CALCULUS & ANALYTICAL GEOMETRY (2) (MATH 132)-WORKSHEET#01

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Answer for OLD Q1

We are given the function

$$f(x) = 1 - x^3,$$

and the region S bounded by the graph of f, the vertical lines $x = 1$ and $x = 5$, and the x-axis.

Step 1: Understanding the Region

Notice that

$$f(1) = 1 - 1^3 = 0,$$

and for $x > 1$,

$$f(x) = 1 - x^3 < 0.$$

Thus, for x in $[1, 5]$, the curve $y = f(x)$ lies below the x-axis. The note in the *lecture 1* file states:

> "If f is a decreasing function, then left endpoints give overestimation."

Since $f(x)$ is decreasing on $[1, 5]$, using the left endpoints will provide an overestimate.

Step 2: Setting Up the Riemann Sum

We approximate the signed area using 4 rectangles. The interval $[1, 5]$ has length

$$5 - 1 = 4,$$

so the width of each rectangle is

$$\Delta x = \frac{4}{4} = 1.$$

The 4 subintervals are:

$$[1, 2], \quad [2, 3], \quad [3, 4], \quad [4, 5].$$

Since f is decreasing, we use the left endpoint of each subinterval to compute the height of the rectangle. The left endpoints are:

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3, \quad x_3 = 4.$$

Step 3: Evaluating f at the Left Endpoints

Compute the function values:

$$\begin{aligned}
f(1) &= 1 - 1^3 = 0, \\
f(2) &= 1 - 2^3 = 1 - 8 = -7, \\
f(3) &= 1 - 3^3 = 1 - 27 = -26, \\
f(4) &= 1 - 4^3 = 1 - 64 = -63.
\end{aligned}$$

Step 4: Compute L_4 Approximation

The area (signed) for each rectangle is given by the height times the width. Thus, the left-endpoint sum is

$$L_4 = \Delta x \cdot \big[f(1) + f(2) + f(3) + f(4)\big].$$

Substituting the values:

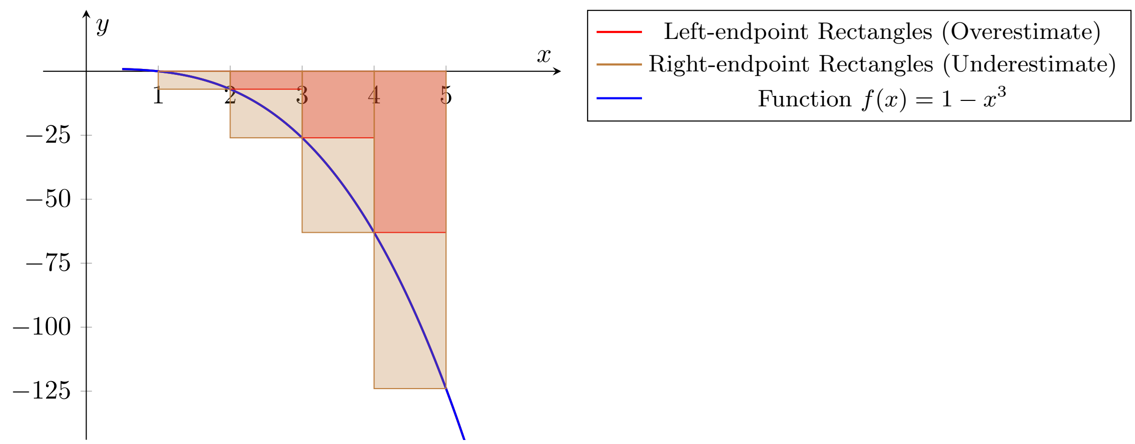
$$L_4 = 1 \cdot (0 + (-7) + (-26) + (-63)) = 0 - 7 - 26 - 63 = -96.$$

Thus, the upper estimation for the signed area is

$$\boxed{S \approx L_4 = -96}.$$

The estimated signed area using the left-endpoint method is -96, while the right-endpoint estimate gives -220.[1]

Graph of $f(x) = 1 - x^3$ with Left and Right Endpoint Rectangles



Answer for NEW Q1

To estimate the area of the region S bounded by the curves $f(x) = 1 - x^3$, the lines $x = -5$ and $x = -1$, and the x-axis, we follow these steps:

Step 1: Determine the Interval and Subintervals

We divide the total width of the interval:

$$-1 - (-5) = 4$$

into 4 equal subintervals:

$$\Delta x = \frac{4}{4} = 1$$

Thus, the subintervals are:

$$[-5, -4], [-4, -3], [-3, -2], [-2, -1]$$

Step 2: Determine Whether $f(x)$ is Decreasing

To check whether $f(x)$ is decreasing, we compute its derivative:

$$f'(x) = \frac{d}{dx}(1 - x^3) = -3x^2$$

Since $x^2 \geq 0$ for all x, it follows that:

$$-3x^2 \leq 0, \quad \text{and for any } x \neq 0, \quad f'(x) < 0$$

[1] In the §OLD Q1, we consider the **signed** area, meaning areas below the x-axis are negative. Since $f(x) = 1 - x^3$ is decreasing on $[1, 5]$, the left-endpoint sum results in a **less negative** value, making it an **overestimate**. However, if we considered the **absolute** area (where all areas are treated as positive), the situation would be reversed: the left-endpoint sum would be smaller, and the right-endpoint sum would be the overestimate.

This confirms that $f(x)$ is strictly decreasing for all x, including on the given interval $[-5, -1]$.

Step 3: Use Left Endpoints for Overestimation

Since $f(x)$ is decreasing, the function values at the left endpoints provide an **overestimate** of the actual area.

The left endpoints are $x = -5, -4, -3, -2$, and evaluating $f(x)$ at these points:

$$f(-5) = 1 - (-5)^3 = 1 + 125 = 126$$

$$f(-4) = 1 - (-4)^3 = 1 + 64 = 65$$

$$f(-3) = 1 - (-3)^3 = 1 + 27 = 28$$

$$f(-2) = 1 - (-2)^3 = 1 + 8 = 9$$

Step 4: Compute the Upper Sum

Each rectangle has width $\Delta x = 1$, so the upper estimate for the area is:

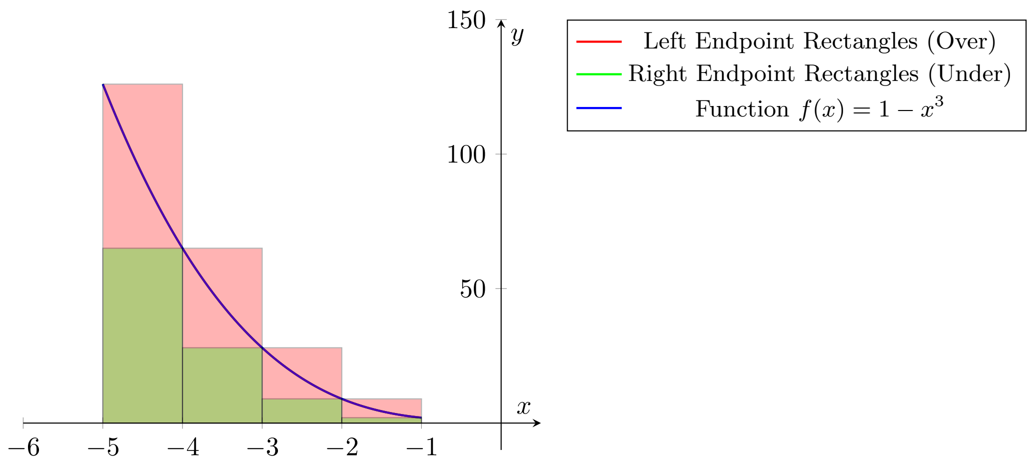
$$S \approx \sum_{i=1}^{4} f(x_i) \cdot \Delta x$$

$$S \approx (126 + 65 + 28 + 9) \times 1$$

$$S \approx 126 + 65 + 28 + 9 = 228$$

Thus, the upper estimation for the area of S using 4 rectangles is **228 square units**.

Graph of $f(x) = 1 - x^3$ with Left and Right Endpoint Rectangles



Answer of Q2

Let the given function be:

$$f(x) = \frac{1}{3 + x^2}$$

The region is bounded by $x = -2$, $x = -1$, and the x-axis. We approximate the area using 3 rectangles.

Step 1: Compute the Partition Width

The interval is $[-2, -1]$ and we divide it into 3 subintervals:

$$\Delta x = \frac{-1 - (-2)}{3} = \frac{1}{3}$$

The left endpoints are given by:

$$x_k = x_0 + k\Delta x, \quad \text{for } k = 0, 1, 2$$

where $x_0 = -2$.

Step 2: Compute the Left Endpoints

$$\begin{aligned} x_0 &= -2, \\ x_1 &= -2 + \frac{1}{3} = -\frac{5}{3}, \\ x_2 &= -2 + \frac{2}{3} = -\frac{4}{3} \end{aligned}$$

Step 3: Compute Function Values at Left Endpoints

$$\begin{aligned} f(x_0) &= f(-2) = \frac{1}{3+4} = \frac{1}{7}, \\ f(x_1) &= f\left(-\frac{5}{3}\right) = \frac{1}{3 + \frac{25}{9}} = \frac{1}{\frac{52}{9}} = \frac{9}{52}, \\ f(x_2) &= f\left(-\frac{4}{3}\right) = \frac{1}{3 + \frac{16}{9}} = \frac{1}{\frac{43}{9}} = \frac{9}{43} \end{aligned}$$

Step 4: Compute L_3 Approximation for S

$$L_3 = \sum_{k=1}^{3} f(x_{k-1})\Delta x$$

$$L_3 = \left(\frac{1}{7} + \frac{9}{52} + \frac{9}{43}\right) \times \frac{1}{3}$$

Converting to a common denominator:

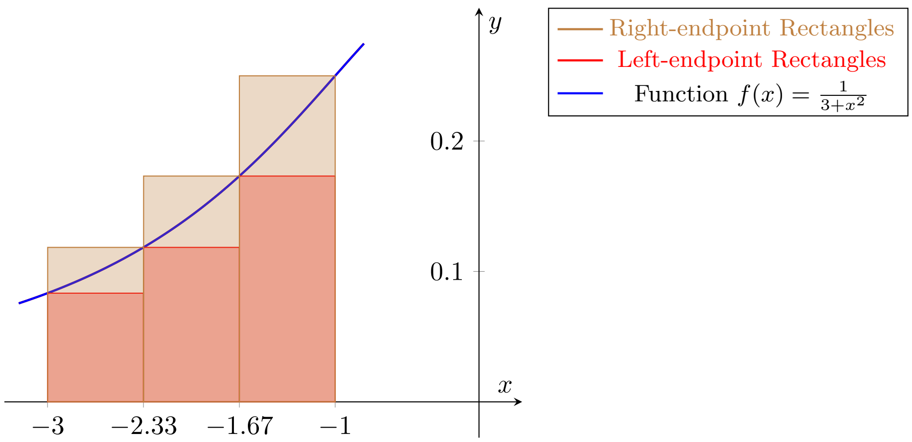
$$\begin{aligned} \frac{1}{7} &= \frac{52}{364}, \\ \frac{9}{52} &= \frac{63}{364}, \\ \frac{9}{43} &= \frac{81}{364} \end{aligned}$$

$$\frac{52}{364} + \frac{63}{364} + \frac{81}{364} = \frac{196}{364}$$

$$\boxed{L_3 = \frac{196}{364} \times \frac{1}{3} = \frac{196}{1092}}$$

Thus, the lower estimation for the area of S using L_3 is *approximately* $\frac{196}{1092}$.

Graph of $f(x) = \frac{1}{3+x^2}$ with Left and Right Endpoint Rectangles



Answer of OLD Q3

We need to lower estimate the area under the curve

$$f(x) = x^2 - 3x + 4$$

over the interval $[0, 4]$ using 4 rectangles.

Step 1: Identify Where $f(x)$ is Increasing or Decreasing

We compute the derivative:

$$f'(x) = 2x - 3.$$

Setting $f'(x) = 0$ to find the critical point:

$$2x - 3 = 0 \Rightarrow x = \frac{3}{2}.$$

- $f(x)$ is **decreasing** on $[0, \frac{3}{2}]$.
- $f(x)$ is **increasing** on $[\frac{3}{2}, 4]$.

Thus, we break the interval at $x = \frac{3}{2}$ and approximate separately:
- On $[0, \frac{3}{2}]$, we use R_2 (right endpoints give the lower estimate).
- On $[\frac{3}{2}, 4]$, we use L_2 (left endpoints give the lower estimate).

Step 2: Compute R_2 on $[0, \frac{3}{2}]$

Divide $[0, \frac{3}{2}]$ into 2 equal subintervals:

$$\left[0, \frac{3}{4}\right], \quad \left[\frac{3}{4}, \frac{3}{2}\right]$$

- Width of each rectangle: $\Delta x = \frac{3}{4}$.
- Right endpoints: $x = \frac{3}{4}$ and $x = \frac{3}{2}$.

Evaluate $f(x)$ at these points:

$$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 4 = \frac{37}{16}.$$

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 4 = \frac{9}{4} - \frac{9}{2} + 4 = \frac{16}{4} - \frac{18}{4} + \frac{9}{4} = \frac{7}{4}.$$

Now, approximate the area using R_2:

$$R_2 = \Delta x \sum_{i=1}^{2} f(x_i) = \left(\frac{3}{4}\right)\left(\frac{37}{16} + \frac{7}{4}\right)$$

$$= \left(\frac{3}{4}\right)\left(\frac{65}{16}\right) = \frac{195}{64}.$$

Step 3: Compute L_2 on $[\frac{3}{2}, 4]$

Divide $[\frac{3}{2}, 4]$ into 2 equal subintervals:

$$\left[\frac{3}{2}, \frac{11}{4}\right], \quad \left[\frac{11}{4}, 4\right]$$

- Width of each rectangle: $\Delta x = \frac{5}{4}$.
- Left endpoints: $x = \frac{3}{2}$ and $x = \frac{11}{4}$.

We already have:

$$f\left(\frac{3}{2}\right) = \frac{7}{4}.$$

Now, evaluate $f(x)$ at $x = \frac{11}{4}$:

$$f\left(\frac{11}{4}\right) = \frac{53}{16}.$$

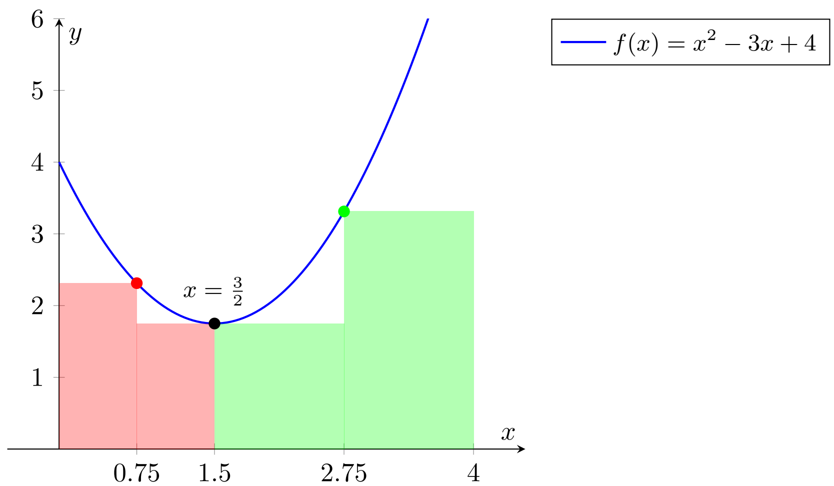
Now, approximate the area using L_2:

$$L_2 = \Delta x \sum_{i=1}^{2} f(x_{i-1}) = \left(\frac{7}{4} + \frac{53}{16}\right)$$

$$= \left(\frac{5}{4}\right)\left(\frac{81}{16}\right) = \frac{405}{64}.$$

Step 4: Compute S Area

$$\boxed{S \approx R_2 + L_2 = \frac{195}{64} + \frac{405}{64} = \frac{600}{64} = \mathbf{9.375}.}$$



Answer of NEW Q3

Let S be the region bounded between the curves:

$$f(x) = x^2 - 3x + 4, \quad x = 2, \quad x = 6, \quad \text{and the } x\text{-axis}.$$

Find the lower estimation for the area of S using 4 rectangles.

Step 1: Identify the Interval and Number of Rectangles

- The smallest point (lower bound) is $a = 2$.
- The largest point (upper bound) is $b = 6$.
- The number of rectangles is $n = 4$.

Step 2: Compute Δx

The width of each rectangle is given by:

$$\Delta x = \frac{b-a}{n} = \frac{6-2}{4} = \frac{4}{4} = 1.$$

Step 3: Determine the x_i Values

The endpoints are given by:

$$x_i = a + i\Delta x$$

For $i = 0, 1, 2, 3, 4$:

$$x_0 = 2, \quad x_1 = 3, \quad x_2 = 4, \quad x_3 = 5, \quad x_4 = 6.$$

Step 4: Evaluate $f(x)$ at the Required Points

The function is:

$$f(x) = x^2 - 3x + 4.$$

Evaluating at the chosen points:

$$f(2) = 2^2 - 3(2) + 4 = 4 - 6 + 4 = 2.$$

$$f(3) = 3^2 - 3(3) + 4 = 9 - 9 + 4 = 4.$$

$$f(4) = 4^2 - 3(4) + 4 = 16 - 12 + 4 = 8.$$

$$f(5) = 5^2 - 3(5) + 4 = 25 - 15 + 4 = 14.$$

$$f(6) = 6^2 - 3(6) + 4 = 36 - 18 + 4 = 22.$$

Step 5: Determine if $f(x)$ is Increasing or Decreasing

Differentiate $f(x)$:

$$f'(x) = 2x - 3.$$

Check the sign of $f'(x)$ on the interval $[2, 6]$:

$$f'(2) = 2(2) - 3 = 1 > 0.$$

Since $f'(x) > 0$ for all $x \in [2, 6]$, $f(x)$ is **increasing** on the interval.

Step 7: Compute the Lower Estimation L_4

For an increasing function:

- Left endpoint approximation gives the **lower estimation**.

Thus, we will use the left endpoint method.

The left endpoint approximation with 4 rectangles is given by:

$$L_4 = \sum_{k=1}^{4} f(x_{k-1}) \Delta x.$$

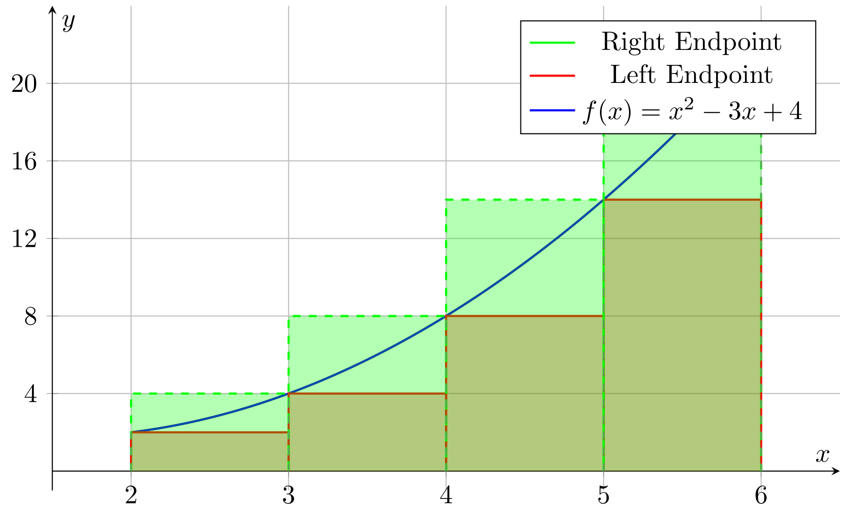
Substituting the values:

$$L_4 = [f(2) + f(3) + f(4) + f(5)] \times 1$$

$$L_4 = (2 + 4 + 8 + 14) \times 1$$

$$L_4 = 28.$$

Left Endpoint and Right Endpoint Approximations with 4 Rectangles



Answer of Q4

We want to find the Riemann sum for the function

$$f(x) = 4 - x^2$$

over the interval $[-1, 2]$ corresponding to the partition:

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{5}{4}, \quad x_4 = 2$$

with sample points:

$$x_1^* = -\frac{1}{4}, \quad x_2^* = \frac{1}{4}, \quad x_3^* = 1, \quad x_4^* = \frac{5}{4}.$$

Step 1: Compute Δx_i

The width of each subinterval is:

$$\Delta x_1 = x_1 - x_0 = 0 - (-1) = 1$$

$$\Delta x_2 = x_2 - x_1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\Delta x_3 = x_3 - x_2 = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$$

$$\Delta x_4 = x_4 - x_3 = 2 - \frac{5}{4} = \frac{3}{4}.$$

Step 2: Evaluate $f(x_i^*)$

Evaluate the function at each sample point:

$$f\left(-\frac{1}{4}\right) = 4 - \left(-\frac{1}{4}\right)^2 = 4 - \frac{1}{16} = \frac{63}{16}$$

$$f\left(\frac{1}{4}\right) = 4 - \left(\frac{1}{4}\right)^2 = 4 - \frac{1}{16} = \frac{63}{16}$$
$$f(1) = 4 - 1 = 3$$
$$f\left(\frac{5}{4}\right) = 4 - \left(\frac{5}{4}\right)^2 = 4 - \frac{25}{16} = \frac{39}{16}.$$

Step 3: Compute the Riemann Sum

The Riemann sum is given by:

$$S = \sum_{i=1}^{4} f(x_i^*)\Delta x_i$$

Substituting the values:

$$S = \left(\frac{63}{16} \times 1\right) + \left(\frac{63}{16} \times \frac{1}{2}\right) + \left(3 \times \frac{3}{4}\right) + \left(\frac{39}{16} \times \frac{3}{4}\right).$$

Evaluating each term:

$$\frac{63}{16} \times 1 = \frac{63}{16}$$
$$\frac{63}{16} \times \frac{1}{2} = \frac{63}{32}$$
$$3 \times \frac{3}{4} = \frac{9}{4}$$
$$\frac{39}{16} \times \frac{3}{4} = \frac{117}{64}.$$

Step 4: Combine the Results

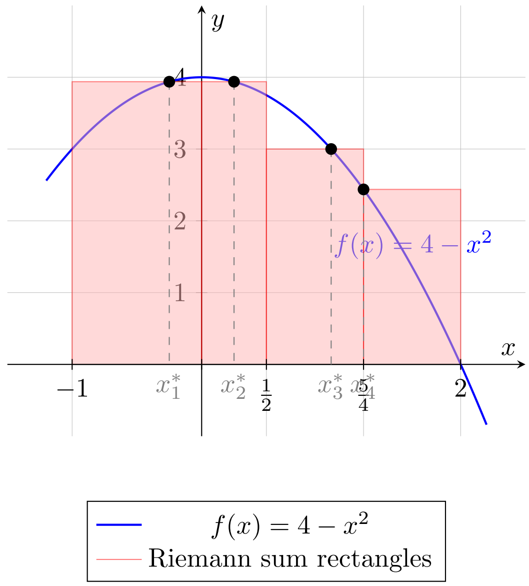
Converting all fractions to a common denominator of 64:

$$\frac{63}{16} = \frac{252}{64}, \quad \frac{63}{32} = \frac{126}{64}, \quad \frac{9}{4} = \frac{144}{64}, \quad \frac{117}{64} \text{ remains unchanged.}$$

Thus:

$$S \approx \frac{252}{64} + \frac{126}{64} + \frac{144}{64} + \frac{117}{64} = \frac{639}{64}.$$

Riemann Sum for $f(x) = 4 - x^2$ on $[-1, 2]$



Answer of Q5

(a)

The area under the curve can be expressed as a limit of Riemann sums:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Step 1: Define the partition.

$$a = 0, \quad b = 2, \quad \Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = a + k\Delta x = 0 + k \cdot \frac{2}{n} = \frac{2k}{n}$$

Step 2: Write the Riemann sum.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{2k}{n}\right) \Delta x$$

Substitute $f(x) = x^3 + x$:

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(\frac{2k}{n}\right)^3 + \frac{2k}{n}\right] \cdot \frac{2}{n}$$

Step 3: Simplify the expression.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{8k^3}{n^3} + \frac{2k}{n}\right] \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{16k^3}{n^4} + \frac{4k}{n^2} \right)$$

Step 4: Separate the sums.

$$= \lim_{n\to\infty} \left[\frac{16}{n^4} \sum_{k=1}^{n} k^3 + \frac{4}{n^2} \sum_{k=1}^{n} k \right]$$

Step 5: Use known summation formulas.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Substitute these into the expression:

$$= \lim_{n\to\infty} \left[\frac{16}{n^4} \left(\frac{n^2(n+1)^2}{4} \right) + \frac{4}{n^2} \cdot \frac{n(n+1)}{2} \right]$$

Step 6: Simplify each term.

$$= \lim_{n\to\infty} \left[\frac{4(n+1)^2}{n^2} + \frac{2(n+1)}{n} \right]$$

Step 7: Expand and combine like terms.

$$= \lim_{n\to\infty} \left[\frac{4(n^2+2n+1)}{n^2} + \frac{2(n+1)}{n} \right]$$

$$= \lim_{n\to\infty} \left(4 + \frac{8}{n} + \frac{4}{n^2} + 2 + \frac{2}{n} \right)$$

(b)

We will approximate this integral using a Riemann sum.

Step 1: Partitioning the Interval

Let the interval $[0, \pi]$ be divided into n subintervals of equal length. Then:

$$a = 0, \quad b = \pi, \quad \Delta x = \frac{\pi}{n}.$$

Step 2: Choosing Sample Points

We choose the right endpoints for the subintervals as the sample points:

$$x_k = a + k\Delta x = 0 + k \cdot \frac{\pi}{n} = \frac{k\pi}{n}, \quad \text{for } k = 1, 2, \ldots, n.$$

Step 3: Expressing the Area as a Limit

To find the exact area, we take the limit of the Riemann sum as $n \to \infty$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\sin\left(\frac{k\pi}{n}\right)} \cdot \frac{\pi}{n}.$$

Factoring out the constant $\frac{\pi}{n}$, we get:

$$= \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sqrt{\sin\left(\frac{k\pi}{n}\right)}.$$

(c)

To find the area under the curve as a limit, we use the definition of a definite integral in terms of Riemann sums:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Step 1: Identify the Interval and Partition

Given:

$$a = -2, \quad b = 0$$

The width of each subinterval (Δx) is:

$$\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{n} = \frac{2}{n}$$

Step 2: Determine the Sample Points

For a left-endpoint Riemann sum, we choose the sample points:

$$x_{n-k} = b - k\Delta x = 0 - k \cdot \frac{2}{n} = -\frac{2k}{n}$$

Step 3: Write the Riemann Sum

Realize that:

$$\sum_{i=1}^{n} a_{i-1} = \sum_{i=1}^{n} a_{n-i}$$

Then Riemann sum becomes:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n-k})\Delta x$$

Substituting $f(x) = x^2 - 3x$:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(-\frac{2k}{n}\right)^2-3\left(-\frac{2k}{n}\right)\right]\cdot\frac{2}{n}$$

Step 4: Simplify the Expression

First, expand and simplify inside the sum:

$$\left(-\frac{2k}{n}\right)^2=\frac{4k^2}{n^2}$$
$$-3\left(-\frac{2k}{n}\right)=\frac{6k}{n}$$

Thus, the sum becomes:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+\frac{6k}{n}\right)\cdot\frac{2}{n}$$

Factor and combine like terms:

$$=\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+\frac{6k}{n}\right)$$
$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{8k^2}{n^3}+\frac{12k}{n^2}\right)$$

Step 5: Use Known Summation Formulas

Recall the standard summation formulas:

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

Substitute these results into the expression:

$$\lim_{n\to\infty}\left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{12}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

Step 6: Final Limit Expression

The final limit expression for the area under the curve is:

$$\boxed{\lim_{n\to\infty}\left(\frac{8(n+1)(2n+1)}{6n^2}+\frac{6(n+1)}{n}\right)}$$

(d)

Step 1: Define the Riemann Sum

The area under the curve can evaluated by the limit of the Riemann sum:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where:

- $a = 0$ and $b = 1$ represent the interval $[0, 1]$.
- $\Delta x = \frac{b-a}{n} = \frac{1}{n}$ is the width of each subinterval.
- $x_k = a + k\Delta x = \frac{k}{n}$ represents the right endpoint of each subinterval.

Step 2: Substitute the Function $f(x)$

Given the function:

$$f(x) = \sin\left(\frac{\pi}{2}x\right),$$

the Riemann sum becomes:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\frac{\pi}{2} \cdot \frac{k}{n}\right) \cdot \frac{1}{n}.$$

Answer of Q6

$$A = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(x_k) = \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(5 + \frac{2k}{n}\right)^{10}$$

- Step size: $\Delta x = \frac{2}{n}$
- Sample points: $x_k = a + k\Delta x = 5 + \frac{2k}{n}$
- Interval: $[5, 7]$
- Function: $f(x) = x^{10}$

Answer of Q7

Step 1: Recognize the Increasing Nature of the Function

The given function is:

$$f(x) = x^4 + 1.$$

The derivative is:

$$f'(x) = 4x^3.$$

Since $f'(x) \geq 0$ for $x \in [0, 1]$, the function $f(x)$ is increasing on the interval $[0, 1]$.

Step 2: Applying the Lower and Upper Sum Approximations

For an increasing function on $[0, 1]$, the following inequality holds for any partition:

$$L_n \leq A \leq R_n,$$

where L_n is the left Riemann sum and R_n is the right Riemann sum.

We choose $n = 3$ subintervals for the partition.

$$\Delta x = \frac{1 - 0}{3} = \frac{1}{3}.$$

Step 3: Compute the Left Riemann Sum L_3

The left Riemann sum is:

$$L_3 = \sum_{k=0}^{2} f\left(\frac{k}{3}\right) \Delta x.$$

Substituting the values:

$$L_3 = \frac{1}{3}\left[f(0) + f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right].$$

Evaluating each term:

$$f(0) = 0^4 + 1 = 1,$$

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^4 + 1 = \frac{1}{81} + 1 = \frac{82}{81},$$

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^4 + 1 = \frac{16}{81} + 1 = \frac{97}{81}.$$

Thus:

$$L_3 = \frac{1}{3}\left[1 + \frac{82}{81} + \frac{97}{81}\right] = \frac{1}{3} \times \frac{260}{81} = \frac{260}{243} \approx 1.07.$$

Step 4: Compute the Right Riemann Sum R_3

The right Riemann sum is:

$$R_3 = \sum_{k=1}^{3} f\left(\frac{k}{3}\right) \Delta x.$$

Substituting the values:

$$R_3 = \frac{1}{3}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1)\right].$$

Evaluating the last term:

$$f(1) = 1^4 + 1 = 2.$$

So:

$$R_3 = \frac{1}{3}\left[\frac{82}{81} + \frac{97}{81} + 2\right] = \frac{1}{3} \times \frac{341}{81} = \frac{341}{243} \approx 1.40.$$

Step 5: Conclude the Bounds for the Area A

Since $f(x)$ is increasing:

$$L_3 \leq A \leq R_3.$$

Substituting the calculated values:

$$1.07 \leq A \leq 1.40.$$

Thus, we have:

$$1 < A < 1.5.$$

Final Answer: $\boxed{\text{True}}$

Answer of Q8

Check the validity of:

$$\sum_{i=1}^{n} \left(i + \sin\left(\frac{i\pi}{2}\right) \right) = \frac{n(n+1)}{2} + \sin\left(\frac{(n+1)\pi}{4}\right)$$

Counterexample: $n = 2$

LHS:

$$1 + \sin\left(\frac{\pi}{2}\right) + 2 + \sin(\pi) = 1 + 1 + 2 + 0 = 4$$

RHS:

$$\frac{2 \times 3}{2} + \sin\left(\frac{3\pi}{4}\right) = 3 + \frac{\sqrt{2}}{2} \approx 3.707$$

Conclusion

Since:

$$\text{LHS} = 4 \neq \text{RHS} \approx 3.707$$

The equation is **false**.

Answer of Q9

Problem Statement

Let A be the area under the graph of an increasing continuous function f from a to b. Let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(a) How are A, L_n, and R_n related?

(b) Show that $R_n - L_n = \dfrac{b-a}{n}\big[f(b) - f(a)\big]$.

(a) Relationship between A, L_n, and R_n

Since f is increasing and continuous on the interval $[a, b]$:

- L_n (Left Riemann Sum) gives a **lower estimation** for A, because it uses the function values at the left endpoints, which are always less than or equal to the function values at the right endpoints in the case of an increasing function.
- R_n (Right Riemann Sum) gives an **upper estimation** for A, since it uses the function values at the right endpoints, which are greater than or equal to the function values at the left endpoints.

Thus, the area A will be bounded by L_n and R_n as follows:

$$L_n \leq A \leq R_n, \quad \text{for all natural numbers } n.$$

(b) Proof that $R_n - L_n = \dfrac{b-a}{n}[f(b) - f(a)]$

Step 1: Recall Definitions

Let $\Delta x = \dfrac{b-a}{n}$ and partition the interval $[a, b]$ into n equal subintervals. Define the points:

$$x_k = a + k\Delta x, \quad k = 0, 1, 2, \ldots, n.$$

The right Riemann sum (R_n) and left Riemann sum (L_n) are defined as:

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x,$$
$$L_n = \sum_{k=0}^{n-1} f(x_k)\Delta x.$$

Step 2: Compute $R_n - L_n$

$$\begin{aligned}
R_n - L_n &= \sum_{k=1}^{n} f(x_k)\Delta x - \sum_{k=0}^{n-1} f(x_k)\Delta x \\
&= \Delta x \left[\sum_{k=1}^{n} f(x_k) - \sum_{k=0}^{n-1} f(x_k) \right] \\
&= \Delta x \left[\sum_{k=1}^{n-1} f(x_k) + f(x_n) - f(x_0) - \sum_{k=1}^{n-1} f(x_k) \right].
\end{aligned}$$

Notice that most terms cancel out in the subtraction, leaving:

$$R_n - L_n = \Delta x \big[f(x_n) - f(x_0) \big].$$

Since $x_n = b$ and $x_0 = a$, this becomes:

$$R_n - L_n = \Delta x \big[f(b) - f(a) \big].$$

Substituting $\Delta x = \frac{b-a}{n}$:

$$R_n - L_n = \frac{b-a}{n}\big[f(b) - f(a)\big].$$

Step 3: Final Result

We have shown that:

$$R_n - L_n = \frac{b-a}{n}\big[f(b) - f(a)\big].$$

Step 4: Proof that $x_n = b$

To confirm the expression for x_n:

$$\begin{aligned} x_n &= a + n\Delta x \\ &= a + n\frac{b-a}{n} \\ &= a + b - a \\ &= b. \end{aligned}$$

This verifies that the endpoint calculations are consistent.

About The Repository

This document is part of the `github.com/3ndlyb/Math132Answers` repository, which contains solutions to Math132 problem sheets. Each sheet is organized in its own folder, including:

- The original problem sheet in PDF format (`Math - 132 - sheetX.pdf`).
- A LaTeX source file (`main.tex`) containing the answers.
- The compiled PDF of the answers (`SheetXAns.pdf`).

To view the answers, simply open the corresponding `SheetXAns.pdf` file. If you wish to edit the answers, you can modify the `main.tex` file and recompile it using:

```
$ pdflatex main.tex
```

Make sure you have a suitable LaTeX distribution installed, such as TeX Live or MiKTeX.

Contributions and Feedback

Contributions to improve the repository are welcome. If you spot any errors or have suggestions, feel free to:

1. Open an issue on the GitHub repository.
2. Fork the repository, make your changes, and submit a pull request.

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Contact

For further questions or feedback, feel free to reach out via Discord at `3ndlybalabyd` .

Acknowledgments

Spotify Playlist

This document wouldn't have been possible without my beloved Spotify playlist.
Click the link below or scan the QR code to listen to it:

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