Generalized and Multilevel Linear Models

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Generalized and Multilevel Linear Models

- In this class we will learn two powerful extensions to the linear model, which we have discussed extensively throughout this course.
- The first extensions is the **Generalized Linear Model** (GLM) which allows the use of distributions other than Gaussian in the outcome variable.
- GLMs can be particularly useful when our outcome variable is binary or bounded to positive values.
- **Multilevel models** (also known as hierarchical or mixed effects models), on the other hand, are useful when there are predictors at different level of variation.
- For example, when studying student performance, we may have information at different levels: individual students (e.g., family background), class-level information (e.g., teacher), and school-level information (e.g., neighborhood) [Gelman et al., 2013].
- Multilevel models extend linear regression to include categorical input variable representing these levels, while allowing intercepts and possibly slopes to vary by level [Gelman and Hill, 2006].

Generalized Linear Models

- The linear regression models of previous classes worked by first assuming a Gaussian distribution over outcomes.
- Then, we replaced the parameter that defines the mean of that distribution, μ, with a linear model.
- This resulted in likelihood definitions of the sort:

$$\begin{aligned} y_i &\sim N(\mu_i, \sigma) && \text{[likelihood]} \\ \mu_i &= \beta_0 + \beta_1 x_i && \text{[linear model]} \end{aligned}$$

- When the outcome variable is either discrete or bounded, a Gaussian likelihood is not the most powerful choice.
- Consider for example a count outcome, such as the number of blue marbles pulled from a bag.
- Such a variable is constrained to be zero or a positive integer.

Generalized Linear Models

- The problem of using a Gaussian model with such a variable is that the model wouldn't know that counts can't be negative.
- So it would happily predict negative values, whenever the mean count is close to zero [McElreath, 2020].
- In linear regression we basically replace the parameter describing the shape of the Gaussian likelihood μ with a linear model.
- The the essence of a Generalized Linear Model (GLM) is to generalize this strategy to probability distributions other than the Gaussian.
- And it results in models that look like this:

$$
\begin{aligned}
y_i &\sim \text{Binomial}(n, p_i) && \text{[likelihood]} \\
f(p_i) &= \beta_0 + \beta_1 x_i && \text{[generalized linear model]}
\end{aligned}
$$

Generalized Linear Models

- The first change we can note is that likelihood is binomial instead of Gaussian.
- For a count outcome y for which each observation arises from n trials and with constant expected value $n * p$, the binomial distribution is the de facto choice.
- The function f represents a **link** function, to be determined separately from the choice of distribution.
- Generalized linear models need a link function, because rarely is there a "μ", a parameter describing the average outcome.
- Parameters are also rarely unbounded in both directions, like μ.
- For example, the shape of the binomial distribution is determined, like the Gaussian, by two parameters.

Generalized Linear Models

- But unlike the Gaussian, neither of these parameters is the mean.
- Instead, the mean outcome is $n * p$, which is a function of both parameters.
- Since n is usually known (but not always), it is most common to attach a linear model to the unknown part, p.
- But p is a probability, so p_i must lie between zero and one.
- But there's nothing to stop the linear model $\beta_0 + \beta_1 x_i$ from falling below zero or exceeding one.
- The **link** function f provides a solution to this common problem.
- The link function that is commonly used when working with binomial GLMs is the logit function.
- It maps a parameter that is defined as a probability p (i.e., $0 \leq p \leq 1$), onto a linear model that can take on any real value.

$$\text{logit}(p_i) = \beta_0 + \beta_1 x_i$$

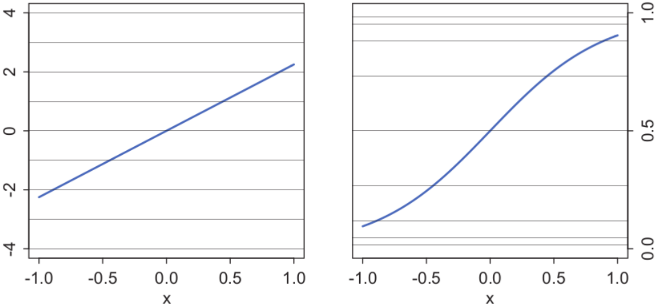
$$\text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right)$$

Generalized Linear Models

- If we solve the logit equation for p_i we get:

$$p_i = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)} \tag{1}$$

  which is known as the **sigmoid** function.
- It is also called the **logistic** or the **inverse-logit** function.

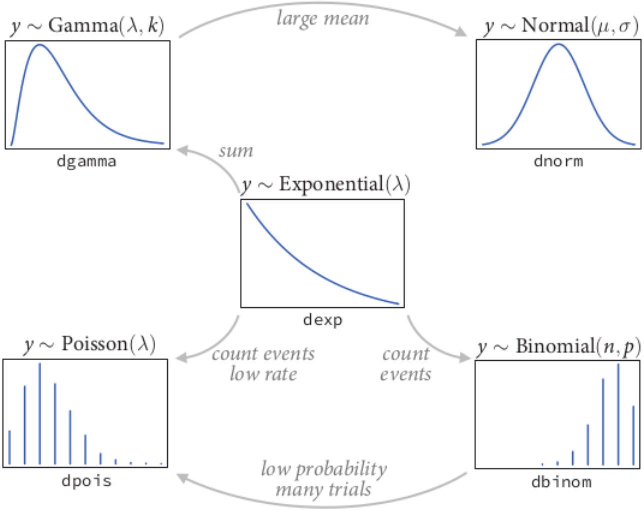


The logit link transforms a linear model (left) into a probability (right).

Generalized Linear Models

- There are two common flavors of GLM that use binomial likelihood functions and logit link functions:
    1. **Logistic regression**: when the data are organized into single-trial cases ($n = 1$), such that the outcome variable can only take values 0 and 1. In this case the likelihood function can also be represented with a Bernoulli distribution.
    2. **Aggregated binomial regression**: when the outcome can take the value zero or any positive integer up to n, the number of trials.
- Other distributions that can be used in GLMs are the Exponential, the Poisson and the Gamma distribution.
- These distributions along with the Gaussian and Binomial are part of the exponential family, which is a set of probability distributions that share some common algebraic properties.
- We won't go into details of the exponential family here.
- A very instructive description is given in [Ng, 2012].

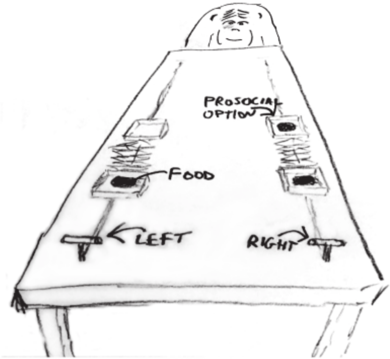
Expontential Family



source: [McElreath, 2020]

Logistic regression: Prosocial chimpanzees.

- Now we will go through an example of logistic regression given in [McElreath, 2020].
- The data for this example come from an experiment aimed at evaluating the **prosocial** tendencies of chimpanzees [Silk et al., 2005].
- A **focal** chimpanzee sits at one end of a long table with two levers, one on the left and one on the right.



Logistic regression: Prosocial chimpanzees.

- On the table are four dishes which may contain desirable food items.
- The two dishes on the right side of the table are attached by a mechanism to the right-hand lever.
- The two dishes on the left side are similarly attached to the left-hand lever.
- When either the left or right lever is pulled by the focal animal, the two dishes on the same side slide towards opposite ends of the table.
- This delivers whatever is in those dishes to the opposite ends.
- In all experimental trials, both dishes on the focal animal's side contain food items.
- But only one of the dishes on the other side of the table contains a food item.
- Therefore while both levers deliver food to the focal animal, only one of the levers delivers food to the other side of the table.

Logistic regression: Prosocial chimpanzees.

- There are two experimental conditions.
- In the **partner** condition, another chimpanzee is seated at the opposite end of the table, as pictured in the figure.
- In the **control** condition, the other side of the table is empty.
- Finally, two **counterbalancing** treatments alternate which side, left or right, has a food item for the other side of the table.
- This helps detect any **handedness** preferences for individual focal animals.
- When human students participate in an experiment like this, they nearly always choose the lever linked to two pieces of food, the prosocial option, but only when another student sits on the opposite side of the table.
- The motivating question is whether a focal chimpanzee behaves similarly, choosing the prosocial option more often when another animal is present.

Conclusions

- Blabla

References I

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