Generalized and Multilevel Linear Models

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October 6, 2021

Generalized and Multilevel Linear Models

- In this class we will learn two powerful extensions to the linear model, which we have discussed extensively throughout this course.
- The first extensions is the **Generalized Linear Model** (GLM) which allows the use of distributions other than Gaussian in the outcome variable.
- GLMs can be particularly useful when our outcome variable is binary or bounded to positive values.
- **Multilevel models** (also known as hierarchical or mixed effects models), on the other hand, are useful when there are predictors at different level of variation.
- For example, when studying student performance, we may have information at different levels: individual students (e.g., family background), class-level information (e.g., teacher), and school-level information (e.g., neighborhood) [Gelman et al., 2013].
- Multilevel models extend linear regression to include categorical input variable representing these levels, while allowing intercepts and possibly slopes to vary by level [Gelman and Hill, 2006].

Generalized Linear Models

- The linear regression models of previous classes worked by first assuming a Gaussian distribution over outcomes.
- Then, we replaced the parameter that defines the mean of that distribution, μ, with a linear model.
- This resulted in likelihood definitions of the sort:

$$\begin{array}{lr} y_i \sim N(\mu_i, \sigma) & \text{[likelihood]} \\ \mu_i = \beta_0 + \beta_1 x_i & \text{[linear model]} \end{array}$$

- When the outcome variable is either discrete or bounded, a Gaussian likelihood is not the most powerful choice.
- Consider for example a count outcome, such as the number of blue marbles pulled from a bag.
- Such a variable is constrained to be zero or a positive integer.

- The problem of using a Gaussian model with such a variable is that the model wouldn't know that counts can't be negative.
- So it would happily predict negative values, whenever the mean count is close to zero [McElreath, 2020].
- In linear regression we basically replace the parameter describing the shape of the Gaussian likelihood μ with a linear model.
- The the essence of a Generalized Linear Model (GLM) is to generalize this strategy to probability distributions other than the Gaussian.
- And it results in models that look like this:

$$\begin{aligned} y_i &\sim \text{Binomial}(n, p_i) && \text{[likelihood]} \\ f(p_i) &= \beta_0 + \beta_1 x_i && \text{[generalized linear model]} \end{aligned}$$

Generalized Linear Models

- The first change we can note is that likelihood is binomial instead of Gaussian.
- For a count outcome y for which each observation arises from n trials and with constant expected value $n * p$, the binomial distribution is the de facto choice.
- The function f represents a link function, to be determined separately from the choice of distribution.
- Generalized linear models need a link function, because rarely is there a "μ", a parameter describing the average outcome.
- Parameters are also rarely unbounded in both directions, like μ.
- For example, the shape of the binomial distribution is determined, like the Gaussian, by two parameters.

Generalized Linear Models

- But unlike the Gaussian, neither of these parameters is the mean.
- Instead, the mean outcome is $n * p$, which is a function of both parameters.
- Since n is usually known (but not always), it is most common to attach a linear model to the unknown part, p.
- But p is a probability, so p_i must lie between zero and one.
- But there's nothing to stop the linear model $\beta_0 + \beta_1 x_i$ from falling below zero or exceeding one.
- The link function f provides a solution to this common problem.

Conclusions

- Blabla

References I

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