Worksheet: Multiple time-scale analysis and amplitude equations

0/ Very short reminder on finite elements

Let us solve the following problem:

$$u - (\partial_{xx}u + \partial_{yy}u) = f$$

$$u = d \text{ on } \Gamma_d$$

$$au + b\partial_n u = c \text{ on } \Gamma_m$$

We consider test functions $\breve{u}$ satisfying $\breve{u} = 0 \text{ on } \Gamma_d$. After multiplying the governing equation by the test-function, we take an integral over the complete domain:

$$\iint \breve{u}\big(u - (\partial_{xx}u + \partial_{yy}u\big)dxdy = \iint \breve{u}f dxdy$$

Integrating by parts, we obtain:

$$\iint \big(\breve{u}u + \partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u\big)dxdy - \int \big(\breve{u}n_x\partial_x u + \breve{u}n_y\partial_y u\big)ds = \iint \breve{u}f dxdy$$

The boundary term is zero on Γ_d because of $\breve{u} = 0$. Therefore, taking into account the boundary condition on Γ_m, we have:

$$\iint \big(\breve{u}u + \partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u\big)dxdy - \int_{\Gamma_m} \breve{u}\left(c - \frac{a}{b}u\right)ds = \iint \breve{u}f dxdy$$

Rearranging:

$$\iint \big(\breve{u}u + \partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u\big)dxdy + \int_{\Gamma_m} \frac{a}{b}\breve{u}u ds = \iint \breve{u}f dxdy + \int_{\Gamma_m} \breve{u}c ds$$

Using for example P2 elements for u and $\breve{u}$, we obtain the following discretized form (taking into account that $u = d \text{ on } \Gamma_d$):

$$Au = b$$

1/ Generate mesh

In DNS/Mesh:

FreeFem++ mesh.edp

2/ Base-flow

The base-flow is solution of the following non-linear equation:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0, \quad \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$$

with the following boundary conditions:

$$(u_0 = 1, v_0 = 0) \text{ on } \Gamma_{in}$$

$$(u_0 = 0, v_0 = 0) \text{ on } \Gamma_{wall}$$

$$\left(-p_0 n_x + \nu\left(n_x \partial_x u_0 + n_y \partial_y u_0\right) = 0, -p_0 n_y + \nu\left(n_x \partial_x v_0 + n_y \partial_y v_0\right) = 0\right) \text{ on } \Gamma_{out}$$

$$\left(\partial_y u_0 = 0, v_0 = 0\right) \text{ on } \Gamma_{lat}$$

The Newton iteration is based on successive solutions of:

$$(\mathcal{N}_w + \mathcal{L})\delta w = -\frac{1}{2}\mathcal{N}(w,w) - \mathcal{L}w \text{ where } \mathcal{N}_w \delta w = \begin{pmatrix} \delta u \cdot \nabla u + u \cdot \nabla \delta u \\ 0 \end{pmatrix}$$

with boundary conditions such that $w + \delta w$ satisfy the above mentioned boundary conditions.

Hence:

$$\begin{aligned} \delta u \partial_x u + \delta v \partial_y u + u \partial_x \delta u + v \partial_y \delta u + \partial_x \delta p - \nu\left(\partial_{xx}\delta u + \partial_{yy}\delta u\right) \\ = -u\partial_x u - v\partial_y u - \partial_x p + \nu\left(\partial_{xx} u + \partial_{yy} u\right) \end{aligned}$$

$$\begin{aligned} \delta u \partial_x v + \delta v \partial_y v + u \partial_x \delta v + v \partial_y \delta v + \partial_y \delta p - \nu\left(\partial_{xx}\delta v + \partial_{yy}\delta v\right) \\ = -u\partial_x v - v\partial_y v - \partial_y p + \nu\left(\partial_{xx} v + \partial_{yy} v\right) \end{aligned}$$

$$-\partial_x \delta u - \partial_y \delta v = \partial_x u + \partial_y v$$

with:

$$(\delta u = 1 - u, \delta v = -\mathrm{v}) \text{ on } \Gamma_{in}$$

$$(\delta u = -u, \delta v = -v) \text{ on } \Gamma_{wall}$$

$$\begin{aligned} (-\delta p n_x + \nu\left(n_x \partial_x \delta u + n_y \partial_y \delta u\right) \\ = p n_x - \nu\left(n_x \partial_x u + n_y \partial_y u\right), -\delta p n_y + \nu\left(n_x \partial_x \delta v + n_y \partial_y \delta v\right) \\ = p n_y - \nu\left(n_x \partial_x v + n_y \partial_y v\right) \text{ on } \Gamma_{out}) \end{aligned}$$

$$(\partial_y \delta u = -\partial_y u, \delta v = -v\ ) \text{ on } \Gamma_{lat}$$

Show that the weak form of these equations is (with $\breve{w}$ as the test-function satisfying $\breve{u} = \breve{v} = 0$ on Γ_{in} and Γ_{wall} and $\breve{v} = 0$ on Γ_{lat})

$$\iint \left(\breve{u}\left(\delta u\partial_x u + \delta v\partial_y u + u\partial_x\delta u + v\partial_y\delta u\right) + \breve{v}\left(\delta u\partial_x v + \delta v\partial_y v + u\partial_x\delta v + v\partial_y\delta v\right)\right.$$
$$- \delta p(\partial_x\breve{u} + \partial_y\breve{v}) + \nu\left(\partial_x\breve{u}\partial_x\delta u + \partial_y\breve{u}\partial_y\delta u + \partial_x\breve{v}\partial_x\delta v + \partial_y\breve{v}\partial_y\delta v\right)$$
$$\left.- \breve{p}(\partial_x\delta u + \partial_y\delta v)\right)dxdy = \iint \left(-\breve{u}\left(u\partial_x u + v\partial_y u\right) - \breve{v}\left(u\partial_x v + v\partial_y v\right)\right.$$
$$+ p\left(\partial_x\breve{u} + \partial_y\breve{v}\right) - \nu\left(\partial_x\breve{u}\partial_x u + \partial_y\breve{u}\partial_y u + \partial_x\breve{v}\partial_x v + \partial_y\breve{v}\partial_y v\right) + \breve{p}(\partial_x u$$
$$\left. + \partial_y v)\right)dxdy$$

After discretization (taking into account all the Dirichlet boundary-conditions), we obtain:

$$A\delta w = b$$

In DNS/BF:

FreeFem++ init.edp

FreeFem++ newton.edp

3/ Direct numerical simulation of cylinder flow at Re=100

We solve the unsteady Navier-Stokes equations in perturbative form ($w \coloneqq w_0 + w$) around a cylinder flow at $Re = \nu^{-1} = 100$. The initial condition is a small amplitude random flowfield. The governing equations are:

$$\mathcal{B}\partial_t w + \mathcal{N}_{w_0} w + \mathcal{L}w = -\frac{1}{2}\mathcal{N}(w, w)$$

with $\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathcal{N}_{w_0} w = \mathcal{N}(w_0, w)$.

A first –order semi-implicit discretization in time yields:

$$\mathcal{B}\frac{w^{n+1} - w^n}{\Delta t} + +\mathcal{N}_{w_0} w^{n+1} + \mathcal{L}w^{n+1} = -\frac{1}{2}\mathcal{N}(w^n, w^n)$$

This may be re-arranged into:

$$\left(\frac{\mathcal{B}}{\Delta t} + \mathcal{N}_{w_0} + \mathcal{L}\right) w^{n+1} = \frac{\mathcal{B}w^n}{\Delta t} - \frac{1}{2}\mathcal{N}(w^n, w^n)$$

Show that the weak form with $\breve{w}$ as the test-function is:

$$\iint \left(\breve{u}\left(\frac{u^{n+1}}{\Delta t} + u^{n+1}\,\partial_x u_0 + v^{n+1}\partial_y u_0 + u_0\partial_x u^{n+1} + v_0\partial_y u^{n+1}\right) - (\partial_x \breve{u})p^{n+1} \right.$$
$$+ \nu\left(\partial_x \breve{u}\partial_x u^{n+1} + \partial_y \breve{u}\partial_y u^{n+1}\right)$$
$$+ \breve{v}\left(\frac{v^{n+1}}{\Delta t} + u^{n+1}\partial_x v_0 + v^{n+1}\partial_y v_0 + u_0\partial_x v^{n+1} + v_0\partial_y v^{n+1}\right) - \left(\partial_y \breve{v}\right)p^{n+1}$$
$$\left. + \nu\left(\partial_x \breve{v}\partial_x v^{n+1} + \partial_y \breve{v}\partial_y v^{n+1}\right) + \breve{p}\left(\partial_x u^{n+1} + \partial_y v^{n+1}\right)\right) dxdy$$
$$= \iint \left(\frac{\breve{u}u^n}{\Delta t} - \breve{u}(u^n \cdot \nabla u^n) + \frac{\breve{v}v^n}{\Delta t} - \breve{v}(v^n \cdot \nabla v^n)\right) dxdy$$

After spatial discretization, we obtain:

$$Aw^{n+1} = b$$

In DNS/DNS:

FreeFem++ init.edp // generate initial condition from small amplitude random noise

FreeFem++ dns.edp // solve Navier-Stokes equations

Octave plotlinlog('out_0.txt',1,2,1) // represent energy as a function of time in fig 1

Octave plotlinlin('out_0.txt',1,4,2) // represent v velocity as a function of time

FreeFem++ plotUvvp.edp // plot snapshot at iteration i=101, 201, 301, …

4/ Van der Pol Oscillator: multiple time-scale analysis

The Van der Pol Oscillator corresponds to the following governing equations:

$$w'' + \omega_0^2 w = 2\tilde{\delta}w' - w^2 w'$$

$$w(0) = w_I, w'(0) = 0$$

where the $(\cdot)'$ is the time-derivative, w_I is the initial condition, ω_0 the frequency and $\tilde{\delta}$ the instability strength. Here, we choose: $\omega_0 = 10$, $\tilde{\delta} = 0.3$ and $w_I = 0.01$.

4a/ Numerical time-integration

We integrate in time the above equations. For this,

In VanDerPol:

Octave pkg load all // load external packages for time integration, Fourier analysis, etc.

Octave vdp // integrate in time unforced Van der Pol equations

4b/ One time-scale approach

We try to approximate the solution by considering a small instability strength: $\tilde{\delta} = \delta\epsilon$, with $\epsilon \ll 1$ and $\delta = O(1)$. We look for an approximation of the solution with an expansion of the form:

$$w = \epsilon^{\frac{1}{2}} y \text{ and } y = y_0 + \epsilon y_1 + \cdots.$$

We first try with only one time-scale: $y(t) = y_0(t) + \epsilon y_1(t) + \cdots$

The second-order solution is given by:

$$w = \left(\tilde{A}e^{i\omega_0 t} + \text{c.c.}\right) + \left(\frac{-3\tilde{A}^3 + 12\tilde{\delta}\tilde{A}}{8\omega_0} i e^{i\omega_0 t} + \frac{\mathrm{i}\tilde{A}^3}{8\omega_0} e^{3i\omega_0 t} - \left(2\tilde{\delta}\tilde{A} - \tilde{A}^3\right)\left(\frac{1 + 2i\omega_0 t}{4\omega_0}\right) i e^{i\omega_0 t} + \text{c.c.}\right)$$

$$\tilde{A} = \frac{w_I}{2}$$

To represent this solution, in VanDerPol:

Octave clf // clear all figures

Octave vdp // integrate in time unforced Van der Pol equations

Octave vdp_tlr // show first and second order approximations with one time-scale

4c/ Two time-scales approach

The two time-scale first-order solution is given by:

$$w(t) = \left(\tilde{A}e^{i\omega_0 t} + \text{c.c.}\right)$$

with:

$$\frac{d\tilde{A}}{dt} = \tilde{\delta}\tilde{A} - \frac{1}{2}\tilde{A}^3$$

$$\tilde{A}(0) = \frac{w_I}{2}$$

To represent this solution, in VanDerPol:

Octave clf // clear all figures

Octave vdp // integrate in time unforced Van Der Pol equations

Octave vdp_tlr // show first and second order approximations with one time-scale

Octave vdp_mts // show first and second order approximations with two time-scales

5/ Global modes of linearized Navier-Stokes equations

The global modes are the structures such that:

$$\lambda\mathcal{B}\widehat{w}+\left(\mathcal{N}_{w_0}+\mathcal{L}\right)\widehat{w}=0,\quad \mathcal{B}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

where $\left(\mathcal{N}_{w_0}+\mathcal{L}\right)$ is the linearized Navier-Stokes operator:

$$\left(\mathcal{N}_{w_0}+\mathcal{L}\right)\widehat{w}=\begin{pmatrix}\hat{u}\partial_x u_0+\hat{v}\partial_y u_0+u_0\partial_x\hat{u}+v_0\partial_y\hat{u}+\partial_x\hat{p}-\nu\left(\partial_{xx}\hat{u}+\partial_{yy}\hat{u}\right)\\ \hat{u}\partial_x v_0+\hat{v}\partial_y v_0+u_0\partial_x\hat{v}+v_0\partial_y\hat{v}+\partial_y\hat{p}-\nu\left(\partial_{xx}\hat{v}+\partial_{yy}\hat{v}\right)\\ -\left(\partial_x\hat{u}+\partial_y\hat{v}\right)\end{pmatrix}$$

$\left(\mathcal{N}_{w_0}+\mathcal{L}\right)$ acts on a subspace of functions $\widehat{w}$ satisfying the following boundary conditions $\widehat{(*)}$

$$(\hat{u}=0,\hat{v}=0)\ \text{ on }\Gamma_{in}\text{ and }\Gamma_{wall}$$

$$\left(-\hat{p}n_x+\nu\left(n_x\partial_x\hat{u}+n_y\partial_y\hat{u}\right)=0,-\hat{p}n_y+\nu\left(n_x\partial_x\hat{v}+n_y\partial_y\hat{v}\right)=0\right)\ \text{ on }\Gamma_{out}$$

$$\left(\partial_y\hat{u}=0,\hat{v}=0\right)\ \text{ on }\Gamma_{lat}$$

Show that the weak form of these equations is (with $\breve{w}$ as the test-function satisfying $\breve{u}=\breve{v}=0$ on Γ_{in} and Γ_{wall} and $\breve{v}=0$ on Γ_{lat}):

$$\begin{aligned}\iint\ \Big(&\breve{u}\left(-\hat{u}\partial_x u_0-\hat{v}\partial_y u_0-u_0\partial_x\hat{u}-v_0\partial_y\hat{u}\right)+(\partial_x\breve{u})\hat{p}-\nu\left(\partial_x\breve{u}\partial_x\hat{u}+\partial_y\breve{u}\partial_y\hat{u}\right)\\ &+\breve{v}\left(-\hat{u}\partial_x v_0-\hat{v}\partial_y v_0-u_0\partial_x\hat{v}-v_0\partial_y\hat{v}\right)+\left(\partial_y\breve{v}\right)\hat{p}-\nu\left(\partial_x\breve{v}\partial_x\hat{v}+\partial_y\breve{v}\partial_y\hat{v}\right)\\ &+\breve{p}\left(\partial_x\hat{u}+\partial_y\hat{v}\right)\Big)\,dxdy=\lambda\iint\ (\breve{u}\hat{u}+\breve{v}\hat{v})dxdy\end{aligned}$$

With a finite element-discretization:

$$A\widehat{w}=\lambda B\widehat{w}$$

In folder DNS/Eigs:

FreeFem++ eigen.edp // compute and show direct global mode.

6/ Definition of adjoint operator.

The adjoint operator $\left(\widetilde{\mathcal{N}}_{w_0}+\tilde{\mathcal{L}}\right)$ is the operator satisfying for all $\widehat{w}$ and $\widetilde{w}$ the following relations:

$$<\widetilde{w},\left(\mathcal{N}_{w_0}+\mathcal{L}\right)\widehat{w}>=<\left(\widetilde{\mathcal{N}}_{w_0}+\tilde{\mathcal{L}}\right)\widetilde{w},\widehat{w}>$$

Here $\widehat{w}$ is in the subspace satisfying the boundary conditions $\widehat{(*)}$.

Determine the adjoint operator $\left(\widetilde{\mathcal{N}_{w_0}}+\tilde{\mathcal{L}}\right)$and the boundary conditions $\widetilde{(*)}$ that $\widetilde{w}$ satisfies.

Solution:

$$\left(\widetilde{\mathcal{N}_{w_0}}+\tilde{\mathcal{L}}\right)\widetilde{w}=\begin{pmatrix}-u_0\partial_x\tilde{u}-v_0\partial_y\tilde{u}+\tilde{u}\partial_x u_0+\tilde{v}\partial_x v_0+\partial_x\tilde{p}-\nu\left(\partial_{xx}\tilde{u}+\partial_{yy}\tilde{u}\right)\\ -u_0\partial_x\tilde{v}-v_0\partial_y\tilde{v}+\tilde{u}\partial_y u_0+\tilde{v}\partial_y v_0+\partial_y\tilde{p}-\nu\left(\partial_{xx}\tilde{v}+\partial_{yy}\tilde{v}\right)\\ -\left(\partial_x\tilde{u}+\partial_y\tilde{v}\right)\end{pmatrix}$$

$$(\tilde{u}=0,\tilde{v}=0)\ \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$\begin{aligned}\big(-\tilde{p}n_x+\nu\partial_x\tilde{u}n_x+\nu\partial_y\tilde{u}n_y=-\tilde{u}u_0n_x-\tilde{u}v_0n_y,-\tilde{p}n_y+\nu\partial_x\tilde{v}n_x+\nu\partial_y\tilde{v}n_y\\ =-\tilde{v}u_0n_x-\tilde{v}v_0n_y\big)\ \text{ on } \Gamma_{out}\end{aligned}$$

$$\left(\partial_y\tilde{u}=0,\tilde{v}=0\right)\ \text{ on } \Gamma_{lat}$$

7/ The adjoint global modes are solution of the following eigen-problem :

$$\lambda\mathcal{B}\widetilde{w}+\left(\widetilde{\mathcal{N}_{w_0}}+\tilde{\mathcal{L}}\ \right)\widetilde{w}=0$$

with the above mentioned boundary conditions.

Show that the weak form of these equations is:

$$\begin{aligned}\iint\ \Big(\breve{u}\left(u_0\partial_x\tilde{u}+v_0\partial_y\tilde{u}-\tilde{u}\partial_x u_0-\tilde{v}\partial_x v_0\right)+(\partial_x\breve{u})\tilde{p}-\nu\left(\partial_x\breve{u}\partial_x\tilde{u}+\partial_y\breve{u}\partial_y\tilde{u}\right)\\ +\breve{v}\left(u_0\partial_x\tilde{v}+v_0\partial_y\tilde{v}-\tilde{u}\partial_y u_0-\tilde{v}\partial_y v_0\right)+\left(\partial_y\breve{v}\right)\tilde{p}-\nu\left(\partial_x\breve{v}\partial_x\tilde{v}+\partial_y\breve{v}\partial_y\tilde{v}\right)\\ +\breve{p}\left(\partial_x\tilde{u}+\partial_y\tilde{v}\right)\Big)\,dxdy\\ -\int_{\Gamma_{out}}\breve{u}\left(\tilde{u}u_0n_x+\tilde{u}v_0n_y\right)ds-\int_{\Gamma_{out}}\breve{v}\left(\tilde{v}u_0n_x+\tilde{v}v_0n_y\right)ds\\ =\lambda\iint\ (\breve{u}\tilde{u}+\breve{v}\tilde{v})dxdy\end{aligned}$$

After discretization, we obtain:

$$\tilde{A}\widetilde{w}=\lambda B\widetilde{w}$$

In folder DNS/Eigs:

FreeFem++ eigenadj.edp // compute and show leading adjoint global mode.

8/ Weakly nonlinear solution of Navier-Stokes equations

We consider the Navier-Stokes equation in perturbative form ($w:=w_0+w$)::

$$\mathcal{B}\partial_t w+\mathcal{N}_{w_0}w+\mathcal{L}w=\tilde{\delta}\mathcal{M}(w_0+w)-\frac{1}{2}\mathcal{N}(w,w).$$

Here:

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix},\ \mathcal{L} = \begin{pmatrix} -\nu_c\Delta() & \nabla() \\ -\nabla\cdot() & 0 \end{pmatrix},\ \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}.$$

The viscosity ν has been replaced by $\nu = \nu_c - \tilde{\delta}$, where ν_c is the critical viscosity which achieves marginal stability of the linear dynamics $Re_c = \nu_c^{-1} = 46.6$.

The base-flow is given by:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0.$$

In the following, we consider a slightly supercritical regime (the Reynolds number is slightly above the critical Reynolds number):

$$\tilde{\delta} = \epsilon\delta,\ \epsilon \ll 1, \delta = O(1).$$

We look for an approximation of the solution under the form:

$$w = \epsilon^{\frac{1}{2}}\left(y_0(t, \tau = \epsilon t) + \epsilon^{\frac{1}{2}} y_{\frac{1}{2}}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots\right)$$

The second-order solution is given by:

$$w = \left(\tilde{A}e^{i\omega_c t} y_A + \text{c.c}\right) + \tilde{\delta} w_\delta + \left(\tilde{A}^2 e^{2i\omega_c t} y_{AA} + \text{c.c.}\right) + \left|\tilde{A}\right|^2 y_{A\bar{A}} + \cdots$$

With :

$$i\omega_c \mathcal{B} y_A + \mathcal{N}_{w_0} y_A + \mathcal{L} y_A = 0$$

$$\mathcal{N}_{w_0} y_\delta + \mathcal{L} y_\delta = \mathcal{M} y_0$$

$$2i\omega_c \mathcal{B} y_{AA} + \mathcal{N}_{w_0} y_{AA} + \mathcal{L} y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0} y_{A\bar{A}} + \mathcal{L} y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

And:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2$$

where:

$$\lambda = < \tilde{y}_A,\ \mathcal{M} y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$

$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$

$$-i\omega_c \mathcal{B}\tilde{y}_A + \widetilde{\mathcal{N}}_{w_0}\tilde{y}_A + \tilde{\mathcal{L}}\tilde{y}_A = 0$$

$$< \tilde{y}_A, \mathcal{B} y_A > \ = 1$$

In AmplEq/Mesh:

FreeFem++ mesh.edp // generate mesh

In AmplEq/BF:

FreeFem++ init.edp // generate initial guess for Newton iterations

FreeFem++ newton.edp // Newton iteration

In AmplEq/Eigs:

FreeFem++ eigen.edp // compute global mode

FreeFem++ eigenadj.edp // compute adjoint global mode

FreeFem++ norm.edp // generate scaled adjoint global mode

In AmplEq/WNL:

FreeFem++ udelta.edp // generate modification of base-flow due to increase in Reynolds number

FreeFem++ uAA.edp // generate second harmonic due to interaction of global mode with himself

FreeFem++ uAAb.edp// generate zero-harmonic due to interaction of global mode with adjoint of himself

FreeFem++ lambda.edp // compute λ coefficient of Stuart-Landau equation

FreeFem++ mu.edp // compute μ coefficient of Suart-Landau equation

FreeFem++ udeltaA.edp // compute λ coefficient (augmented method)

FreeFem++ uAAAb.edp // compute μ coefficient (augmented method)