Probability

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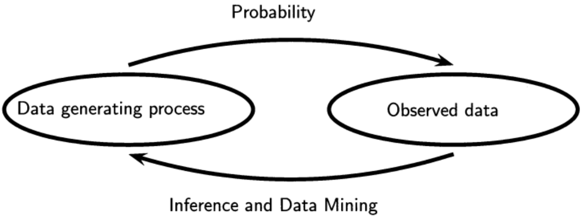
Probability and Statistics

- Probability is the language of **uncertainty** that is also the basis for statistical inference [Poldrack, 2019].
- It forms an important part of the foundation for statistics, because it provides us with the mathematical tools to describe uncertain events.
- The study of probability arose in part due to interest in understanding games of chance, like cards or dice.
- These games provide useful examples of many statistical concepts, because when we repeat these games the likelihood of different outcomes remains (mostly) the same.

Probability and Statistics

- The problem studied in probabilities is: given a data generating process, which are the properties of the outputs?
- The problem studied in statistical inference, data mining and machine learning is: given the outputs, what can we say about the process that generates the observed data?



[1] Figure taken from [Wasserman, 2013]

What is Probability?

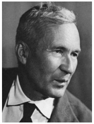
- We think of probability as a number that describes the likelihood of some event occurring, which ranges from zero (impossibility) to one (certainty).
- Probabilities can also be expressed in percentages: when the weather forecast predicts a twenty percent chance of rain today.
- In each case, these numbers are expressing how likely that particular event is, ranging from absolutely impossible to absolutely certain.

Probability Concepts

- A **random experiment** in the act of measuring a process whose output is uncertain.
- Examples: flipping a coin, rolling a 6-sided die, or trying a new route to work to see if it's faster than the old route.
- The set with all possible outputs of a random experiment is the **sample space** Ω (it can be discrete or continuous).
- For a coin flip $\Omega = \{\text{heads, tails}\}$, for the 6-sided die $\Omega = \{1, 2, 3, 4, 5, 6\}$, and for the amount of time it takes to get to work Ω is all possible real numbers greater than zero.
- An **event** $E \subseteq \Omega$ corresponds to a subset of those outputs.
- For example, $E = \{2, 4, 6\}$ is the event of observing an even number when rolling a die.

Probability

- Now we can outline the formal features of a probability, which were first defined by the Russian mathematician Andrei Kolmogorov.



- A probability $\mathbb{P}$ is a real-valued function defined over Ω that satisfies the following properties:

Properties

1. For any event $E \subseteq \Omega$, $0 \leq \mathbb{P}(E) \leq 1$.
2. The probability of the sample space is 1: $\mathbb{P}(\Omega) = 1$
3. Let $E_1, E_2, \ldots, E_k \in \Omega$ be disjoint sets

$$\mathbb{P}(\bigcup_{i=1}^{k} E_i) = \sum_{i}^{k} P(E_i)$$

Probabilities cannot be negative or greater than 1.

Interpretation of Probabilities

- The are two common interpretations of probabilities: frequencies and degrees of beliefs.
- In the frequency interpretation, $\mathbb{P}(E)$ is the long run proportion of times that E is true in repetitions.
- For example, if we say that the probability of heads is 1/2, we mean that if we flip the coin many times then the proportion of times we get heads tends to 1/2 as the number of tosses increases.
- The degree-of-belief interpretation is that $\mathbb{P}(E)$ measures an observer's strength of belief that E is true.
- In either interpretation, we require that properties 1 to 3 hold.
- The difference in interpretation will not matter much until we deal with statistical inference.
- There, the differing interpretations lead to two schools of inference: the frequentist and the Bayesian schools.

Random Variable

- A **random variable** is a mapping (or function)

$$X : \Omega \to \mathbb{R}$$

  which assigns a real value $X(e)$ to any event of Ω.
- Example: We flip a fair coin 10 times. The outcome of each toss is a head H or a tail T.
- Let $X(e)$ be the number of heads in the sequence of outcomes.
  - If $e = HHTHHHTHHTT$, then $X(e) = 6$

Example

- We flip a coin 2 times. Let X be the number of tails obtained.
- The random variable and its distribution is summarized as:

e	$\mathbb{P}(e)$	$X(e)$
HH	1/4	0
HT	1/4	1
TC	1/4	1
TT	1/4	2

x	$\mathbb{P}(X = x)$
0	1/4
1	1/2
2	1/4

R.V Definitions

- Let X be a R.V , we define **cumulative distribution function** (CDF) or $F_X : \mathbb{R} \to [0, 1]$ as:

$$F_X(x) = \mathbb{P}(X \leq x)$$

Discrete Random Variables

- A R.V X is **discrete** if it maps the outputs to a countable set.
- We define the **probability function** or **probability mass function** of a discrete R.V X as $f_X(x) = \mathbb{P}(X = x)$.
- Then $f_X(x) \geq 0 \ \forall x \in \mathbb{R}$, and $\sum_i f_X(x_i) = 1$
- The CDF of X is related to f_X as follows:

$$F_X = \mathbb{P}(X \leq x) = \sum_{x_i \leq x} f_X(x_i)$$

R.V Definitions II

Continuous Random Variable

- A R.V X is continuous if:
- there exists a function f_X such that $f_X(x) \geq 0 \; \forall x$, $\int_{-\infty}^{\infty} f_X(x)dX = 1$

$$\int_{-\infty}^{\infty} f_X(x)dX = 1$$

- For all $a \geq b$:

$$\mathbb{P}(a < X < b) = \int_a^b f_X(x)dx$$

- The function f_X is called the probability density function (PDF).
- The PDF is related to the CDF as follows:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

- Then $f_X(x) = F'_X(x)$ at all points x where F_X is differentiable.
- For continuous distributions the probability that X takes a particular value x is always zero.

Some Properties

1. $\mathbb{P}(x < X \leq y) = F(y) - F(x)$
2. $\mathbb{P}(X > x) = 1 - F(x)$
3. If X is continuous:

$$\begin{aligned} F(b) - F(a) &= \mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b) \\ &= \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X \leq b) \end{aligned}$$

Quantiles

- Let X be a R.V with CDF F. The inverse CDF or quantile function is defined as

$$F^{-1}(q) = inf\,\{x : \; F(x) > q\}$$

- For $q \in [0, 1]$ if F is strictly increasing and continuous, $F^{-1}(q)$ is the only real value such that $F(x) = q$.
- Then $F^{-1}(1/4)$ is the first quartile, $F^{-1}(1/2)$ the median (or second quartile) and $F^{-1}(3/4)$ the third quartile.

Some distributions

	Probability Function	Parameters
Normal	$f_x = \frac{1}{\sqrt{2\pi\sigma}} \exp^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$	μ, σ
Binomial	$f_x = \binom{n}{x} p^x (1-p)^{n-x}$	n, p
Poisson	$f_x = \frac{1}{x!}\lambda^x \exp^{-\lambda}$	λ
Exponential	$f_x = \lambda \exp^{-\lambda x}$	λ
Gamma	$f_x = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp^{-\lambda x}$	λ, α
Chi-square	$f_x = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(\frac{k}{2}-1)} \exp^{-x/2}$	k

Binomial Distribution

- The binomial distribution is a discrete distribution that provides a way to compute the probability of some number of successes out of a number of trials.
- In each trial there is either success or failure and nothing in between (known as "Bernoulli trials") given some known probability of success on each trial.
- Let n be the number of trials, x the number of successes, and p the probability of a success, the probability mass function of the Binomial distribution is as follows:

$$f_x(n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

- The binomial coefficient $\binom{n}{x}$ describes the number of different ways that one can choose x items out of n total items.

Normal Distribution

- X has a Normal or Gaussian distribution of parameters μ and σ, $X \sim N(\mu, \sigma^2)$ if

$$f_x = \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

- Where $\mu \in \mathbb{R}$ is the “center” or the “mean” of the distribution and $\sigma > 0$ is the “standard deviation”.
- When $\mu = 0$ and $\sigma = 1$ we have a **Standard Normal Distribution** denoted by Z.
- We refer to the PDF by $\phi(z)$ and to the CDF of a Standard Normal by $\Phi(z)$.
- The values of $\Phi(z)$, $\mathbb{P}(Z \leq z)$ are tabulated.

Useful Properties

1. If $X \sim N(\mu, \sigma^2)$, then $Z = (X - \mu)/\sigma \sim N(0, 1)$
2. If $Z \sim N(0, 1)$, then $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$
3. Let $X_i \sim N(\mu_i, \sigma_i^2)$,$i = 1, \ldots, n$ be independt R.Vs:

$$\sum_{i=1}^{n} X_i \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)$$

Example Normal

- In R we can access the PDF, CDF, quantile function and random number generation of the distributions.
- For a Normal distribution the R commands are:

```
dnorm(x, mean = 0, sd = 1, log = FALSE)
pnorm(q, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
qnorm(p, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
rnorm(n, mean = 0, sd = 1)
```

Example

Let $X \sim N(3,5)$, calculate $\mathbb{P}(X > 1)$

$\mathbb{P}(X > 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(Z < \frac{1-3}{\sqrt{5}}) = 1 - \Phi(-0.8944) = 0.81$

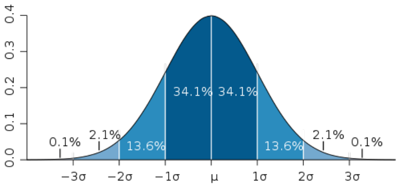
In R:

```
 > 1-pnorm(q=(1-3)/sqrt(5))
[1] 0.8144533
```

Or directly:

```
> 1-pnorm(q=1,mean=3,sd=sqrt(5))
[1] 0.8144533
```

The 68-95-99.7 rule of a Normal Distribution



Let X be a R.V *simN*(μ, σ^2).

- $\mathbb{P}(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.6827$
- $\mathbb{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9545$
- $\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.9973$

In R for $X \sim N(0, 1)$:

```
> pnorm(1)-pnorm(-1)
[1] 0.6826895
> pnorm(2)-pnorm(-2)
[1] 0.9544997
> pnorm(3)-pnorm(-3)
[1] 0.9973002
```

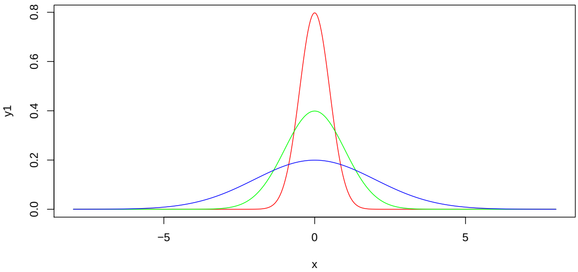
Symmetry of the Normal Distribution

- The PDF of a normal is symmetric around μ.
- Then $\phi(z) = \phi(-z)$
- $\Phi(z) = 1 - \Phi(-z)$

```
> dnorm(1)
[1] 0.2419707
> dnorm(-1)
[1] 0.2419707
> pnorm(0.95)
[1] 0.8289439
> 1-pnorm(-0.95)
[1] 0.8289439
```

Plotting the PDF of Normals with different variance in R

```
x=seq(-8,8,length=400)
y1=dnorm(x,mean=0,sd=0.5)
y2=dnorm(x,mean=0,sd=1)
y3=dnorm(x,mean=0,sd=2)
plot(y1~x,type="l",col="red")
lines(y2~x,type="l",col="green")
lines(y3~x,type="l",col="blue")
```



Joint and Conditional Probabilities

- The notion of probability function (mass or density) can be **extended** to more than one R.V.
- Let X Y be two V.A, $\mathbb{P}(X, Y)$ represents the joint probability function.
- X and Y are independent of each other, if

$$\mathbb{P}(X, Y) = \mathbb{P}(X) \times \mathbb{P}(Y)$$

- The **conditional probability** for Y given X is defined as:

$$\mathbb{P}(Y|X) = \frac{\mathbb{P}(X, Y)}{\mathbb{P}(X)}$$

- If X and Y are independent $\mathbb{P}(Y|X) = \mathbb{P}(Y)$

Joint and Conditional Probabilities (2)

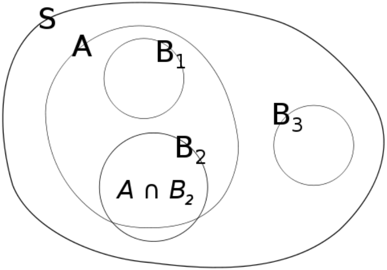


Figure: Source:
en.wikipedia.org/wiki/Conditional_probability

- Let S be the sample space, A and B_n events.
- The probabilities are proportional to the area.
- $\mathbb{P}(A) \sim 0.33$, $\mathbb{P}(A|B_1) = 1$
- $\mathbb{P}(A|B_2) \sim 0.85$ y $\mathbb{P}(A|B_3) = 0$

Bayes' Theorem and Total Probabilities

- The conditional probability $\mathbb{P}(Y|X)$ and $\mathbb{P}(X|Y)$ can be expressed as a function of each other using Bayes' theorem.

$$\mathbb{P}(Y|X) = \frac{\mathbb{P}(X|Y)\mathbb{P}(Y)}{\mathbb{P}(X)}$$

- $P(Y|X)$ can be interpreted as the fraction of times Y occurs when X is known to occur.
- Then let $\{Y_1, Y_2, \ldots, Y_k\}$ be a set of mutually exclusive events of the sample space of a R.V X, the denominator of Bayes' theorem can be expressed as:

$$\mathbb{P}(X) = \sum_{i=1}^{k} \mathbb{P}(X, Y_i) = \sum_{i=1}^{k} \mathbb{P}(X|Y_i)\mathbb{P}(Y_i)$$

Example

- I split my emails into three categories: A_1="'spam", A_2="'low priority", A_3="'high priority".'
- We know that $\mathbb{P}(A_1) = 0.7$, $\mathbb{P}(A_2) = 0.2$ and $\mathbb{P}(A_3) = 0.1$, clearly $0.7 + 0.2 + 0.1 = 1$.
- Let B be the event that the mail contains the word "free".
- We know that $\mathbb{P}(B|A_1) = 0.9$ $\mathbb{P}(B|A_2) = 0.01$ y $\mathbb{P}(B|A_3) = 0.01$ clearly $0.9 + 0.01 + 0.01 \neq 1$
- What is the probability that an email with the word "free" in it is "spam"?
- Using Bayes and Total Probabilities:

$$\mathbb{P}(A_1|B) = \frac{0.9 \times 0.7}{(0.9 \times 0.7) + (0.01 \times 0.2) + (0.01 \times 0.1)} = 0.995$$

Expectation

- Let X be a R.V, we define its **expectation** or **first-order moment** as:

$$\mathbb{E}(X) = \begin{cases} \sum_x (x \times f(x)) & \text{If } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x \times f(x)) dx & \text{If } X \text{ is continuous} \end{cases}$$

- The expectation is the weighted average of all the possible values that a random variable can take.
- For the case of tossing a coin twice with X the number of heads:

$$\begin{aligned} \mathbb{E}(X) &= (0 \times f(0)) + (1 \times f(1)) + (2 \times f(2)) \\ &= (0 \times (1/4)) + (1 \times (1/2)) + (2 \times (1/4)) = 1 \end{aligned}$$

- Let the random variables $X_1, X_2, \ldots, X_n$ and the constants $a_1, a_2, \ldots, a_n$,

$$\mathbb{E}\left(\sum_i a_i X_i\right) = \sum_i a_i \mathbb{E}(X_i)$$

Variance

- The variance measures the "dispersion" of a distribution.
- Lex X be a R.V of mean μ, we define the variance of X denoted as σ^2, σ_X^2 or $\mathbb{V}(X)$ as:

$$\mathbb{V}(X) = \mathbb{E}(X-\mu)^2 = \begin{cases} \sum_{i=1}^{n} f_X(x_i)(x_i-\mu)^2 & \text{If } X \text{ is discrete} \\ \int (x-\mu)^2 f_X(x)dx & \text{If } X \text{ is continuous} \end{cases}$$

- The **standard deviation** σ is defined as $\sqrt{\mathbb{V}(X)}$

Properties

- $\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \mu^2$
- If a and b are constants, then $\mathbb{V}(aX+b) = a^2\mathbb{V}(X)$
- If $X_1, \ldots, X_n$ are independent and $a_1, \ldots, a_n$ are constants, then

$$\mathbb{V}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathbb{V}(X_i)$$

Law of the Large Numbers

Weak Form

- Let $X_1, X_2, \ldots X_n$ be IID random variables of mean μ and variance σ^2.
- The mean $\overline{X_n} = \frac{\sum_{i=1}^{n} X_i}{n}$ onverges in probability to μ, $\overline{X_n} \xrightarrow{P} \mu$
- This is equivalent to saying that for all $\epsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(|\overline{X_n} - \mu| < \epsilon) = 1$$

- Then the distribution of $\overline{X_n}$ becomes centered around μ as n grows.

Example

- Let be the experiment of flipping a coin where the probability of heads is p.
- For a Bernoulli distributed R.V $E(X) = p$.
- Let be $\overline{X_n}$ the fraction of heads after n tosses.
- The law of large numbers tells us that $\overline{X_n}$ converges in probability to p.
- This does not imply that $\overline{X_n}$ is numerically equal to p.
- But if n in large enough, the distribution of $\overline{X_n}$ will be centered around p.

Central Limit Theorem

- While the law of large numbers tells us that $\overline{X_n}$ approaches μ as n grows.
- This is not sufficient to say anything about the distribution of $\overline{X_n}$.

Central Limit Theorem (CLT)

- Let X_1, X_n be IID random variables of mean μ and variance σ^2.
- Let $\overline{X_n} = \frac{\sum_{i=1}^{n} X_i}{n}$

$$Z_n \equiv \frac{\overline{X_n} - \mu}{\sqrt{\mathbb{V}(\overline{X_n})}} = \frac{\overline{X_n} - \mu}{\frac{\sigma}{\sqrt{n}}} \rightsquigarrow Z$$

where $Z \sim N(0,1)$
- This is equivalent to:

$$\lim_{n \to \infty} \mathbb{P}(Z_n \leq z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Central Limit Theorem (2)

- The theorem allows us to approximate the distribution of $\overline{X_n}$ to a Gaussian distribution when n is large.
- Even if we do not know the distribution of X_i, we can approximate the distribution of its mean.

Alternative notations showing that Z_n converges to a Normal

$$\begin{aligned}
Z_n &\approx N(0,1)\\
\overline{X_n} &\approx N\left(\mu, \frac{\sigma^2}{n}\right)\\
\overline{X_n} - \mu &\approx N\left(0, \frac{\sigma^2}{n}\right)\\
\sqrt{n}(\overline{X_n} - \mu) &\approx N(0, \sigma^2)\\
\frac{\overline{X_n} - \mu}{\frac{\sigma}{\sqrt{n}}} &\approx N(0,1)
\end{aligned}$$

Central Limit Theorem (3)

- Suppose that the number of errors of a computer program follows a Poisson distribution with parameter $\lambda = 5$
- If $X \sim Poisson(\lambda)$, $\mathbb{E}(X) = \lambda$ and $\mathbb{V}(X) = \lambda$.
- If we have 125 independent programs $X_1, \ldots, X_{125}$ we would like to approximate $\mathbb{P}(\overline{X_n} < 5.5)$
- Using the CLT we have that

$$\begin{aligned}
\mathbb{P}(\overline{X_n} < 5.5) &= \mathbb{P}\left(\frac{\overline{X_n} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{5.5 - \mu}{\frac{\sigma}{\sqrt{n}}}\right) \\
&\approx \mathbb{P}\left(Z < \frac{5.5 - 5}{\frac{\sqrt{5}}{\sqrt{125}}}\right) = \mathbb{P}(Z < 2.5) = 0.9938
\end{aligned}$$

References I

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