Adjoint of linearized Navier-Stokes equations

First equation : $\iint \tilde{u}\left(\hat{u}\partial_x u_0 + \hat{v}\partial_y u_0 + u_0\partial_x\hat{u} + v_0\partial_y\hat{u} + \partial_x\hat{p} - \nu\left(\partial_{xx}\hat{u} + \partial_{yy}\hat{u}\right)\right)dxdy =$
$\iint \left(\tilde{u}\hat{u}\partial_x u_0 + \tilde{u}\hat{v}\partial_y u_0\right) + \iint \left(\tilde{u}u_0\partial_x\hat{u} + \tilde{u}v_0\partial_y\hat{u} + \tilde{u}\partial_x\hat{p} - \nu\left(\tilde{u}\partial_{xx}\hat{u} + \tilde{u}\partial_{yy}\hat{u}\right)\right)dxdy =$
$\iint \left(\tilde{u}\hat{u}\partial_x u_0 + \tilde{u}\hat{v}\partial_y u_0\right) + \int \left(\tilde{u}u_0 n_x\hat{u} + \tilde{u}v_0 n_y\hat{u} + \tilde{u}n_x\hat{p} - \nu\left(\tilde{u}n_x\partial_x\hat{u} + \tilde{u}n_y\partial_y\hat{u}\right)\right)ds -$
$\iint \left(\partial_x(\tilde{u}u_0)\hat{u} + \partial_y(\tilde{u}v_0)\hat{u} + (\partial_x\tilde{u})\hat{p} - \nu\left(\partial_x\tilde{u}\partial_x\hat{u} + \partial_y\tilde{u}\partial_y\hat{u}\right)\right)dxdy$

$$
\begin{aligned}
&= \iint \left(\tilde{u}\hat{u}\partial_x u_0 + \tilde{u}\hat{v}\partial_y u_0 - \partial_x(\tilde{u}u_0)\hat{u} - \partial_y(\tilde{u}v_0)\hat{u} - (\partial_x\tilde{u})\hat{p} - \nu\left(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\right)\hat{u}\right)dxdy \\
&\qquad + \int \left(\tilde{u}u_0 n_x\hat{u} + \tilde{u}v_0 n_y\hat{u} + \tilde{u}n_x\hat{p} - \nu\left(\tilde{u}n_x\partial_x\hat{u} + \tilde{u}n_y\partial_y\hat{u}\right)\right. \\
&\qquad \left. + \nu\left(\partial_x\tilde{u}n_x\hat{u} + \partial_y\tilde{u}n_y\hat{u}\right)\right)ds
\end{aligned}
$$

With second equation:

$$
\begin{aligned}
&= \iint \left(\tilde{v}\hat{u}\partial_x v_0 + \tilde{v}\hat{v}\partial_y v_0 - \partial_x(\tilde{v}u_0)\hat{v} - \partial_y(\tilde{v}v_0)\hat{v} - \left(\partial_y\tilde{v}\right)\hat{p} - \nu\left(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}\right)\hat{v}\right)dxdy \\
&\qquad + \int \left(\tilde{v}u_0 n_x\hat{v} + \tilde{v}v_0 n_y\hat{v} + \tilde{v}n_y\hat{p} - \nu\left(\tilde{v}n_x\partial_x\hat{v} + \tilde{v}n_y\partial_y\hat{v}\right)\right. \\
&\qquad \left. + \nu\left(\partial_x\tilde{v}n_x\hat{v} + \partial_y\tilde{v}n_y\hat{v}\right)\right)ds
\end{aligned}
$$

With third equation:

$$
\iint \tilde{p}\left(-\partial_x\hat{u} - \partial_y\hat{v}\right)dxdy = \int \left(\tilde{p}\left(-n_x\hat{u} - n_y\hat{v}\right)\right)ds + \iint \left((\partial_x\tilde{p})\hat{u} + \left(\partial_y\tilde{p}\right)\hat{v}\right)dxdy +
$$

Boundary term:

$$
\begin{aligned}
&\tilde{u}u_0 n_x\hat{u} + \tilde{u}v_0 n_y\hat{u} + \tilde{u}n_x\hat{p} - \nu\left(\tilde{u}n_x\partial_x\hat{u} + \tilde{u}n_y\partial_y\hat{u}\right) + \nu\left(\partial_x\tilde{u}n_x\hat{u} + \partial_y\tilde{u}n_y\hat{u}\right) + \tilde{v}u_0 n_x\hat{v} + \tilde{v}v_0 n_y\hat{v} \\
&\qquad + \tilde{v}n_y\hat{p} - \nu\left(\tilde{v}n_x\partial_x\hat{v} + \tilde{v}n_y\partial_y\hat{v}\right) + \nu\left(\partial_x\tilde{v}n_x\hat{v} + \partial_y\tilde{v}n_y\hat{v}\right) - \tilde{p}\left(n_x\hat{u} + n_y\hat{v}\right) = 0
\end{aligned}
$$

Rearranged into:

$$
\begin{aligned}
&\left(\tilde{u}u_0 n_x + \tilde{u}v_0 n_y - \tilde{p}n_x + \nu\partial_x\tilde{u}n_x + \nu\partial_y\tilde{u}n_y\right)\hat{u} + \left(\tilde{u}n_x + \tilde{v}n_y\right)\hat{p} - \nu\tilde{u}n_x\partial_x\hat{u} - \nu\tilde{u}n_y\partial_y\hat{u} \\
&\qquad + \left(\tilde{v}u_0 n_x + \tilde{v}v_0 n_y - \tilde{p}n_y + \nu\partial_x\tilde{v}n_x + \nu\partial_y\tilde{v}n_y\right)\hat{v} - \nu\tilde{v}n_x\partial_x\hat{v} - \nu\tilde{v}n_y\partial_y\hat{v} = 0
\end{aligned}
$$

Ccl:

On Γ_w and Γ_{in}: $\hat{u} = \hat{v} = 0 => \tilde{u} = \tilde{v} = 0$

On Γ_{lat} : $\partial_y\hat{u} = \hat{v} = 0$

$$
\left(\tilde{u}v_0 + \nu\partial_y\tilde{u}\right)\hat{u} + \tilde{v}\hat{p} - \nu\tilde{v}\partial_y\hat{v} = 0
$$

Hence: $\tilde{v} = \partial_y\tilde{u} = 0$ (since $v_0 = 0$)

On Γ_{out}:

$$
-\hat{p}n_x + \nu\left(n_x\partial_x\hat{u} + n_y\partial_y\hat{u}\right) = 0, -\hat{p}n_y + \nu\left(n_x\partial_x\hat{v} + n_y\partial_y\hat{v}\right) = 0 \text{ on } \Gamma_{out}
$$

$$
\begin{aligned}
&\left(\tilde{u}u_0 n_x + \tilde{u}v_0 n_y - \tilde{p}n_x + \nu\partial_x\tilde{u}n_x + \nu\partial_y\tilde{u}n_y\right)\hat{u} + \left(\tilde{v}u_0 n_x + \tilde{v}v_0 n_y - \tilde{p}n_y + \nu\partial_x\tilde{v}n_x + \nu\partial_y\tilde{v}n_y\right)\hat{v} \\
&\qquad = 0
\end{aligned}
$$

So that:

$$-\tilde{p}n_x + \nu\partial_x\tilde{u}n_x + \nu\partial_y\tilde{u}n_y = -\tilde{u}u_0n_x - \tilde{u}v_0n_y$$

$$-\tilde{p}n_y + \nu\partial_x\tilde{v}n_x + \nu\partial_y\tilde{v}n_y = -\tilde{v}u_0n_x - \tilde{v}v_0n_y$$

Equations:

$$= \iint \Big[\Big(\tilde{u}\partial_x u_0 + \tilde{v}\partial_x v_0 - \partial_x(\tilde{u}u_0) - \partial_y(\tilde{u}v_0) + \partial_x\tilde{p} - \nu\big(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\big)\Big)\hat{u} - \big(\partial_x\tilde{u} + \partial_y\tilde{v}\big)\hat{p} \\ + \Big(\tilde{v}\partial_y v_0 + \tilde{u}\partial_y u_0 - \partial_x(\tilde{v}u_0) - \partial_y(\tilde{v}v_0) - \nu\big(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}\big) + \big(\partial_y\tilde{p}\big)\Big)\hat{v}\Big]\,dxdy$$

Hence:

$$\tilde{u}\partial_x u_0 + \tilde{v}\partial_x v_0 - \partial_x(\tilde{u}u_0) - \partial_y(\tilde{u}v_0) + \partial_x\tilde{p} - \nu\big(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\big)$$

$$\tilde{v}\partial_y v_0 + \tilde{u}\partial_y u_0 - \partial_x(\tilde{v}u_0) - \partial_y(\tilde{v}v_0) + \partial_y\tilde{p} - \nu\big(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}\big)$$

$$-\big(\partial_x\tilde{u} + \partial_y\tilde{v}\big)$$

Or:

$$-u_0\partial_x\tilde{u} - v_0\partial_y\tilde{u} + \tilde{u}\partial_x u_0 + \tilde{v}\partial_x v_0 + \partial_x\tilde{p} - \nu\big(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\big)$$

$$-u_0\partial_x\tilde{v} - v_0\partial_y\tilde{v} + \tilde{u}\partial_y u_0 + \tilde{v}\partial_y v_0 + \partial_y\tilde{p} - \nu\big(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}\big)$$

$$-\big(\partial_x\tilde{u} + \partial_y\tilde{v}\big)$$

$$\big(\tilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\big)\tilde{w} = \begin{pmatrix} -u_0\partial_x\tilde{u} - v_0\partial_y\tilde{u} + \tilde{u}\partial_x u_0 + \tilde{v}\partial_x v_0 + \partial_x\tilde{p} - \nu\big(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\big) \\ -u_0\partial_x\tilde{v} - v_0\partial_y\tilde{v} + \tilde{u}\partial_y u_0 + \tilde{v}\partial_y v_0 + \partial_y\tilde{p} - \nu\big(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}\big) \\ -\big(\partial_x\tilde{u} + \partial_y\tilde{v}\big) \end{pmatrix}$$