Cairo University
Faculty of Science
Department of Mathematics

Calculus and analytical geometry (2)
(MATH 132)-Worksheet #02

3rd week (February 22, 2025 - February 27, 2025)

Full Name: ....
Code number.: ....
Group number: ....

Theoretical Background

Theorem: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Theorem: If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2) $\int_a^a f(x)dx = 0$

3) For any constant c
$\int_a^b c\,dx = c(b-a)$

4) For any constant c
$\int_a^b cf(x)dx = c\int_a^b f(x)dx$

5) For any values $a, b, and\ c \in \mathbb{R}$
$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

6) $$\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

7) If $f(x) \geq 0$ for $a \leq x \leq b$, then
$\int_a^b f(x)dx \geq 0$

8) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

9) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

10) If $f(x)$ is an odd function, then
$$\int_{-a}^a f(x)dx = 0$$

11) If $f(x)$ is an even function, then
$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

Solve the following questions:

Q1) Evaluate the following integral by interpreting the integral as a limit of a Riemann sum:

$$\int_1^5 (4 - 2x)\, dx$$

Q2) Evaluate the following integral by interpreting the integral as a limit of a Riemann sum:

$$\int_1^5 (x^2 - 4x + 2)\, dx$$

Q3) Prove that $\int_a^b x^2 dx = \frac{b^3-a^3}{3}$.

Q4) Express the integral as a limit of Riemann sums. Do not evaluate the limit.

$$\int_1^2 \sqrt{4-x^2}\, dx$$

Q5) Express the integral as a limit of Riemann sums. Do not evaluate the limit.

$$\int_2^5 \left(x^2 + \frac{1}{x}\right) dx$$

Q6) Express each of the following limits as integrals:

a) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{3k}{n} + 2\right)$

b) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{1+\left(\frac{k}{n}\right)^2}\right)$

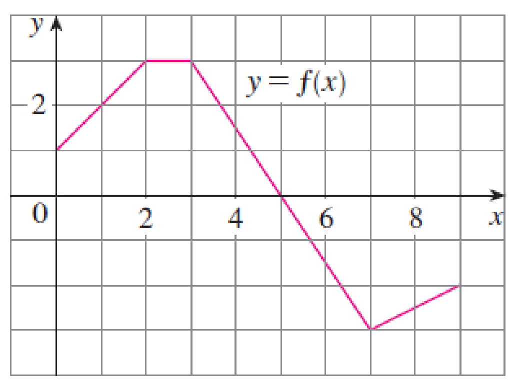
Q7) The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

a) $\int_0^2 f(x)dx$

b) $\int_0^5 f(x)dx$

c) $\int_5^7 f(x)dx$

d) $\int_0^9 f(x)dx$



Q8) Find the value of the integral $\int_{-3}^{3} \sqrt{9-x^2}dx$ by regarding it as the area under the graph of an appropriately chosen function and using an area formula from plane geometry.

Q9) Find the value of the integral $\int_{-2}^{2} (4-|x|)dx$ by regarding it as the area under the graph of an appropriately chosen function and using area formulas from plane geometry.

Q10) If $\int_{2}^{8} f(x)dx$ =7.3 and $\int_{2}^{4} f(x)dx$=5.9, then find $\int_{4}^{8} f(x)dx$.

Q11) If $\int_{-3}^{2} f(x)dx$ =10, $\int_{2}^{3} f(x)dx = -4$, and $\int_{3}^{-3} g(x)dx = -2, then$ find

$$\int_{-3}^{3} (4f(x) - 2g(x) + 5x^3)\, dx$$

Q12) Give an example of two functions f, g which are not integrable on $[0, 1]$ such that $f + g$ is integrable on that interval.

Q13) Find a function $f(x)$ non-integrable on $[0, 1]$ such that $|f(x)|$ is integrable.

Q14) Use the properties of integrals to verify the inequality without evaluating the integrals.

$$2 \leq \int_{-1}^{1} \sqrt{1 + x^2}\, dx \leq 2\sqrt{2}$$

Q15) If f is continuous on $[a, b]$, show that

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Use this result to show that

$$\left|\int_0^{2\pi} f(x)\sin 2x\, dx\right| \leq \int_0^{2\pi} |f(x)|dx$$

Hint: $-|f(x)| \leq f(x) \leq |f(x)|$