive skew
> skewness(c(1,1,2,3))
[1] 0.4933822
#symmetrical
> skewness(c(1,2,3))
[1] 0
#negative skew
> skewness(c(1,2,3,3))
[1] -0.4933822
```


Kurtosis

Kurtosis

- The Kurtosis describes the “tailedness” of a distribution [Westfall, 2014].

$$\text{kurtosis}(x) = \frac{\sum_{i=1}^n (x_i - \bar{x})^4}{(n-1)sd(x)^4}$$

- Let's see the main three types of kurtosis.
 - 1 **Mesokurtic**: This is the normal distribution (kurtosis ≈ 3)
 - 2 **Leptokurtic**: This distribution has fatter tails than a normal distribution (kurtosis > 3). Consequently, outliers are more likely to occur than in a normal distribution.
 - 3 **Platykurtic**: The distribution has thinner tails than a normal distribution (kurtosis < 3). Consequently, outliers are less likely to occur than in a normal distribution.

Kurtosis

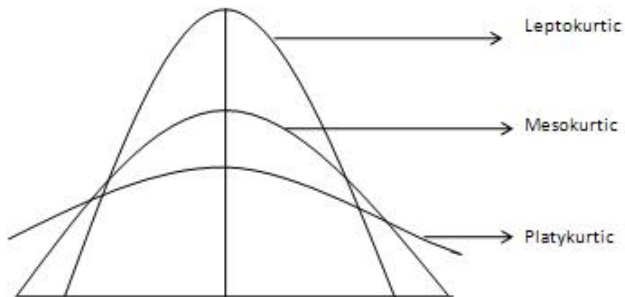
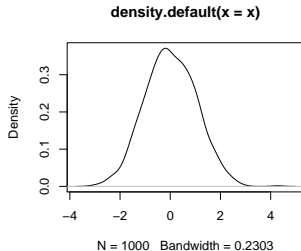


Figure: Source: tutorialspoints

Kurtosis

- In R, we can calculate kurtosis with the formula **kurtosis** from the library **moments**.
- Let's calculate kurtosis for normal data:

```
> x <- rnorm(1000, 0, 1)
> plot(density(x))
> kurtosis(x)
[1] 2.946869
```
- As expected we got a value close to 3.

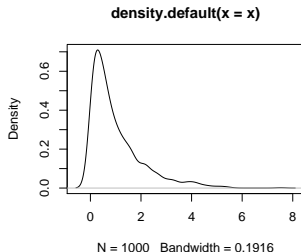


Kurtosis

- Now using random data generated with an exponential long-tailed distribution:

```
> x<-rexp(1000)
> plot(density(x))
> kurtosis(x)
[1] 6.839485
```

- As expected we get a positive excess kurtosis (i.e. greater than 3) since the distribution has fatter tails (Leptokurtic).

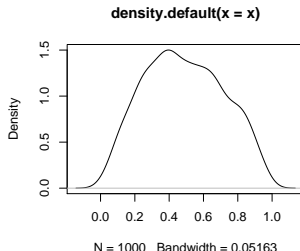


Kurtosis

- Now using data generated with a thinner-tailed Beta distribution with hyperparameters 5, 5:

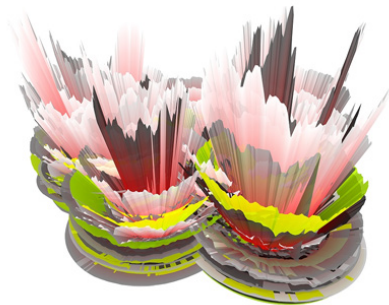
```
> x <-rbeta(1000,2,2)
> plot(density(x))
> kurtosis(x)
[1] 2.1046
```

- As expected we get a negative excess kurtosis (i.e. less than 3) since the distribution has thinner tails (Platykurtic).



Data Visualization

- Data visualization is the transformation of a dataset into a visual format that allows people to identify the characteristics and relationships between examples in the dataset.
- Visualization allows people to recognize patterns or trends based on their judgment or expertise in the particular domain.



Representation

- Representation is understood as the mapping of data into a visual format
- Examples, their attributes and relationships are translated into graphical elements such as points, lines, shapes and colors.
- Examples are usually represented as points.
- Attribute values are represented as the position of the points or the characteristics of the points, e.g. color, size and shape.
- When using position to represent values it is simple to detect if groups of objects are formed or the presence of outliers.

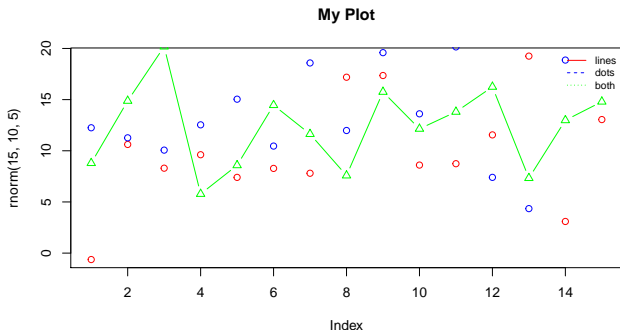
Plotting in R

- In R the most frequent display function is `plot`.
- `t` is a generic function whose result depends on the nature of the variables given as input.
- To all plots we can add additional parameters such as: `main` for the title, `xlab` and `ylab` for the name of the x-axis and y-axis.
- Other properties are: `col` to define the color, `type` to define the chart type: (p) for points or (l) for lines.
- We can also add new layers to a plot with the command `lines`.
- To save an image to a file we can use Rstudio's **export** button.
- To do it from the R command line:

```
png("imagen.png")  
plot(1:10)  
dev.off()
```


Example

```
plot(rnorm(15,10,5),col="red",type="p",pch=1)  
lines(rnorm(15,10,5),col="blue",type="p",pch=1)  
lines(rnorm(15,10,5),col="green",type="b",pch=2)  
title(main="My Plot")  
legend('topright', c("lines","dots","both") ,  
      lty=1:3, col=c("red", "blue","green"), bty='n', cex=.75)
```

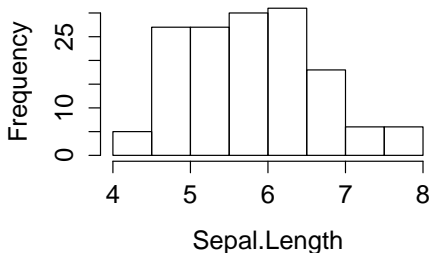


Histograms

- They show the distribution of the values of a variable.
- The values of the elements are divided into bins and bar charts are created for each of them.
- The height of each bar indicates the number of examples in the corresponding bin.
- In R they are created with the command `hist`.

```
> hist(Sepal.Length)
```

Histogram of Sepal.Length

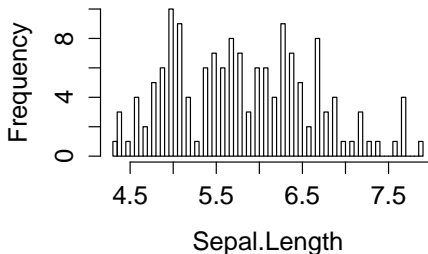


Histograms (2)

- The shape of the histogram depends on the number of bins.
- In R this number can be defined with the parameter `nclass`.

```
> hist(Sepal.Length, nclass=100)
```

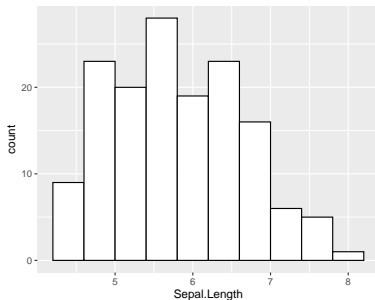
Histogram of Sepal.Length



Histograms (3)

- A very popular library for making visualizations in R, which is part of tidyverse, is *ggplot2*.
- It is based on the idea of decomposing the plot into semantic components such as scales and layers.

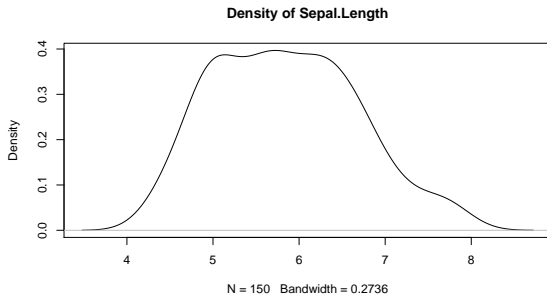
```
>install.packages("ggplot2")  
>library(ggplot2)  
>ggplot(iris, aes(x=Sepal.Length))  
+ geom_histogram(bins = 10, color="black", fill="white")
```



Density

- Another way to visualize how the data are distributed is to estimate a density.
- These are calculated using nonparametric statistical techniques called **kernel** density estimation.
- The density is a smoothed version of the histogram and allows us to determine more clearly if the observed data behaves like a known density e.g. Gaussian.
- In R they are created with the command `density`, and then visualized with the command `plot`.

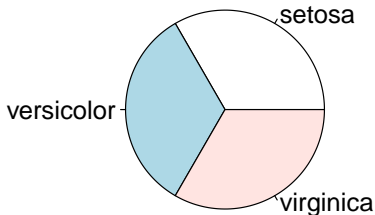
```
plot(density(iris$Sepal.Length), main="Density of Sepal.Length")
```



Pie Charts

- The pie charts represent the frequency of the elements in a circle.
- Each category has a share proportional to its relative frequency.
- They are generally used for categorical variables:

```
pie(table(iris$Species))
```



Boxplots

- Boxplots are built from the percentiles.
- A rectangle is constructed between the first and third quartiles (Q_1 and Q_3).
- The height of the rectangle is the interquartile range RIC ($Q_3 - Q_1$).
- The median is a line that divides the rectangle.
- Each end of the rectangle is extended with a line or arms of length $Q_1 - 1.5 \cdot \text{IQR}$ for the lower line and $Q_3 + 1.5 \cdot \text{IQR}$ for the upper line.
- Values more extreme than the length of the arms are considered outliers.
- The boxplot gives us information about the symmetry of the data distribution.
- If the median is not in the center of the rectangle, the distribution is not symmetrical.
- They are useful to detect the presence of outliers.

Boxplots (2)

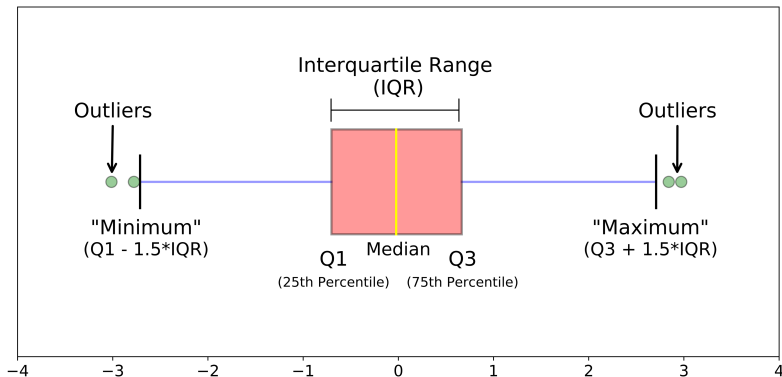
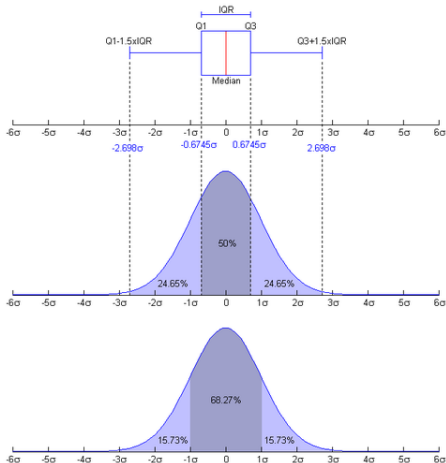


Figure: <https://www.laboneconsultoria.com.br/wp-content/uploads/2018/07/Boxplot-04.png>

Boxplots (3)

- The length of the arms as well as the criteria for identifying outliers is based on the behavior of a normal distribution.

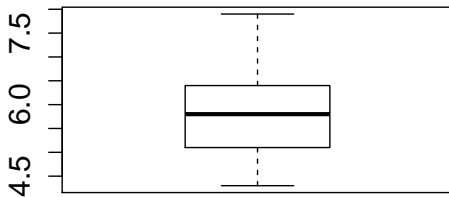


Boxplots (4)

- In R boxplots are plotted with the command `boxplot`:

```
> boxplot(Sepal.Length, main="Boxplot Sepal.Length")
```

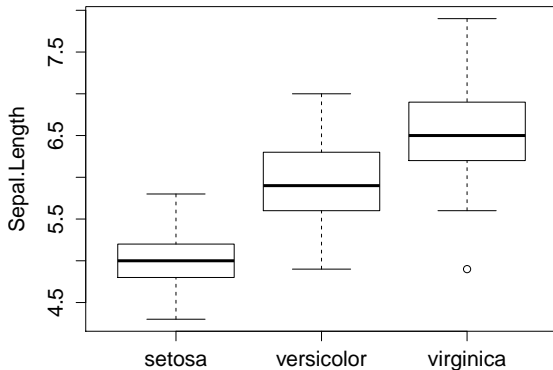
Boxplot Sepal.Length



Boxplots (4)

- If we have a factor variable we can create a boxplot for each category as follows:

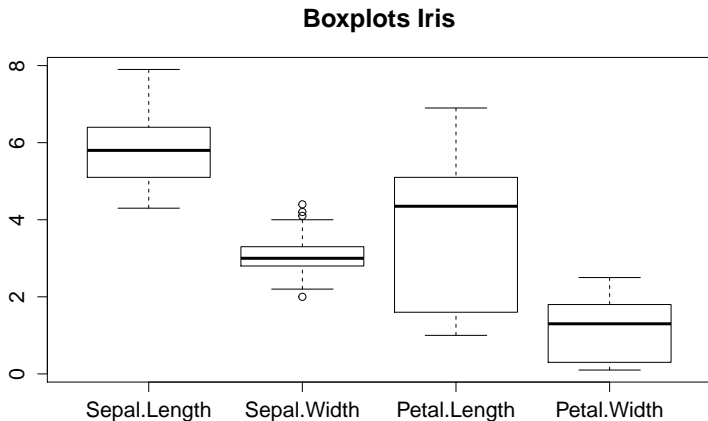
```
> boxplot(Sepal.Length~Species,ylab="Sepal.Length")
```



Boxplots (5)

- We can also compare several boxplots in the same plot:

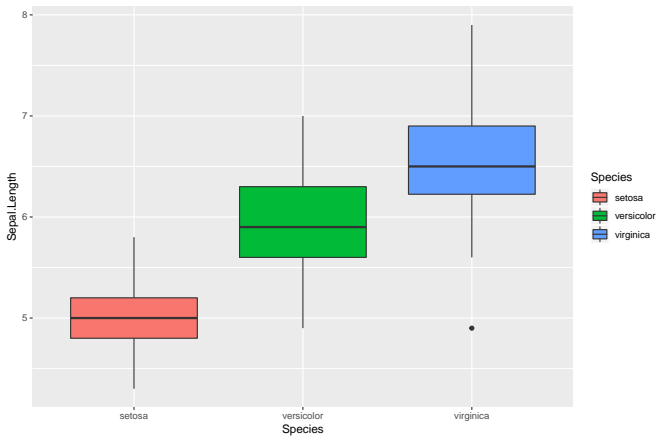
```
> boxplot(x=iris[,1:4],main="Boxplots Iris")
```



Boxplots (6)

- Now using *ggplot2*:

```
> ggplot(iris, aes(x = Species, y = Sepal.Length,  
  fill = Species)) + geom_boxplot()
```



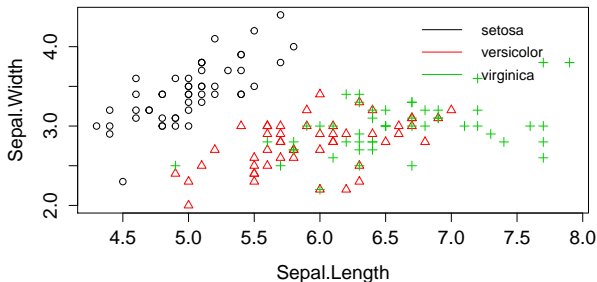
Scatter Plots

- Scatter plots use Cartesian coordinates to display the values of two numerical variables of the same length.
- The values of the attributes determine the position of the elements.
- Other attributes can be used to define the size, shape or color of objects.
- In R we can plot a scatterplot of two numeric variables using the command `plot(x, y)`, which would be y vs x .
- We can also define formulas $f(x) = y$ using the notation $y \sim x$.
- Thus, the command `plot(y ~ x)` is equivalent to `plot(x, y)`.
- If we have a `data.frame` or a numerical matrix, we can see the scatterplots of all pairs using the command `pairs(x)`.

Scatter Plots (2)

- Examples:

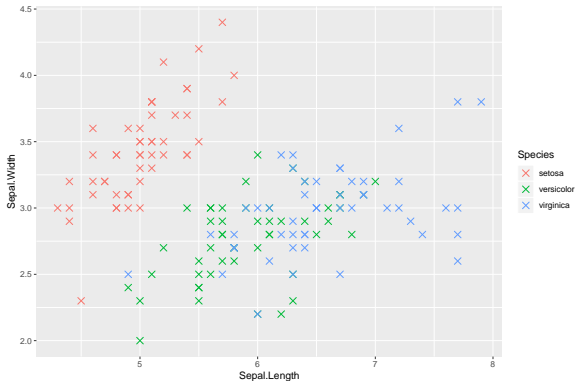
```
# Sepal width vs. sepal length
plot(Sepal.Width~Sepal.Length, col=Species)
# Equivalent
plot(Sepal.Length, Sepal.Width,col=Species,
     pch=as.numeric(Species))
# We add a legend
legend('topright', levels(Species) ,
      lty=1, col=1:3, bty='n', cex=.75)
```



Scatter Plots (3)

- The same using *ggplot2*:

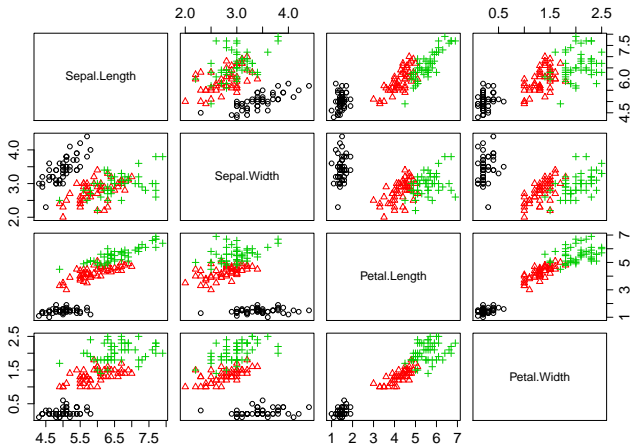
```
ggplot(iris, aes(x=Sepal.Length,  
y=Sepal.Width, color=Species)) +  
geom_point(size=3, shape=4)
```



Scatter Plots (4)

- All pairs of the 4 variables of the iris dataset using a different color and character for each species:

```
pairs(iris[,1:4],pch=as.numeric(iris$Species),col=iris$Species)
```



Scatter Plots (5)

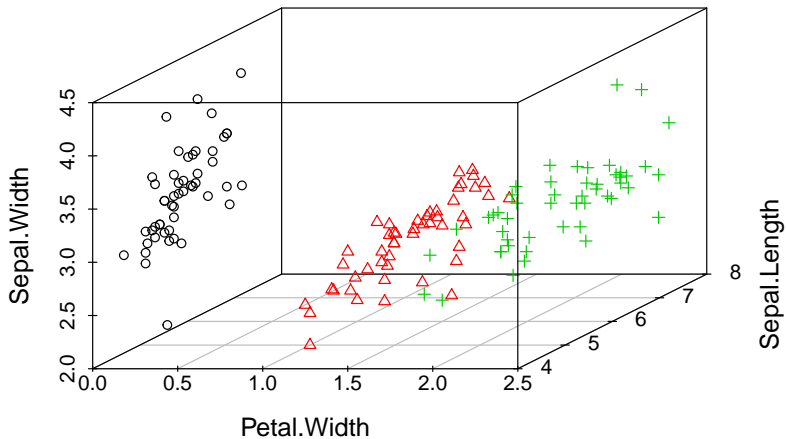
- Three-dimensional scatterplots can also be created.
- The `scatterplot3d` library must be installed using the following command:

```
install.packages("scatterplot3d", dependencies=T)
```

- Then load the library by typing `library(scatterplot3d)`.
- A 3d scatterplot for petal width, sepal length and sepal width:

```
scatterplot3d(iris$Petal.Width, iris$Sepal.Length,  
              iris$Sepal.Width, color=as.numeric(iris$Species),  
              pch=as.numeric(iris$Species))
```

Scatter Plots (6)



Parallel Coordinate Plots

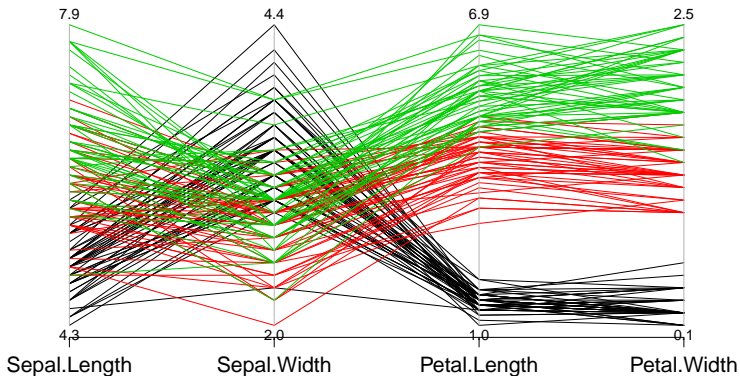
- Parallel coordinate plots are another way to visualize multi-dimensional data.
- Instead of using perpendicular axes (x-y-z) we use several axes parallel to each other.
- Each attribute is represented by one of the parallel axes with its respective values.
- The values of the different attributes are scaled so that each axis has the same height.
- Each observation represents a line that joins the different axes according to their values.
- In this way, examples similar to each other tend to be grouped in lines with similar trajectory.
- In many occasions it is necessary to re-order the axes in order to visualize a pattern.

Parallel Coordinate Plots (2)

- In R we can create parallel coordinate graphs with the command `parcoord` from the `MASS` library.

- Example:

```
library(MASS)
parcoord(iris[1:4], col=iris$Species, var.label=T)
```



Conclusions

- We have learned how to describe a dataset using various metrics and visualizations.
- These techniques give us a general understanding of the data.

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