Vector

Vector: Object끼리 더해지고, Scalar에 의해 곱해질 수 있는 Object.

Vector의 종류

① Geometric Vector. $\vec{x}$, $\vec{x}+\vec{y}=\vec{z}$ $\lambda\vec{x}=\vec{w}$

② Polynomial

③ Signal.

④ $\mathbb{R}^n$ - $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

⑤ $\mathbb{R}^{m\times n}$

"Closure" : Vector 끼리의 Scaling이나 Adding은 다른 Vector Space 생성.

Systems of Linear Equations

m equation n unknown variables.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \text{Solution} \begin{cases} \text{Unique} \\ \text{No} \\ \text{Infinite} \end{cases}$$

가진 해.

Matrix.

- $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$ 일 때

  $A+B :=$ 성분끼리의 합 $\in \mathbb{R}^{m\times n}$: Element-wise Sum.

- $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times k}$ 일 때,

  $C = AB \in \mathbb{R}^{m\times k}$ 에서 성분 $c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$

  ↳ Inner Product of i th row of A and j th column of A

- Properties of matrix addition and multiplication
  - Associativity : $(AB)C = A(BC) = ABC$
  - Distributivity : $(A+B)C = AC+BC$
  - Multiplication with the identity matrix: $\forall A \in \mathbb{R}^{m\times n}$: $I_m A = A I_n = A$

- Matrix Inverse : $AB = I_n = BA$ when $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ $\Rightarrow A = B^{-1}, B = A^{-1}$
  - 역행렬 존재시: Regular, Invertible, Nonsingular ⇔ Noninvertible, Singular

Matrix.

- Transpose : $A \in R^{m \times n}$, $B \in R^{n \times m}$ $b_{ij} = a_{ji}$ is called transpose of A $B = A^T$
  - $(AB)^T = B^T A^T$

- Symmetric Matrix.
  - $A^T = A$ and $A \in R^{n \times n}$
  - when A and B are symmetric, $A+B$ is also symmetric

- Multiplication
  - by Scalar.
    $A \in R^{m \times n}$ and $\lambda \in R$, $\Rightarrow \lambda A = K$ $\quad K_{ij} = \lambda A_{ij}$
  - Compact representation of Systems of linear equations.
    $Ax = b \rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\Rightarrow A \in R^{m \times n}$, $B \in R^m$ 일때, How to Solve?

    ① $x = A^{-1} b$ when $A \in R^{m \times m}$ is Non Singular $\Rightarrow$ 매우 오래걸림 (m이 클수록)
    ② Make Simpler form
      - 1. Exchange two equation (row 끼리)
      - 2. Multiplicate with Costant $\lambda \neq 0$ to an equation (row에 상수배)
      - 3. Addition of two equations

    ⇓

    Row-Echelon Form.

    ① All-Zero Row는 Matrix의 제일 아래로
    ② Pivot이 앞의 열에서 존재할수록 가장 위로 배치
       Pivot: first non zero number from the left

    ex) $\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$ ● : Pivot

    Echelon (계단형)

  - Gaussian Elimination에 의해 Row-Echelon Form 형성.
    $Ax = b \Leftrightarrow [A|b] \rightsquigarrow R$ (Row-Echelon Form)
    Gaussian Elimination
    Augmented Matrix = Concatenation with A and B

Gaussian Elimination.

- example

$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & a \end{array}\right]$: Row-Echelon Form ⇒ $\begin{cases} \text{when } a=0, \text{ at least one solution} \\ \text{when } a\neq 0, \text{ no solution} \end{cases}$

⇓

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + x_2 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$ 에서 x_2는 free Variable (λ_1, λ_2)

not pivot column

x_1과 x_3 은 basic Variable

↳ Particular Solution ($b=0$일 때, 존재X := Homogeneous system)

- Free Variable: 자유롭게 결정되는 Variable (주로 lambda로 표현) ⇒ Solution Set 결정.
- Basic Variable: Particular Solution에 의해 결정되는 Variable
  ↳ Row-Echelon Form에서 Pivot Column에 대응하는 Variable

※ Unique Solution인 경우, Free Variable이 존재X.
Infinite Solution인 경우 Free Variable이 존재 O, ⇐역도 성립.

Back Substitution.

- After Gaussian Elimination, lowest row로부터 구하는 방법.

③ Make Another Simpler Form. (Reduced-Echelon Form).

- Row-Echelon Form
- Every Pivot = 1
- Pivot의 위, 아래 행의 요소는 모두 0 (Pivot is the only nonzero entity in its column)
  ⇒ back propagation process에 사용 but Gaussian Elimination보다 오래걸려 사용X.

- 만약 Nonsingular (Invertible) 하다면 A의 Inverse Matrix를 구하는 방법임.
  ↳ $[A \mid I_n] \rightsquigarrow \cdots \to [I_n \mid A^{-1}]$

Solving System of Linear Equation.

- by using the Inverse.

① If A is invertible,

$x = A^{-1}b$ and x is Unique Solution.

② If A has linearly Independent columns. (Assumption)

$Ax = b \Leftrightarrow A^TAx = A^Tb \Leftrightarrow x = (A^TA)^{-1}A^Tb$

$\hookrightarrow (A^TA^{-1})A^T$: Moore-Penrose pseudo-inverse of A

↳ Why?

if $A \in R^{k \times 2}$ (when $k > 2$), A is singular matrix. (and No Square Matrix B)

A^TA is square matrix (Can be Nonsingular)

$x = (A^TA)^{-1}A^Tb$ 와 같은 Solution은

$\|Ax-b\|^2$를 최소화하는 Approximation Solution으로 대응됨. (Linear Regression)

※ Linearly Independent.

↳ $\left\{ x_1 \underset{\alpha_1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} + x_2 \underset{\alpha_2}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} \cdots x_m \underset{\alpha_k}{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}} = b \right.$ 에서

$b=0$을 만족하는

$x_1 \ldots x_m$이 모두 0일때만 성립시

$\alpha = \{\alpha_1, \alpha_2, \ldots \alpha_k\}$인 Vector가

서로 선형독립 (Linearly Independent)