chapter 3 Analytic Geometry.

- Norm (mapping의 일종) → lenght $\|x\|$
  - L1 norm $= \sum_{i=1}^{n} |x_i|$
  - L2 norm $= \sum_{i=1}^{n} \sqrt{x^T x}$.

특성
- Absolutely homogeneous : $\|\lambda x\| = |\lambda| \|x\|$
- Triangle inequality : $\|x+y\| \le \|x\| + \|y\|$
- Positive definite.

- Bilinear mapping : 각각의 arguement에 대해 linear

$$\Omega(\lambda x + \psi \lambda, z) = \lambda \Omega(x, z) + \psi \Omega(y, z)$$

- Inner product → Bilinear의 일종. , 몇가지 특성이 추가된 것.

  └ = A positive definite, symmetric bilinear mapping $\Omega: V \times V \to R$. ⇒ $\langle x, y \rangle$

  ex) scalar / dot product : $x^T y = \sum_{i=1}^{n} x_i y_i$

- symmetric, positive definite matrix.

$B = (b_1 \cdots b_n)$ (ordered basis)

$$x = \sum_{i=1}^{n} \psi_i b_i \in V, \quad y = \sum_{j=1}^{n} \lambda_j b_j \in V$$

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{n} \psi_i b_i, \sum_{j=1}^{n} \lambda_j b_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_i \langle b_i, b_j \rangle \lambda_j = \hat{x}^T A \hat{y}$$

Coordinate of x.y

- Inner product induces a norm. (모든 norm을 induce x)
- Distance : $d(x, y) = \|x - y\| = \sqrt{\langle x-y, x-y \rangle}$

  metrics의 특성
  - positive definite
  - Symmetric
  - Triangle inequality

- Angle between 2 vectors : $\cos w = \dfrac{\langle x, y \rangle}{\|x\| \|y\|}$
- Orthogonality → $\langle x, y \rangle = 0$

  ↳ Orthogonal matrix : $A^T A = I \to A^{-1} = A^T$

  Orthogonal matrix로 인한 transformation $\|Ax\|^2 = \|x\|^2$ → length 보존

  $$\cos w = \frac{(Ax)^T (Ay)}{\|Ax\| \|Ay\|} = \frac{x^T y}{\|x\| \|y\|}$$ → angle 보존.

  = Inner product 보존.

- orthonormal basis $\rightarrow \begin{cases} \langle b_i, b_j \rangle = 0 \\ \langle b_i, b_i \rangle = 1. \end{cases}$

- orthogonal complement

$V \in \mathbb{R}^D,\ U \in \mathbb{R}^M \subseteq V \rightarrow$ orthogonal complement $U^{\perp} \in \mathbb{R}^{D-M}$

$\Rightarrow$ U에 orthogonal 한 V에 모든 vector들.

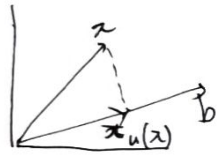
$$x \in V = \sum_{m=1}^{M} \underbrace{\lambda_m b_m}_{\text{basis of } U} + \sum_{j=1}^{D-M} \psi_j \underbrace{b_j^{\perp}}_{\text{basis of } U^{\perp}}$$

$\{b_m, b_i^{\perp}\}$: V의 basis

- Projection : $\pi^2 = \pi \circ \pi = \pi$.

Projection matrices : $P_{\pi}^2 = P_{\pi}$.

- orthogonal projections.



$$\pi_U(x) = \lambda b$$

$$\langle x - \pi_U(x), b \rangle \xleftrightarrow{\pi_U(x) = \lambda b} \langle x - \lambda b, b \rangle = 0.$$

$$\rightarrow \langle x, b \rangle - \lambda \langle b, b \rangle = 0. \rightarrow \lambda = \frac{\langle x, b \rangle}{\langle b, b \rangle} = \frac{\langle b, x \rangle}{\|b\|}$$

$$\pi_U(x) = \frac{b^T x}{\|b\|^2} b = \underbrace{\frac{b b^T}{\|b\|^2}}_{P_{\pi}} x$$

P_{π} : symmetric matrix.

- orthogonal projections onto a general subspace U.

$$\pi_U(x) = B\lambda \quad \begin{cases} B = [b_1 \cdots b_m] \in \mathbb{R}^{n \times m} \\ \lambda = [\lambda_1 \cdots \lambda_m]^T \in \mathbb{R}^m \end{cases} \Rightarrow \begin{array}{l} b_1^T (x - \pi_U(x)) = 0 \\ \vdots \\ b_m^T (x - \pi_U(x)) = 0 \end{array}$$

$$\Rightarrow \begin{bmatrix} b_1^T \\ \vdots \\ b_m^T \end{bmatrix} \begin{bmatrix} x - B\lambda \end{bmatrix} = 0 \iff B^T (x - B\lambda) = 0 \iff B^T B \lambda = B^T x. \rightarrow \lambda = (B^T B)^{-1} B^T x$$

$$\therefore \pi_U(x) = \underbrace{B (B^T B)^{-1} B^T}_{P_{\pi}} x.$$

- Gram-schmidt orthogonalization process.

$$u_1 := b_1$$

$$u_k := b_k - \pi_{span(u_1 \cdots u_{k-1})}(b_k)$$

- Projection onto affine subspaces. ⇒ affine subspace를 origin으로 move → projection → 다시 affine subspace로 옮기면서 x_0를 더한다.

- Rotation : linear mapping. : θ만큼 원점을 중심으로 돌아감. = orthogonal matrix.

$$R(\theta) = \begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$