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The hydrodynamic mechanism of whistling

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Key Words:

1. Introduction

Tone produced by jets, holes : human whistling, bird call, corrugated pipes etc...

This class of situations is characterized by a jet instability, but varicose mode (opposite to flutes : sinuous mode).

Controversy regarding the respective roles of the jet and the acoustic resonator for frequency selection : Rayleigh, Bouasse, ...

Controversy regarding the role of the vortices (Cyclones de Lootens...)

Billon, Valeau, & Sakout (2005) : show that the two kinds of feedback can actually occur in a related configuration (planar jet / slot).

Langthjem and Nakano (2010) : still think that the feedback is acoustic !

The objective is to clarify the respective roles of hydrodynamics and acoustics for the selection of frequency.

2. Global stability approach (incompressible case)

Spatial instability : for sure !

Global stability : one has to distinguish two cases according to the boundary conditions.

Case 1 : zero flow rate, at inlet, zero stress at outlet

We find global stability for the variety of cases considered : single hole, double hole (with or without lateral walls), 2 holes + wall in front, etc...

Case 2 : zero pressure (or more properly zero stress) at both inlet and outlet

In this case we do find instability ! But what does the zero-pressure condition mean ?

3. Modelling the coupling with a resonator : role of the impedance

Def. Impedance

Which one ? The better seems to me :

$Z = \Delta p/q_m$; where q_m = mass flow rate = $\rho s u$ where u is the mean velocity through the aperture. Dimension : $[Z] = m/s$.

Impedance of the upstream cavity :

$$Z_R = \frac{p_i}{q} = -\frac{c^2}{i\omega V}$$

Impedence of the outlet (matching with open space)

$$Z_O = \frac{p_s}{q} = -\frac{i\omega\delta}{cs} + \frac{\beta}{s}\left(\frac{R\omega}{c}\right)$$

where δ is the open-end correction, and β the radiation coefficient. In the case of an infinite flare : $\delta = 0.8R$, $\beta = 1/2$.

What remains to do is compute the impedance Z_a of the aperture.

3.1. *Classical modelling*

Classical modeling : Bernouilli

$$\rho\left(\frac{1}{l_a}\frac{\partial u}{\partial t} + \alpha U u\right) = p_i - p_s$$

Where α is a veina contracta coefficient.

This results in the following expression :

$$Z_a = \frac{p_i - p_s}{sq} = \frac{-\rho i\omega\delta}{cs} + \frac{\beta c}{s}\left(\frac{R\omega}{c}\right)^2$$

The total impedance is :

$$Z_t = Z_O + Z_A - Z_R$$

Which can also be written :

$$Z_t = \frac{-il_t}{\omega}\left(\omega^2 - \omega_0^2\right) + \left(\alpha U + \beta\omega^2 R^2/c\right)$$

With $l_t = l_a + \delta$, and $\omega_0^2 = \frac{c^2 s}{V l_t}$.

The real part is positive (resistive impedance) so the system is stable.

Assuming the real part to be small compared to the imaginary part, the natural complex frequency (eigenfrequency) is :

$$\omega = \omega_0 + \frac{iV\omega_0^2\left(\beta\omega_0^2 R^2 + \alpha U c\right)}{2sc^3}$$

3.2. *Computation of the impedance through a linearized approach*

For the two-hole configuration, we find negative impedance in some ranges of ω.

4. Modelling with a top-hat jet

We work here with a simple, inviscid model which considers the jet with a top-hat profile :

$$u(r) = \begin{cases} U & \text{if } r < R; \\ 0 & \text{if } r > R. \end{cases} \tag{4.1}$$

This corresponds to a cylindrical shear layer. The stability analysis of this flow consists of adding small perturbations in potential form, both inside (ϕ_o) and outside (ϕ_i) the jet. These perturbations are searched in eigenvalue form as follows :

$$\phi_i = A I_m(kr) e^{i(kx-\omega t)}; \quad \phi_o = B K_m(kr) e^{i(kx-\omega t)}; \quad \eta = C e^{i(kx-\omega t)} \tag{4.2}$$

Where $r = R + \eta$ is the location of the jet edge.

The matching conditions at $r = R$ are continuity of the pressure ($p_i = p_o$) , and kinematical conditions connecting the temporal derivative of η to the radial velocity $\partial\phi/\partial r$. Hence :

$$i(\omega - kU)\phi_i = i\omega\phi_o,$$
$$-i\omega\eta = \partial\phi_o/\partial r,$$
$$i(kU - \omega)\eta = \partial\phi_i/\partial r.$$

Eliminating constants A, B, C, we get the following dispersion relation :

$$D(\omega, k, m) = (\omega - kU)^2 + L_m(kR)\omega^2 = 0 \tag{4.3}$$

Where

$$L_m(k) = -\frac{I'_m(k)K_m(k)}{I_m(k)K'_m(k)}$$

Note that this dispersion relation generalizes the classical one for Kelvin-Helmholtz instability of a infinitely thin shear layer (obtained by replacing $L_m(k)$ by one). It is also close to the one obtained for a planar jet of width $2R$. The latter case has two kinds of modes : varicose (with $L_v(k) = tanh(k)$) and sinuous (with $L_s(k) = cotanh(k)$).

In the present case, $L_m(k)$ is always close to 1 for $m > 0$. The case $m = 0$ differs : $L_m(k)$ tends to 1 for k large, but tends to zero for small k with the following asymptotic behavior :

$$L_0(k) \approx \frac{k^2}{2}\left(\log\frac{2}{k} - \gamma\right)$$

4.1. *Temporal stability analysis*

The temporal analysis gives :

$$c = \frac{\omega}{k} = U\frac{1 \pm i\sqrt{L}}{1 + L}$$

Note that for $m > 0$, L is always close to 1 so the real part of c is close to $U/2$, in accordance with the classical case.

For $m = 0$, the real part of c also tends to $U/2$ for large k, but for small k it tends to U, meaning that long-wavelength varicose perturbation propagate with the velocity of the jet. Note that varicose perturbations of a planar jet also display this feature.

4.2. *Spatial stability analysis*

The spatial analysis (solving for k as function of ω real) classically yields two kinds of solutions. The branches called k^+ are relevant to perturbations propagating in the downstream direction, while the branches called k^- are relevant to perturbations propagating in the upstream direction.

Figure 1 shows the two main branches of solutions found with negative k_i, for $m = 0$ and $m = 1$. The branch k^+ corresponds to a perturbation which is amplified in the downstream direction, while the branch k^- is a perturbation which is damped in the upstream direction.

4.3. *Spatial stability analysis of a forced jet passing through a second aperture*

We have to use both k^+ and k^- branches to solve the problem.

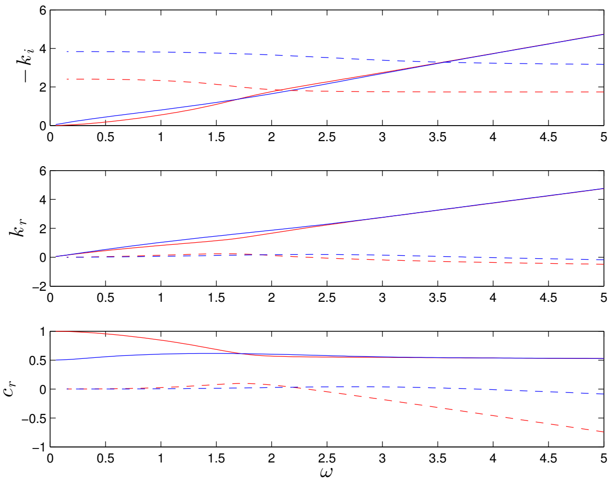

FIGURE 1. Properties of the spatial branches for $m = 0$ (red) and $m = 1$ (blue). Plain line : branch k^+ ; dashed line : branch k^-. Upper plot : spatial amplification rate $-k_i$; Middle plot : spatial oscillation rate k_r ; Lower plot : real part of the wave velocity $c_r = Re(\omega/k)$.